Solid mechanics equations for experiments

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Revision History

Revision	**Date**	**Author(s)**	**Description**
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1 Introduction

This document aims to regroup the equations of mechanics traditionally used in experiments. A notation is proposed. It also serves at support for the functions developped in Basin Tools project.

2 Example of a model

The segmented hull model is presented in Figure 1.

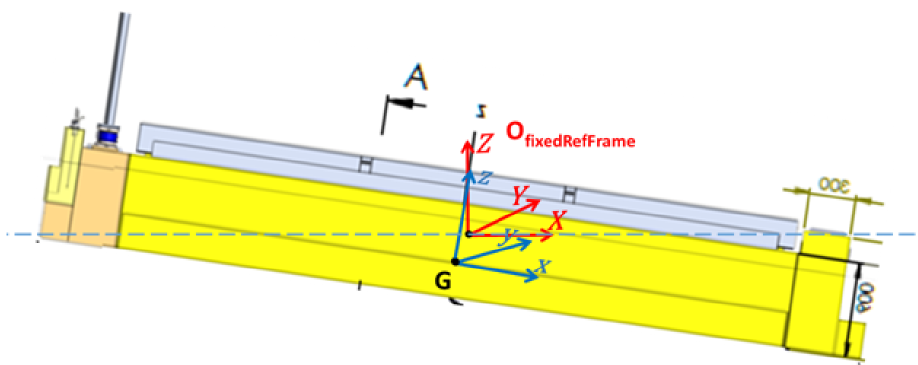

Figure 1: Sketch of the BGF platform

3 Solid kinematics

3.1 Reference system

The galilean reference system is given as $(O, \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, the body-fixed reference system R_b is given as $(O_b, \mathbf{x}, \mathbf{y}, \mathbf{z})$. The three rotations Yaw ψ, Pitch θ and Roll ϕ angles are Tait-Bryan angles.

3.2 Vector notation

The mathematical notation that allows to identify position, velocity and acceleration of different of points of interest of the mock-up must establish to express them in different frames. For instance, for a generic point of interest P on the mock-up (the purpose of P and $R_f(O_f, \mathbf{x_f}, \mathbf{y_f}, \mathbf{z_f})$ in this explanation are only use to described the sample notation):

1. $\mathbf{r}_p^f$ denotes the position of P with respect to a frame R_f:

$$\mathbf{r}_P^f = \mathbf{O_f P} \tag{1}$$

$$\mathbf{r}_P^f = x_P^f \mathbf{x}_f + y_P^f \mathbf{x}_y + z_P^f \mathbf{x}_z \equiv \begin{bmatrix} x_P^f \\ y_P^f \\ z_P^f \end{bmatrix} = [x_P^f, y_P^f, z_P^f]^T \tag{2}$$

2. $\mathbf{v}_P^f$ denotes the velocity of P with respect to a frame R_f.

$$\mathbf{v}_P^f = \mathbf{v}(P/R_f) = \left.\frac{d\mathbf{O_f P}}{dt}\right|_{R_f} \tag{3}$$

3. $\dot{\mathbf{v}}_P^f$ denotes the acceleration of P with respect to a frame R_f
4. $\mathbf{\Theta}_{0f}$ is vector of Euler angles Tait bryan that take the R_0 into the orientation of R_f
5. ω_{0f}^f denotes the relative angular velocity of the R_f with respect to R_0, decomposed in the R_f An other notation could be $\omega_{0f} = \omega(R_f/R_0)$

With those notations the

$$\mathbf{v}_P^f = \mathbf{v}(P/R_f) = \left.\frac{d\mathbf{O_f P}}{dt}\right|_{R_f} \tag{4}$$

The transformation of derivation basis is given with:

$$\left.\frac{d\mathbf{OP}}{dt}\right|_{R_0} = \left.\frac{d\mathbf{OP}}{dt}\right|_{R_f} + \omega_{0f} \times \mathbf{OP} \tag{5}$$

If A and B are two points fixed with respect to the reference frame R_f what happens for moving point in f

$$\mathbf{v}_B^0 = \left.\frac{d\mathbf{OB}}{dt}\right|_{R_0} = \left.\frac{d\mathbf{OA}}{dt}\right|_{R_0} + \omega_{0f} \times \mathbf{AB} \tag{6}$$

