Numerical methods for ODEs : Week 1

$y(t)$ = number of atoms at time t

λ = constant related to half-life

$y'(t)$ = change in the number of atoms at time t

$y'(t) = -\lambda y(t)$ ← ODE : differential equation
ordinary as it involves
only derivatives w.r.t.
variable

$y(t) = A e^{-\lambda t}$ initial value: $y(0) = y_0$

ODE + initial value = initial value problem (IVP)

finding A: $y(0) = A e^{-\lambda 0} = A \overset{!}{=} y_0$

$\Rightarrow A = y_0$

Relation λ to half-life:

$$y(t_c) = \frac{1}{2} y_0$$

$$\Leftrightarrow \quad y_0 e^{-\lambda t_c} = \frac{1}{2} y_0$$

$$\Leftrightarrow \quad e^{-\lambda t_c} = \frac{1}{2}$$

$$\Leftrightarrow \quad -\lambda t_c = \log(1/2)$$

$$\Leftrightarrow \quad t_c = -\frac{\log(1/2)}{\lambda}$$

$$\boxed{\underbrace{y'''}_{\text{3rd order ODE}}(t) = -\lambda y(t)}$$

General form of first-order IVPs

$$y'(t) = f(t, y(t))$$

radioactive decay: $f(t, y(t)) = \underbrace{f(y(t)) = -\lambda y(t)}_{\text{autonomous IVP}}$

Another example:

$$y'(t) = \sin(t) - y(t) \; ; \; f(t, y(t)) = \sin(t) - y(t)$$

$$y(t) = A e^{-t} + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)$$

Check: $y'(t) = -A e^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$

$$= \sin(t) - A e^{-t} + \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t)$$

$$= \sin(t) - \left(A e^{-t} + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) \right)$$

$$= \sin(t) - y(t)$$

☺

Fix A according to $y(0) = y_0$:

$$y(0) = A - \frac{1}{2} \stackrel{!}{=} y_0 \Rightarrow \underline{\underline{A = y_0 + \frac{1}{2}}}$$

1 Basics of the Theory of Ordinary Differential Equations

This section introduces the analytical foundations for the study of Ordinary Differential Equations (ODEs) and Inital Value Problems (IVPs).

Careful: x and t play the same role!!

1.1 Ordinary Differential Equations

An *ordinary differential equation of m-th order* is an equation of the following form:

$$F(x, y, y', \dots, y^{(m)}) = 0,$$

$y'(x) = f(x, y(x))$

where F is a suitable real-valued function. One can think of x as the independent variable and of y as the function to be found, which depends on x.

or

Usually, we look for a *solution function* $y : I \to \mathbb{R}$ of the ODE above defined on a certain interval $I \subseteq \mathbb{R}$, such that y is m-times continuously differentiable on I and

$y'(t) = f(t, y(t))$

$$F(x, y(x), y'(x), \dots, y^{(m)}(x)) = 0 \quad \text{for all} \quad x \in I.$$

1 Example. The equation $y' + 2xy = 0$ is an ordinary differential equation of first order since the highest derivative of y is of order one ($m = 1$). Note that this equation matches the form shown above since $F\colon \mathbb{R}^3 \to \mathbb{R}$, $F(x, y, z) := z + 2xy$ fulfills $F(x, y, y') = 0$.

$\cancel{x'(t) = f(t, x(t))}$

A solution of this ODE is $y(x) := e^{-x^2}$, because $y'(x) = e^{-x^2} \cdot (-2x) = -2xy(x)$ for all $x \in \mathbb{R}$ and thus

$$F(x, y, y') = y' + 2xy = -2xy + 2xy = 0. \tag{1.1}$$

This solution is not unique since any solution of the form $y(x) = ce^{-x^2}$, for any $c \in \mathbb{R}$, will satisfy the equation.

Usually, an m-th order differential equation can be presented in an *explicit form*, i.e.

$$y^{(m)} = f(x, y, y', \dots, y^{(m-1)}).$$

The function f is often called the *right-hand side function* since it defines the ODE's right-hand side.

