3D Geometric Modelin and Processing (6 cfu) 23-24

Notes

University of Pisa

M.Sc. in Computer Science

Contents

Chapter 1

Representing Surfaces

We talk about modeling surfaces because most of the object we see are opaque, thus from the point of view of rendering them we are only interested in their external surface **since we never see what is inside them**.

Formally the main aim of Computer Graphics is representing the boundary surface between the objects and non-objects.

1.1 Defining Surfaces

There are two methods of defining surfaces: Analytic definitions and Approximated definitions.

1.1.1 Analytic Definitions

In the analytic definitions that try to define the surface by using a mathematical (exact) notation.

There are two way to represent a surface in a 3D space using mathematical notation:

- **Parametric representation:** it consists in a function $S : \mathbb{R}^2 \to \mathbb{R}^3$ that maps points on a 2D domain over a 3D surface.
- **Implicit representation:** I define the surface S as the zeros of a function $f : \mathbb{R}^3 \to \mathbb{R}$ s.t.:
$$S = \{p \in \mathbb{R}^3 : f(p) = 0\}$$

1.1.2 Drawbacks of Analytic Definitions

The main limitation of analytic definitions is that realistic surfaces are not made of simple shape, making intractable defining a parametric/implicit representation of them.

To overcome this limitation we introduce the concept of mesh.

1.2 Meshes

Before defining meshes we need to introduce the concept of cellular complex.

1.2.1 Cellular Complex

A cellular complex, in mathematics and topology, refers to a structure composed of cells (convex polytope) of various dimensions, such as vertices (0-dimensional cells), edges (1-dimensional cells), faces (2-dimensional cells), and higher-dimensional cells. These cells are glued together according to certain rules or combinatorial data. After the second dimension we begin to consider solid 3D object (meaning their inside is not empty), usually used in simulation scenarios.

Properties of Cell Complex

- The order of a cell is the number of its sides. Sides are defined as boundaries of the higher-dimensional cells within the complex.
- a complex is a k-complex if the maximum of the order of its cells is k.

1.2.2 Definition of Meshes

Meshes are based upon the formalisation of cellular complex. Meshes are a set of polygons (usually they are triangles or four-sided polygons) that approximate my surface.

We say that a cellular complex is a mesh if and only if it satisfies the following rules:

- Every face of a cell belongs to the complex
- $\forall$ cells $C \wedge C'$ their intersection is either empty or is a common face of both. This means that two cells either do not touch, or they must have at most edges/vertices/faces in their boundaries in common (I cannot have a faces defined inside another face).
- a cell is maximal if it is not a face of another cell.

1.2.3 Maximal Cell Complex

Given a cell complex we can say that it is a maximal cell complex if and only if all of its maximal cells have order k. In general, since we mostly will talk about 2 dimension cellular complex, it will amount in checking if there are no dangling edges.

1.3 Simplicial Complex

Simplicial Complex are a specific case of Cellular Complex where I limit the faces to be triangles or 2-simplex.

Triangles are really convenient because they have some convenient properties:

- The number of edges in a triangles is always three.
- A triangle has three vertices and **three points in a topological space always define a plane**. So every 2-simplex defines a plane. Consider for points in a space, these four points does not imply the existence of a plane that touches all of them (to do it I have to make assumption and defining some rules).

1.3.1 Face

Given a simplex σ, we say that the simplex $\sigma^{'}$ is a face (or sub-simplex) of σ if it is defined by a subset of vertices of σ. If $\sigma^{'}$ is a face of σ and $\sigma \neq \sigma^{'}$, then it is a proper face.

1.3.2 Collection of simplexes

A collection of simplexes Σ is a simplicial k-complex if and only if:

- $\forall \sigma_1, \sigma_2 \in \Sigma . \sigma_1 \cap \sigma_2 \neq 0 \Rightarrow \sigma_1 \cap \sigma_2$ is a simplex of Σ.
- $\forall \sigma \in \Sigma$ all the faces of σ belong to Σ.
- k is the maximum degree (order) of simplexes in Σ.

