0053

Question 1 (10)

Use the integral test to determine the convergence or divergence of the series. Be sure to check that the conditions of the integral test are satisfied.

$$\sum_{n=2}^{\infty} \frac{n-4}{n^2-2n+1} \quad \Rightarrow f(n) = \frac{n-4}{(n-1)^2}$$

① f is continuous for all $x \geq 2$. $\underline{2}$

② f is decreasing so $f'(x) < 0$.

$$f'(x) = \frac{(n-1)^2(1) - (n-4)\,2(n-1)}{(n-1)^4}$$

$$f'(n) = \frac{\cancel{(n-1)}\left[(n-1) - 2n + 8\right]}{(n-1)^{\cancel{4}\,3}} = \frac{7-n}{(n-1)^3}$$

$f'(n) < 0$ whenever $n \geq 8$. $\underline{2}$

③ $$\int_8^{+\infty} \frac{n-1-3}{(n-1)^2}\, dn = \int_8^{+\infty} \frac{\cancel{n-1}}{(n-1)^{\cancel{2}}}\, dn - 3\int \frac{dn}{(n-1)^3}$$

$$= \ln|n-1| - 3\,\frac{(n-1)^{-2}}{-2}$$ $\underline{4}$

$$= \ln|n-1| + \frac{3}{2}\,\frac{1}{(n-1)^2}\Bigg|_8^{\infty} = \infty$$

divergent

Since integral is divergent so series is ~~also~~ divergent.

Question 2A (10)

Find the area between the curves

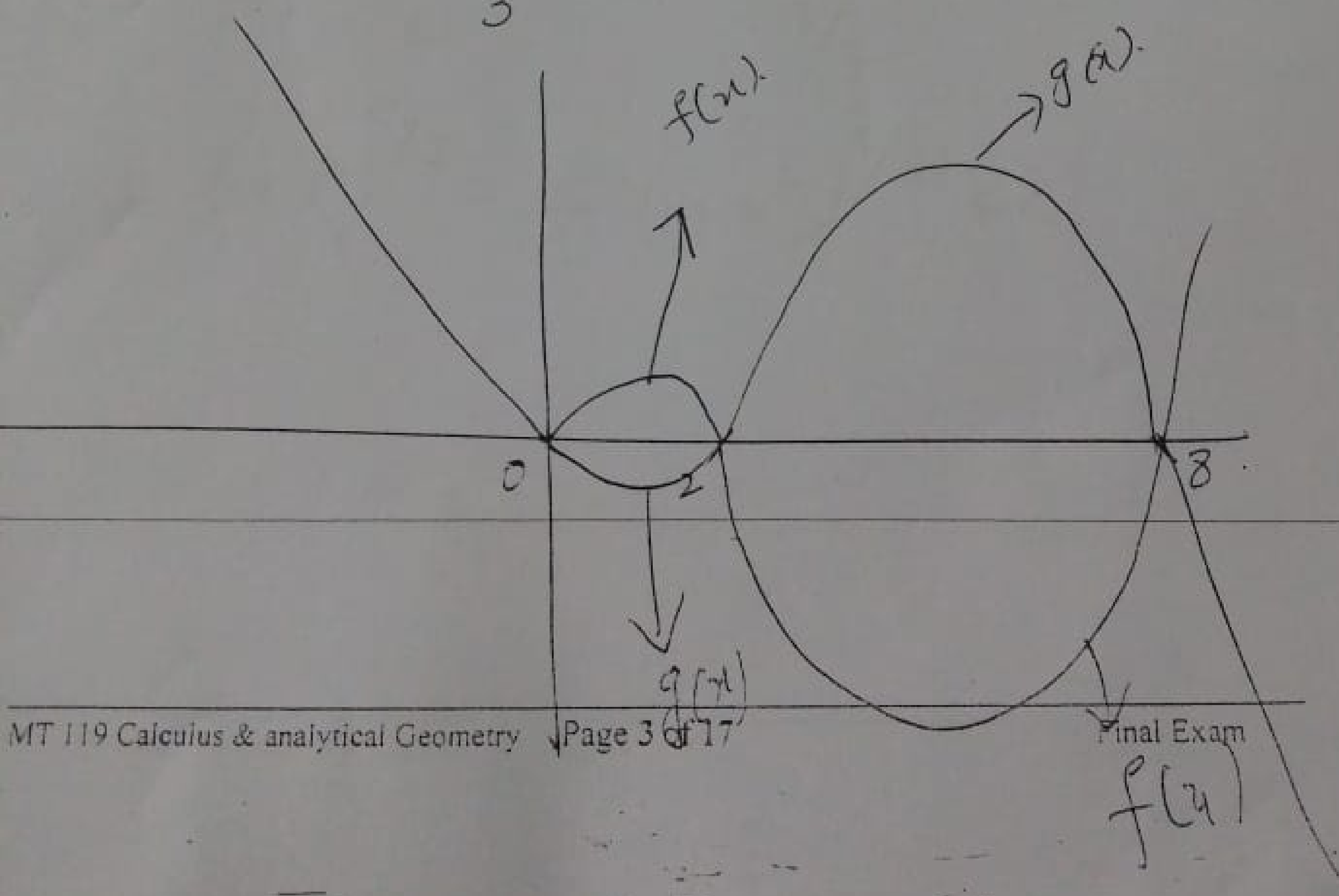
$$y = 16x - 10x^2 + x^3. \quad = f(x)$$
$$y = -16x + 10x^2 - x^3. \quad = g(x)$$

