EXAMPLE 6 **Business: Minimizing Inventory Costs.** A retail appliance store sells 2500 television sets per year. It costs \$10 to store one set for a year. To reorder, there is a fixed cost of \$20, plus a fee of \$9 per set. How many times per year should the store reorder, and in what lot size, to minimize inventory costs?

Solution Let x = the lot size. Inventory costs are given by

$$C(x) = (\text{Yearly carrying costs}) + (\text{Yearly reorder costs}).$$

We consider each component of inventory costs separately.

a) *Yearly carrying costs.* The average amount held in stock is $x/2$, and it costs \$10 per set for storage. Thus,

$$\text{Yearly carrying costs} = \begin{pmatrix}\text{Yearly cost}\\ \text{per item}\end{pmatrix} \cdot \begin{pmatrix}\text{Average number}\\ \text{of items}\end{pmatrix}$$

$$= 10 \cdot \frac{x}{2}.$$

b) *Yearly reorder costs.* We know that x is the lot size, and we let N be the number of reorders each year. Then $Nx = 2500$, and $N = 2500/x$. Thus,

$$\text{Yearly reorder costs} = \begin{pmatrix}\text{Cost of each}\\ \text{order}\end{pmatrix} \cdot \begin{pmatrix}\text{Number of}\\ \text{reorders}\end{pmatrix}$$

$$= (20 + 9x)\frac{2500}{x}.$$

c) Thus, we have

$$C(x) = 10 \cdot \frac{x}{2} + (20 + 9x)\frac{2500}{x}$$

$$= 5x + \frac{50{,}000}{x} + 22{,}500 = 5x + 50{,}000x^{-1} + 22{,}500.$$

d) To find a minimum value of C over $[1, 2500]$, we first find $C'(x)$:

$$C'(x) = 5 - \frac{50{,}000}{x^2}.$$

e) $C'(x)$ exists for all x in $[1, 2500]$, so the only critical values are those x-values such that $C'(x) = 0$. We solve $C'(x) = 0$:

$$5 - \frac{50{,}000}{x^2} = 0$$

$$5 = \frac{50{,}000}{x^2}$$

$$5x^2 = 50{,}000$$

$$x^2 = 10{,}000$$

$$x = \pm 100.$$

Since there is only one critical value in $[1, 2500]$, that is, $x = 100$, we can use the second derivative to see whether it yields a maximum or a minimum:

$$C''(x) = \frac{100{,}000}{x^3}.$$

$C''(x)$ is positive for all x in $[1, 2500]$, so we have a minimum at $x = 100$. Thus, to minimize inventory costs, the store should order $2500/100$, or 25, times per year. The lot size is 100 sets.

3.83. Example. A monopolistic firm has a total revenue function

$$R(Q) = -AQ^2 + BQ$$

(Sec. 3.2, Prob. 7) and a total cost function

$$C(Q) = aQ^2 + bQ + c$$

(Sec. 3.22a), where the coefficients A, B, a, b, c are all positive constants and $B > b$. The government wishes to levy an excise tax on the commodity produced by the firm. What tax rate should the government impose on the firm's output to maximize the tax revenue $T = rQ$, knowing that the firm will add the tax to its costs and adjust its output to maximize the profit after taxes?

SOLUTION. The cost and profit after taxes are

$$C_T(Q) = C(Q) + rQ = aQ^2 + (b + r)Q + c$$

and

$$\Pi_T(Q) = R(Q) - C_T(Q) = -(A + a)Q^2 + (B - b - r)Q - c \tag{5}$$

(Sec. 3.2, Prob. 10). Differentiating (5) with respect to Q, with r regarded as a constant, and setting the result equal to zero, we get the equation

$$\frac{d\Pi_T(Q)}{dQ} = -2(A + a)Q + (B - b - r) = 0,$$

whose only solution is

$$Q_0 = \frac{B - b - r}{2(A + a)}. \tag{6}$$

Since

$$\frac{d^2\Pi_T(Q)}{dQ^2} = -2(A + a) < 0,$$

it follows from the second derivative test that the output level (6) actually maximizes the firm's profit after taxes, at the tax rate r.