There are the following properties:

- A simplex σ is maximal in a simplicial complex Σ if it is not a proper face of another simplex $\sigma^{'}$ of Σ.
- A simplician k-complex Σ is maximal if all its maximal simplex are of order k (there are no dangling lower dimensional pieces).

1.4 Meshes

When talking about triangle meshes the intended meaning is a maximal 2-simplicial complex, but in reality is not always true (in most cases I have some extra vertices in our space).

1.5 Manifoldess

Manifoldess is one of the main property to check in a simplicial complex.

Given a surface S iff $\forall$ point in S if I take its neighborhood, then the neighborhood is homeomorphic to the Euclidean space in two dimension. In practice this means that the neighborhood of each point needs to be homeomorphic to a disk (or a semidisk if the surface has boundaries).

There are two interesting cases where a surface is not manifold:

- When in the surface there are at least three triangles sharing a common edge.
- When in the surface there is at least an hourglass waist, meaning two triangles shares just a vertex (or two cubes touching in just one edge).

1.6 Orientability

A surface S is orientable if it is possible to make a consistent choice for the normal vector. In practice it means that starting from a face with a normal, I should be able to propagate this normal for all the faces near it and the faces near the faces near it.

1.7 Incidency

Given two simplexes $\sigma, \sigma^{'}$, if σ is a proper face of $\sigma^{'}$ then $\sigma, \sigma^{'}$ are incident.

1.8 Adjacency

Given two k-simplexes $\sigma, \sigma^{'}$ and $m < k$, if there exists a m-simplex that is a proper face of both $\sigma, \sigma^{'}$, then $\sigma, \sigma^{'}$ are m-adjacent.

Note:

- Two triangles sharing an edge are 1-adjacent.
- Two triangles sharing a vertex are 0-adjacent.

There are three classes of adjacency relations, identified by a pair of letters where each of them refers to one of the entity involved in the relation:

- **FF:** refers to an adjacency between triangular **F**aces (so they touch in an edge). It corresponds to the 1-adjacency.
- **EE:** it moves from one **E**dge to another following the vertices that connects them. It corresponds to the 0-adjacency.
- **FE:** it refers to the proper subfaces of a **F**ace that have dimension 1.
- **FV:** adjacency from **F**aces to **V**ertices (e.g. the vertices composing a face). It refers to the proper subfaces of a **F**ace with dimension 0.
- **EV:** given an **E**dge in returns its two **V**ertices. It refers to the propert subface of the **E**dge with dimension 0.
- **VF:** adjacency from a **V**ertex to a **F**ace (e.g. the triangles incident on a vertex). It refers to the $\{F \in Faces | V$ is a proper subface of $F\}$.
- **VE:** given a vertex it returns all the edges that utilize it. Its the set $\{E \in Edges :$ V is a proper subface of $E\}$
- **EF:** given an **E**dge it returns all the **F**aces that utilize it. If there are more than two **F**aces for an edge, then our mesh is not manifold. Its the set $\{F \in Faces | E$ is a proper subface of $F\}$.
- **VV:** given a vertex it returns all the vertices that have edges in common. its the set $\{V' \in Vertices | \exists E : (V, V')$.

These relation are usually stored in the data structures of meshes. The idea is to store a subset of these relations (usually FF, FV and VF) and procedurally generate the rest.

1.9 Partial adiacency

For the sake of conciseness it can be useful to keep only partial sets of relations. This is done not only to save space, but also to have non-redundant sets that in complex situations are difficult to keep updating.

For example VF^* memorize only a reference from a vertex to a face and then surfs over the surface using FF to find the other faces incident on V.

1.10 Bounds of Adjacency Relation

Most of the adjacency relation we have seen have bounds well defined.

