itleIntroduction to Artificial Intelligence

Exercise session 3

Kolmogorov's axioms

A probability space is a sample space equipped with a probability function, i.e. an assignment $P : \mathcal{P}(\Omega) \to \mathbb{R}$ such that

1. $P(\omega) \in \mathbb{R}, 0 \leq P(\omega)$ for all $\omega \in \Omega$.
2. $P(\Omega) = 1$.
3. $P(\{\omega_1, ..., \omega_n\}) = \sum_{i=1}^{n} P(\omega_i)$.

with Ω a sample space, $\omega \in \Omega$ a sample point and $\mathcal{P}(\Omega)$ the power set of Ω.

Probability rules

The product rule states that

$$P(a, b) = P(b)P(a \mid b) = P(a)P(b \mid a)$$

from which Bayes' formula is derived as

$$P(a \mid b) = \frac{P(b \mid a)P(a)}{P(b)}$$

with $P(a, b)$ being the joint probability of events a and b, $P(b)$ - $P(a)$ the marginals and $P(a \mid b)$ - $P(b \mid a)$ the conditionals.

The chain rule states that

$$P(x_1, .., x_n) = \prod_{i=1}^{n} P(x_i \mid x_1, ..., x_{i-1}).$$

extbfIn session exercises: Ex. 1, Ex. 2, Ex. 3, Ex. 5

Exercise 1 Beliefs

Let A and B be two *events* over the probability space Ω. An agent holds the beliefs $P(A) = 0.4$ and $P(B) = 0.3$. What ranges of probabilities would it be rational for the agent to hold for the events $A \cup B$ and $A \cap B$? What if the agent also believes that $P(A|B) = 0.5$?

1. $P(A) = 0.4$, $P(B) = 0.3$

หาช่วงความน่าจะเป็นของ $P(A \cup B)$ และ $P(A \cap B)$

$P(A \cap B)$

$\min = 0$

$\max = 0.3 \to (P(B) = 0.3, B \subset A)$

$\therefore 0 \leq P(A \cap B) \leq 0.3$

$P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.4 + 0.3 - P(A \cap B)$$
$$= 0.7 - P(A \cap B)$$

$\min = P(A \cap B)\ \max = 0.3 \quad = 0.7 - 0.3 = 0.4$

$\max = P(A \cap B)\ \min = 0 \quad = 0.7 - 0 \quad = 0.7$

$\therefore 0.4 \leq P(A \cup B) \leq 0.7$

2. $P(A|B) = 0.5$ หาความน่าจะเป็นของ $A \cup B$ และ $A \cap B$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A|B) \times P(B) = 0.5 \times 0.3 = 0.15$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.3 - 0.15 = 0.55$$

Exercise 2 (AIMA, Ex 13.8)

	toothache		no toothache	
	catch	no catch	catch	no catch
cavity	0.108	0.012	0.072	0.008
no cavity	0.016	0.064	0.144	0.576

Given the hereabove probability table, compute the following probabilities:

1. $P(toothache)$
2. $P(cavity)$
3. $P(toothache|cavity)$
4. $P(cavity|toothache \cup catch)$

1. $P(toothache) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$

2. $P(cavity) = 0.108 + 0.012 + 0.072 + 0.008 = 0.2$

3. $P(toothahe | cavity)$

$P(toothahe \cap cavity) = 0.108 + 0.012 = 0.12$

$$P(toothahe | cavity) = \frac{P(toothache \cap cavity)}{P(cavity)} = \frac{0.12}{0.2} = 0.6$$

4. $P(cavity | toothache \cup catch)$

$P(catch) = 0.108 + 0.072 + 0.016 + 0.144 = 0.34$

$P(toothache \cap catch) = 0.108 + 0.016 = 0.124$

$P(toothache \cup catch) = P(tootache \cap) + P(catch) - P(toothache \cap catch)$

$= 0.2 + 0.34 - 0.124 = 0.416$

$P(cavity \cap (toothache \cup catch)) = 0.108 + 0.012 + 0.072 = 0.192$

$$P(cavity | toothache \cup catch) = \frac{0.192}{0.416} \approx 0.4615$$

Exercise 3 (AIMA, Ex 13.15)

After your yearly checkup, the doctor has bad news and good news. The bad news is that you tested positive for a serious disease and that the test is 99 % accurate, *i.e.* the probability of testing positive when you do have the disease is 0.99 and the probability of testing positive when you don't is 0.01. The good news is that this is a rare disease, striking only 1 in 10 000 people of your age. Why is it good news that the disease is rare? What are the chances that you actually have the disease?

