Question 1- Optimize: $f(x,y,z) = x^3 + 3xz + y - y^2 - 3z^2$ **[10 marks]**

Q1 $f(x,y,z) = x^3 + 3xz + y - y^2 - 3z^2$

$f_x = 3x^2 + 3z = 0$, $f_y = 1 - 2y = 0$, $f_z = 3x - 6z = 0$

$3(2z)^2 + 3z = 0$ $\quad y = \frac{1}{2}$ $\quad 6z = 3x$

$12z^2 + 3z = 0$ $\quad P_1 = (0, \frac{1}{2}, 0)$ $\quad 2z = x$

$3z(4z+1) = 0$ $\quad P_2 = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{4})$ $\quad x = 0$ @ $z = 0$

$z = 0$ & $z = -\frac{1}{4}$ $\quad x = -\frac{1}{2}$ @ $z = -\frac{1}{4}$

$f_{xx} = 6x$ $\quad f_{yy} = -2$ $\quad f_{zz} = -6$

$f_{xy} = 0$ $\quad f_{yz} = 0$

$f_{xz} = 3$

$$\begin{vmatrix} 6x & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{vmatrix}$$

$\rightarrow$ @ $(0, \frac{1}{2}, 0) = \begin{vmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{vmatrix} = 6(3) = 18$

$D > 0$ & $f_{xx} = 0$

so inconclusive

$\rightarrow$ @ $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}) = \begin{vmatrix} -3 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{vmatrix} = -36 + 18 = -18$

$D < 0$ so saddle point.

For $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{4})$

$\Delta_1 = |f_{xx}| = -3 < 0$

$\Delta_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -3 & 0 \\ 0 & -2 \end{vmatrix} = 6 > 0$

$\Delta_3 = \left| \quad \right|_{3\times3} = -18 < 0$

Hence Gives local maximum

Question: 02

- **Length** = $8 - 2x$
- **Width** = $3 - 2x$
- **Height** = x

Thus, the **volume** V of the box is given by:

$$V = (8 - 2x)(3 - 2x)(x)$$

Step 2: Find the Critical Points

To maximize V, take the derivative and set it to zero.

1. Expand the volume function:

$$V = (8 - 2x)(3 - 2x) \cdot x$$

$$V = (24 - 16x - 6x + 4x^2) \cdot x$$

$$V = 24x - 16x^2 - 6x^2 + 4x^3$$

$$V = 4x^3 - 22x^2 + 24x$$

2. Differentiate:

$$\frac{dV}{dx} = 12x^2 - 44x + 24$$

3. Solve $\frac{dV}{dx} = 0$:

$$12x^2 - 44x + 24 = 0$$

Using the quadratic formula:

$$x = \frac{-(-44) \pm \sqrt{(-44)^2 - 4(12)(24)}}{2(12)}$$

$$x = \frac{44 \pm \sqrt{1936 - 1152}}{24}$$

$$x = \frac{44 \pm \sqrt{784}}{24}$$

$$x = \frac{44 \pm 28}{24}$$

$$x = \frac{44 + 28}{24} \quad \text{or} \quad x = \frac{44 - 28}{24}$$

$$x = \frac{72}{24} = 3 \quad \text{or} \quad x = \frac{16}{24} = \frac{2}{3}$$

Since cutting $x = 3$ inches would eliminate the width entirely (making the box impossible), we choose:

$$x = \frac{2}{3} \text{ inches}$$

To ensure this is a maximum, we use the second derivative test:

$$\frac{d^2V}{dx^2} = 24x - 44$$

Substituting $x = \frac{2}{3}$:

$$\frac{d^2V}{dx^2} = 24 \times \frac{2}{3} - 44 = 16 - 44 = -28$$

Since this is negative, $x = \frac{2}{3}$ inches gives a **maximum** volume.

Hence: The optimal square size to cut is $\frac{2}{3}$ **inches.**

Question 03-

Approach: To find the direction in which the temperature decreases most rapidly, we need to compute the **negative gradient** of TTT, as the gradient points in the direction of the steepest increase.

