Sketch the graph of the function $f(x) = \dfrac{2x^2}{x^2 - 1}$.

▾ Solution

a. The domain is $\{x : x^2 - 1 \neq 0\} = \{x : x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

b. There is an x-intercept at $x = 0$. The y intercept is $y = 0$.

c. $f(-x) = f(x)$, so f is an even function (symmetric about y-axis)

d. $\lim\limits_{x\to\pm\infty} \dfrac{2x^2}{x^2 - 1} = \lim\limits_{x\to\pm\infty} \dfrac{2}{1 - 1/x^2} = 2$, so $y = 2$ is a horizontal asymptote. Now the denominator is 0 at $x = \pm 1$, so we compute:

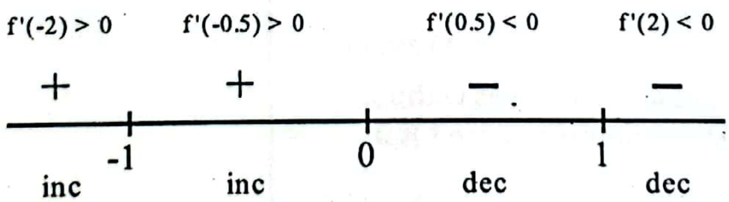
$$\lim_{x\to 1^+} \frac{2x^2}{x^2 - 1} = +\infty, \quad \lim_{x\to 1^-} \frac{2x^2}{x^2 - 1} = -\infty, \quad \lim_{x\to -1^+} \frac{2x^2}{x^2 - 1} = -\infty, \quad \lim_{x\to -1^-} \frac{2x^2}{x^2 - 1} = +\infty.$$

So the lines $x = 1$ and $x = -1$ are vertical asymptotes.

e. For critical values we take the derivative:

$$f'(x) = \frac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}.$$

Note that $f'(x) = 0$ when $x = 0$ (the top is zero). Also, $f'(x) = DNE$ when $x = \pm 1$ (the bottom is zero). As $x = \pm 1$ is *not* in the domain of $f(x)$, the only critical point is $x = 0$ (recall that to be a critical point we need it to be in the domain of the original function). Drawing a number line and including *all* of the split points of $f'(x)$ we have:

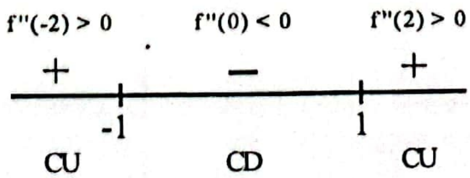


Thus f is increasing on $(-\infty, -1) \cup (-1, 0)$ and decreasing on $(0, 1) \cup (1, \infty)$. By the First Derivative Test, $x = 0$ is a relative max.

f. For possible inflection points we take the second derivative:

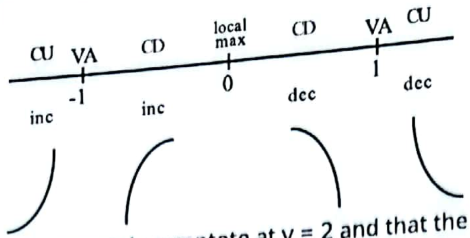
$$f''(x) = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

The top is never zero. Also, the bottom is only zero when $x = \pm 1$ (neither of which are in the domain of $f(x)$). Thus, there are no possible inflection points to consider. Drawing a number line and including *all* of the split points of $f''(x)$ we have:

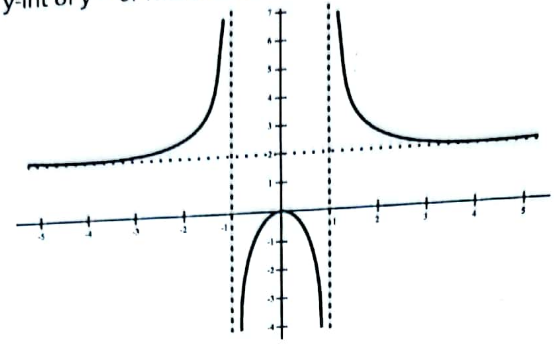


Hence f is concave up on $(-\infty, -1) \cup (1, \infty)$, concave down on $(-1, 1)$.

