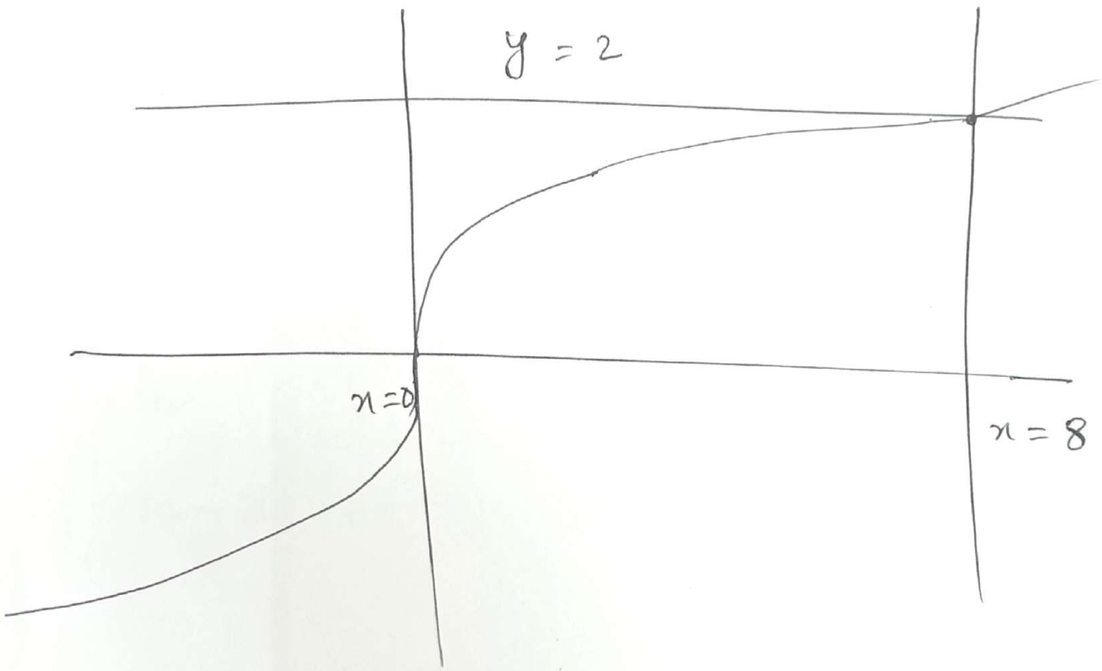
01

Q #1.1

05 marks

$$\int_0^8 \int_{x^{1/3}}^2 f(x,y)\,dy\,dx = \int_0^2 \int_0^{y^3} f(x,y)\,dx\,dy.$$

3



2

کیم سکینر سے سکین کیا گیا۔

Q # 1.3

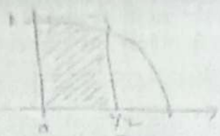
Q. Sol

$\iint 2xy\, dA \qquad R: \{(x,y) \mid 0 \le x \le \frac{1}{2}$

$\qquad 0 \le y \le \sqrt{1-x}$

$y=\sqrt{1-x}$
$y^2 = 1-x$
$1-y^2 = x$

$$V = \int_0^{1/2}\int_0^{\sqrt{1-x}} 2xy\, dy\, dx$$



$$= \int_0^{1/2} xy^2 \Big|_0^{\sqrt{1-x}} dx = \int_0^{1/2} [x(1-x) - 0]\, dx$$

$$= \int_0^{1/2} (x - x^2)\, dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^{1/2} = \frac{1}{12}$$ → 2 → 4

$$\int_0^{1/2}\int_0^{\sqrt{1-x}} dy\, dx = \int_0^{1/2} y \Big|_0^{\sqrt{1-x}} dx$$

$$= \int_0^{1/2} \sqrt{1-x}\, dx$$

$$= -\int_0^{1/2} (1-x)^{1/2} \cdot (-1)\, dx$$

$$= -\left[\frac{(1-x)^{3/2}}{3/2}\right]_0^{1/2} = -\frac{2}{3}(1-x)^{3/2}$$

$$= -\frac{2}{3}\left(1-\frac{1}{2}\right) + \frac{2}{3}$$

$$= -\frac{2}{\cancel{3}} + \frac{\cancel{2}}{3\cdot \cancel{2}} + \frac{2}{\cancel{3}} = \frac{1}{3}$$ → 3