3.3 Rotations

The mock-up attitude orientation is defined by the orientation of the body-fixed reference system R_b relative to R_0. This is given by three instrinsic rotations that take the R_0 into R_b defined by three angles roll ϕ, pitch θ, and yaw ψ. These rotations are called Tait-Bryan angles or Euler angles defined as:

$$\mathbf{\Theta}_{0b} \triangleq \begin{bmatrix} \psi \\ \theta \\ \phi \end{bmatrix} \tag{7}$$

The vector coordinates between different frames can be transform via appropriate matrices. Following [?], the generic vector $\mathbf{t}$ can be address either in frame 0 or the frame b as:

$$\mathbf{t} = \begin{bmatrix} x_t^0 \\ y_t^0 \\ z_t^0 \end{bmatrix}^0 = \begin{bmatrix} x_t^b \\ y_t^b \\ z_t^b \end{bmatrix}^b \tag{8}$$

This lead to the transformation matrix with notation $R_{b0} \triangleq R_{b0}(\mathbf{\Theta}_{0b})$, which can be expressed in the b-frame to the 0-frame as:

$$\mathbf{t}^0 = R_{b0} \cdot \mathbf{t}^b \tag{9}$$

where the rotation matrix R_{b0} is obtained with three consecutive rotations about the successive axes obtained during intrisic rotations:

$$R_{0b} \triangleq \mathbf{R}_{x_b,\phi}\,\mathbf{R}_{y',\theta}\,\mathbf{R}_{z_0,\psi} \tag{10}$$

Where [?],

$$\mathbf{R}_{x_b,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}, \quad \mathbf{R}_{y',\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad \mathbf{R}_{z_0,\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$R_{b0} = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\cos\phi\sin\theta \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\phi\sin\theta\sin\psi & -\cos\psi\sin\phi + \sin\phi\cos\phi\sin\theta \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \tag{12}$$

And,

$$R_{b0} = R_{0b}^{-1} = R_{0b}^T \tag{13}$$

The transformation of velocities $\mathbf{v^b_{O_b}} = (u, v, w)$ of origin of expressed in R_b and $\mathbf{v^0_{O_b}}$ the time derivative of the position in R_0 can be expressed as:

$$\begin{bmatrix} \dot{x_{O_b}} \\ \dot{y_{O_b}} \\ \dot{z_{O_b}} \end{bmatrix}^0 = R_{b0} \begin{bmatrix} u \\ v \\ w \end{bmatrix}^b \tag{14}$$

The vector of angular velocity ω_{0b}^b in the fixed body frame, b-frame related to the time rate of change of the Euler angles $\mathbf{\Theta}_{0b}$ can be expressed as:

$$\dot{\mathbf{\Theta}}_{0b} = \mathbf{T}_{b0}\, \omega_{0b}^b, \quad \text{or} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = T_{b0} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{15}$$

The notation $\mathbf{T}_{b0}$ has to be understood that it transformsan angular velocity expressed clearly in R_b in a velocity expressed in the construction basis so “b0”.

The transformation matrix $\mathbf{T}_\Theta(\mathbf{\Theta}_{0b})$ can be derived from [**?**]:

$$\omega_{0b}^b = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}_{x_b,\phi}^T \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{R}_{x_b,\phi}^T\, \mathbf{R}_{y',\theta}^T \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \triangleq \mathbf{T}_{0b} \dot{\mathbf{\Theta}}_{0b} \tag{16}$$

Where $\mathbf{T}_{b0}$ is the transformation matrix and its inverse given by:

$$\mathbf{T}_{b0} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix}, \quad \mathbf{T}_{0b} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \tag{17}$$

3.3.1 I don’t know where this goes

Therefore, the position orientation vector is defined as:

$$\eta \triangleq \begin{bmatrix} \mathbf{r}_{O_b}^0 \\ \mathbf{\Theta}_{0b} \end{bmatrix} = [x_G, y_G, z_G, \psi, \theta, \phi]^T \tag{18}$$

While the linear and angular velocity vector of the body are conveniently expressed in b-frame as:

$$\nu \triangleq \begin{bmatrix} \mathbf{v}_G^b \\ \omega_{0b}^b \end{bmatrix} = [u, v, w, p, q, r]^T \tag{19}$$

Where $\mathbf{v}_G^b = [u, v, w]^T$ is the linear velocity of the point G expressed in the b-frame, and $\omega_{0b}^b = [p, q, r]^T$ is the angular velocity of the b-frame with respect to 0-frame expressed in the b-frame. Expression of p, q, r is detailed in the next section.

