EXAMPLE 3 Area

Find the area of the region enclosed by $y = x^2 - 2x$ and $y = x$ on $[0, 4]$.

SOLUTION Verify that the two graphs cross at $x = 0$ and $x = 3$. Because the first graph is a parabola opening upward, the parabola must be below the line between 0 and 3 and above the line between 3 and 4. See Figure 26. (The greater function could also be identified by checking a point between 0 and 3, such as 1, and a point between 3 and 4, such as 3.5. For each of these values of x, we could calculate the corresponding value of y for the two functions and see which is greater.) Because the graphs cross at $x = 3$, the area is found by taking the sum of two integrals as follows.

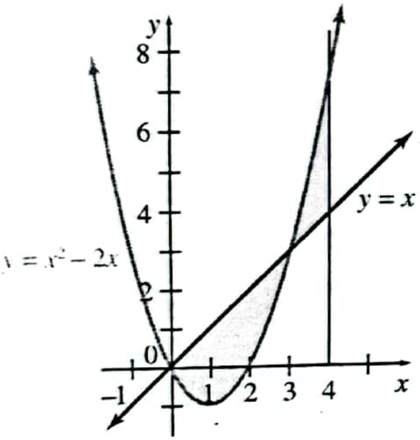


FIGURE 26

$$\begin{aligned}
\text{Area} &= \int_0^3 [x - (x^2 - 2x)]\,dx + \int_3^4 [(x^2 - 2x) - x]\,dx \\
&= \int_0^3 (-x^2 + 3x)\,dx + \int_3^4 (x^2 - 3x)\,dx \\
&= \left(\frac{-x^3}{3} + \frac{3x^2}{2}\right)\Bigg|_0^3 + \left(\frac{x^3}{3} - \frac{3x^2}{2}\right)\Bigg|_3^4 \\
&= \left(-9 + \frac{27}{2} - 0\right) + \left(\frac{64}{3} - 24 - 9 + \frac{27}{2}\right) \\
&= \frac{19}{3}
\end{aligned}$$

YOUR TURN 3 Repeat Example 3 for $y = x^2 - 3x$ and $y = 2x$ on $[0, 6]$.

TRY YOUR TURN 3

In the remainder of this section we will consider some typical applications that require finding the area between two curves.

EXAMPLE 4 Savings

A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time t (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2,$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time t (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t.$$

(a) For how many years will the company realize savings?

APPLY IT

SOLUTION Figure 27 shows the graphs of the rate-of-savings and rate-of-cost functions. The rate of cost (marginal cost) is increasing, while the rate of savings (marginal savings) is decreasing. The company should use this new process until the difference between these quantities is zero; that is, until the time at which these graphs intersect. The graphs intersect when

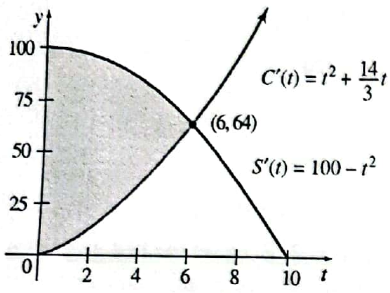


FIGURE 27

$$C'(t) = S'(t),$$

or

$$t^2 + \frac{14}{3}t = 100 - t^2.$$

Solve this equation as follows.

$$0 = 2t^2 + \frac{14}{3}t - 100$$

$$0 = 3t^2 + 7t - 150 \qquad \textbf{Multiply by } \tfrac{3}{2}.$$

$$= (t - 6)(3t + 25) \qquad \textbf{Factor.}$$

Set each factor equal to 0 separately to get

$$t = 6 \qquad \text{or} \qquad t = -25/3.$$

Only 6 is a meaningful solution here. The company should use the new process for 6 years.

(b) What will be the net total savings during this period?