g. We put this information on a single number line to
this information on a single number line to
certain intervals:



Note that there is a horizontal asymptote at y = 2 and that the curve has x-int of x = 0 and y-int of y = 0. Therefore, a sketch of f(x) is as follows:



Example5.87. Curve Sketching. A 1

Sketch the graph of the function

$$f(x) = x^3 - 6x^2 + 9x + 2.$$

▾ Solution

Obtain the following information on the graph of f.

a. The domain of f is $(-\infty, \infty)$.

b. By setting x = 0, we find that the y-intercept is 2. The x-intercept is found by setting y = 0, which in this case leads to a cubic equation. Since the solution is not readily found, we will not use this information.

c. Since f is a cubic polynomial, we expect odd symmetry. This will become more obvious once we analyze f' and f''.

d. We now look for any asymptotes of f:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x^3 - 6x^2 + 9x + 2) = \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x^3 - 6x^2 + 9x + 2) = -\infty$$

We see that f decreases without bound as x decreases and increases without bound as x increases. Therefore, f has no horizontal asymptotes. Since f is a polynomial,there are no vertical asymptotes.

e. $f'(x) = 3x^2 - 12x + 9 = 3(x-3)(x-1)$

Setting $f'(x) = 0$ gives x = 1 or x = 3 as our only critical points. The following sign diagram for f shows that f is increasing on the intervals $(-\infty, 1)$ and $(3, \infty)$ and decreasing on the interval (1, 3).

+		−		+
inc	1	dec	3	inc

Since the sign of f' changes as we move across the critical point x = 1, a relative maximum occurs at (1, f(1)) = (1, 6). Similarly, a relative minimum of f occurs at (3, 2).