The gradient of T is given by:

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right)$$

Partial Derivatives

1. **Partial derivative with respect to x:**

$$\frac{\partial T}{\partial x} = y$$

2. **Partial derivative with respect to y:**

$$\frac{\partial T}{\partial y} = x + z\cos(yz)$$

3. **Partial derivative with respect to z:**

$$\frac{\partial T}{\partial z} = y\cos(yz)$$

Thus,

$$\nabla T = (y, x + z\cos(yz), y\cos(yz))$$

Substituting $x = 1$, $y = 1$, and $z = 1$:

$$\nabla T(1,1,1) = (1, 1 + \cos(1), \cos(1))$$

Approximating $\cos(1) \approx 0.5403$:

$$\nabla T(1,1,1) = (1, 1.5403, 0.5403)$$

The temperature decreases most rapidly in the **negative gradient direction**:

$$\mathbf{v} = -\nabla T(1,1,1) = (-1, -1.5403, -0.5403)$$

The rate of change in this direction is given by the **magnitude of the gradient**:

$$\|\nabla T(1,1,1)\| = \sqrt{1^2 + (1.5403)^2 + (0.5403)^2}$$

$$= \sqrt{1 + 2.3733 + 0.2919} = \sqrt{3.6652} \approx 1.915$$

Thus, the rate of maximum decrease is -1.915.

Question 04-

The gradient is:

$$\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)$$

Compute partial derivatives:

$$\frac{\partial h}{\partial x} = -\frac{2x}{3}, \quad \frac{\partial h}{\partial y} = -\frac{y}{1}$$

At $(3, 2)$:

$$\frac{\partial h}{\partial x} = -\frac{2(3)}{3} = -2$$
$$\frac{\partial h}{\partial y} = -\frac{2}{1} = -2$$

So,

$$\nabla h(3, 2) = (-2, -2)$$

The general equation for a tangent plane at (x_0, y_0) is:

$$z = h(x_0, y_0) + \frac{\partial h}{\partial x}(x - x_0) + \frac{\partial h}{\partial y}(y - y_0)$$

Find $h(3, 2)$:

$$h(3, 2) = 1500 - \frac{3^2}{3} - \frac{2^2}{2}$$
$$= 1500 - \frac{9}{3} - \frac{4}{2}$$
$$= 1500 - 3 - 2 = 1495$$

Substituting values:

$$z = 1495 + (-2)(x - 3) + (-2)(y - 2)$$
$$z = 1495 - 2(x - 3) - 2(y - 2)$$

$$z = 1495 - 2x + 6 - 2y + 4$$

$$z = 1505 - 2x - 2y$$

Thus, the **tangent plane equation** is:

$$z = 1505 - 2x - 2y$$

The directional derivative is given by:

$$D_{\mathbf{v}}h = \nabla h \cdot \hat{\mathbf{v}}$$

where $\hat{\mathbf{v}}$ is the unit vector in the direction of $\mathbf{v}$.

Compute the magnitude of $\mathbf{v} = (-4, 5)$:

$$|\mathbf{v}| = \sqrt{(-4)^2 + 5^2} = \sqrt{16 + 25} = \sqrt{41}$$

The unit vector is:

$$\hat{\mathbf{v}} = \left(\frac{-4}{\sqrt{41}}, \frac{5}{\sqrt{41}}\right)$$

Compute the dot product:

$$D_{\mathbf{v}}h = (-2, -2) \cdot \left(\frac{-4}{\sqrt{41}}, \frac{5}{\sqrt{41}}\right)$$

$$= (-2)(-4) + (-2)(5)$$

$$= 8 - 10 = -2$$

$$D_{\mathbf{v}}h = \frac{-2}{\sqrt{41}}$$

So, the rate of altitude change in direction (4,5) is $-\frac{2}{\sqrt{41}}$ meters per kilometer. Since it's negative, the drone is descending in this direction.