01

Average value $= \dfrac{1/12}{1/3} = \dfrac{1}{12} \times \dfrac{3}{1} = \dfrac{1}{4}.$

Q # 1.2

$\rightarrow 3/$

$$V = \iiint_Q dV = \int_{-1}^{1} \int_{3x^2}^{4-x^2} \int_0^{6-z} dy\, dz\, dx = \int_{-1}^{1} \int_{3x^2}^{4-x^2} [y]_0^{6-z}\, dz\, dx$$

2

$$= \int_{-1}^{1} \int_{3x^2}^{4-x^2} (6-z)\, dz\, dx \overset{2/}{=} \int_{-1}^{1} \left[6z - \tfrac{1}{2}z^2\right]_{3x^2}^{4-x^2} dx$$

2/

$$= \int_{-1}^{1} (16 - 20x^2 + 4x^4)\, dx = \frac{304}{15} \approx 20.3.$$

01

wrong direction = 04

full Attempt

کیم سکینر سے سکین کیا گیا۔

Q # 2.3

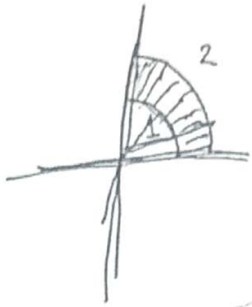
07 marks

Solution

Q.1

$$A=\int_0^2\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} 1\cdot dx\,dy$$

2



$$A=\int_0^{\pi/2}\int_1^2 r\,dr\,d\theta$$

$$A=\int_0^{\pi/2}\left\{\left[\frac{r^2}{2}\right]_1^2\right\}d\theta=\left(\frac{4-1}{2}\right)\frac{\pi}{2}=\frac{3\pi}{4} \quad [\cancel{3}]$$

$$V=\int_0^{\pi/2}\left[\int_1^2 (4r^2+1)^{1/2}\,r\,dr\right]d\theta=\int_0^{\pi/2}\left[\frac{2(4r^2+1)^{3/2}}{24}\right]_1^2 d\theta$$

5

$$V=\frac{1}{12}\int_0^{\pi/2}\left[(17)^{3/2}-(5)^{3/2}\right]d\theta$$

$$V=\left[\frac{1}{12}\left\{(17)^{3/2}-(5)^{3/2}\right\}\right]\left[\theta\right]_0^{\pi/2}=\frac{\pi}{24}\left((17)^{3/2}-(5)^{3/2}\right)$$

[2]

$$av(f)=\frac{1}{Area}\times\iint_R f(x,y)\,dA=\frac{4}{3\pi}\times\frac{\pi}{24}\left[(17)^{3/2}-(5)^{3/2}\right]$$

$$=\frac{1}{18}\left[(17)^{3/2}-(5)^{3/2}\right]$$

S2 MVC Questions and Solution

Q #2.1

Muhammad Adnan

~~Question 1:~~

10 marks.

A heated storage room is shaped like a rectangular box and has a volume of 1000 cubic feet. Because warm air rises, the heat loss per unit of area through the ceiling is five times as great as the heat loss through the floor. The heat loss through the four walls is three times as great as the heat loss through the floor. Determine the room dimensions that will minimize heat loss and therefore minimize heating costs. Determine the room dimensions that will minimize heat loss using the method of Lagrange multipliers.

Solution

Let:

- x = width of the room
- y = depth of the room
- h = height of the room

The constraint is the volume of the room: ✓

$$V = xyh = 1000.$$

$\nabla f = \lambda \nabla g$

The heat loss through different surfaces:

- **Floor**: Area $A_f = xy$, heat loss $= xy$.
- **Ceiling**: Area $A_c = xy$, heat loss $= 5xy$.
- **Walls**: Two walls of area xh and two walls of area yh, heat loss $= 3(2xh + 2yh) = 6xh + 6yh$.