point of intersection $x = 0, \; y = 0$
$x = 2, \; y = 0$
$x = 8, \; y = 0.$] 3

$$\int_0^2 \{f(x) - g(x)\}.dx + \int_2^8 \{g(x) - f(x)\}.dx \quad | \; 3$$

$$\int_0^2 (32x - 20x^2 + 2x^3)dx + \int_2^8 (-32x + 20x^2 - 2x^3)dx.$$

$$= \frac{1136}{3} = 378.667. \qquad \underline{2}$$



$\underline{2}$

Question 2B (05)

Evaluate the indefinite integral

$$\int x^3\sqrt{1+x^2}\,dx$$

$$= \int x \cdot x^2 \sqrt{1+x^2}\,dx$$

let $w = 1+x^2 \Rightarrow dw = 2x\,dx$ $\underline{2}$

$$= \frac{1}{2}\int \sqrt{w}\,(w-1)\,dw$$

$$= \frac{1}{2}\int (w^{3/2} - w^{1/2})\,dw$$

$$= \frac{1}{2}\frac{w^{5/2}}{5/2} - \frac{1}{2}\frac{w^{3/2}}{3/2} + c$$

$$= \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + c$$

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Question 3A (05)

Evaluate the indefinite integral by using integration by parts.

$$\int x^2 lnx dx.$$

$= \ln x \dfrac{x^3}{3} - \int \dfrac{x^{\cancel{3}\,2}}{3} \times \dfrac{1}{\cancel{x}} dx$

05

$= \ln x \dfrac{x^3}{3} - \dfrac{1}{3}\dfrac{x^3}{3} + c$

Question 3B (05)

Solve the following by trigonometric substitution.

$$\int_0^{\frac{\sqrt{3}}{2}} \left(\frac{4x^2}{(1-x^2)^{3/2}}\right) dx$$

$x = \sin\theta, \quad , \quad 0 \leq \theta < \pi/3$] 1

$dx = \cos\theta \, d\theta$

$$\int_0^{\pi/3} \frac{4\sin^2\theta \cos\theta}{(1-\sin^2\theta)^{3/2}} d\theta$$

$$= \int_0^{\pi/3} \frac{4\sin^2\theta \cancel{\cos\theta} \, d\theta}{\cos^{\cancel{3}\,2}\theta}$$] 2