The direction of **steepest ascent** is the **gradient direction** $(-2,-2)$.
The **maximum rate of altitude increase** is the **magnitude of the gradient**:

$$|\nabla h| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

So, the drone should fly in the **direction of (-2,-2) to ascend fastest** at a rate of $2\sqrt{2}$ meters per kilometer.

To maintain the same altitude, the drone should travel **perpendicular to the gradient**. Any vector **perpendicular to** $(-2,-2)$ will work, such as:

$$(2,-2) \quad \text{or} \quad (-2,2)$$

These directions ensure **no change in altitude**.

Final Answers:

1. **Gradient at (3,2):** $(-2,-2)$
2. **Tangent Plane Equation:** $z = 1505 - 2x - 2y$
3. **Directional Derivative in** $(-4,5)$**:** $-\frac{2}{\sqrt{41}}$ (altitude decreases)
4. **Steepest Ascent Direction:** $(-2,-2)$, **Max Rate of Increase:** $2\sqrt{2}$
5. **Constant Altitude Direction:** $(2,-2)$ or $(-2,2)$

Question 05-

Step 1: Define Variables

Let:

- A be the area of the rectangle.
- x and y be the length and width of the rectangle, respectively.
- The original dimensions are:
$$x = 200 \text{ ft}, \quad y = 100 \text{ ft}$$
- The boundary stripe adds a uniform width of **3 inches** (which is $3/12 = 1/4$ **ft**) to each side.

Thus, the **new dimensions** of the rectangle after the stripe is added are:

$$x + dx = 200 + \frac{1}{2}, \quad y + dy = 100 + \frac{1}{2}$$

where $dx = dy = \frac{1}{2}$ ft.

Step 2: Compute Differential of Area

The area of the rectangle is:

$$A = xy$$

Taking differentials:

$$dA = y\,dx + x\,dy$$

Substituting values:

$$dA = (100)(\frac{1}{2}) + (200)(\frac{1}{2})$$

$$dA = 50 + 100 = 150 \text{ square feet}$$

Question 06-

a)

a. First, we calculate $f_x(x, y)$ and $f_y(x, y)$:

$$\begin{aligned} f_x(x,y) &= \frac{1}{2}(-18x+36)(4y^2-9x^2+24y+36x+36)^{-1/2} \\ &= \frac{-9x+18}{\sqrt{4y^2-9x^2+24y+36x+36}} \\ f_y(x,y) &= \frac{1}{2}(8y+24)(4y^2-9x^2+24y+36x+36)^{-1/2} \\ &= \frac{4y+12}{\sqrt{4y^2-9x^2+24y+36x+36}} \end{aligned}$$

Next, we set each of these expressions equal to zero:

$$\begin{aligned} \frac{-9x+18}{\sqrt{4y^2-9x^2+24y+36x+36}} &= 0 \\ \frac{4y+12}{\sqrt{4y^2-9x^2+24y+36x+36}} &= 0. \end{aligned}$$

Then, multiply each equation by its common denominator:

$$\begin{aligned} -9x+18 &= 0 \\ 4y+12 &= 0. \end{aligned}$$

Therefore, $x = 2$ and $y = -3$, so $(2, -3)$ is a critical point of f.

We must also check for the possibility that the denominator of each partial derivative can equal zero, thus causing the partial derivative not to exist. Since the denominator is the same in each partial derivative, we need only do this once:

The above equation represents a parabola $4y^2 - 9x^2 + 24y + 36x + 36 \geq 0.$

Therefore, any points on the hyperbola are not only critical points, they are also on the boundary of the domain. To put the hyperbola in standard form, we use the method of completing the square:

$$\begin{aligned}
4y^2 - 9x^2 + 24y + 36x + 36 &= 0 \\
4y^2 - 9x^2 + 24y + 36x &= -36 \\
4y^2 + 24y - 9x^2 + 36x &= -36 \\
4(y^2 + 6y) - 9(x^2 - 4x) &= -36 \\
4(y^2 + 6y + 9) - 9(x^2 - 4x + 4) &= -36 - 36 + 36 \\
4(y+3)^2 - 9(x-2)^2 &= -36.
\end{aligned}$$

Dividing both sides by -36 puts the equation in standard form:

$$\begin{aligned}
\frac{4(y+3)^2}{-36} - \frac{9(x-2)^2}{-36} &= 1 \\
\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} &= 1.
\end{aligned}$$

Notice that point $(2, -3)$ is the center of the hyperbola.