Thus, the total heat loss function is:

$$H(x, y, h) = 6xy + 6xh + 6yh.$$

Using $\nabla H = \lambda \nabla V$, we compute gradients:

$$\nabla H = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial h}\right) = (6y + 6h, 6x + 6h, 6x + 6y).$$

$$\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial h}\right) = (yh, xh, xy).$$

Setting $\nabla H = \lambda \nabla V$:

$$6y + 6h = \lambda yh, \quad 6x + 6h = \lambda xh, \quad 6x + 6y = \lambda xy.$$

From the first two equations:

$$\lambda = \frac{6y + 6h}{yh} = \frac{6x + 6h}{xh}.$$

$$(6y + 6h)xh = (6x + 6h)yh.$$

$$6yxh + 6hxh = 6xyh + 6hyh.$$

کیم سکینر سے سکین کیا گیا۔

Canceling $6xyh$:

$$6hxh = 6hyh \Rightarrow xh = yh \Rightarrow x = y.$$

Similarly, from the third equation:

$$\lambda = \frac{6x+6y}{xy} = \frac{12x}{x^2} = \frac{12}{x}.$$

$$\frac{12}{x} = \frac{6x+6h}{xh}.$$

$$12h = 6x + 6h.$$

$$6h = 6x \Rightarrow h = x.$$

Using the volume constraint:

$$x^3 = 1000 \Rightarrow x = 10, \quad y = 10, \quad h = 10.$$

The dimensions that minimize heat loss are:

$$\boxed{10 \times 10 \times 10 \text{ feet.}}$$

Question 2. 2

Consider the quadratic function:

$$f(x_1, x_2) = 2x_1^2 + x_1x_2 + \frac{3}{2}x_2^2 - x_1 - 2x_2.$$

The initial point is given as:

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Perform two iterations using the steepest descent method.

Solution

Iteration 1

The gradient of the function is computed as:

$$\nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 + x_2 - 1 \\ x_1 + 3x_2 - 2 \end{bmatrix}.$$

At the initial point $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\nabla f(x_0) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

The search direction is given by:

$$S_0 = -\nabla f(x_0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The Hessian matrix is given by:

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}.$$

The optimal step length is given by:

$$\lambda_0 = \frac{S_0^T S_0}{S_0^T H S_0}.$$

First, compute the numerator:

$$(1,2)^T(1,2) = 1 + 4 = 5.$$

Compute the denominator:

$$(1,2)^T \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = (1 \times 6 + 2 \times 7) = 20.$$

Thus,

$$\lambda_0 = \frac{5}{20} = \frac{1}{4}.$$

Update x_1:

$$x_1 = x_0 + \lambda_0 S_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}.$$

4

Iteration 2

Compute the gradient at x_1:

$$\nabla f(x_1) = Hx_1 - c = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}.$$

The search direction is:

$$S_1 = -\nabla f(x_1) = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix}.$$

Compute the optimal step length: Compute step size λ_1:

$$\lambda_1 = \frac{S_1^T S_1}{S_1^T H S_1}.$$

Compute numerator:

$$S_1^T S_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{4} + \frac{1}{16} = \frac{5}{16}.$$

Compute denominator:

$$HS_1 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{7}{4} \\ \frac{1}{4} \end{bmatrix}.$$

$$S_1^T H S_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) \begin{bmatrix} -\frac{7}{4} \\ \frac{1}{4} \end{bmatrix} = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}.$$

$$\lambda_1 = \frac{5}{16} \div \frac{15}{16} = \frac{1}{3}.$$

Update x_2:

$$x_2 = x_1 + \lambda_1 S_1 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix}.$$

$$x_2 = \begin{bmatrix} \frac{1}{4} - \frac{1}{6} \\ \frac{1}{2} + \frac{1}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} \\ \frac{7}{12} \end{bmatrix}.$$

4