Given a two manifold simplicial 2-complex in $\mathbb{R}^3$:

- FV, FE, FF, EF, EV have bounded degree (meaning they are constants if there are no borders):
  - $|FV| = 3, |EV| = 2, |FE| = 3$
  - $|FF| \leq 2$
  - $|EF| \leq 2$
- VV, VE, VF, EE have some average estimations:
  - $|VV| \sim |VE| \sim |VF| \sim 6$
  - $|EE| \sim 10$
  - The number of **F**aces is usually double the number of **V**ertices.

Appendix A

Elemental Geometry Concepts

A.1 Topological Space

In mathematics, a topological space is, roughly speaking, a geometrical space in which closeness is defined but cannot necessarily be measured by a numeric distance. More specifically, a topological space is a set whose elements are called points, along with an additional structure called a topology, which can be defined as a set of neighbourhoods for each point that satisfy some axioms formalizing the concept of closeness. It is the most general type of a mathematical space that allows for the definition of limits, continuity, and connectedness. Common types of topological spaces include Euclidean spaces, metric spaces and manifolds.

A.2 Polytope

In elementary geometry, a polytope is a geometric object with flat sides (faces). Polytopes are the generalization of three-dimensional polyhedra to any number of dimensions. Polytopes may exist in any general number of dimensions n as an n-dimensional polytope or n-polytope. For example, a two-dimensional polygon is a 2-polytope and a three-dimensional polyhedron is a 3-polytope.

A.3 Curve

In mathematics, a curve (also called a curved line in older texts) is an object similar to a line, but that does not have to be straight.

Formally a curve is the image of an interval to a topological space by a continuous function. In some contexts, the function that defines the curve is called a parametrization, and the curve is a parametric curve. In this article, these curves are sometimes called topological curves to distinguish them from more constrained curves such as differentiable curves. This definition encompasses most curves that are studied in mathematics; notable exceptions are level curves (which are unions of curves and isolated points), and algebraic curves (see below). Level curves and algebraic curves are sometimes called implicit curves, since they are generally defined by implicit equations.

A.4 Convex Hull

A convex hull (also known as convex envelope or convex closure) of a shape is the smallest convex set that contains it. The convex hull may be defined either as the intersection of all convex sets containing a given subset of a Euclidean space, or equivalently as the set of all convex combinations of points in the subset. For a bounded subset of the plane, the convex hull may be visualized as the shape enclosed by a rubber band stretched around the subset.

A.5 D-simplex

A d-simplex is the convex hull of d+1 point that are linearly dependent in d dimensions.

A.6 Plane

In mathematics, a plane is a two-dimensional space or flat surface that extends indefinitely. A plane is the two-dimensional analogue of a point (zero dimensions), a line (one dimension) and three-dimensional space.

A.7 Topological Characterization

How the elements are combinatorially connected.

A.8 Geometric Characteriation

Where the vertices are actually placed in space.

A.9 Open Set

An open set is a set that, along with every point P, contains all points that are sufficiently near to P (that is, all points whose distance to P is less than some value depending on P).

More generally, an open set is a member of a given collection of subsets of a given set, a collection that has the property of containing every union of its members, every finite intersection of its members, the empty set, and the whole set itself. A set in which such a collection is given is called a topological space, and the collection is called a topology. These conditions are very loose, and allow enormous flexibility in the choice of open sets. For example, every subset can be open (the discrete topology), or no subset can be open except the space itself and the empty set (the indiscrete topology).

In practice, however, open sets are usually chosen to provide a notion of nearness that is similar to that of metric spaces, without having a notion of distance defined. In particular, a topology allows defining properties such as continuity, connectedness, and compactness, which were originally defined by means of a distance.

A.10 Neighborhood of a point

The neighborhood of a point is a concept of topological space.

Given a topological space X and a point $p \in X$, then a neighbourhood of p is a subset V of X that includes an open set U containing p s.t.:

$$p \in U \subseteq V \subseteq X$$

Bibliography