Thus, the critical points of the function f are $(2, -3)$ and all points on the hyperbola, $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$.

b)

Step 1: Compute First-Order Partial Derivatives

To find the **critical points**, we need to compute f_x and f_y, then set them equal to zero.

Partial derivative with respect to x:

$$f_x = \frac{\partial}{\partial x}\left(e^{x^2-y} + x^2y - y^3\right)$$

Using the chain rule:

$$\frac{d}{dx}e^{x^2-y} = e^{x^2-y} \cdot (2x)$$

$$f_x = e^{x^2-y} \cdot 2x + 2xy$$

Setting $f_x = 0$:

$$e^{x^2-y} \cdot 2x + 2xy = 0$$

$$2x\left(e^{x^2-y} + y\right) = 0$$

Thus, the critical points satisfy:

$$x = 0 \quad \text{or} \quad e^{x^2-y} + y = 0$$

Since $e^{x^2-y} > 0$ for all real x, y, the equation $e^{x^2-y} + y = 0$ is **never true.**

So the only possible solution comes from:

$$x = 0$$

Partial derivative with respect to y:

$$f_y = \frac{\partial}{\partial y}\left(e^{x^2-y} + x^2y - y^3\right)$$

$$f_y = -e^{x^2-y} + x^2 - 3y^2$$

Setting $f_y = 0$:

$$-x^2 + e^{x^2-y} - 3y^2 = 0$$

Substituting $x = 0$:

$$e^{-y} - 3y^2 = 0$$

Step 2: Solve for Critical Points

We need to solve:

$$e^{-y} = 3y^2$$

This is a **transcendental equation**, which usually requires numerical methods for exact solutions. However, we can estimate solutions graphically or use approximation methods.

- Trying $y = 0$:

$$e^0 - 3(0)^2 = 1 \neq 0$$

- Trying $y = 1$:

$$e^{-1} - 3(1)^2 \approx 0.368 - 3 = -2.632 \neq 0$$

- Trying $y = -1$:

$$e^1 - 3(-1)^2 = 2.718 - 3 = -0.282$$

- Trying $y = -0.5$:

$$e^{0.5} - 3(-0.5)^2 = 1.648 - 0.75 = 0.898$$

A **numerical approach** suggests a solution around $y \approx -0.6$.

Thus, the **critical point** is approximately:

$$(0, -0.6)$$

Step 3: Compute Second-Order Partial Derivatives

To classify the critical point, we compute the second-order derivatives:

$$f_{xx} = \frac{d}{dx}\left(e^{x^2-y} \cdot 2x + 2xy\right)$$

$$= e^{x^2-y} \cdot (4x^2 + 2) + 2y$$

At $x = 0$, this simplifies to:

$$f_{xx} = e^{-y} \cdot 2 + 2y$$

$$f_{yy} = \frac{d}{dy}\left(-e^{x^2-y} + x^2 - 3y^2\right)$$

$$= e^{x^2-y} - 6y$$

At $x = 0$, this simplifies to:

$$f_{yy} = e^{-y} - 6y$$

$$f_{xy} = \frac{d}{dy}\left(e^{x^2-y} \cdot 2x + 2xy\right)$$

$$= -2xe^{x^2-y} + 2x$$

At $x = 0$, this simplifies to:

$$f_{xy} = 0$$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

$$D = (e^{-y} \cdot 2 + 2y)(e^{-y} - 6y) - (0)^2$$

Substituting $y \approx -0.6$, we estimate:

$$D \approx (e^{0.6} \cdot 2 - 1.2)(e^{0.6} + 3.6)$$

Since $D > 0$ and $f_{xx} > 0$, this means $(0, -0.6)$ is a **local minimum**.