The velocity $\mathbf{v}^0$ of any point of the solid in the galilean frame is expressed with:

$$\mathbf{v}^0 = \mathbf{v}_G^0 + \omega_{0b}^b \times \mathbf{r}^b \tag{20}$$

3.4 Velocity transformations

3.4.1 Definitions

The movement of a solid 'b' with respect to a reference frame R_0 defined by the set of rotations R_{b0} and translations $\mathbf{d}(t)$ assembled into the homogeneous transformation $[T(t)] = [R_{b0}(t)\,\mathbf{d}(t)]$.

$\mathbf{d}(t)$ corresponds to the position of the origin of the moving frame in R_0 . $\mathbf{v}^0_{O_b} = \dot{\mathbf{d}}$ and $\mathbf{a}^0_{O_b} = \ddot{\mathbf{d}}$ are respectively the velocity and the acceleration of the origin O_b of the moving frame R_b.

3.4.2 Position

If $\mathbf{r^b_P}$' are the coordinates of a point "P" attached to the solid body measured in the moving reference frame R_b, then the trajectory of this point traced in R_0' is given by:

$$\mathbf{r^0_P}(t) = [T(t)]\mathbf{r^b_P}(t) \tag{21}$$

It is convenient to expand T in the following way.

$$\begin{bmatrix} \mathbf{r^0_P} \\ 1 \end{bmatrix} = \begin{bmatrix} R_{b0} & \mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r^b_P} \\ 1 \end{bmatrix}. \tag{22}$$

This equation for the trajectory of "P" can be inverted to compute the coordinate vector "'p"' in R_b as:

$$\mathbf{r^b_P} = [T(t)]^{-1}\mathbf{r^0_P} \tag{23}$$

$$\begin{bmatrix} \mathbf{r^b_P} \\ 1 \end{bmatrix} = \begin{bmatrix} R^T_{b0} & -R^T_{b0}\mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r^0_P} \\ 1 \end{bmatrix}. \tag{24}$$

This expression uses the fact that the transpose of a rotation matrix is also its inverse, that is:

$$R^T_{b0}\, R_{b0} = I. \tag{25}$$

3.4.3 Velocity

The velocity of the point "P" along its trajectory "'P"'(t) is obtained as the time derivative of this position vector

$$\mathbf{v}^0_P = [\dot{T}(t)]\mathbf{r^b_P} \tag{26}$$

$$\begin{bmatrix} \mathbf{v}^0_P \\ 0 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} R_{b0} & \mathbf{d}(t) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r^b_P} \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{R_{b0}} & \dot{\mathbf{d}}(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r^b_P} \\ 1 \end{bmatrix}. \tag{27}$$

This would mean $\frac{d\mathbf{r^b_P}}{dt} = 0$ so that means P is fixed in R_b The dot denotes the derivative with respect to time; because "'p"' is constant, its derivative is zero.

$$\mathbf{v^0_P} = [\dot{T}(t)][T(t)]^{-1}\mathbf{r^0_P}(t) \tag{28}$$

$$
\begin{bmatrix} \mathbf{v_P^0} \\ 0 \end{bmatrix} = \begin{array}{l}
\begin{bmatrix} \dot{R_{b0}} & \dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{b0} & \mathbf{d} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{r_P^0}(t) \\ 1 \end{bmatrix} = \\
\begin{bmatrix} \dot{R_{b0}} & \dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} R^{-1} \begin{bmatrix} 1 & -\mathbf{d} \\ 0 & R_{b0} \end{bmatrix} \begin{bmatrix} \mathbf{r_P^0}(t) \\ 1 \end{bmatrix} = \\
\begin{bmatrix} \dot{R_{b0}} R_{b0}^{-1} & -\dot{R_{b0}} R_{b0}^{-1}\mathbf{d} + \dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r_P^0}(t) \\ 1 \end{bmatrix} = \\
\begin{bmatrix} \dot{R_{b0}} R_{b0}^{T} & -\dot{R_{b0}} R_{b0}^{T}\mathbf{d} + \dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r_P^0}(t) \\ 1 \end{bmatrix}
\end{array} \tag{29}
$$

$$
\mathbf{v_P^0} = [S]\mathbf{r_P^0}. \tag{30}
$$

The matrix [S] is given by:

$$
[S] = \begin{bmatrix} \Omega & -\Omega\mathbf{d} + \dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} \tag{31}
$$

where

$$
[\Omega] = \dot{R_{b0}} R_{b0}^T, \tag{32}
$$

is the angular velocity matrix.