D = เป็นโรค , $\neg D$ = ไม่เป็นโรค

T = ตรวจพบเชื้อ (Positive test)

$$P(D) = \frac{1}{10000} = 0.0001$$

$$P(\neg D) = 1 - 0.0001 = 0.9999$$

$$P(T|D) = 0.99$$

$$P(T|\neg D) = 0.01$$

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$

$$P(T) = ?$$

$$P(T) = P(T|D)P(D) + P(T|\neg D)P(\neg D)$$

$$P(T) = (0.99 \times 0.0001) + (0.01 \times 0.9999)$$

$$P(T) = 0.000099 + 0.009999 = 0.010098$$

$$P(D|T) = \frac{0.99 \times 0.0001}{0.010098} = \frac{0.000099}{0.010098} \approx 0.0098$$

Ans โอกาสที่จะเป็นโรคนี้จริง 0.98%

ผลตรวจแม่นยำ 99% จึงเป็นข่าวดี

Exercise 4 (AIMA, Ex 13.13)

For each of the following statements, either prove it is valid or give a counterexample.

1. If $P(a|b, c) = P(b|a, c)$, then $P(a|c) = P(b|c)$.
2. If $P(a|b, c) = P(a)$, then $P(b|c) = P(b)$.
3. If $P(a|b) = P(a)$, then $P(a|b, c) = P(a|c)$.

1. if $P(a|b,c) = P(b|a,c)$ then $P(a|c) = P(b|c)$

Valid

$$P(x|y) = \frac{P(x,y)}{P(y)}$$

$$\frac{P(a,b,c)}{P(b,c)} = \frac{P(b,a,c)}{P(a,c)}$$

$$\frac{1}{P(b,c)} = \frac{1}{P(a,c)}$$

$$P(a,c) = P(b,c)$$

$$P(a|c) = P(b,c)$$

2. if $P(a|b,c) = P(a)$, then $P(b|c) = P(b)$

Invalid

a = โยนเหรียญออกหัว

b = การพกร่ม

c = ฝนตก

$P(a|b,c) = P(a)$ เป็นจริง

ฝนตก หรือ พกร่มไม่ส่งผลต่อการโยนเหรียญ

$P(b|c) = P(b)$ เป็นเท็จ

ฝนตกกับพกร่มเป็นอิสระต่อกัน

3. if $P(a|b) = P(a)$, then $P(a|b,c) = P(a|c)$

Invalid

a = สปริงเกอร์ทำงาน

b = ฝนตก

c = หญ้าเปียก

$P(a|b) = P(a)$ เป็นจริง

สปริงเกอร์กับฝนเป็นอิสระต่อกัน

เหตุ จริง

$P(a|b,c) = P(a|c)$ เป็นเท็จ

สปริงเกอร์ทำงาน เมื่อรู้ว่า หญ้าเปียก และ ฝนตก

Exercise 5 Bag of coins

We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 20 %, 60 %, and 80 %, respectively. One coin is drawn randomly from the bag (with equal probability of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes X_1, X_2 and X_3 (heads or tail).

1. Draw the Bayesian network corresponding to this setup and define the corresponding conditional probability tables (CPTs).
2. Determine which coin (a, b or c) is most likely to have been drawn from the bag if the observed tosses came out heads twice and tail once.

1. $C \to X_1$

$C \to X_2$

$C \to X_3$

CPTs

Table C

C	$P(C)$
a	1/3
b	1/3
c	1/3

Table X_i

C	$P(X_i = \text{Head} \mid C)$	$P(X_i = \text{Tails} \mid C)$
a	0.2	0.8
b	0.6	0.4
c	0.8	0.2