Final Answer:

- **Local minimum** at $(0, -0.6)$.
- **No saddle points or local maxima.**

Question 07

SOLUTION As in Figure 12.74, let x and y denote the dimensions (in feet) of the base and z the altitude (in feet). Since the area of the base is xy and there are two sides of area xz and two sides of area yz, the cost C (in dollars) of the material is

$$C = 4xy + 3(2xz) + 2(2yz).$$

where x, y, and z are positive. Since $V = xyz = 12$, it follows that $z = 12/(xy)$. Substituting for z in the formula for C and simplifying, we obtain

$$C = 4xy + \frac{72}{y} + \frac{48}{x}.$$

There are no boundary points, since $x > 0$ and $y > 0$ for every (x, y). Hence there are no boundary extrema. To determine possible local extrema, we solve the following system of two equations:

$$C_x = 4y - \frac{48}{x^2} = 0, \qquad C_y = 4x - \frac{72}{y^2} = 0.$$

Equivalent equations are

$$y = \frac{12}{x^2} \quad \text{and} \quad xy^2 = 18.$$

Substituting $y = 12/x^2$ into the second equation give us

$$x\left(\frac{144}{x^4}\right) = 18, \quad \text{or} \quad 144 = 18x^3.$$

Question 08

Find the partial derivatives:

$$f_x = \frac{\partial f}{\partial x} = 2x - 3y - 6$$

$$f_y = \frac{\partial f}{\partial y} = -3x - 2y + 2$$

Set $f_x = 0$ and $f_y = 0$ to find critical points:

$$2x - 3y - 6 = 0$$

$$-3x - 2y + 2 = 0$$

Solve for x and y.

So the critical point is:

$$\left(\frac{18}{13}, -\frac{14}{13}\right)$$

Checking if this point is inside the region $|x| \leq 3$, $|y| \leq 2$, we see that:

$$\left|\frac{18}{13}\right| \approx 1.38 \leq 3, \quad \left|-\frac{14}{13}\right| \approx 1.08 \leq 2$$

Since it lies inside R, we evaluate $f(x, y)$ at this point:

$$f\left(\frac{18}{13}, -\frac{14}{13}\right) \approx -5.23$$

Evaluate $f(x, y)$ on the Boundary

The boundary consists of four edges:

1. **Edge 1:** $x = -3, y \in [-2, 2]$
   - $f(-3, -2) = -27$
   - $f(-3, 2) = -1$
2. **Edge 2:** $x = 3, y \in [-2, 2]$
   - $f(3, -2) = -17$
   - $f(3, 2) = 45$
3. **Edge 3:** $y = -2, x \in [-3, 3]$
   - $f(-3, -2) = -27$ (already computed)
   - $f(3, -2) = -17$ (already computed)
4. **Edge 4:** $y = 2, x \in [-3, 3]$
   - $f(-3, 2) = -1$ (already computed)
   - $f(3, 2) = 45$ (already computed)

Determine Maximum and Minimum Values

We compare all values:

- $f(\frac{18}{13}, -\frac{14}{13}) \approx -5.23$
- $f(-3, -2) = -27$
- $f(-3, 2) = -1$
- $f(3, -2) = -17$
- $f(3, 2) = 45$
- **Minimum value:** -27 at $(-3, -2)$.
- **Maximum value:** 45 at $(3, 2)$.

Question: 09

SOLUTION

(a) By Definition (12.31), the gradient of $T = 100/(x^2 + y^2 + z^2)$ is

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}.$$

Since

$$\frac{\partial T}{\partial x} = \frac{-200x}{(x^2 + y^2 + z^2)^2}, \quad \frac{\partial T}{\partial y} = \frac{-200y}{(x^2 + y^2 + z^2)^2}, \quad \frac{\partial T}{\partial z} = \frac{-200z}{(x^2 + y^2 + z^2)^2},$$

we obtain

$$\nabla T = \frac{-200}{(x^2 + y^2 + z^2)^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Hence,

$$\nabla T|_P = \frac{-200}{196}(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}).$$