Assignment 3.

Q1) A).

x = length of base (feet)

y = Width of the base (feet)

z = height of container (feet)

Volume of constraint $= V = xyz = 480$ ---- (A)

$$g(x,y,z) = xyz - 480 = 0 \Rightarrow V(x,y,z).$$

The cost function to minimize is.

Bottom area: xy costing 5\$ per sq. foot.

Top area: xy costing 3\$ per sq. foot

Side areas: $2(xz + yz)$. costing 3\$ per sq. foot.

Total cost function

$$C(x,y,z) = 5xy + 3xy + 3(2xz + 2yz)$$

$$= 8xy + 6xz + 6yz \Rightarrow f(x,y,z)$$

Now by using Lagrange Multiplier.

$$\nabla C = \lambda \nabla V$$

$$(8y + 6z)i + (8x + 6z)j + (6x + 6y)k$$

$$= \lambda(yz\, i + xz\, j + xy\, k).$$

Now comparing

$8y + 6z = \lambda yz$; $8x + 6z = \lambda xz$; $6x + 6y = \lambda xy$.

$8y + 6z - \lambda yz = 0$; $8x + 6z - \lambda xz = 0$; $6x + 6y - \lambda xy = 0$

---- (1) ---- (2) ---- (3)

compare (1) and (2)

$$8y + 6z - \lambda yz = 8x + 6z - \lambda xz$$

$$8y - 8x + \cancel{6z} - \lambda yz - \cancel{6z} + \lambda xz = 0.$$

$$8(y - x) - \lambda z(y - x) = 0$$

$$(y-x)\ (8-hz) = 0$$

$$y - x = 0 \quad \text{or} \quad 8 - hz = 0$$

$$y = x \quad \text{or} \quad z = \frac{8}{h}$$

Now from eqn. (A)

$$x \cdot x \cdot z = 480$$

$$z = \frac{480}{x^2}$$

put in eqn (3)

$$6x + 6x - hx^2 = 0$$

$$12x = hx^2$$

$$h = \frac{12}{x}$$

Now put in eqn (1)

$$8y + 6z - hyz = 0$$

$$8x + 6\left(\frac{480}{x^2}\right) - \frac{12}{x} \cdot x \cdot \frac{480}{x^2} = 0$$

$$8x + \frac{2880}{x^2} = \frac{5760}{x^2}$$

$$8x^3 = 5760 - 2880$$

$$x^3 = \frac{2880}{8} = 360$$

$$x = (360)^{1/3}$$

$$\boxed{x = 7.11 \text{ feet}}$$

$$\boxed{y = 7.11 \text{ feet}}$$

$$z = \frac{480}{(7.11)^2} = \boxed{9.49 \text{ feet} = z}$$

These dimensions gives the minimum construction cost for the cargo container.

Q2) 13) x and y be the dimension of rectangular cross section. (width and height)

z = length of package.

$$V = xyz.$$

The constraint equation is.

$$2(x+y) + z = 108$$

$$0 = 2(x+y) + z - 108 = g.$$

So $\nabla V = \lambda \nabla g$

$$yz\,i + xz\,j + xy\,k = \lambda(2i + 2j + k)$$

$yz = 2\lambda$; $xz = 2\lambda$; $xy = \lambda$.

Now

$$yz = xz$$

$$yz - xz = 0$$

$$z(y-x) = 0$$

Since $z \neq 0$ so $y - x = 0 \Rightarrow y = x$.

$$2(x+x) + z = 108$$

$$z = 108 - 4x.$$

So $x(108 - 4x) = 2\lambda$.

$$x \cdot x = \frac{x(108-4x)}{2}$$

$$2x^2 = 108x - 4x^2$$

$$6x^{\cancel{2}} = 108\cancel{x}$$

$$x = \frac{108}{6} = 18 \Rightarrow \boxed{y = x = 18}$$

$$z = 108 - 4(18)$$
$$= 36.$$

$$V = x \times y \times z$$
$$= 18 \times 18 \times 36$$

$$V = 11664$$ cubic inches.

Q1) Volume of the silo.