Knowing that the firm will maximize its profit after taxes, the government chooses its tax rate r to maximize the revenue

$$T = rQ_0 = \frac{(B - b - r)r}{2(A + a)}, \tag{7}$$

calculated at the output level (6). To maximize T as a function of the tax rate r, which is now regarded as variable, we differentiate (7) with respect to r, obtaining

$$\frac{dT}{dr} = \frac{B - b - 2r}{2(A + a)}.$$

The optimum tax rate r_0 is the solution of the equation $dT/dr = 0$, namely

$$r_0 = \frac{B - b}{2}.$$

By the second derivative test, r_0 actually maximizes the government's revenue, since

$$\frac{d^2T}{dr^2} = -\frac{1}{A + a} < 0.$$

3.82. Example. An island lies l miles offshore from a straight beach. Down the beach h miles from the point nearest the island, there is a group of vacationers who plan to get to the island by using a beach buggy going α mi/hr, trailing a motorboat which can do β mi/hr. At what point of the beach should the vacationers transfer from the buggy to the boat in order to get to the island in the shortest time?

SOLUTION. The geometry of the problem is shown in Figure 15, where the vacationers start at A, the island is at C, and x is the distance between the point P at which they launch the boat and the point B of the beach nearest the island. The time it takes to get to the island is given by the formula

$$T = T(x) = \frac{|AP|}{\alpha} + \frac{|PC|}{\beta}$$
$$= \frac{1}{\alpha}(h - x) + \frac{1}{\beta}\sqrt{x^2 + l^2} \qquad (0 \leqslant x \leqslant h) \tag{3}$$

(the time taken equals the distance travelled divided by the speed), where the boat leaves from the starting point A if $x = h$ and from the point B nearest the island if $x = 0$. Differentiating (3) with respect to x, we get

$$\frac{dT}{dx} = \frac{1}{\beta}\left(\frac{x}{\sqrt{x^2 + l^2}} - k\right),$$

where $k = \beta/\alpha$. If $k \geqslant 1$, that is, if $\beta \geqslant \alpha$, then dT/dx is negative for all $x \in (0, h)$, and hence T is decreasing in $[0, h]$, by Sec. 3.54. In this case, T takes its global minimum in $[0, h]$ at $x = h$, so that the vacationers should forget about the buggy and go straight to the island by boat.

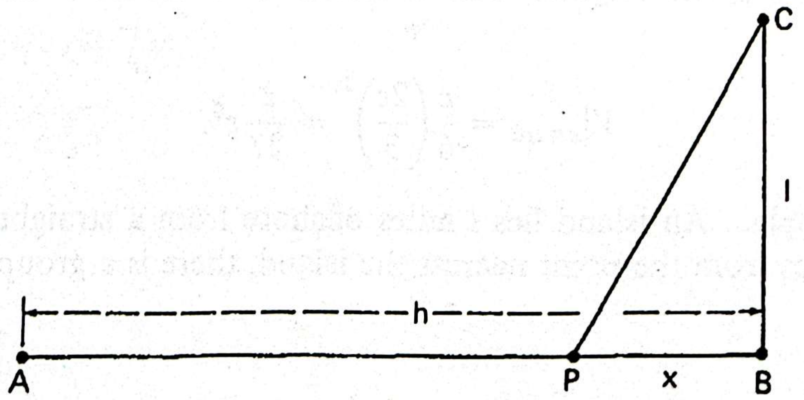
The same is true if $k < 1$, provided that

$$x_0 = \frac{kl}{\sqrt{1 - k^2}} \geqslant h. \tag{4}$$

In fact, if $k < 1$, the equation

$$\frac{x}{\sqrt{x^2 + l^2}} - k = 0$$

has the unique solution $x = x_0$, where x_0 lies outside the interval $(0, h)$ if (4) holds. Therefore dT/dx is again negative at every point of $(0, h)$, since dT/dx cannot change sign in $(0, h)$ and dT/dx is clearly negative for small enough x. But then T is again decreasing in $[0, h]$. Thus, in this case too, the vacationers should go straight to the island by boat.