$$= 4\int_0^{\pi/3} \frac{(1-\cos^2\theta)}{\cos^2\theta} d\theta$$

$$= 4\int_0^{\pi/3} (\sec^2\theta - 1) \, d\theta$$

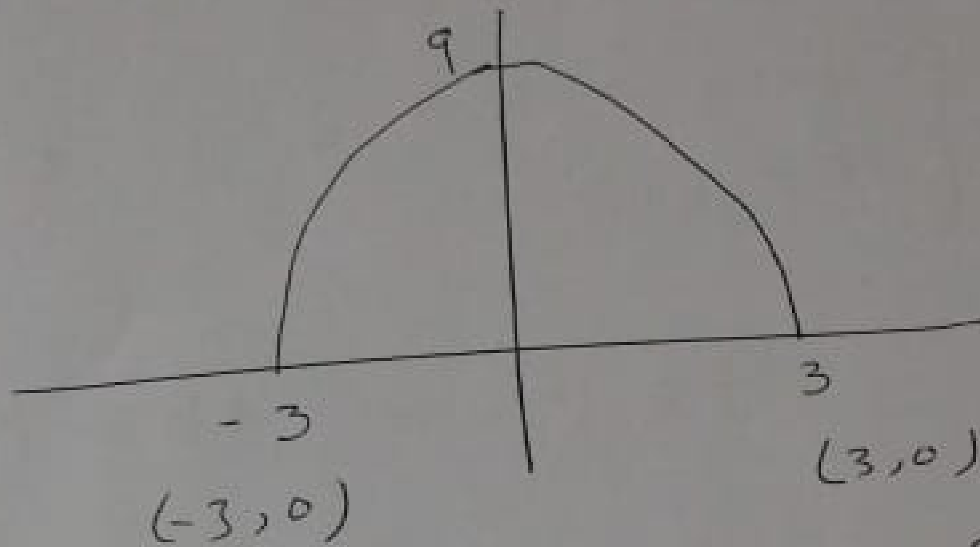
$$= 4\left[\tan\theta - \theta \Big|_0^{\pi/3}\right] = 4\sqrt{3} -$$

Question 4A (05)

We need to build a door in the shape of a parabola. The door is to be of width 6 feet and height of 9 feet. What is the surface area of the door?

01

$f(x) = 9 - x^2$



01

$\int_{-3}^{3} (9 - x^2)\, dx$

02

$= 2\int_{0}^{3} (9 - x^2)\, dx$

$= 2\left[9x - \frac{x^3}{3}\Big|_0^3\right] = 2\left[27 - \frac{\cancel{27}^{\,9}}{\cancel{3}}\right] = 36$

sq fe

01

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Question 4B (05)

Suppose that baseball is thrown upward from the roof of a 100 meter high building. It hits the street eight second later. What was the initial velocity of the baseball and how high did it rise above the street before beginning its descent?

$$\frac{d^2s}{dt^2} = -9.8, \qquad \left.\begin{array}{l} s(0) = 100 \\ s(8) = 0 \end{array}\right] 02$$

$$\frac{ds}{dt} = -\int 9.8\,dt = -9.8t + c$$

$$s = -9.8\frac{t^2}{2} + c_1 t + c_2$$

$$s = -4.9t^2 + c_1 t + c_2$$

$t = 0, \ s = 100$ 01

$$\boxed{c_2 = 100}$$

$$s = -4.9t^2 + c_1 t + 100$$

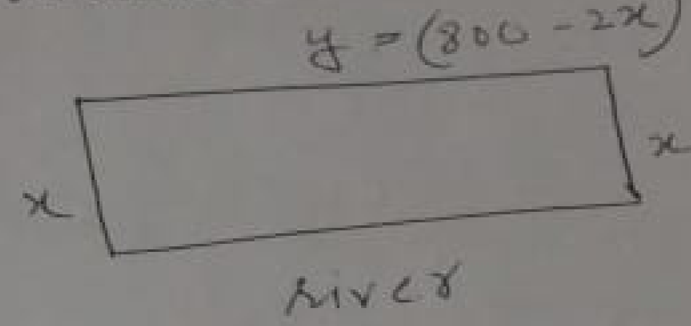
$= 8, \quad 0 = s(8) = -4.9(8)^2 + c_1(8) + 10$

$$c_1 = (4.9) \times 64 - 100$$

8

Question 5A (10)

A rectangular plot of a farmland will be bounded on one side by a river and on the other three sides by a single strand electric fence. With 800 m of the wire at your disposal, what is the largest area you can enclose and what are its dimensions?



