Travelling Sales Person

Given n cities and the distances d_{ij} between any two of them, we wish to find the shortest tour going through all cities and back to the starting sity. Usually the TSP is given as a $G = (V, D)$ where $V = \{1, 2, \ldots, n\}$ is the set of cities, and D is the adjacency distance matrix, with $\forall i, j \in V, i \neq j, d_{i,j} > 0$, the probem is to find the tour with minimal distance weight, that starting in 1 goes through all n cities and returns to 1.

The TSP is a well known and difficult problem, that can be solved in $O(n!) \sim O(n^n e^{-n})$ steps.

Characterization of the optimal solution

Given $S \subseteq V$ with $1 \in S$ and given $j \neq 1, j \in S$, let $C(S, j)$ be the shortest path that starting at 1, visits all nodes in S and ends at j.

Notice:

- If $|S| = 2$, then $C(S, k) = d_{1,k}$ for $k = 2, 3, \ldots, n$
- If $|S| > 2$, then $C(S, k) =$ the optimal tour from 1 to m, $+d_{m,k}$,

  $\exists m \in S - \{k\}$

Recursive definition of the optimal solution

$$C(S, k) = \begin{cases} d_{1,m} & \text{if } S = \{1, k\} \\ \min_{m \neq k, m \in S}[C(S - \{k\}, m) + d(m, k)] & \text{otherwise} \end{cases}$$

The optimal solution

function algorithm $TSP(G, n)$

    for $k := 2$ **to** n **do**

        $C(\{i, k\}, k) := d_{1,k}$

    end for

    for $s = 3$ **to** n **do**

        for all $S \subseteq \{1, 2, \ldots, n\}||S|| = s$ **do**

            for all $k \in S$ **do**

                $\{C(S, k) = \min_{m \neq k, m \in S}[C(S - \{k\}, m) + d_{m,k}]\}$

                $opt := \min_{k \neq 1}[C(\{1, 2, 3, \ldots, n\}, k) + d_{1,k}$

            end for

        end for

    end for;

    return (opt)

end

Complexity:

Time: $(n-1)\sum_{k=1}^{n-3}\binom{n-2}{k} + 2(n-1) \sim O(n^2 2^n) << O(n!)$

Space: $\sum_{k=1}^{n-1} k\binom{n-1}{k} = (n-1)2^{n-2} \sim O(n2^n)$