Multiplying by the operator [S], the formula for the velocity $\mathbf{v_P^0}$ takes the form:

$$
\mathbf{v_P^0} = \Omega(\mathbf{r_P^0} - \mathbf{d}) + \dot{\mathbf{d}} = \omega_{0b} \times \mathbf{r_P^b} + \mathbf{v}_{O_b}^0, \tag{33}
$$

where the vector ω_{0b} is the angular velocity vector obtained from the components of the matrix Ω; the vector

$$
\mathbf{r^P}_{R_b} = \mathbf{r_P^0} - \mathbf{d}, \tag{34}
$$

is the position of "P" relative to the origin "O" of the moving frame R_b.

3.4.4 Acceleration

The acceleration of a point "P" in a moving body "B" is obtained as the time derivative of its velocity vector:

$$
\mathbf{a}_P^0 = \frac{d}{dt}\mathbf{v_P^0} = \frac{d}{dt}\big([S]\mathbf{r_P^0}\big) = [\dot{S}]\mathbf{r_P^0} + [S]\dot{\mathbf{r}}_\mathbf{P}^\mathbf{0} = [\dot{S}]\mathbf{r_P^0} + [S][S]\mathbf{r_P^0}. \tag{35}
$$

This equation can be expanded firstly by computing

$$
[\dot{S}] = \begin{bmatrix} \dot{\Omega} & -\dot{\Omega}\mathbf{d} - \Omega\dot{\mathbf{d}} + \ddot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \dot{\Omega} & -\dot{\Omega}\mathbf{d} - \Omega\mathbf{v}_{O_b}^0 + \mathbf{a}_{O_b}^0 \\ 0 & 0 \end{bmatrix} \tag{36}
$$

and

$$
[S]^2 = \begin{bmatrix} \Omega & -\Omega\mathbf{d} + \mathbf{v}_{O_b}^0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} \Omega^2 & -\Omega^2\mathbf{d} + \Omega\mathbf{v}_{O_b}^0 \\ 0 & 0 \end{bmatrix}. \tag{37}
$$

The formula for the acceleration $\mathbf{A}_P$ can now be obtained as:

$$
\mathbf{a}_P^0 = \dot{\Omega}(\mathbf{r_P^0} - \mathbf{d}) + \mathbf{a}_{O_b}^0 + \Omega^2(\mathbf{r_P^0} - \mathbf{d}), \tag{38}
$$

or

$$\mathbf{a}_P^0 = \mathbf{a}_{O_b}^0 + \frac{d\omega_{0b}}{dt} \times \mathbf{r^P}_{R_b} + \omega_{0b} \times \omega_{0b} \times \mathbf{r^P}_{R_b}, \tag{39}$$

where $\frac{d\,\omega_{0b}}{dt}$ is the angular acceleration vector obtained from the derivative of the angular velocity matrix;

3.5 Newton/Euler equation of motion on a moving frame

The 6DOF motions of a generic body are computed by solving the standard Euler law in the body-fixed reference frame R_b with the origin O not coinciding with the center of gravity G, which position is defined by the vector $\mathbf{r^G}$ ([1]):

$$\begin{aligned}
&m\left(\frac{d\mathbf{V}_{R_b}}{dt} + \mathbf{\Omega}_{R_b} \times \mathbf{V}_{R_b} + \frac{d\mathbf{\Omega}_{R_b}}{dt} \times \mathbf{r^G}_{R_b} + \mathbf{\Omega}_{R_b} \times (\mathbf{\Omega}_{R_b} \times \mathbf{r^G}_{R_b})\right) = \mathbf{F}_{R_b} \\
&I\frac{d\mathbf{\Omega}_{R_b}}{dt} + \mathbf{\Omega}_{R_b} \times I\mathbf{\Omega}_{R_b} + m\mathbf{r^G}_{R_b} \times \left(\frac{d\mathbf{V}_{R_b}}{dt} + \mathbf{\Omega}_{R_b} \times \mathbf{V}_{R_b}\right) = \mathbf{M}_{R_b}
\end{aligned} \tag{40}$$