SO

$$\text{Area} = xy = x(800-2x)$$

$$A = 800x - 2x^2 \quad (2)$$

$$\frac{dA}{dx} = 800 - 4x \quad \Rightarrow \frac{dA}{dx} = 0$$

$$(02)$$

$$800 - 4x = 0 \quad \Rightarrow 4x = 800 \quad (02)$$

$$\boxed{x = 200} \text{ m}$$

$$y = 800 - 2(200) = 400 \text{ m}$$

$$(02)$$

$$\frac{d^2(A)}{dx^2} = -4 < 0 \quad \text{so (maximum value of the Area)}.$$

$$(02)$$

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Question 5B (05)

Use any method to determine if the series converges or diverges. Give reasons for your answer.

$$\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)ln\,(n+1)}$$

$$a_n = \frac{n5^n}{(2n+3)\ln(n+1)}, \quad a_{n+1} = \frac{(n+1)5^{n+1}}{(2n+5)\ln(n+2)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\cancel{5^n}.5}{(2n+5)\ln(n+2)} \times \frac{(2n+3)\ln(n+1)}{n\,\cancel{5^n}}$$

$$= 5\lim_{n\to\infty}\frac{(n+1)}{(n)} \lim_{n\to\infty}\frac{(2n+3)}{(2n+5)} \lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n+2)}$$

$$= 5\,\frac{\cancel{n}(1+\cancel{1/n})}{\cancel{n}}\;\frac{\cancel{n}(2+\cancel{3/n})}{\cancel{n}(2+\cancel{5/n})}\;\lim_{n\to\infty}\frac{1/n+1}{1/n+2}$$

$$= 5\lim_{n\to\infty}\frac{n+2}{n+1}$$

$$= 5 > 1 \quad \text{divergent}$$

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3

Question 6A (05)

When driving for uber there is a base fare which is a flat fee you are charged at the beginning of the ride. Suppose that the base fee is \$2 and you are charged 0.40 per mile. The amount of the total fare charge is a function of the total miles driven. Find the domain and range up to the 10 miles.

$y = 0.40x + 2$ 02

Miles Driven : x

1.5

D :- $\{0 \le x \le 10\}$.

1.5

R :- $\{2 \le y \le 6\}$

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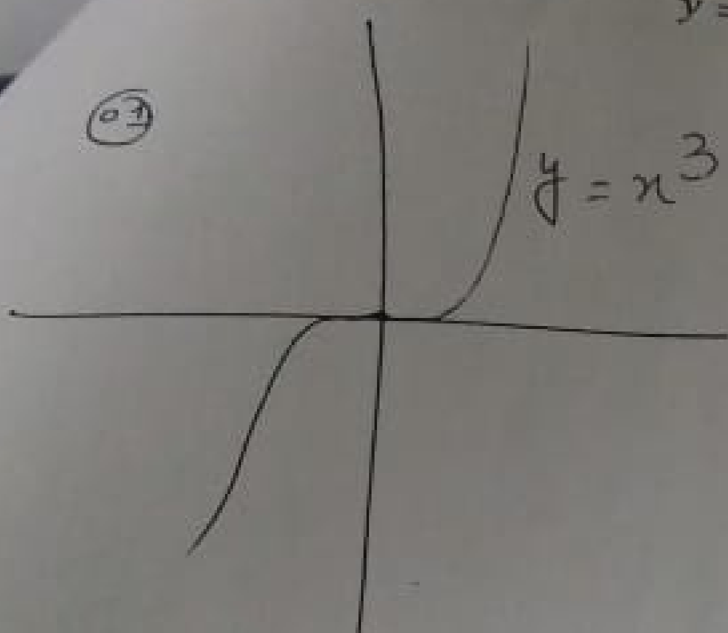
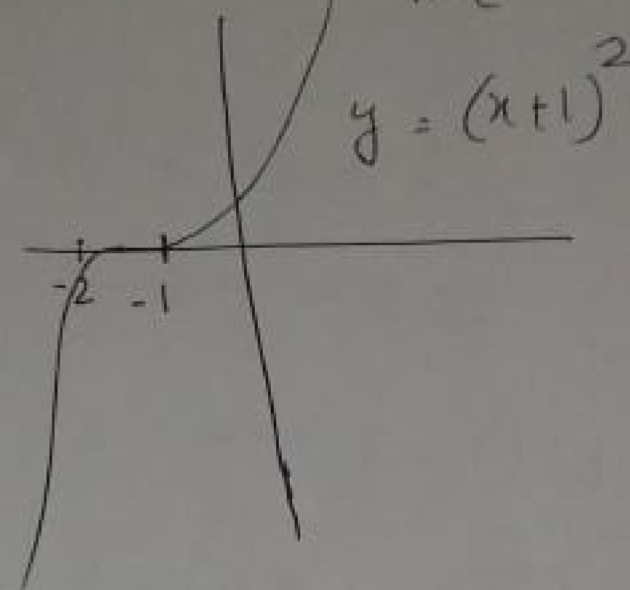
Question 6B (05)