$V =$ Volume of cylinder + Volume of Hemisphere

$$V = \pi r^2 h + \frac{2}{3}\pi r^3.$$

Surface area of silo.

$S =$ curved surface of cylinder + curved surface of Hemisphere + Base.

$$= 2\pi r h + 2\pi r^2 + \pi r^2 = 2\pi r h + 3\pi r^2 = S.$$

$$2\pi r h + 3\pi r^2 - S = 0 \Rightarrow g.$$

Now

$$\nabla V = \lambda \nabla g.$$

$$\left(\pi 2 r h + \frac{2}{3}\cdot\pi\cdot 3r^2\right) i + (\pi r^2) j$$

$$= \lambda\left((2\pi h + 6\pi) i + 2\pi r j\right)$$

$$2\pi r h + 2\pi r^2 = \lambda(2\pi h + 6\pi r) \quad ; \quad \cancel{\pi} r^{\cancel{2}} = \lambda 2 \cancel{\pi} \cancel{r}$$

$$\ldots (1) \qquad \boxed{\lambda = \frac{r}{2}}$$

put in (1)

$$2\pi r h + 2\pi r^2 = \frac{r}{2}(2\pi h + 6\pi r).$$

$$2\pi r h + 2\pi r^2 = \pi r h + 3r^2\pi$$

$$2\pi r h - \pi r h = 3r^2\pi - 2r^2\pi$$

$$\cancel{\pi} r h = \cancel{\pi} r^2$$

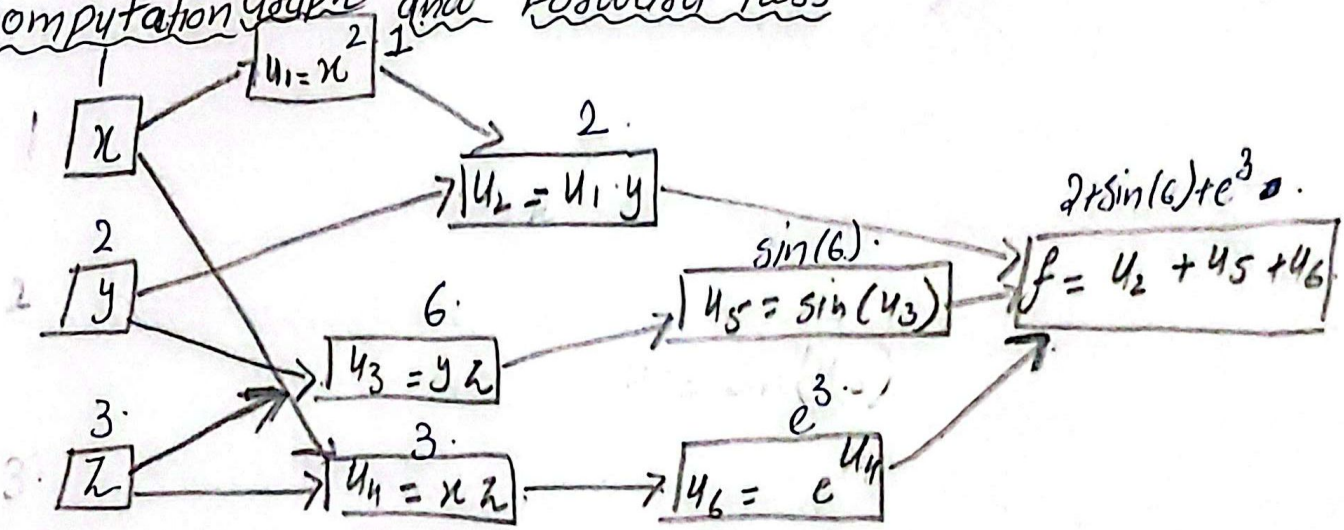
$$\boxed{h = r}$$

proved.