2 Example. Consider again the ODE $y' + 2xy = 0$. Written in explicit form it becomes

$$y'(x) = -2xy(x) =: f(x, y(x)). \tag{1.2}$$

The function $y(x) = e^{-x^2}$ of course remains a solution since

$$y'(x) = e^{-x^2} \cdot (-2x) = -2xy(x) = f(x, y(x)). \tag{1.3}$$

1.2 Systems of Ordinary Differential Equations

Let $D \subseteq \mathbb{R} \times \mathbb{R}^m$, $f\colon D \to \mathbb{R}^m$ be a continuous function and

$$y' = f(x, y)$$

be a system of m first order differential equations. A solution $y = (y_1, \dots, y_m)^T\colon I \to \mathbb{R}^m$ defined on a certain interval $I \subseteq \mathbb{R}$ is necessarily continuously differentiable on I and must fulfill $(x, y(x)) \in D$ and

$$y'(x) = f(x, y(x)) \quad \text{for all} \quad x \in I.$$

Written componentwise as a system, this reads

$$\begin{aligned} y_1' &= f_1(x, y_1, \dots, y_m), \\ y_2' &= f_2(x, y_1, \dots, y_m), \\ &\vdots \\ y_m' &= f_n(x, y_1, \dots, y_m). \end{aligned}$$

Systems of differential equations are common in many applications, e.g. compartmental models in chemical process engineering.

3 Example (Prey-predator model)**.** Let y_1 be the number of a prey population and y_2 the number of a predator population. Equations modeling the evolution of prey and predator populations can be stated as

$$\begin{aligned} y_1' &= a y_1 - b y_1 y_2 \\ y_2' &= c y_1 y_2 - d y_2, \end{aligned}$$

where a, b, c, d are positive constants.

4 Remark. Any explicit differential equation of m-th order can be written as a system of m first order differential equations:

Consider the m-th order equation $y^{(m)} = f(x, y, y', \dots, y^{(m-1)})$. Introducing the notation

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} := \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(m-1)} \end{pmatrix}$$

the equation can be stated as $y_m' = f(x, y_1, y_2, \dots, y_m)$, which is a first order differential equation. Hence, the new notation leads to the following first order system of m ordinary differential equations:

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
&\vdots \\
y_m' &= f(x, y_1, \ldots, y_m).
\end{aligned}
$$

5 Lemma. *The function $y\colon I \to \mathbb{R}$ is a solution of the m-th order differential equation if*

$$I \ni x \mapsto \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(m-1)}(x) \end{pmatrix}$$ *is a solution for the m-dimensional first order system.*

6 Example (Harmonic oscillator)**.** Newton's law states that mass times acceleration equals force. Consider a small mass $\mathbf{m}$ on a spring that can oscillate up and down. Let $y(t)$ be the distance from its resting position at some time t. Then, according to Hooke's law, the mass is exposed to a restoring force

$$F = -\mathbf{k}y(t)\ , \tag{1.4}$$

where $\mathbf{k}$ is the spring constant. Since $y'(t)$ is the velocity and $y''(t)$ is the acceleration of the mass, Newton's law in this case becomes

$$\mathbf{m}y''(t) = -\mathbf{k}y(t). \tag{1.5}$$

Defining $y_1 = y$ and $y_2 = y'$ the equation transforms into the following two-dimensional first order ODE system:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = f(\mathbf{y}) = \begin{pmatrix} y_2 \\ -\frac{\mathbf{k}}{\mathbf{m}} y_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\mathbf{k}}{\mathbf{m}} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $\mathbf{y} = (y_1, y_2)^T$. Solutions of first order systems can also be found as solutions of integral equations. This point of view will be explored later.

1.3 Well-posedness of ODE Systems

The following is the key theorem regarding existence and uniqueness of solutions of ODE systems. Together with the continuous dependence of the solution on the initial value and other data, these give conditions for well-posedness.

7 Lemma. *Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be open, $f\colon D \to \mathbb{R}^n$ continuous. Let $I \subseteq \mathbb{R}$ be an interval, $y\colon I \to \mathbb{R}^n$, $x_0 \in I$, $y_0 \in \mathbb{R}^n$. Then, the following statements are equivalent:*

initial value
parameters

1. Solution exists
2. Solution is unique
3. Solution depends continuously on "data"

(a) y is the solution for the initial value problem (IVP)

$$y' = f(x, y), \quad y(x_0) = y_0.$$

(b) y is continuous, $(x, y(x)) \in D$ for all $x \in I$, and y satisfies the integral equation

$$y(x) = y_0 + \int\limits_{x_0}^{x} f(s, y(s))\, ds \quad \textit{for all} \quad x \in I.$$

Lemma 7 is the key to existence (and uniqueness) of solutions of initial value problems.