However, if

$$x_0 = \frac{kl}{\sqrt{1 - k^2}} < h,$$

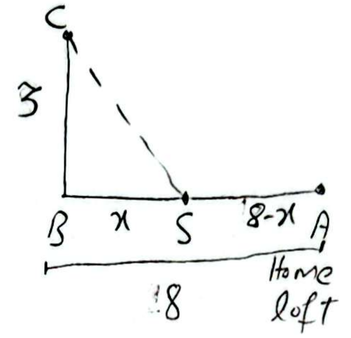
then x_0 lies in the interval $(0, h)$. Moreover, in this case, dT/dx is negative at every point of $(0, x_0)$ and positive at every point of (x_0, h). Therefore T must have a local minimum at x_0, by the first derivative test (Theorem 3.64a). But then T takes its global minimum in $[0, h]$ at x_0 (why?). This means that the vacationers should now stop the buggy and launch the boat at the point with coordinate x_0, as measured from B.

1) pigeons avoid flying over water. Suppose a homing pigeon is released island at pt C, which is 3mi directly out in the water from B on shore. Pt B is 8mi downshore from the pigeon's home loft pt A. Assuming that a pigeon flying over water uses energy at a e 1.28 times the rate over land Toward what pt S downshore from A uld the pigeon fly inorder to minimize total energy required to et to home loft at A.

Sol Total energy = (Energy rate over water).(Distance over water)

+(" " " land).(" " land)

Let 'x' be the distance from A to S



$$E(x) = 1.28 \times \sqrt{9+x^2} + 1.(8-x)$$

$$E(x) = 1.28\sqrt{9+x^2} + (8-x) \quad [0, 8]$$

$$E'(x) = (1.28).\frac{1}{2}(9+x^2)^{-1/2}.(2x) - 1$$

$$E'(x) = \frac{1.28x}{\sqrt{9+x^2}} - 1$$

$$E'(x) = 0$$

$$\frac{1.28x}{\sqrt{9+x^2}} = 1$$

$$1.28x = \sqrt{9+x^2}$$

$$(1.28x)^2 = 9+x^2.$$

$$1.6384x^2 = 9+x^2$$

$$0.6384x^2 = 9$$

$$x^2 = 14.0977$$

$$x = \pm 3.75$$

$E(0) = 11.84$

$E(3.75) = 10.39$ $\quad 8-3.75 = 4.25$

$E(8) = 10.93$

S should be at 4.25mi inorder to minimize total energy required to get to home at A.

3.81. Example. A square box with no top is made by cutting little squares out of the four corners of a square sheet of metal c inches on a side, and then folding up the resulting flaps, as shown in Figure 14. What size squares should be cut out to make the box of largest volume?

SOLUTION. Let x be the side length of each little square. Then the volume of the box in cubic inches is just