f. We find

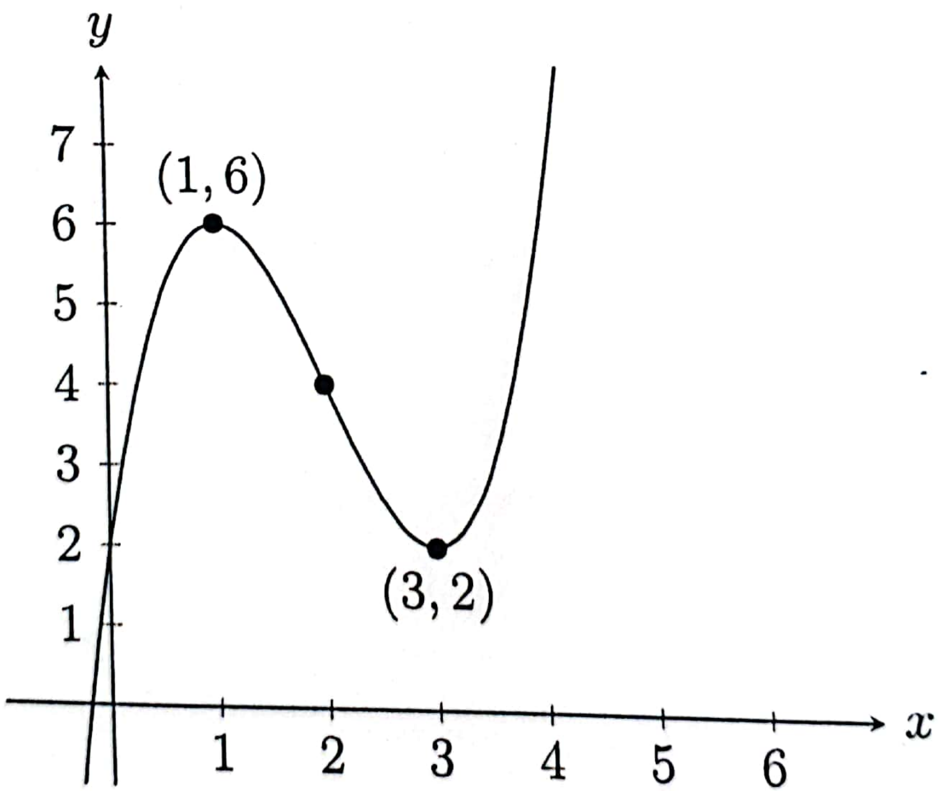
$$f''(x) = 6x - 12 = 6(x-2),$$

which is equal to zero when x = 2. The sign diagram for f″,

$-$		$+$
CD	2	CU

shows that f is concave downward on (−∞, 2) and concave upward on (2, ∞). Since the sign of f″ changes sign as we move across x = 2, f must have an inflection point at (2, f(2)) = (2, 4). In fact, we can show that f exhibits odd symmetry about this point.

g. Putting all of the above information together, we arrive at the following graph of f(x).



Q :- $y = x^3 - 3x^2 - 9x - 5$

Sol

Domain: $(-\infty, \infty)$

Intercepts : $(0, -5)$, $(5, 0)$, $(-1, 0)$

$y' = 3x^2 - 6x - 9$

$3x^2 - 6x - 9 = 0$

$x = -1, 3$

Interval	$-\infty < x < -1$	$-1 < x < 3$	$3 < x < \infty$
Sign of y'	+ive	−ive	+ive
Behavior of y	Inc	Dec	Inc