8 Definition (Lipschitz continuous functions)**.** Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$. A function $f\colon D \to \mathbb{R}^n$ is said to be Lipschitz continuous if f fulfills the *Lipschitz condition* (with respect to y), i.e. if there exists an $L \geq 0$, such that

$$\|f(x, y) - f(x, \tilde{y})\| \leq L\|y - \tilde{y}\|$$

for all $x \in \mathbb{R}$, $y, \tilde{y} \in \mathbb{R}^n$ with $(x, y), (x, \tilde{y}) \in D$.[1]

9 Remark. A Lipschitz continuous function is continuous.

10 Example. Let $D \subset \mathbb{R} \times \mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ be linear in the second argument, that is $f(x, y) = Ay$ for some matrix $A \in \mathbb{R}^{n \times n}$. Then, f is Lipschitz continuous in y with Lipschitz constant $L = \|A\|$.[2]

Proof.

$$\|f(x, y) - f(x, \tilde{y})\| = \|Ay - A\tilde{y}\| = \|A(y - \tilde{y})\| \leq \|A\| \, \|y - \tilde{y}\| \,. \tag{1.6}$$

□

11 Example. Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be compact. If $f : D \to \mathbb{R}^n$ is continuously differentiable on D, then it is also Lipschitz continuous.

Proof. Since D is compact and f' is continuous,

$$L := \max_{(x,y^{(1)}),\ldots,(x,y^{(n)}) \in D} \left\| \begin{pmatrix} (f_1)_y(x, y^{(1)}) \\ \vdots \\ (f_n)_y(x, y^{(n)}) \end{pmatrix} \right\| < \infty \tag{1.7}$$

is a well-defined real number. Here $f_i(x, y)$ denotes the i-th component function of f and $(f_i)_y = \left(\frac{\partial f_i}{\partial y_j}\right)_{1 \leq j \leq n}$ its derivative (or gradient) with respect to y considered as a row

[1] In the whole script $\|\cdot\|$ denotes some not further specified vector norm on some $\mathbb{R}^n$, $n \in \mathbb{N}$. For example, the Euclidean norm $\|v\| = \|v\|_2 = \sqrt{v_1^2 + \ldots + v_n^2}$, or the maximum norm $\|v\| = \|v\|_\infty = \max\limits_i |v_i|$ might be chosen. In the 1-dimensional case ($n = 1$), we demand $\|\cdot\| = |\cdot|$.

[2] $\|A\|$ is the matrix norm induced by the given vector norm, i.e. $\|A\| = \sup\{\|Ax\| \,; x \in \mathbb{R}^n, \|x\| = 1\}$.

vector of length n. Let $(x,y),(x,\tilde{y}) \in D$. By the generalized mean value theorem, for each component $i \in \{1,\dots,n\}$ there is a point $\xi^{(i)}$ on the line connecting y and $\tilde{y}$ such that $f_i(x,y) - f_i(x,\tilde{y}) = f_i'(x,\xi^{(i)})(y-\tilde{y})$. Note carefully that $\xi^{(i)} \neq \xi^{(j)}$ may hold true for $i \neq j$. Using the definition of L we conclude

$$\|f(x,y) - f(x,\tilde{y})\| = \left\| \begin{pmatrix} (f_1)_y(x,\xi^{(1)}) \\ \vdots \\ (f_n)_y(x,\xi^{(n)}) \end{pmatrix} (y-\tilde{y}) \right\| \le L\,\|y-\tilde{y}\|\,. \tag{1.8}$$

□

The *Lipschitz condition* is the key requirement of the Picard-Lindelöf Theorem, which proves local existence and uniqueness of solutions for IVPs.

12 Theorem (Picard-Lindelöf's Theorem)**.** *Let $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^n$, $a,b > 0$, and let $f\colon [x_0, x_0+a] \times \{y \in \mathbb{R}^n;\ \|y-y_0\| \le b\} \to \mathbb{R}^n$ be continuous fulfilling a Lipschitz condition with respect to y. Then the initial value problem*

$$y' = f(x,y), \quad y(x_0) = y_0 \tag{1.9}$$

has a unique solution in a subinterval $[x_0, x_0+\delta] \subseteq [x_0, x_0+a]$, where $0 < \delta \le a$.

13 Remark. This subinterval can be taken as $I = [x_0, x_0+\delta]$ with $\delta = \min\{a, \frac{b}{\|f\|_\infty}\}$.

To summarize, if the right-hand side function $f(x,y)$ of the first order IVP (1.9) is Lipschitz continuous, we know that there exists a solution and this solution is unique in an interval. If the function $f(x,y)$ is continuous but not Lipschitz continuous with respect to y, then existence of solutions can sill be proved. However, solutions my no longer be unique, i.e. the IVP (1.9) may have several distinct solutions in this case.