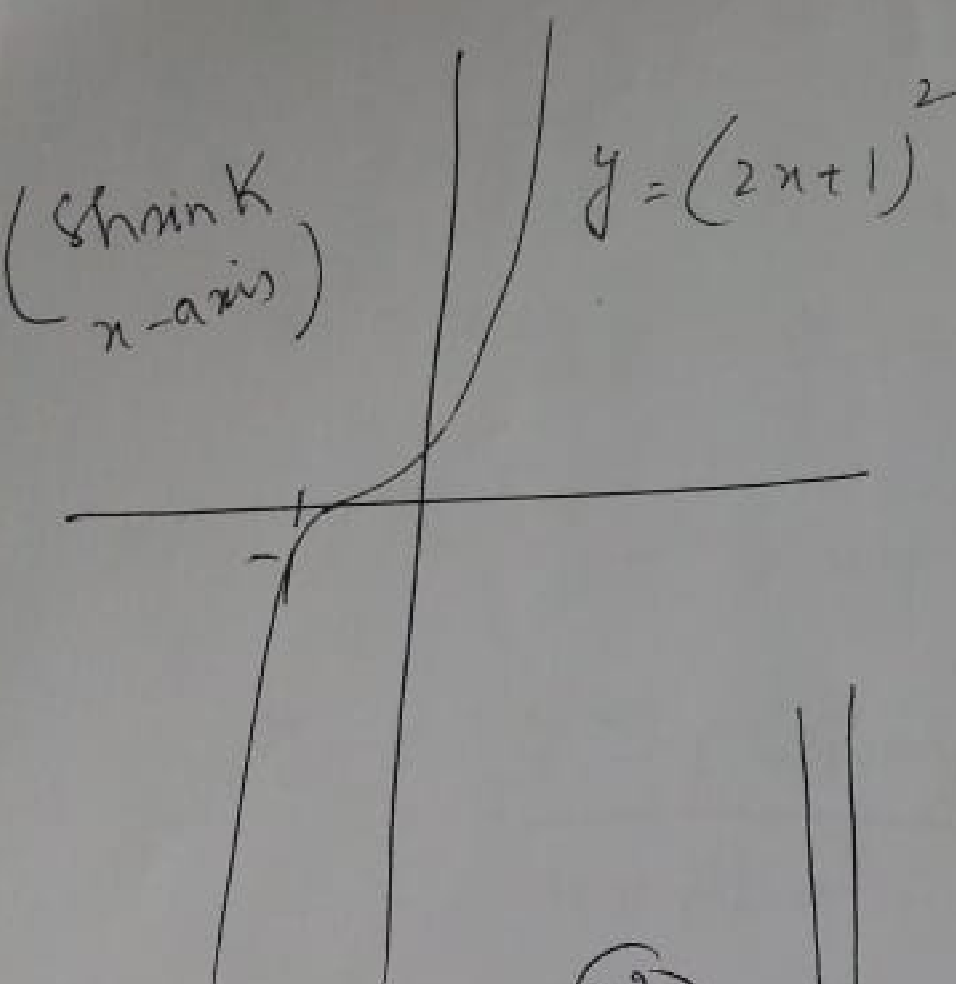
Graph the function, not by plotting points, but by starting with the graph of one of the standard functions and applying appropriate transformations.