SOLUTION Since the total savings over the 6-year period is given by the area under the rate-of-savings curve and the total additional cost is given by the area under the rate-of-cost curve, the net total savings over the 6-year period is given by the area between the rate-of-cost and the rate-of-savings curves and the lines $t = 0$ and $t = 6$. This area can be evaluated with a definite integral as follows.

$$\text{Total savings} = \int_0^6 \left[(100 - t^2) - \left(t^2 + \frac{14}{3}t\right)\right] dt$$

$$= \int_0^6 \left(100 - \frac{14}{3}t - 2t^2\right) dt$$

$$= \left(100t - \frac{7}{3}t^2 - \frac{2}{3}t^3\right)\Bigg|_0^6$$

$$= 100(6) - \frac{7}{3}(36) - \frac{2}{3}(216) = 372$$

The company will save a total of \$372,000 over the 6-year period.

The answer to a problem will not always be an integer. Suppose in solving the quadratic equation in Example 4 we found the solutions to be $t = 6.7$ and $t = -7.3$. It may not be

If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region R.

Solution

The region is depicted in the following figure.

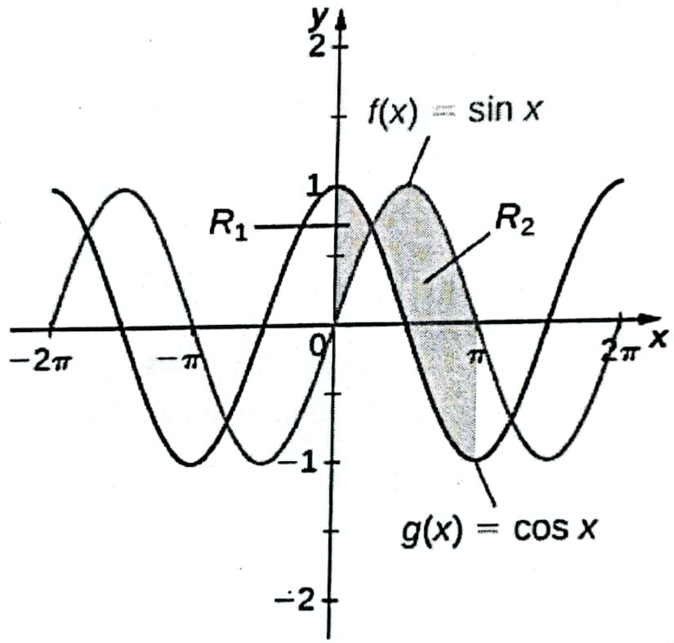


Figure 2.6 The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4]$, $\cos x \geq \sin x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x.$$

On the other hand, for $x \in [\pi/4, \pi]$, $\sin x \geq \cos x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then

$$\begin{aligned}
A &= \int_a^b |f(x) - g(x)| dx \\
&= \int_0^{\pi} |\sin x - \cos x| dx = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\
&= [\sin x + \cos x] \Big|_0^{\pi/4} + [-\cos x - \sin x] \Big|_{\pi/4}^{\pi} \\
&= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}.
\end{aligned}$$

The area of the region is $2\sqrt{2}$ units^2.

2.3 If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[\pi/2, 2\pi]$, find the area of region R.

Example 2.4

Finding the Area of a Complex Region

Consider the region depicted in **Figure 2.7**. Find the area of R.

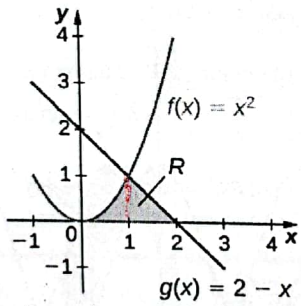


Figure 2.7 Two integrals are required to calculate the area of this region.

Solution

As with **Example 2.3**, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the x-axis, so we have

$$A_1 = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the x-axis, so we have

$$A_2 = \int_1^2 [2 - x - 0] dx$$

$$A_2 = 2x - \frac{x^2}{2} \Big|_1^2 = \left[4 - \frac{4}{2} - 2 + \frac{1}{2}\right] = \frac{1}{2}$$

$$A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

EXAMPLE 6.2.5 **(Multiple Curves, Multiple Regions).** Find the area of the region enclosed by the graphs of $y = 8 - x^2$, $y = 7x$, and $y = 2x$ in the first quadrant.