Q2)

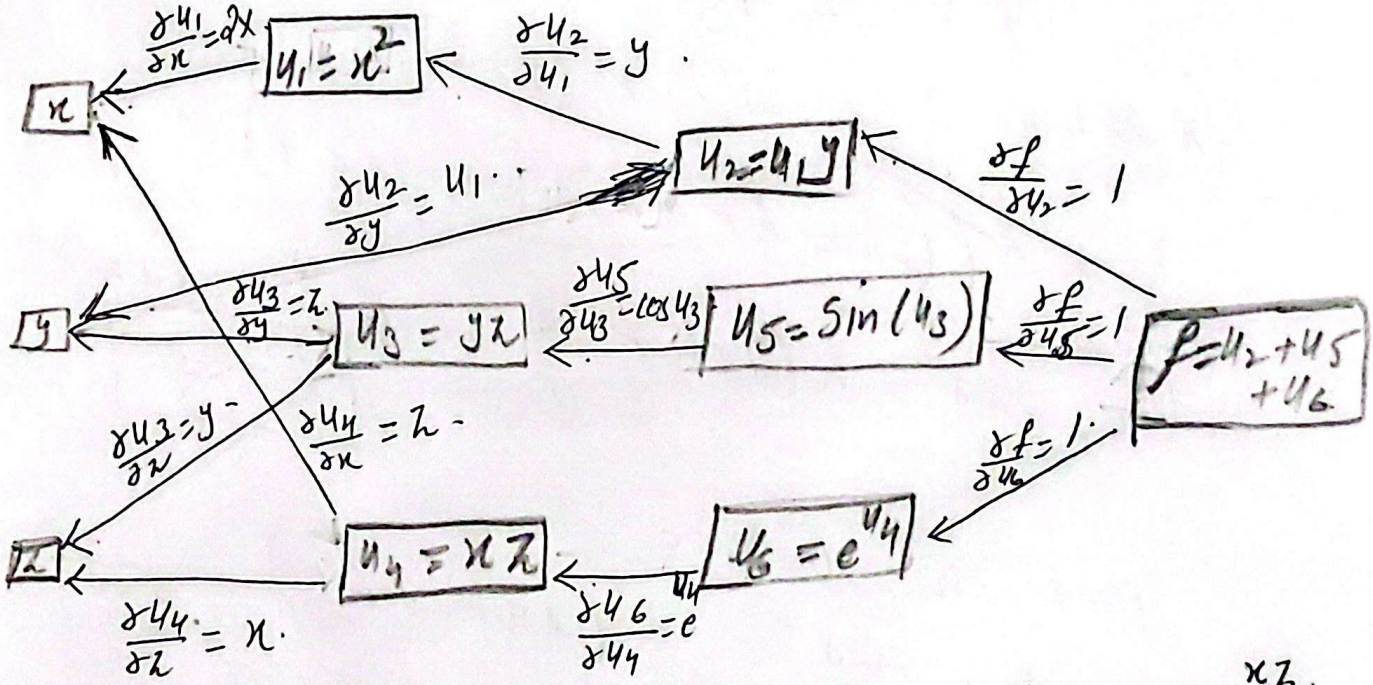
A) $f(x,y,z) = x^2 y + \sin y z + e^{xz}$. $(x,y,z) = (1,2,3)$.

Computation Graph and Forward Pass.



Backward Pass

This is used to compute gradients using chain rule.



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u_2} \times \frac{\partial u_2}{\partial u_1} \times \frac{\partial u_1}{\partial x} + \frac{\partial f}{\partial u_6} \times \frac{\partial u_6}{\partial u_4} \times \frac{\partial u_4}{\partial x} = 2xy + ze^{xz}$$
$$= 4 + 3e^3.$$