2. case a

$P(H, H, T \mid C=a)\,P(C=a) = (0.2 \times 0.2 \times 0.8) \times \frac{1}{3} = 0.032 \times \frac{1}{3} \approx 0.0107$

case b

$P(H, H, T \mid C=b)\,P(C=b) = (0.6 \times 0.6 \times 0.4) \times \frac{1}{3} = 0.144 \times \frac{1}{3} = 0.0480$

case c

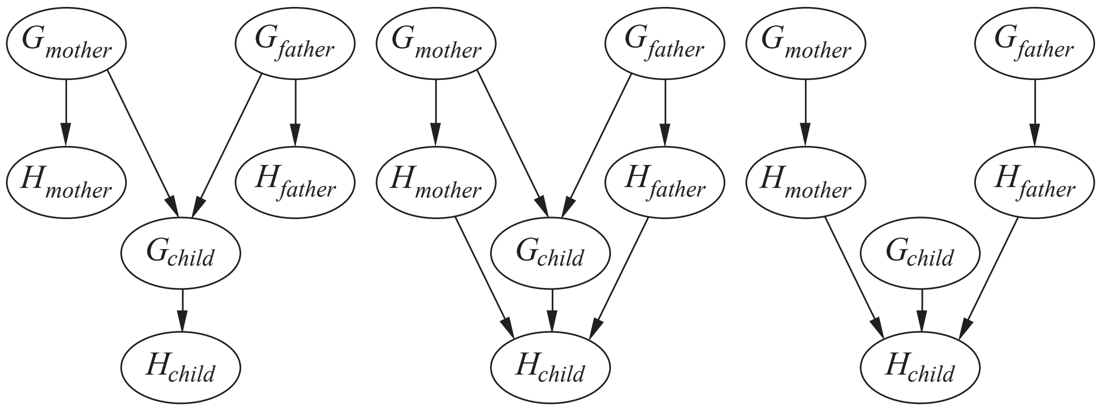
$P(H, H, T \mid C=c)\,P(C=c) = (0.8 \times 0.8 \times 0.2) \times \frac{1}{3} = 0.128 \times \frac{1}{3} \approx 0.0427$

Exercise 6 Handedness (AIMA, Ex 14.6)

Let H_x be a random variable denoting the handedness of an individual x, with possible values l or r. A common *hypothesis* is that left or right-handedness is inherited by a simple mechanism; that is, there is a gene G_x, also with values l or r, and H_x turns out the same as G_x with some probability s.

Furthermore, the gene itself is equally likely to be inherited from either of the individual's parents, with a small nonzero probability m of a random mutation flipping the gene.

1. Which of the following networks claim that $P(G_{child}|G_{mother}, G_{father}) = P(G_{child})$?



2. Which of the networks make independence claims that are consistent with the hypothesis about the inheritance of handedness ?
3. Write down the CPT for the G_{child} nodes in first network, in terms of s and m.
4. Suppose that $P(G_{mother} = l) = P(G_{father} = l) = q$. In the first network, derive an expression for $P(G_{child} = l)$ in terms of s, m and q.
5. Under conditions of genetic equilibrium, we expect the distribution of genes to be the same across generations. Use this to calculate the value of q, and, given what you know about handedness in humans, explain why the hypothesis described at the beginning of this question must be wrong.

1. Ans ไม่มีของสักโครงข่าย

2. Ans โครงข่ายที่ 1 เหมาะสุด)

3. Ans

G_{mother}	G_{father}	$P(G_{child}=l)$	$P(G_{child}=r)$
l	l	1 − m	m
l	r	0.5	0.5
r	l	0.5	0.5
r	r	m	1 − m

4. Ans $P(G_{child} = l) = q(1-2m)+m$

5. Ans $2q = 1 \rightarrow q = 0.5$ สมมติฐานนี้น่าจะผิด เพราะโมเดล 50/50 แต่ในความเป็นจริงมีคนถนัดซ้ายเพียง 10%.

Quiz

The Bayes' rule states that ...

- ☐ $P(x \mid y) = \frac{P(y|x)P(y)}{P(x)}$.
- ☐ $P(y \mid x) = \frac{P(x|y)P(x)}{P(y)}$.
- ☐ $P(x) = \frac{P(x|y)P(y)}{P(x)}$.

Which of the following is true?

- ☐ $P(x, y \mid z) = \frac{P(x|y,z)}{P(y,z|x)}$.
- ☐ $P(x, y, z) = P(x)P(y)P(z)$.
- ☐ $P(x \mid y, z) = \frac{P(x,y,z)}{P(y)P(z)}$.

Naive Bayes model assumes ...

- ☑ pairwise independence between effects.
- ☐ independence between effects and cause.
- ☐ conditional independence between each effect and the cause, given another effect.

The Bayes' rule can be read as "The posterior is equal to ...

- ☑ the product of the likelihood and the prior".
- ☐ the ratio between the posterior distribution and the evidence, times the prior".
- ☐ the likelihood to prior ratio multiplied by the evidence".