Solution. This time there are three curves to contend with. Since the curves are relatively simple (an upside-down parabola and two lines through the origin, it is relatively easy to make a sketch of the region. See Figure 6.9. Let $f(x) = 8 - x^2$, $g(x) = 7x$, and $h(x) = 2x$. A wedge-shaped region is determined by all three curves. Notice that the 'top' curve of the region switches from $g(x)$ to $f(x)$. We find the intersections of the pairs of graphs:

$f(x) = g(x) \Rightarrow 8 - x^2 = 7x \Rightarrow x^2 + 7x - 8 = 0 \Rightarrow (x-1)(x+8) = 0 \Rightarrow x = 1$ (not -8).

$f(x) = h(x) \Rightarrow 8 - x^2 = 2x \Rightarrow x^2 + 2x - 8 = 0 \Rightarrow (x-2)(x+4) = 0 \Rightarrow x = 2$ (not -4).

$g(x) = h(x) \Rightarrow 7x = 2x \Rightarrow 5x = 0 \Rightarrow x = 0.$

The region is thus divided into two subregions and the graph gives the relative positions of the curves. Since all the functions are continuous Theorem 6.1.1 applies.

$$
\begin{aligned}
\text{Area enclosed by } f, g, \text{ and } h &= \int_0^1 [7x - 2x]\,dx + \int_1^2 [(8 - x^2) - 2x\,dx \\
&= \int_0^1 [5x]\,dx + \left(8x - \frac{x^3}{3} - x^2\right)\Bigg|_1^2 \\
&= \left(\frac{5x^2}{2}\right)\Bigg|_0^1 + \left(\left[16 - \frac{8}{3} - 4\right] - \left[8 - \frac{1}{3} - 1\right]\right) \\
&= \left(\frac{5}{2} - 0\right) + \left(\frac{8}{3}\right) = \frac{31}{6}.
\end{aligned}
$$

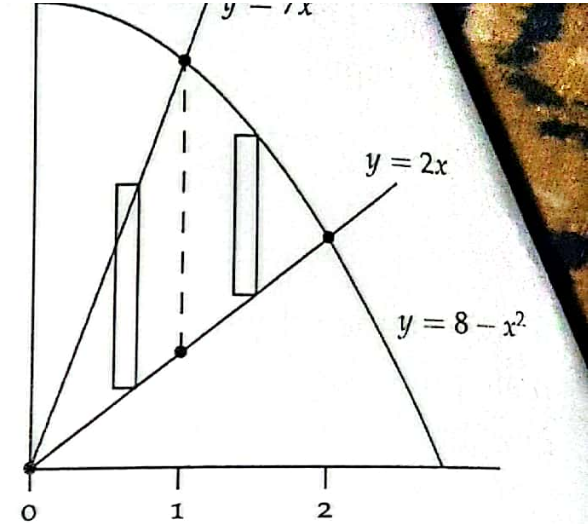
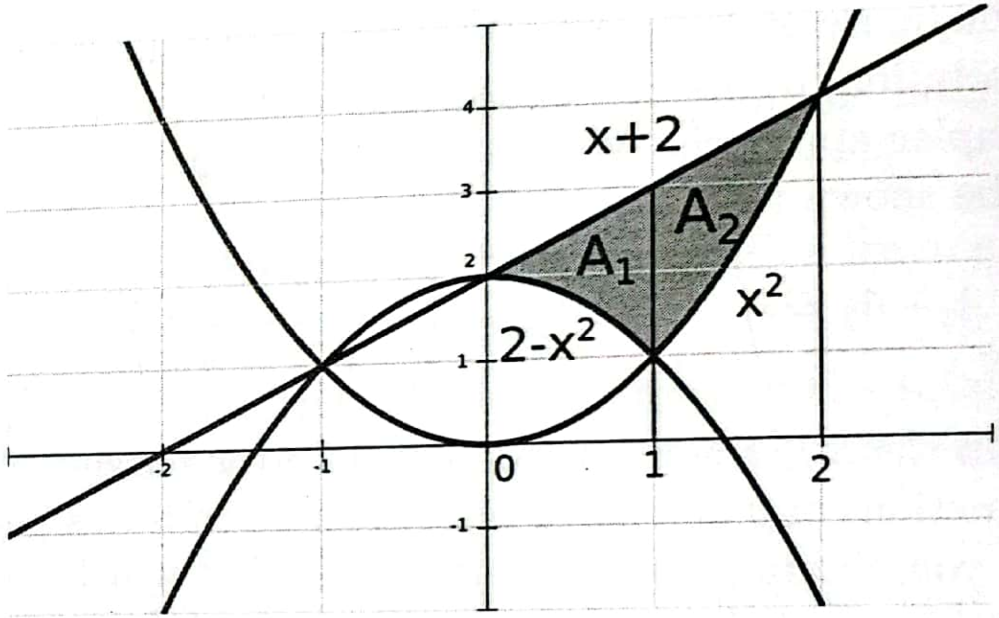