where m and I are the mass and moments of inertia tensor of the hull, respectively. $\mathbf{V}_{R_b} = (u,v,w)^T$ at the origin O and $\mathbf{\Omega}_{R_b} = (p,q,r)^T$ are the vectors of translational (surge, sway, heave) and angular velocities (roll, pitch, yaw). $\mathbf{F}_{Rb}^{ext} = (F_x, F_y, F_z)^T$ and $\mathbf{M}_{Rb}^{ext} = (M_x, M_y, M_z)^T$ are the vectors of force and moment with respect to O acting on the body. $\mathbf{r}_{R_b}^G = (x_G, y_G, z_G)$ is the position of the center of gravity G in the body fixed coordinate system. Note that the superscript T denotes the transpose of a vector or a matrix. $\mathbf{F}_{Rb}^{ext}$ and $\mathbf{M}_{Rb}^{ext}$ represents the hydrostatic and hydrodynamic forces and moments together with the weight.

3.6 Relation between the load balance measurements in segments and shear stress and bending moments

With the experimental setup used the ATI balance measures directly the shear force and the bending moment at the intersegment 4. However, the load balances c_i installed on each segment between the beam b_i and the hull segment h_i (see Figure **??**) at the position O_i $(x_{si}, 0, z_{si})$ does not directly measure the external hydrodynamic and hydrostatic forces and moments acting on this segment. For each segment, the application of Euler/Newton equation of motion (Eq. 40) on h_i which is everything below the sensor (and contain the hull) gives:

$$\begin{aligned}
&m_{h_i}(\frac{d\mathbf{V_{h_i}}}{dt} + \mathbf{\Omega} \times \mathbf{V_{h_i}}) = \mathbf{P}_{h_i} + \mathbf{F}_{hydro/h_i} + \mathbf{F}_{c_i/h_i} \quad \text{for each segment } i \\
&I_{h_i}\frac{d\mathbf{\Omega}}{dt} + \mathbf{\Omega} \times I_{h_i}\mathbf{\Omega} = \mathbf{M}_{P_{h_i}} + \mathbf{M}_{hydro/h_i} + \mathbf{M}_{c_i/h_i}^{G_h} \quad \text{for each segment } i
\end{aligned} \tag{41}$$

where m_{h_i} and I_{h_i} are mass and inertia tensor for h_i, $\mathbf{P}_{h_i}$ and $\mathbf{M}_{P_{h_i}}$ are the weight and the moment exerted by the weight in O_i, $\mathbf{F}_{hydro/h_i}$ and $\mathbf{M}_{hydro/h_i}$ the hydrodynamic force and moments. $\mathbf{F}_{c_i/h_i}$ and $\mathbf{M}_{c_i/h_i}^{G_h}$ are related to the measurement on the sensor $\mathbf{T}_{c_i}$ defined as:

$$\begin{aligned}
&\mathbf{F}_{c_i/b_i} = -\mathbf{F}_{c_i/h_i} = \mathbf{T}_{c_i} = T_{c_i}\mathbf{z} \\
&\mathbf{M}_{c_i/b_i} = -\mathbf{M}_{c_i/h_i}^{O_i} = \mathbf{R}_{c_i} = R_{c_i}^y\mathbf{y} + R_{c_i}^x\mathbf{x}
\end{aligned} \tag{42}$$

with

$$\mathbf{M}_{c_i/h_i}^{G_h} = \mathbf{M}_{c_i/h_i}^{O_i} + \mathbf{G_h O_i} \times \mathbf{F}_{c_i/h_i} \tag{43}$$

This leads to the formulation of hydrodynamic forces and moments acting on each segment:

$$\begin{aligned} \mathbf{F}_{hydro/h_i} &= m_{h_i}(\frac{d\mathbf{V_{h_i}}}{dt} + \mathbf{\Omega} \times \mathbf{V_{h_i}}) - \mathbf{P}_{h_i} + \mathbf{T}_{c_i} \\ \mathbf{M}_{hydro/h_i} &= I_{h_i}\frac{d\mathbf{\Omega}}{dt} + \mathbf{\Omega} \times I_{h_i}\mathbf{\Omega} + \mathbf{R}_{c_i} - \mathbf{M}_{P_{h_i}} \end{aligned} \tag{44}$$

