FINM 32000: Homework 5

Due Friday May 3, 2024 at 11:59pm

Problem 1

Let r be the constant interest rate. Let $0 < T_1 < T_2$.

(a) Let F_t be the time-t forward price for T_2-delivery of some arbitrary underlying S, not necessarily tradeable. Recall from FINM 33000, that a *forward price* is *not the same thing* as the *value of a forward contract.* By definition of the time-t *forward price* F_t for T_2-delivery:

a forward contract paying $S_{T_2} - F_t$ at time T_2 has time-t value 0.

Let f_t be the time-t value of a T_2-forward contract on the same underlying, but with some delivery price K (not necessarily equal to F_t).

Express f_t in terms of K and F_t and a discount factor.

Hint: consider a portfolio long one (K, T_2)-forward contract and short one (F_t, T_2)-forward contract. The portfolio has (in terms of f_t) what value at time t? The portfolio pays how much at expiration?

(b) If S is a *stock* paying no dividends, the forward price must be $F_t = S_t e^{r(T_2-t)}$; otherwise, arbitrage would exist.

> If, say, $F_t > S_t e^{r(T_2-t)}$, then arbitrage would exist: at time t, borrow S_t dollars, buy the stock, and short the forward (with delivery price F_t and time-t value 0). At time T_2, deliver the stock, and receive F_t, which is more than enough to cover your accumulated debt of $S_t e^{r(T_2-t)}$ dollars.

However, if S is the spot price of a barrel of crude oil (so, for all t, the time-t price for time-t delivery is S_t per barrel), then this argument fails. Explain briefly (one or two sentences, no math) why *this specific arbitrage* does not apply to crude oil, by specifically pinpointing, in the quote above, why we cannot simply replace "stock" with "crude oil".

Hint: Consider practical complications.

So we need more assumptions to relate F_t and S_t (here and in (c,d,e,f,g), the S denotes spot crude oil, and F_t denotes the time-t forward price for T_2-delivery crude oil). One approach is to model the risk-neutral dynamics of S. Assume that S satisfies

$$
\begin{aligned}
S_t &= \exp(X_t) \\
\mathrm{d}X_t &= \kappa(\alpha - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t.
\end{aligned}
$$

where W is Brownian motion, under risk-neutral measure.

Then, since r is constant and $\mathbb{E}_t(e^{-r(T_2-t)}(S_{T_2} - F_t))$ must be 0, one can calculate

$$F_t = \mathbb{E}_t(S_{T_2}) = \exp\left[e^{-\kappa(T_2-t)}\log S_t + \left(1 - e^{-\kappa(T_2-t)}\right)\alpha + \frac{\sigma^2}{4\kappa}\left(1 - e^{-2\kappa(T_2-t)}\right)\right],$$

where $\mathbb{E}_t$ is time-t conditional expectation. Suppose $\kappa = 0.472$, $\alpha = 4.4$, $\sigma = 0.368$, $r = 0.05$, and the time-0 spot price is $S_0 = 106.9$.

Let C be the time-0 price of a K-strike T_1-expiry European call on F. So this call pays $(F_{T_1} - K)^+$. Let the call option have strike $K = 103.2$ and expiration $T_1 = 0.5$. Let the forward have delivery date $T_2 = 0.75$. See the `ipynb` file.

(c) Estimate $C(S_0)$ using Monte Carlo simulation of S with 100 timesteps on $[0, T_1]$. Choose the number of paths large enough that the standard error [the sample standard deviation, divided by the square root of the number of paths] is less than 0.05. Report the standard error. Don't use any variance reduction technique.

(d) Estimate $\partial C/\partial S$ by using Monte Carlo simulation to calculate $(C(S_0 + 0.01) - C(S_0))/0.01$. For the $C(S_0+0.01)$ calculation, *reuse* the same normal random variables which you generated for the $C(S_0)$ calculation. (Do not re-*generate* random variables to compute $C(S_0 + 0.01)$)

(e) Calculate analytically $\partial f_0/\partial S$, where f_0 is the time-0 value of a position long one forward contract on a barrel of crude oil, with delivery date T_2 and some fixed delivery price K.

(f) Suppose you want to hedge a position short one call (so your hedge portfolio should replicate a position long one call), by continuously rebalancing a position in T_2-delivery forward contracts. Your hedge portfolio at time 0 should be long how many forward contracts? Your final answer should be a number.

The delivery price K of the forward contracts is irrelevant to the answer here; it would affect only how many units of the bank account to carry in the portfolio (which I am not asking you to compute).

(g) Consider the following "purchase agreement" contract. The holder of this contract receives time-T_2 delivery of θ barrels of crude oil, and pays, at time T_2, a delivery price of K dollars per barrel. The θ is chosen at time T_1 by the holder of the purchase agreement, subject to the restriction that $4000 \leq \theta \leq 5000$; in particular, $\theta = 0$ is not a valid choice, because the contract is a commitment to purchase at least 4000 barrels. Using your answer to (c), without running any new simulations, find the time-0 value of this contract.

Here K, T_1, T_2 have the same values as on the previous page.

Hint: Assume the holder acts optimally; thus θ is either 4000 or 5000, depending on F_{T_1}.

Problem 2

Suppose that a non-dividend-paying stock has dynamics

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}W_t \tag{1}$$

where W is Brownian motion under risk-neutral probabilities, and where the time-dependent but non-random volatility function $\sigma : [0,T] \to \mathbb{R}$ is piecewise continuous and sufficiently integrable. L2.13 shows that this particular type of local volatility function σ (to be specific: the type of σ function that depends on t but does not depend on S, nor on anything else that is random) has an explicit relationship with the Black-Scholes implied volatility σ_{imp}.

(a) Are the dynamics (1) capable of generating a non-constant (with respect to T) term-structure of implied volatility? Are they capable of generating an implied volatility skew (non-constant with respect to K)? Explain briefly.

(b) Let $S_0 = 100$ and $r = 0.05$. At time 0, you observe the prices of at-the-money (this means $K = 100$) European calls at 0.1-year, 0.2-year, and 0.5-year expiries to be 5.25, 7.25, and 9.5, respectively. First find the Black-Scholes implied volatilities of the three options. Then find (calibrate) a time-varying local volatility function $\sigma : [0, 0.5] \to \mathbb{R}$ consistent with these option prices. A step function suffices (but other answers are also acceptable).

(c) Consistently with your local volatility function σ from part (b), find the time-0 price of an at-the-money European call with expiry 0.4, and find the time-0.1 implied volatility of that call (the European call with expiry $T = 0.4$). Do not use a tree or finite difference or Monte Carlo calculation.