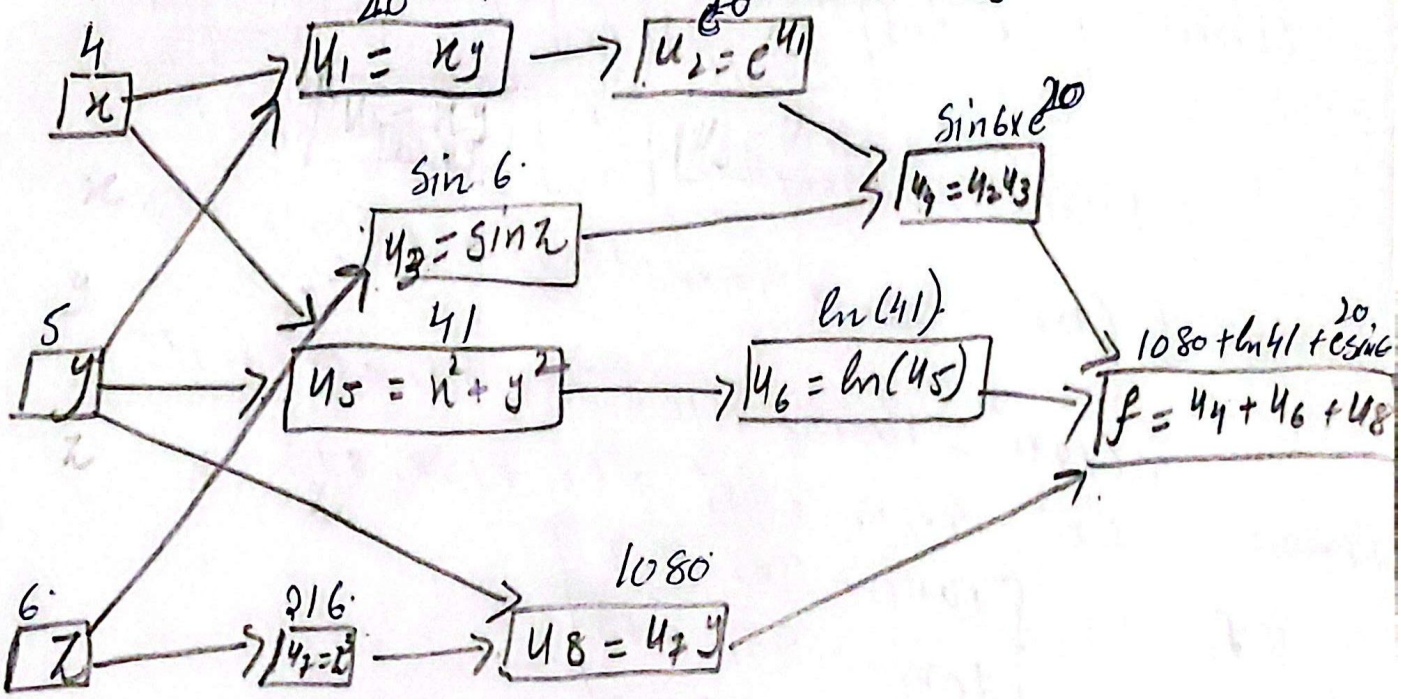
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u_2} \times \frac{\partial u_2}{\partial y} + \frac{\partial f}{\partial u_5} \times \frac{\partial u_5}{\partial u_3} \times \frac{\partial u_3}{\partial y} = x^2 + z\cos yz$$
$$= 1 + 3\cos(6).$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u_5} \times \frac{\partial u_5}{\partial u_3} \times \frac{\partial u_3}{\partial z} + \frac{\partial f}{\partial u_6} \times \frac{\partial u_6}{\partial u_4} \times \frac{\partial u_4}{\partial z} = z\cos(yz) + xe^{xz}$$
$$= 2\cos 6 + e^3.$$