$x = -1$ is Local max

$x = 3$ is " min according to 2^{nd} derivative test

$y(-1) = 0$

$y(3) = -32$

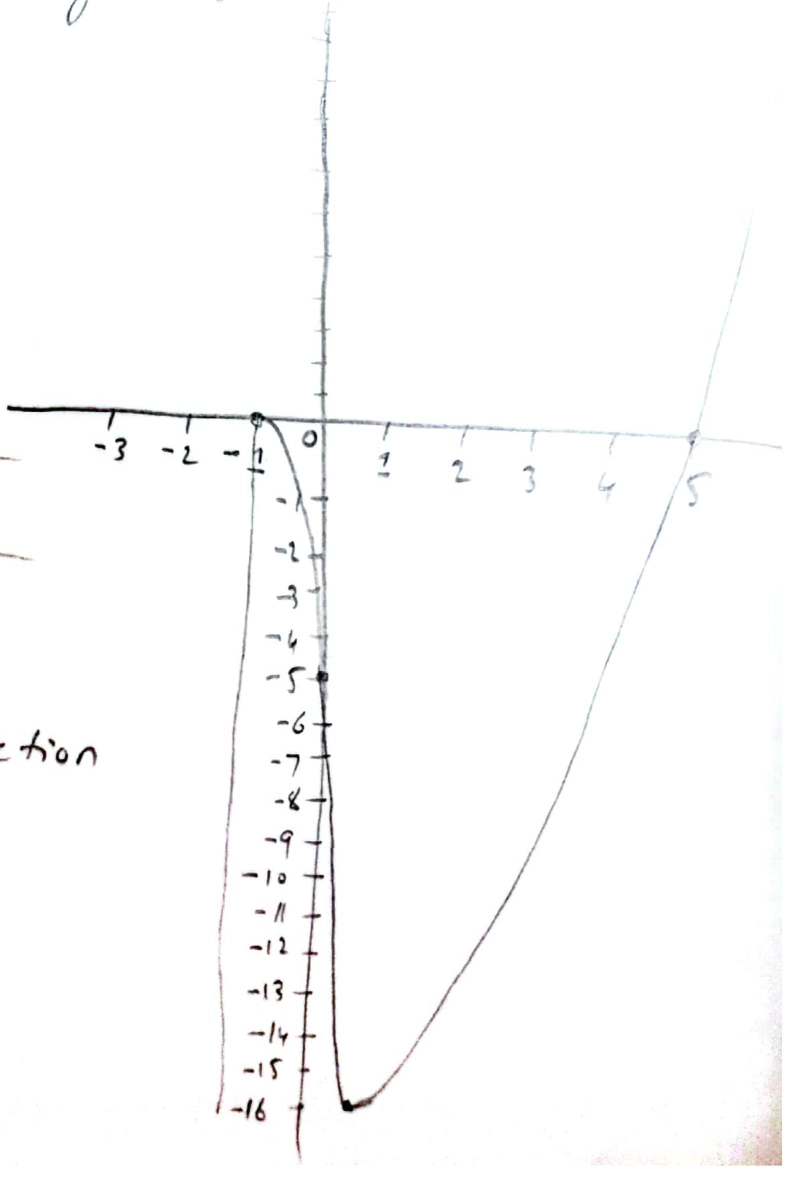
$y'' = 6x - 6 = 0$

$x = 1$

Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of y''	−ive	+ive
Concavity	C-D	C-up

$x = 1$ is pt of inflection

$y(1) = -16$



Sketch a graph of $f(x) = (x-1)^2(x+2)$.

Solution

Step 1. Since f is a polynomial, the domain is the set of all real numbers.

Step 2. When $x = 0$, $f(x) = 2$. Therefore, the y-intercept is $(0, 2)$. To find the x-intercepts, we need to solve the equation $(x-1)^2(x+2)=0$, gives us the x-intercepts $(1,0)$ and $(-2,0)$

Step 3. We need to evaluate the end behavior of f. As $x \to \infty$, $(x-1)^2 \to \infty$ and $(x+2) \to \infty$. Therefore, $\lim\limits_{x\to\infty} f(x) = \infty$. As $x \to -\infty$, $(x-1)^2 \to \infty$ and $(x+2) \to -\infty$. Therefore, $\lim\limits_{x\to\infty} f(x) = -\infty$.

Step 4. Since f is a polynomial function, it does not have any vertical asymptotes.

Step 5. The first derivative of f is

$f'(x) = 3x^2 - 3.$

Therefore, f has two critical numbers: $x = 1, -1$. Divide the interval $(-\infty, \infty)$ into the three smaller intervals: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. Then, choose test points $x = -2$, $x = 0$, and $x = 2$ from these intervals and evaluate the sign of $f'(x)$ at each of these test points, as shown in the following table.

Interval	Test Point	Sign of Derivative $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$	Conclusion
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing.
$(-1, 1)$	$x = 0$	$(+)(-)(+) = -$	f is decreasing.
$(1, \infty)$	$x = 2$	$(+)(+)(+) = +$	f is increasing.

From the table, we see that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. Evaluating $f(x)$ at those two points, we find that the local maximum value is $f(-1) = 4$ and the local minimum value is $f(1) = 0$.