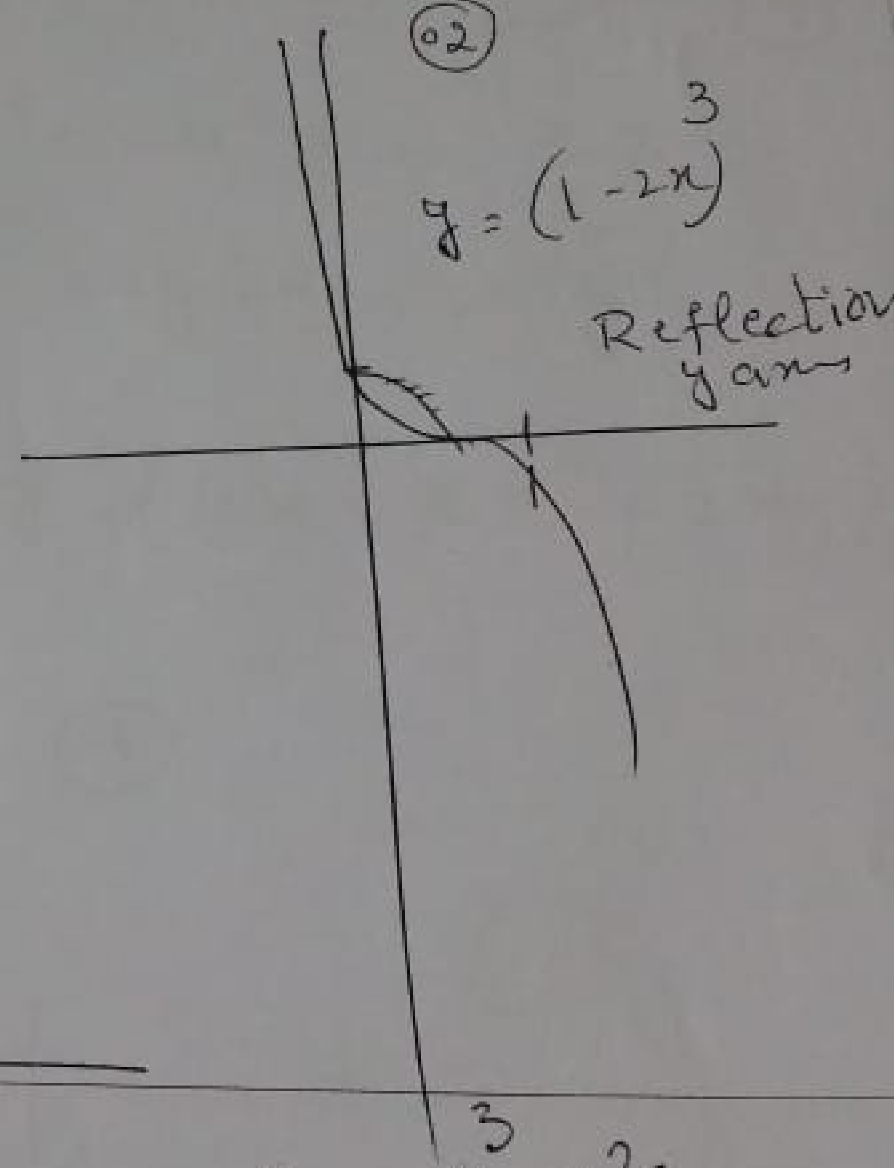
$$y = (1 - 2x)^3 - 2$$

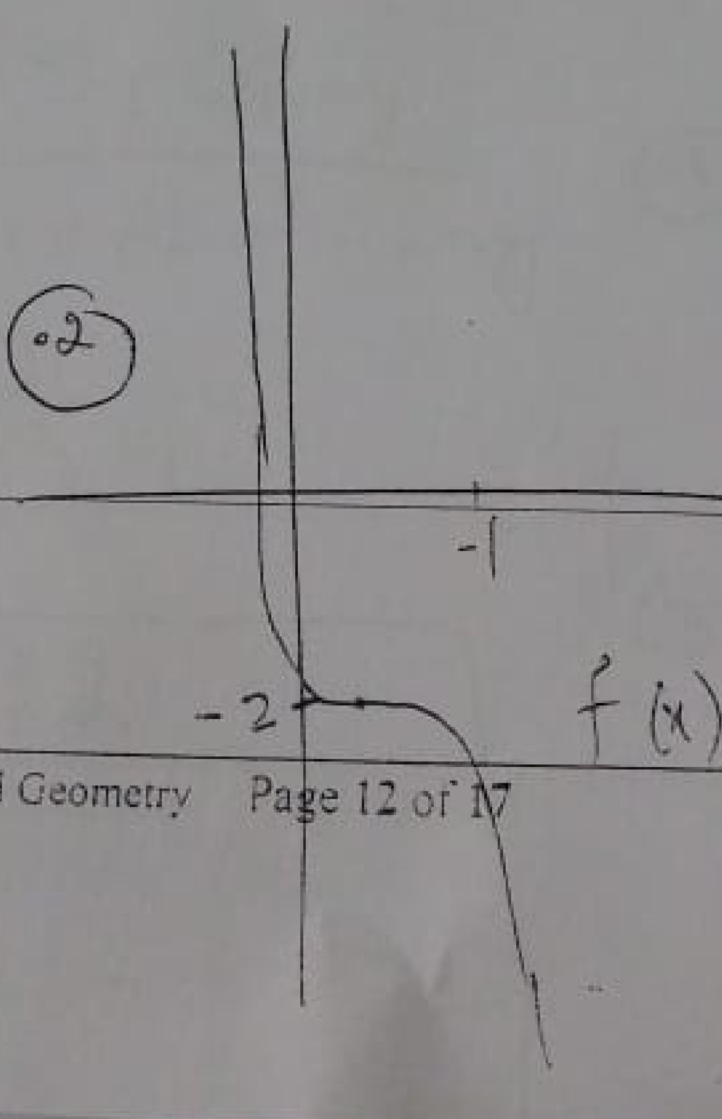
$D = (-\infty, +\infty)$
$R = (-\infty, +\infty)$











$f(x) = (1-2x)^3 - 2$

Question 7A (12)

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Consider the relation determined by the equation

$$y^2x - 5x = 2(y^2 + x^2y - 16).$$

Find the equation of all tangents drawn to the graph of the relation at $x = 3$.

$\underline{x=3}$:- $3y^2 - 15 = 2(y^2 + 9y - 16)$

$\Rightarrow 3y^2 - 15 = 2y^2 + 18y - 32$

$\Rightarrow y^2 - 18y + 17 = 0 \Rightarrow (y-1)(y-17) = 0$

$\Rightarrow y = 1, 17$

$(3,1)\ (3,17)$ 03+1 = 04

$xy^2 - 5x = 2y^2 + 2x^2y - 32$

$y^2 + x\,2yy' - 5 = 4yy' + 2y(2x) + 2x^2y'$

$\Rightarrow y^2 - 5 - 4xy = y'(4x + 2x^2 - 2xy)$

$$y' = \frac{y^2 - 5 - 4xy}{4x + 2x^2 - 2xy}$$ 04

$y'\Big|_{(3,17)} = -5$ 01 , $y'\Big|_{(3,1)} = -1$ 01

$y = -5x + 32$

$y = -x + 2$

02 $y - y_1 = m(x - x$

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Question 7B (08)

Find $a, b \in R$ such that the function

$$f(x) = \begin{cases} \log(1+x), & \text{if } -1 < x < 0 \\ aSinx + bCosx, & \text{if } 0 < x < \frac{\pi}{2} \\ x, & \text{if } x \geq \frac{\pi}{2}. \end{cases}$$

Is continuous on its domain.

$D =]-1, +\infty[$

$]-1, 0[, \ [0, \pi/2[, \ [\pi/2, \infty)$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \log(1+x) = 0$$

$$\lim_{x \to 0^+} (a\sin x + b\cos x) = b$$

Hence f is continuous $\boxed{b = 0}$

04

$$\lim_{x \to \pi/2^-} = \lim_{x \to \pi/2^-} (a\sin x + b\cos x) = a.$$

$$\lim_{x \to \pi/2^+} x = \pi/2.$$

04

and $\boxed{a = \pi/2}$

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Question 9 (10)

A car is travelling at 50 mph due south at a point $\frac{1}{2}$ miles north of an intersection. A police car is travelling 40 miles per hour due west at a point $\frac{1}{4}$ miles east of same intersection. At that instant the radar in police car measurement the rate at which the distance between two cars is changing. What does the radar gun register? If the police car is not moving does this make the radar guns measurement more accurate?