14 Theorem (Peano's Theorem)**.** *Let $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^n$, $a,b > 0$, and suppose that $f\colon [x_0, x_0+a] \times \{y \in \mathbb{R}^n;\ \|y-y_0\| \le b\} \to \mathbb{R}^n$ is continuous. Then the initial value problem*

$$y' = f(x,y), \quad y(x_0) = y_0$$

has at least one solution on a subinterval $[x_0, x_0+\delta] \subseteq [x_0, x_0+a]$, where $0 < \delta \le a$.

15 Example. We consider the following initial value problem

$$y' = y^{1/3}, \quad y(0) = 0.$$

In addition to the trivial solution $y = 0$, there are two other solutions $y(x) = \left(\frac{2}{3}x\right)^{3/2}$ and $y(x) = -\left(\frac{2}{3}x\right)^{3/2}$. Note that the right-hand side function $f(x,y) = y^{1/3}$ does not depend on x and is not Lipschitz continuous with respect to y.

16 Remark. If f is k-times continuously differentiable, then the solution y is $(k+1)$-times continuously differentiable.

Roughly:

$y(t_0) = y_0$

f is Lipschitz continuous $\Rightarrow$ $y' = f(t, y(t))$ well-posed for $[t_0, t_0 + \delta]$

Lipschitz continuous:

$$\| f(y) - f(\tilde{y}) \| \leq \underbrace{L}_{\text{constant}} \| y - \tilde{y} \|$$

Examples:

- $f(y) = Ay$: $\| f(y) - f(\tilde{y}) \| = \| Ay - A\tilde{y} \|$

$$= \| A(y - \tilde{y}) \|$$

$$\leq \|A\| \cdot \| y - \tilde{y} \|$$

- f differentiable and f' bounded.

$$\| f(y) - f(\tilde{y}) \| = \| f'(\theta)(y - \tilde{y}) \|$$

$$\leq \| f'(\theta) \| \cdot \| y - \tilde{y} \|$$

$$\leq C \| y - \tilde{y} \|$$

Without L-continuity, IVP need not be well-posed!

$$y' = y^{(1/3)} \quad ; \quad y(0) = 0$$

We have three solutions:

1) $y(t) = 0$

2) $y(t) = \left(\frac{2}{3}t\right)^{3/2}$

3) $y(t) = -\left(\frac{2}{3}t\right)^{3/2}$

Continuous Dependence on Data

Let y be a solution to the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0,$$

and $\tilde{y}$ a solution to the initial value problem

$$y' = \tilde{f}(x, y), \quad y(\tilde{x}_0) = \tilde{y}_0.$$

If $\tilde{f}$ is "close" to f, $\tilde{x}_0$ close to x_0, and $\tilde{y}_0$ close to y_0, then $\tilde{y}$ is "close" to y in the following sense:

17 Theorem. *Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be open, $f\colon D \to \mathbb{R}^n$ be continuous and Lipschitz continuous w.r.t. y, and $(x_0, y_0) \in D$. Furthermore, let $I \subseteq \mathbb{R}$ be a bounded, closed interval such that $y\colon I \to \mathbb{R}^n$ is the solution to the initial value problem*

$$y' = f(x, y), \quad y(x_0) = y_0$$

on I. Then there is a $C \geq 0$, such that the following holds true: If $\tilde{f}\colon D \to \mathbb{R}^n$ is continuous and bounded, $\tilde{I} \subseteq I$ an interval with $\tilde{x}_0 \in \tilde{I}$, $(\tilde{x}_0, \tilde{y}_0) \in D$, and $\tilde{y}\colon \tilde{I} \to \mathbb{R}^n$ is a solution to the initial value problem

$$\tilde{y}' = \tilde{f}(x, y), \quad \tilde{y}(\tilde{x}_0) = \tilde{y}_0,$$

then

$$\|\tilde{y} - y\|_{\infty,\tilde{I}} \leq C\Big(|\tilde{x}_0 - x_0| + \|\tilde{y}_0 - y_0\| + \big\|\tilde{f} - f\big\|_{\infty,D}\Big),$$

where

$$\|\varphi\|_{\infty,A} := \sup\{\|\varphi(a)\|_\infty\,;\ a \in A\}.$$

18 Remark. For continuously differentiable f one can show the continuously differentiable dependence of the solution on the data x_0 and y_0.

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