Step 6. The second derivative of f is

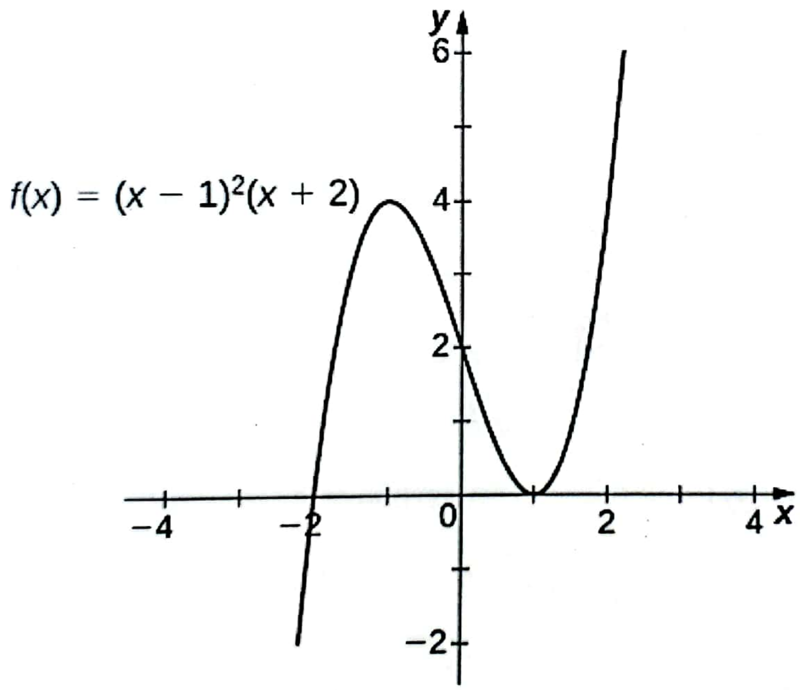
$f''(x) = 6x.$

The second derivative is zero at $x = 0$. Therefore, to determine the concavity of f, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$ and $(0, \infty)$, and choose test points $x = -1$ and $x = 1$ to determine the concavity of f on each of these smaller intervals as shown in the following table.

Interval	Test Point	Sign of $f''(x) = 6x$	Conclusion
$(-\infty, 0)$	$x = -1$	$-$	f is concave down.
$(0, \infty)$	$x = 1$	$+$	f is concave up.

We note that the information in the preceding table confirms the fact, found in step 5, that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. In addition, the information found in step 5—namely, f has a local maximum at $x = -1$ and a local minimum at $x = 1$, and $f'(x) = 0$ at those points—combined with the fact that $f''(x)$ changes sign only at $x = 0$ confirms the results found in step 6 on the concavity of f.

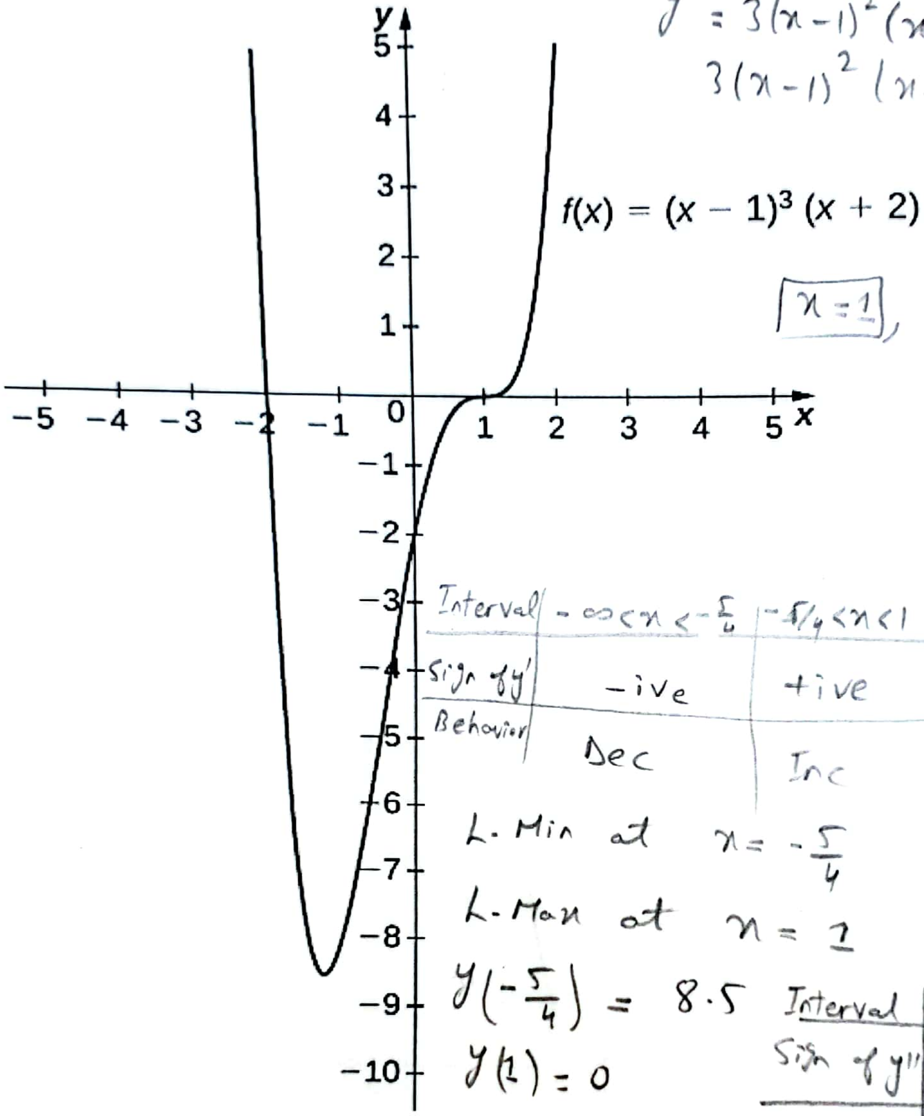
Combining this information, we arrive at the graph of $f(x) = (x-1)^2(x+2)$ shown in the following graph.