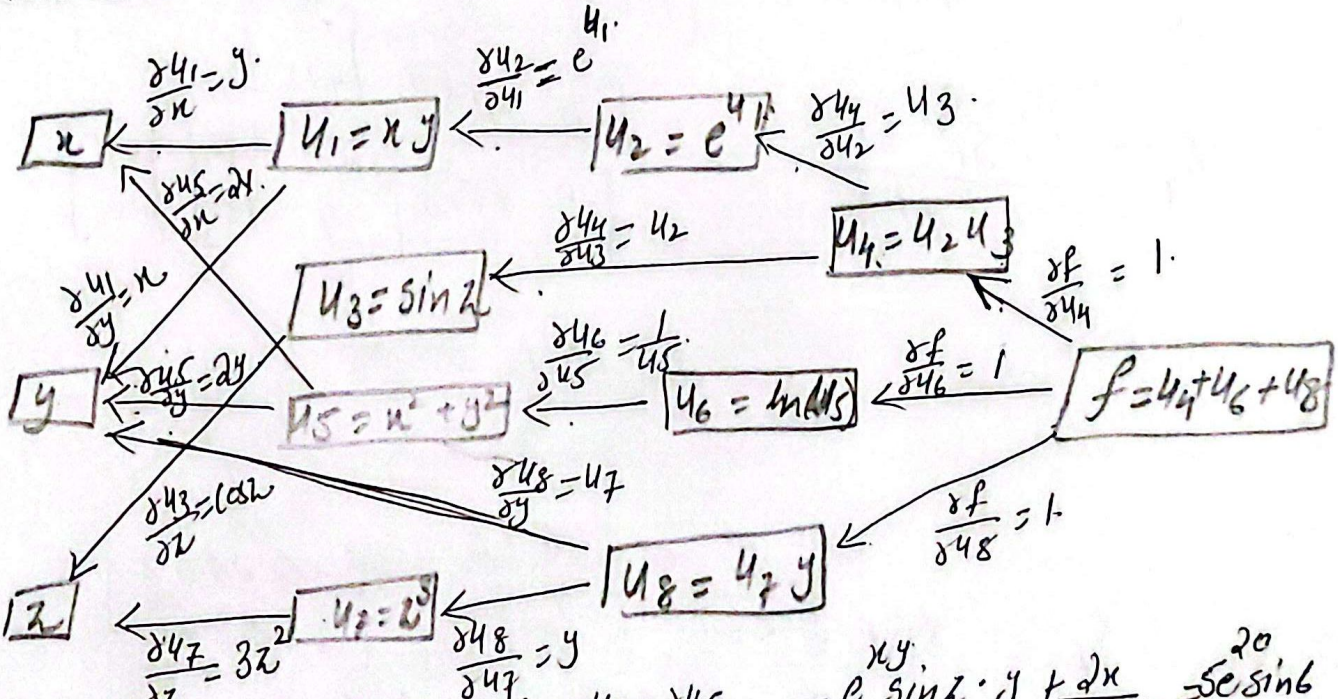
Q2)

B) $f(x,y,z) = e^{xy} \sin z + \ln(x^2+y^2) + yz^3$ $\quad (x,y,z)=(4,5,6)$

Computation Graph and Forward Pass.



Backward Pass.



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u_4} \times \frac{\partial u_4}{\partial u_2} \times \frac{\partial u_2}{\partial u_1} \times \frac{\partial u_1}{\partial x} + \frac{\partial f}{\partial u_6} \times \frac{\partial u_6}{\partial u_5} \times \frac{\partial u_5}{\partial x} = e^{xy} \sin z \cdot y + \frac{2x}{x^2+y^2} = 5e^{20} \sin 6 + \frac{8}{41}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u_4} \times \frac{\partial u_4}{\partial u_2} \times \frac{\partial u_2}{\partial u_1} \times \frac{\partial u_1}{\partial y} + \frac{\partial f}{\partial u_6} \times \frac{\partial u_6}{\partial u_5} \times \frac{\partial u_5}{\partial y} + \frac{\partial f}{\partial u_8} \times \frac{\partial u_8}{\partial y} = xe^{xy} \sin z + \frac{2y}{x^2+y^2} + z^3$$

$$= 4e^{20} \sin 6 + \frac{10}{41} + 216.$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u_4} \times \frac{\partial u_4}{\partial u_3} \times \frac{\partial u_3}{\partial z} + \frac{\partial f}{\partial u_8} \times \frac{\partial u_8}{\partial u_7} \times \frac{\partial u_7}{\partial z} = e^{xy} \cos z + 3yz^2 = e^{20} \cos 6 + 540$$

Q3) Steepest Descent Method.

A) $F(X) = \frac{1}{2} X^T H X + C^T X + 13$ $\quad H = \begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix}$.