3.7 Linear momentum

General definition of momentum $\mathbf{r_P^b}$ of the solid R is given as:

$$\mathbf{r_P^b} = \int_R \mathbf{v}^0 \,\mathrm{dm} \tag{45}$$

where $\mathbf{v}^0$ is the velocity of the elementary element dm in the galilean reference frame. The momentum at the center of gravity can be establish as:

$$\begin{aligned} \mathbf{r_P^b} &= \int_R \mathbf{v} \,\mathrm{dm} && (46) \\ &= \int_R \frac{d\,\mathbf{r}^0}{dt} \,\mathrm{dm} && (47) \\ &= \frac{d}{dt}\left(\int_R \mathbf{r}^0 \,\mathrm{dm}\right) && (48) \\ &= \frac{d}{dt}(m\,\mathbf{r}_G^0) && (49) \\ &= m\mathbf{v}_G^0 && (50) \end{aligned}$$

Noticed that the time derivative in is put outside the integral whose region of integration R depend on time. This manipulation is justified because center of gravity of rigid body behaves as material point as explained in [**?**, **?**].

3.8 Angular momentum and moment of inertia

The angular momentum of a rigid body relative to its center of gravity and the fixed origin of the reference 0-frame 0 are denoted respectively as $\mathbf{h}$ and $\mathbf{H}_0$. By definition:

$$\mathbf{h} = \int_R \mathbf{r}^b \times \mathbf{v}^0 \,\mathrm{dm}, \quad \mathbf{h}_0 = \int_R \mathbf{r}^0 \times \mathbf{v}^0 \,\mathrm{dm} \tag{51}$$

Expending the angular momentum $\mathbf{H}$:

$$\mathbf{h} = \int_R \mathbf{r}^b \times (\mathbf{v}_G^0 + \omega_{0b}^b \times \mathbf{r}^b)\,\mathrm{dm}, \tag{52}$$

$$= \int_R \mathbf{r}^b \times \mathbf{v}_G^0\,\mathrm{dm} + \int_R \mathbf{r}^b \times (\omega_{0b}^b \times \mathbf{r}^b)\,\mathrm{dm}, \tag{53}$$

$$= \int_R \mathbf{r}^b \times (\omega_{0b}^b \times \mathbf{r}^b)\,\mathrm{dm}, \tag{54}$$

Expending the angular momentum $\mathbf{h}_0$ Check:

$$\mathbf{h}_0 = \int_R \mathbf{r}^0 \times \mathbf{v}^0\,\mathrm{dm} \tag{55}$$

$$= \int_R (\mathbf{r}_G^0 + \mathbf{r}^b) \times \mathbf{v}^0\,\mathrm{dm} \tag{56}$$

$$= \mathbf{h} + \mathbf{r}_G^0 \times \mathbf{r}_\mathbf{P}^\mathbf{b} \tag{57}$$

Equation 52 can also be rewritten as:

$$\mathbf{h} = \int_R \mathbf{r}^b \times (\omega_{0b}^b \times \mathbf{r}^b)\,\mathrm{dm} = \int_R ((\mathbf{r}^b \cdot \mathbf{r}^b)\omega_{0b}^b - (\mathbf{r}^b \cdot \omega_{0b}^b)\mathbf{r}^b)\,\mathrm{dm} \tag{58}$$

Using

$$\mathbf{r}^b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix}^b \tag{59}$$

the expression can be developped in

$$\mathbf{h} = \omega_{0b}^b \int_R ({x_b}^2 + {y_b}^2 + {z_b}^2)\,\mathrm{dm} + \int_R \mathbf{r}^b (x_b p + y_b q + z_b r)\,\mathrm{dm} \tag{60}$$

$$= \int_R \begin{bmatrix} (y_b^2 + z_b^2) & -x_b y_b & -x_b z_b \\ -x_b z_b & (x_b^2 + z_b^2) & -y_b z_b \\ -x_b z_b & -y_b z_b & (x_b^2 + y_b^2) \end{bmatrix} \omega_{0b}^b \tag{61}$$

where the matrix of inertia with respect to the center of gravity $\mathbf{I}_g^b$ appears:

$$\mathbf{h} = \mathbf{I}_g^b\ \omega_{0b}^b \tag{62}$$

4 Newton/Euler 2nd law

The motions of the mock-up in 3 DOF therefore are computed by solving the standard Euler's law in the b-frame with the G coinciding with the COG:

$$\frac{d\mathbf{r}_\mathbf{P}^\mathbf{b}}{dt} = m\left(\frac{d\,\mathbf{v}_G^b}{dt} + \omega_{0b}^b \times \mathbf{v}_G^b\right) = \mathbf{F}_b \tag{63}$$

$$\frac{d\mathbf{h}}{dt} = \mathbf{I}_g^b\left(\frac{d\,\omega_{0b}^b}{dt} + \omega_{0b}^b \times \mathbf{I}_g^b\ \mathbf{v}_G^b\right) = \mathbf{M}_b \tag{64}$$

Where m is the mass and moments of inertia I of the mock-up defined as:

$\mathbf{F}_b = (F_x, F_y, F_z)$ and $\mathbf{M}_b = (Mx, My, M_z)$ are the vectors of force and moment acting on the mock-up.