B2

Sketch a graph of $f(x) = (x-1)^3(x+2)$.

Solution



Domain: $(-\infty, \infty)$

x-intercept $(1, 0)$ $(-2, 0)$

y-intercept $(0, -2)$

$y' = 3(x-1)^2(x+2) + (x-1)^3$

$3(x-1)^2(x+2) + (x-1)^3 = 0$

$(x-1)^2[3(x+2)+(x-1)] = 0$

$\boxed{x = 1}$, $3x+6+x-1 = 0$

$4x + 5 = 0$

$\boxed{x = -\frac{5}{4}}$

Interval	$-\infty < x < -\frac{5}{4}$	$-\frac{5}{4} < x < 1$	$1 < x < \infty$
Sign of y'	$-$ive	$+$ive	$+$ive
Behavior	Dec	Inc	Inc

L. Min at $x = -\frac{5}{4}$

L. Max at $x = 1$

$y(-\frac{5}{4}) = 8.5$

$y(1) = 0$

$y'' = 12x^2 - 6x - 6$

$12x^2 - 6x - 6 = 0$

$6(2x^2 - x - 1) = 0$

$2x^2 - 2x + x - 1 = 0$

$2x(x-1) + 1(x-1) = 0$

$(2x+1)(x-1) = 0$

$x = -\frac{1}{2}, \; x = 1$

Interval	$-\infty < x < -\frac{1}{2}$	$-\frac{1}{2} < x < 1$	$1 < x < \infty$
Sign of y''	$+$ive	$-$ive	$+$ive
	C.up	C.D	C.up

$y(-\frac{1}{2}) = -\frac{81}{16} = 5.06$

Sketching a Rational Function

G1

Sketch the graph of $f(x) = \frac{x^2}{(1-x^2)}$.

Solution

Step 1. The function f is defined as long as the denominator is not zero. Therefore, the domain is the set of all real numbers x except $x = \pm 1$.

Step 2. Find the intercepts. If $x = 0$, then $f(x) = 0$, so 0 is an intercept. If $y = 0$, then $\frac{x^2}{(1-x^2)} = 0$, which implies $x = 0$. Therefore, $(0, 0)$ is the only intercept.

Step 3. Evaluate the limits at infinity. Since f is a rational function, divide the numerator and denominator by the highest power in the denominator: x^2. We obtain

$$\lim_{x \to \pm\infty} \frac{x^2}{1-x^2} = \lim_{x \to \pm\infty} \frac{1}{\frac{1}{x^2}-1} = -1.$$

Therefore, f has a horizontal asymptote of $y = -1$ as $x \to \infty$ and $x \to -\infty$.

Step 4. To determine whether f has any vertical asymptotes, first check to see whether the denominator has any zeroes. We find the denominator is zero when $x = \pm 1$. To determine whether the lines $x = 1$ or $x = -1$ are vertical asymptotes of f, evaluate $\lim\limits_{x \to 1} f(x)$ and $\lim\limits_{x \to -1} f(x)$. By looking at each one-sided limit as $x \to 1$, we see that

$$\lim_{x \to 1^+} \frac{x^2}{1-x^2} = -\infty \text{ and } \lim_{x \to 1^-} \frac{x^2}{1-x^2} = \infty.$$