$$V = V(x) = x(c - 2x)^2. \tag{1}$$

Moreover,

$$0 \leqslant x \leqslant \frac{c}{2}, \tag{2}$$

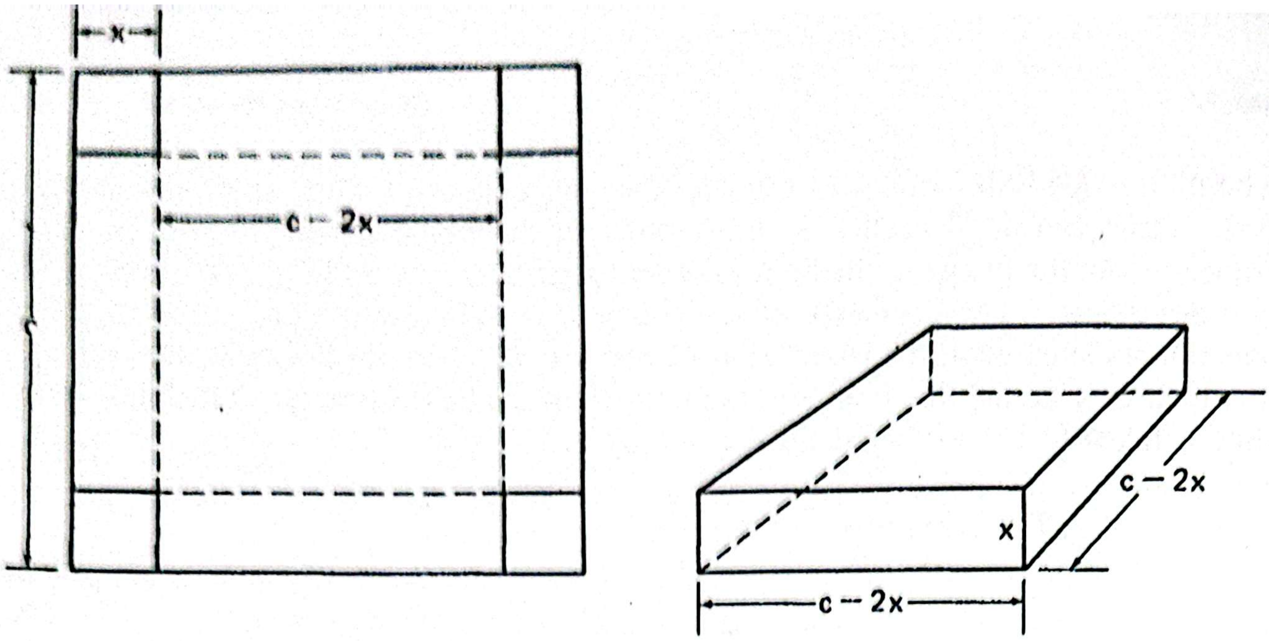


Figure 14.

since it is impossible to cut away either overlapping squares or squares of negative side length. Our problem is thus to determine the value of x at which the function (1) takes its global maximum in the interval (2). Since V is differentiable for all x, the only critical points of V, and hence, by Sec. 3.63c, the only points at which V can have a local extremum in the whole interval $(-\infty, \infty)$ are the solutions of the equation

$$\frac{dV}{dx} = c^2 - 8cx + 12x^2 = (c - 6x)(c - 2x) = 0,$$

namely $x = c/6$ and $x = c/2$. Moreover, since V is positive in the open interval

$$0 < x < \frac{c}{2}$$

and vanishes at the end points $x = 0$ and $x = c/2$, the global maximum of V in the closed interval (2), guaranteed by Theorem 3.32c and by the "physical meaning" of the problem, must be at an interior point of (2), and hence must be a local maximum of V. But $x = c/6$ is the only interior point of (2) at which V can have a local extremum, and therefore it is apparent without any further tests that V takes its maximum in (2) at the point $x = c/6$. This can be confirmed by noting that

$$\left.\frac{d^2V}{dx^2}\right|_{x=c/6} = (-8c + 24x)|_{x=c/6} = -4c < 0,$$

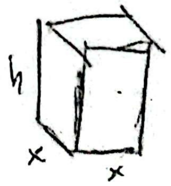
and then applying the second derivative test (Theorem 3.65a).

Thus, finally, the largest box is obtained by cutting squares of side length $c/6$ out of the corners of the original sheet of metal. The volume of the resulting box equals

$$V|_{x=c/6} = \frac{c}{6}\left(\frac{2c}{3}\right)^2 = \frac{2}{27}c^3.$$

50 by $66\frac{2}{3}$ each   A = 2000 -

5) Engineers are designing a box-shaped aquarium with a square bottom and an open top. The aquarium must hold 500 ft³ of water. What dimensions should they use to create an acceptable aquarium with the least amount of glass?   200 -



10 ft by 10 ft by 5 ft

$V = x^2 h$

$500 = x^2 h$

$SA = 4xh + x^2$

$\frac{500}{x^2} = h \quad (0, \infty)$

$SA = \frac{4x \cdot 500}{x^2} + x^2$

$\frac{2000}{x} + x^2$

$\cancel{SA}\ 0 = 2x - \frac{2000}{x^2}$

$2000 = 2x^3$

$x = 10$

$h = 5$