$C = \begin{bmatrix} 4 \\ -15 \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$F(x_1 x_2) = \frac{1}{2} [x_1 \ x_2] \begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [4 \ -15] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 13$$

$$= \frac{1}{2} [10x_1 - 9x_2 \quad -9x_1 + 10x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 4x_1 - 15x_2 + 13$$

$$= \frac{1}{2} (10x_1^2 - 9x_1x_2 - 9x_1x_2 + 10x_2^2) + 4x_1 - 15x_2 + 13$$

$$= \frac{1}{2} (10x_1^2 - 18x_1x_2 + 10x_2^2) + 4x_1 - 15x_2 + 13$$

$$f(x_1 x_2) = 5x_1^2 - 9x_1x_2 + 5x_2^2 + 4x_1 - 15x_2 + 13$$

$$\nabla f_1 = \begin{bmatrix} 10x_1 - 9x_2 + 4 \\ 10x_2 - 9x_1 - 15 \end{bmatrix} =$$

$$S_1 = -\nabla f_1(X_1) = \begin{bmatrix} -4 \\ +15 \end{bmatrix}$$

$$h_1 = \frac{S_i^T S_i}{S_i^T H S_i} = \frac{[-4 \ +15] \begin{bmatrix} -4 \\ +15 \end{bmatrix}}{[-4 \ +15] \begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix} \begin{bmatrix} -4 \\ +15 \end{bmatrix}}$$

$$= \frac{16 + 225}{3490}$$

$$= 0.0691$$

$$X_2 = X_1 + 0.0691 \begin{bmatrix} -4 \\ -15 \end{bmatrix} = \begin{bmatrix} -0.2762 \\ 1.0358 \end{bmatrix}$$

Now $\nabla f(X_2) = \begin{bmatrix} -8.0845 \\ -2.1559 \end{bmatrix} \neq 0$

$$S_2 = -\nabla f_2(X_2) = \begin{bmatrix} 8.0845 \\ 2.1559 \end{bmatrix}$$

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$$h_2 = \frac{S_2^T S_2}{S_2^T H S_2} = \frac{\begin{bmatrix} 8.0845 & 2.1559 \end{bmatrix} \begin{bmatrix} 8.0845 \\ 2.1559 \end{bmatrix}}{\begin{bmatrix} 8.0845 & 2.1559 \end{bmatrix} \begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix} \begin{bmatrix} 8.0845 \\ 2.1559 \end{bmatrix}}$$

$$h_2 = 0.1812$$

$$X_3 = X_2 + h_2 S_2 = \begin{bmatrix} -0.2762 \\ 1.0358 \end{bmatrix} + 0.1812 \begin{bmatrix} 8.0845 \\ 2.1559 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1887 \\ 1.4265 \end{bmatrix}$$

$$\nabla f(X_3) = \begin{bmatrix} 3.0490 \\ -11.4338 \end{bmatrix} \neq 0$$

So $S_3 = -\nabla f(X_3) = \begin{bmatrix} -3.0490 \\ 11.4338 \end{bmatrix}$

$$h_3 = \frac{S_3^T S_3}{S_3^T H S_3} = \frac{\begin{bmatrix} -3.0490 & 11.4338 \end{bmatrix} \begin{bmatrix} -3.0490 \\ 11.4338 \end{bmatrix}}{\begin{bmatrix} -3.0490 & 11.4338 \end{bmatrix} \begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix} \begin{bmatrix} -3.0490 \\ 11.4338 \end{bmatrix}}$$

$$= 0.0691$$

$$X_4 = X_3 + h_3 S_3 = \begin{bmatrix} 1.1887 \\ 1.4265 \end{bmatrix} + 0.0691 \begin{bmatrix} -3.0490 \\ 11.4338 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 0.9782 \\ 2.2160 \end{bmatrix}$$

Q3 B) $f(x,y) = 5x^2 + 5y^2 - xy - 11x + 11y + 11$. $(x,y) = (0,0)$

$$\nabla f = \begin{bmatrix} 10x - y - 11 \\ 10y - x + 11 \end{bmatrix}$$

$$S_1 = -\nabla f(X_1) = -\begin{bmatrix} -11 \\ 11 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \end{bmatrix} \qquad H = \begin{bmatrix} 10 & -1 \\ -1 & 10 \end{bmatrix}$$

$$h_1 = \frac{S_1^T S_1}{S_1^T H S_1} = \frac{[11 \quad -11]\begin{bmatrix}11\\-11\end{bmatrix}}{[11 \quad -11]\begin{bmatrix}10 & -1\\-1 & 10\end{bmatrix}\begin{bmatrix}11\\-11\end{bmatrix}} = 0.0909$$