$$d^2(t) = x^2(t) + y^2(t) \quad 02$$

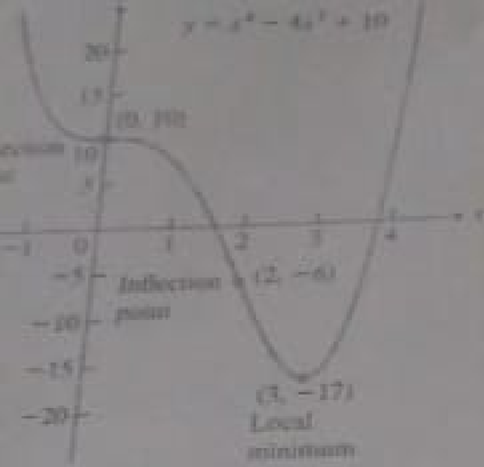
$$\cancel{2}\, d(t)\, d'(t) = \cancel{2}\, x(t)\, x'(t) + \cancel{2}\, y(t)\, y'(t)$$

$$\Rightarrow d'(t) = \frac{x(t)\, x'(t) + y(t)\, y'(t)}{d(t)} \quad 02$$

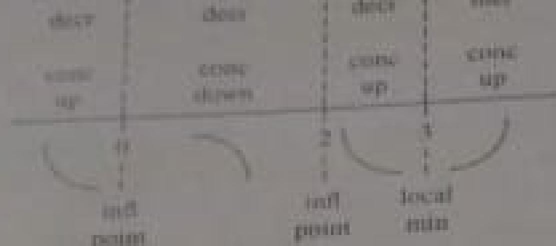
$$x(t) = 1/4, \quad y(t) = 1/2. \quad 02$$

$$d'(t) = \frac{\frac{1}{4}(-40) + \left(\frac{1}{2}\right)(-50)}{\sqrt{\left(\frac{1}{4}\right)^2 + \left(1/2\right)^2}} = \underline{-62.6} \quad 04$$

The End



E 4.30 The graph of $f(x) =$ ... $x^3 + 10$ (Example 7).



(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 4.30 shows the graph of f. ■

The steps in Example 7 give a procedure for graphing the key features of a function. Asymptotes were defined and discussed in Section 2.6. We can find them for rational functions, and the methods in the next section give tools to help find them for more general functions.

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

EXAMPLE 8 Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

x-intercept at $x = -1$, y-intercept ($y = 1$) at $x = 0$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2\cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f''(x) = \frac{(1+x^2)^2\cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

After some algebra

3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$, yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$, yielding a relative maximum by the Second Derivative test.

4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.

5. *Inflection points.* Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.

6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} \qquad \text{Expanding numerator}$$
$$= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1} \qquad \text{Dividing by } x^2$$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \le y \le 2$).

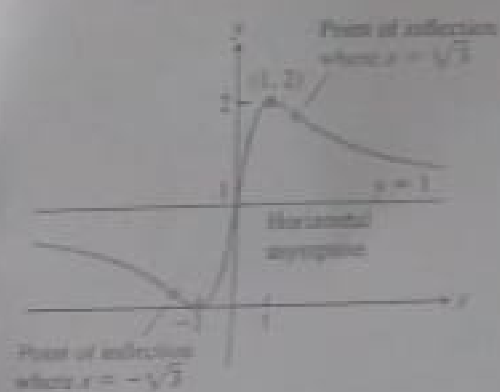


FIGURE 4.31 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 8).

7. The graph of f is sketched in Figure 4.31. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$. ■

EXAMPLE 9 Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$.

Solution

1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.

2. We calculate the derivatives of the function, but first rewrite it in order to simplify our computations:

$$f(x) = \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x} \qquad \text{Function simplified for differentiation}$$
$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2} \qquad \text{Combine fractions to solve easily } f'(x) = 0$$
$$f''(x) = \frac{4}{x^3} \qquad \text{Exists throughout the entire domain of } f$$

3. The critical points occur at $x = \pm 2$ where $f'(x) = 0$. Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.