In addition, by looking at each one-sided limit as $x \to -1$, we find that

$$\lim_{x \to -1^+} \frac{x^2}{1-x^2} = \infty \text{ and } \lim_{x \to -1^-} \frac{x^2}{1-x^2} = -\infty.$$

Step 5. Calculate the first derivative:

$$f'(x) = \frac{(1-x^2)(2x)-x^2(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}.$$

Critical numbers occur at points x where $f'(x) = 0$ or $f'(x)$ is undefined. We see that $f'(x) = 0$ when $x = 0$. The derivative f' is not undefined at any point in the domain of f. However, $x = \pm 1$ are not in the domain of f. Therefore, to determine where f is increasing and where f is decreasing, divide the interval $(-\infty, \infty)$ into four smaller intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$, and choose a test point in each interval to determine the sign of $f'(x)$ in each of these intervals. The values $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are good choices for test points as shown in the following table.

Interval	Test Point	Sign of $f'(x) = \frac{2x}{(1-x^2)^2}$	Conclusion
$(-\infty, -1)$	$x = -2$	$-/+ = -$	f is decreasing.
$(-1, 0)$	$x = -1/2$	$-/+ = -$	f is decreasing.
$(0, 1)$	$x = 1/2$	$+/+ = +$	f is increasing.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing.

From this analysis, we conclude that f has a local minimum at $x = 0$ but no local maximum.

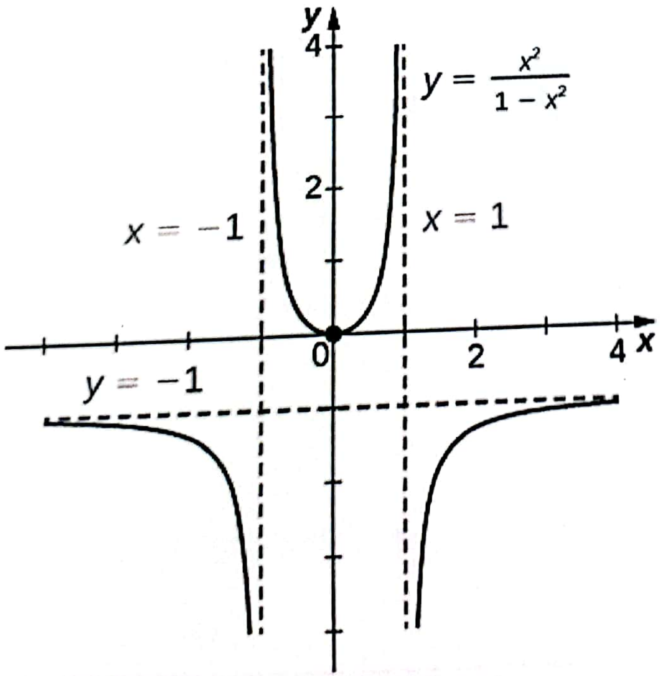
Step 6. Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{(1-x^2)^2(2) - 2x(2(1-x^2)(-2x))}{(1-x^2)^4} \\ &= \frac{(1-x^2)\left[2(1-x^2)+8x^2\right]}{(1-x^2)^4} \\ &= \frac{2(1-x^2)+8x^2}{(1-x^2)^3} \\ &= \frac{6x^2+2}{(1-x^2)^3}. \end{aligned}$$

To determine the intervals where f is concave up and where f is concave down, we first need to find all points x where $f''(x) = 0$ or $f''(x)$ is undefined. Since the numerator $6x^2 + 2 \neq 0$ for any x, $f''(x)$ is never zero. Furthermore, $f''(x)$ is not undefined for any x in the domain of f. However, as discussed earlier, $x = \pm 1$ are not in the domain of f. Therefore, to determine the concavity of f, we divide the interval $(-\infty, \infty)$ into the three smaller intervals $(-\infty, -1)$, $(-1, -1)$, and $(1, \infty)$, and choose a test point in each of these intervals to evaluate the sign of $f''(x)$. in each of these intervals. The values $x = -2$, $x = 0$, and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{6x^2+2}{(1-x^2)^3}$	Conclusion
$(-\infty, -1)$	$x = -2$	$+/- = -$	f is concave down.
$(-1, -1)$	$x = 0$	$+/+ = +$	f is concave up.
$(1, \infty)$	$x = 2$	$+/- = -$	f is concave down.