$$X_2 = X_1 + h_1 S_1$$

$$= \begin{bmatrix}0\\0\end{bmatrix} + 0.0909\begin{bmatrix}11\\-11\end{bmatrix} = \begin{bmatrix}0.9999\\-0.9999\end{bmatrix}$$

$$\nabla f(X_2) = \begin{bmatrix}-0.0001\\0.0001\end{bmatrix} \approx 0$$

Solution converges in single iteration

Q4) A)

$$\int_0^2\int_0^2 (16 - x^2 - y^2)\,dy\,dx$$

$$= \int_0^2 \left[16y - x^2 y - \frac{y^3}{3}\right]_0^2 dx$$

$$= \int_0^2 \left(16(2) - x^2(2) - \frac{2^3}{3}\right) dx$$

$$= \int_0^2 \left(32 - 2x^2 - \frac{8}{3}\right) dx$$

$$= \int_0^2 \left(\frac{96-8}{3} - 2x^2\right) dx = \int_0^2 \left(\frac{88}{3} - 2x^2\right) dx$$

$$= \frac{88}{3}x - \frac{2x^3}{3}\Big|_0^2 = \frac{88}{3}(2) - \frac{2(2)^3}{3}$$

$$= \frac{176}{3} - \frac{16}{3} = \frac{160}{3}$$

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Q4) B) Find volume.

$$3x + 2y - z = 0$$

$$z = 3x + 2y.$$

$y = x^2$ and $x = y^2$

$$y = x^2$$

$$y = (y^2)^2 \Rightarrow y = y^4 \Rightarrow y^4 - y = 0.$$

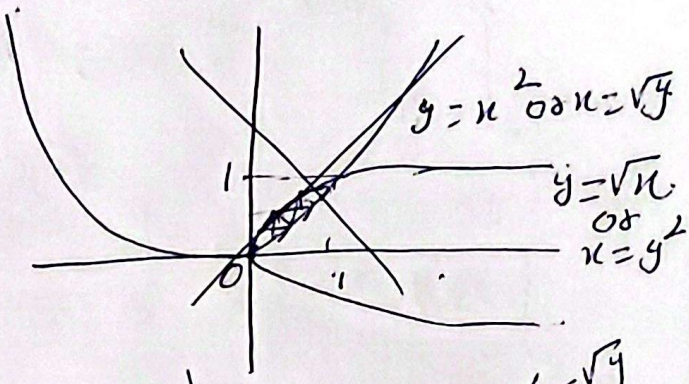
$$y(y^3 - 1) = 0.$$

$$y = 0; \; y^3 = 1$$

$$y = 1.$$

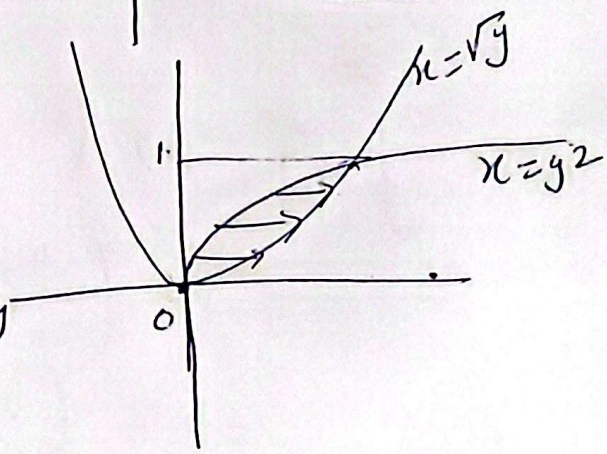
Now $x = y^2$

and $y = \pm\sqrt{x}$



$$\int_0^1 \int_{y^2}^{\sqrt{y}} (3x + 2y)\, dx\, dy$$

$$\int_0^1 \frac{3x^2}{2} + 2xy \Big|_{y^2}^{\sqrt{y}} dy$$



$$\int_0^1 \left(\frac{3}{2}(\sqrt{y})^2 + 2\sqrt{y}\cdot y\right) - \left(\frac{3y^4}{2} + 2y^3\right) dy$$