Figure 6.9: The area enclosed by $y = 8 - x^2$, $y = 7x$, and $y = 2x$ in the first quadrant. There are two representative rectangles because the top curve changes.

Example 4. Find the area of the region between $y = 2 - x^2$, $y = x^2$, and $y = x + 2$ in the first quadrant.

Solution. Graph the functions first and identify the region which area you need to determine. Note that it has to be divided into two regions with areas A_1 and A_2. The curve $y = x + 2$ is upper both on A_1 and on A_2. On A_1, $y = 2 - x^2$ is lower and on A_2, $y = x^2$ is lower.



Find the intersections to determine the bounds of integration. The curves intersect at the following points.

(1) $2 - x^2 = x^2 \Rightarrow 2 = 2x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

(2) $2 - x^2 = x + 2 \Rightarrow x^2 + x = 0 \Rightarrow x(x+1) = 0 \Rightarrow x = 0$ and $x = -1$.

(3) $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2$ and $x = -1$.

Considering the graph in the first quadrant, you can see that the relevant intersections are $x = 0$, $x = 1$ and $x = 2$. The bounds for A_1 are 0 and 1 and the bounds for A_2 are 1 and 2. Thus,

$A_1 = \int_0^1 (x + 2 - (2 - x^2))dx = \int_0^1 (x + x^2)dx = \frac{x^2}{2} + \frac{x^3}{3}x^3\Big|_0^1 = \frac{5}{6}$.

$A_2 = \int_1^2 (x + 2 - x^2)dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_1^2 = 2 + 4 - \frac{8}{3} - \frac{1}{2} - 2 + \frac{1}{3} = \frac{7}{6}$.

The total area $A = A_1 + A_2 = \frac{5}{6} + \frac{7}{6} = \frac{12}{6} = 2$.

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) $y = x,\ y = 2x,\ y = 6 - x$

(k) Find the three intersections first. (1) $x = 6 - x \Rightarrow 2x = 6 \Rightarrow x = 3$. (2) $2x = 6 - x \Rightarrow 3x = 6 \Rightarrow x = 2$. (3) $2x = x \Rightarrow x = 0$. The relevant region consists of two parts: the area

A_1 on interval [0,2] between the upper curve $y = 2x$ and the lower $y = x$, and the area A_2 on interval [2,3] between the upper curve $y = 6 - x$ and the lower $y = x$.
Thus, the total area can be computed as $A = A_1 + A_2 = \int_0^2 (2x - x)dx + \int_2^3 (6 - x - x)dx = \frac{x^2}{2}\Big|_0^2 + (6x - x^2)\Big|_2^3 = 2 + (18 - 9 - 12 + 4) = 3.$