Combining all this information, we arrive at the graph of f shown below. Note that, although f changes concavity at $x = -1$ and $x = 1$, there are no inflection points at either of these places because f is not continuous at $x = -1$ or $x = 1$.



Sketch the graph of $f(x) = \frac{x^2}{(x-1)}$ C2

Solution

Step 1. The domain of f is the set of all real numbers x except $x = 1$.

Step 2. Find the intercepts. We can see that when $x = 0$, $f(x) = 0$, so $(0, 0)$

is the only intercept.

Step 3. Evaluate the limits at infinity. Since the degree of the numerator is one more than the degree of the denominator, f must have an oblique asymptote. To find the oblique asymptote, use long division of polynomials to write

$$f(x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

Since $1/(x-1) \to 0$ as $x \to \pm\infty$, $f(x)$ approaches the line $y = x + 1$ as $x \to \pm\infty$. The line $y = x + 1$ is an oblique asymptote for f.

Step 4. To check for vertical asymptotes, look at where the denominator is zero. Here the denominator is zero at $x = 1$. Looking at both one-sided limits as $x \to 1$, we find

$$\lim_{x\to1^+} \frac{x^2}{x-1} = \infty \text{ and } \lim_{x\to1^-} \frac{x^2}{x-1} = -\infty.$$

Therefore, $x = 1$ is a vertical asymptote, and we have determined the behavior of f as x approaches 1 from the right and the left.

Step 5. Calculate the first derivative:

$$f'(x) = \frac{(x-1)(2x)-x^2(1)}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2}.$$

We have $f'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$. Therefore, $x = 0$ and $x = 2$ are critical numbers. Since f is undefined at $x = 1$, we need to divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$, and choose a test point from each interval to evaluate the sign of $f'(x)$ in each of these smaller intervals. For example, let $x = -1$, $x = \frac{1}{2}$, $x = \frac{3}{2}$, and $x = 3$ be the test points as shown in the following table.

Interval	Test Point	$f'(x) = \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$	Conclusion
$(-\infty, 0)$	$x=-1$	$(-)(-)/+=+$	f is increasing.
$(0,1)$	$x=1/2$	$(+)(-)/+=-$	f is decreasing.
$(1,2)$	$x=3/2$	$(+)(-)/+=-$	f is decreasing.
$(2,\infty)$	$x=3$	$(+)(+)/+=+$	f is increasing.

From this table, we see that f has a local maximum at $x=0$ and a local minimum at $x=2$. The value of f at the local maximum is $f(0)=0$ and the value of f at the local minimum is $f(2)=4$. Therefore, $(0,0)$ and $(2,4)$ are important points on the graph.

Step 6. Calculate the second derivative:

$$\begin{aligned}
f''(x) &= \frac{(x-1)^2(2x-2)-(x^2-2x)(2(x-1))}{(x-1)^4} \\
&= \frac{(x-1)\left[(x-1)(2x-2)-2(x^2-2x)\right]}{(x-1)^4} \\
&= \frac{(x-1)(2x-2)-2(x^2-2x)}{(x-1)^3} \\
&= \frac{2x^2-4x+2-(2x^2-4x)}{(x-1)^3} \\
&= \frac{2}{(x-1)^3}.
\end{aligned}$$

We see that $f''(x)$ is never zero or undefined for x in the domain of f. Since f is undefined at $x=1$, to check concavity we just divide the interval $(-\infty,\infty)$ into the two smaller intervals $(-\infty,1)$ and $(1,\infty)$, and choose a test point from each interval to evaluate the sign of $f''(x)$ in each of these intervals. The values $x = 0$ and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{2}{(x-1)^3}$	Conclusion
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is concave down.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is concave up.

From the information gathered, we arrive at the following graph for f.