$$\int_0^1 \left(\frac{3}{2}y + 2y^{3/2} - \frac{3}{2}y^4 - 2y^3\right) dy$$

$$\left(\frac{3}{2}\frac{y^2}{2} + \frac{2y^{3/2+1}}{3/2+1} - \frac{3}{2}\frac{y^5}{5} - \frac{2y^4}{4}\right)_0^1$$

$$= \frac{3}{4} + \frac{2}{5/2} - \frac{3}{10} - \frac{1}{2} = 3/4.$$

Q4) c) $\int_0^2 \int_{\frac{x^2}{2}}^2 \sqrt{y}\cos y \, dy \, dx.$

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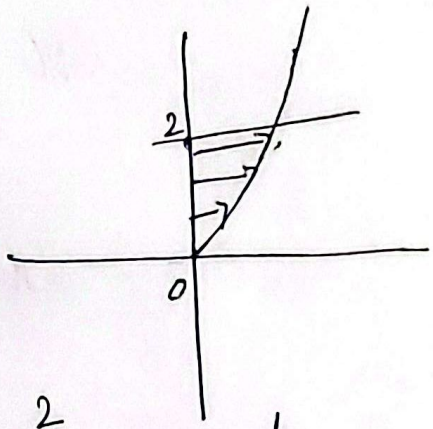
$y = 2 \; ; \; y = \frac{x^2}{2} \; ; \; x = 0 \; ; \; x = 2$

$y = 2 \; ; \; x^2 = 2y$ put $x = 0$, $y = 0$.

$x = \sqrt{2y}.$

So $y = 0$ to 2 and $x = 0$ to $x = \sqrt{2y}$

$$\int_0^2 \int_0^{\sqrt{2y}} \sqrt{y}\cos y \, dx \, dy.$$



$$\int_0^2 \sqrt{y}\cos y \, x \Big|_0^{\sqrt{2y}} dy$$

$$\int_0^2 \sqrt{y}\cos y \left(\sqrt{2y} - 0\right) dy = \sqrt{2}\int_0^2 y\cos y \, dy$$

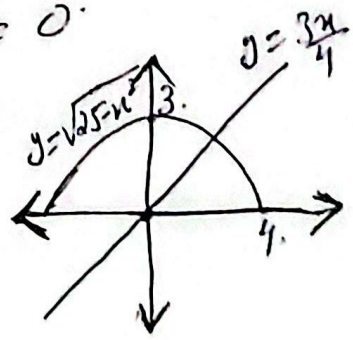
$$= \sqrt{2}\left(y\sin(y) + \cos y\right)\Big|_0^2$$

$$= \sqrt{2}\left(2\sin 2 + \cos 2 - 1\right)$$

Q4) D) Sector of circle in first Quadrant

$y = \sqrt{25 - x^2}$, $3x - 4y = 0$, $y = 0$.

$y = \frac{3x}{4}.$



$\left(\frac{3x}{4}\right)^2 = 25 - x^2$

$9x^2 + x^2 = 25$

16

$\frac{25}{16}x^2 = 25$

$x^2 = 16$

$\boxed{x = 4}$ $y = 3$

Integral in dxdy order.

$$\int_0^4 \int_{\frac{3x}{4}}^{\sqrt{25-x^2}} x \, dy \, dx.$$

Integral in dydx order

$$\int_0^3 \int_{\frac{4y}{3}}^{\sqrt{25-y^2}} x \, dx \, dy.$$

Now Evaluate above.

$$\int_0^3 \left[\frac{x^2}{2}\right]_{\frac{4y}{3}}^{\sqrt{25-y^2}} dy$$

$$\frac{1}{2}\int_0^3 \left(\frac{25-y^2}{} - \frac{16y^2}{9}\right) dy.$$

$$\frac{1}{2}\int_0^3 \left(25 - \frac{25}{9}y^2\right) dy$$

$$= \frac{1}{2}\left(25y - \frac{25}{9} \cdot \frac{y^3}{3}\right)_0^3$$

$$= 25$$