5 Energy conservation

The definition of the kinetic energy is given by

$$E_k = \frac{1}{2} \int_R \mathbf{v}^2 \mathrm{dm} \tag{65}$$

it becomes with the previous definitions:

$$E_k = \frac{1}{2} m \mathbf{v}_G^b \cdot \mathbf{v}_G^b + \frac{1}{2} \omega_{0b}^b \cdot \mathbf{I}_g^b \omega_{0b}^b \tag{66}$$

6 Other writing to compare

Here applied on mass over the balance

$$\begin{array}{l} m\mathbf{a}_G^0 = m\mathbf{g} + \mathbf{F}_{ext} - \mathbf{F}_m \\ \mathbf{M^B} = \mathbf{M^B_{P_m}} + \mathbf{M^B_{F_{ext}}} + \mathbf{M^B_{-F_m}} \end{array} \tag{67}$$

$$\mathbf{M^B} = \int_{\text{body}} \mathbf{BM} \times \mathbf{a}_M^0 dm \tag{68}$$

$$\mathbf{a}_M^0 = \mathbf{a}_G^0 + \frac{d\omega_{0b}}{dt} \times \mathbf{GM} + \omega_{0b} \times (\omega_{0b} \times \mathbf{GM}), \tag{69}$$

As ω_{0b} and $\mathbf{a}_G^0$ do not depend on the point M, and introducing the inertia operator around G $\mathbf{I_G}$ as:

$$\mathbf{I_G}\, \mathbf{X} = \int_{\text{body}} \mathbf{GM} \times (\mathbf{X} \times \mathbf{GM}) dm \tag{70}$$

The relation 68 can be rewritten in:

$$\mathbf{M^B} = \mathbf{BM} \times m\mathbf{a}_G^0 + \mathbf{I_G} \frac{d\omega_{0b}}{dt} + \int_{\text{body}} \mathbf{GM} \times \omega_{0b} \times (\omega_{0b} \times \mathbf{GM}) dm \tag{71}$$

$$\begin{array}{ll} \int_{\text{body}} \mathbf{GM} \times \omega_{0b} \times (\omega_{0b} \times \mathbf{GM}) dm & = -\int_{\text{body}} \mathbf{GM} \times (\omega_{0b} \times \mathbf{GM}) \times \omega_{0b} dm \\ & = \omega_{0b} \times \mathbf{I_G}\, \omega_{0b} \end{array} \tag{72}$$

The final expression for $\mathbf{M^B}$ is

$$\mathbf{M^B} = \mathbf{BM} \times m\mathbf{a}_G^0 + \mathbf{I_G} \frac{d\omega_{0b}}{dt} + \omega_{0b} \times \mathbf{I_G}\, \omega_{0b} \tag{73}$$

$$\mathbf{M^B_{F_{ext}}} = \mathbf{BM} \times m\mathbf{a}_G^0 + \mathbf{I_G} \frac{d\omega_{0b}}{dt} + \omega_{0b} \times \mathbf{I_G}\, \omega_{0b} - \mathbf{M^B_{P_m}} - \mathbf{M^B_{-F_m}} \tag{74}$$

7 Accelerometers

An accelerometer measures $\mathbf{a} - \mathbf{g}$ in the fixed body coordinates.

8 Gyroscopes

Gyroscopes should be able to measure ω_{0b}

9 Other things

9.1 Instantaneous centre of rotation

$$\mathbf{V_A} = \mathbf{V_B} + \mathbf{AB} \times \omega_{0b}^{b} \tag{75}$$

if A=G and we are looking for instantaneous centre of rotation B where $\mathbf{V_B} = \mathbf{0}$. Let's consider $\mathbf{GB} = [x, y, z]^T$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^b \times \begin{bmatrix} p \\ q \\ r \end{bmatrix}^b \tag{76}$$

References

[1] Thor I Fossen. *Handbook of marine craft hydrodynamics and motion control.* John Wiley & Sons, 2011.