MAPM312 Project - Parabolic PDEs

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April 29, 2014

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1 Introduction

1.1 Problem Statement

The following set of coupled partial differential equations will be analyised at the hand of the Crank-Nicolson model:

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + u(1-u) - \frac{auv}{1+\lambda u}$$

$$\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} - \frac{v}{ab} + \frac{auv}{b(1+\lambda u)}$$

$$a \leq x \leq b, \quad 0 \leq t$$
$$u_x(a,t) = 0, \quad u_x(b,t) = 0$$

2 Derivation

From the problem statement we have the following system of PDEs.

$$u_t = D_u u_{xx} + f(u, v) \tag{1a}$$

$$v_t = D_v v_{xx} + g(u, v) \tag{1b}$$

where $a \leq x \leq b$, $0 \leq t \leq T$, and

$$f(u, v) = u(1 - u) - \frac{auv}{1 + \lambda u} \tag{2a}$$

$$g(u, v) = -\frac{v}{ab} + \frac{auv}{b(1 + \lambda u)} \tag{2b}$$

We use the following finite difference approximations, derived from the Taylor series expansion, in place of the u_t, u_{xx}, v_t, and v_{xx} in equations (1). The spacial and temporal domains are descretized using N and M points respectively and h and k are the related step lengths.

$$u_t = \frac{u(x, t + k) - u(x, t)}{k} - \frac{k}{2} u_{tt}(x, \tau) \tag{3a}$$

$$v_t = \frac{v(x, t + k) - v(x, t)}{k} - \frac{k}{2} v_{tt}(x, \sigma) \tag{3b}$$

$$u_{xx} = \frac{u(x - h, t) - 2u(x, t) + u(x + h, t)}{h^2} - \frac{h^2}{12} u_{xxxx}(\xi, t) \tag{3c}$$

$$v_{xx} = \frac{v(x - h, t) - 2v(x, t) + v(x + h, t)}{h^2} - \frac{h^2}{12} v_{xxxx}(\epsilon, t) \tag{3d}$$

Let $n = 1 \ldots N$ and $m = 1 \ldots M$. Discard the truncation terms from equations (3) and let U_n^m and V_n^m approximate the true solutions $u(x_n, t_m)$ and $v(x_n, t_m)$. We can then replace equations (1) with the following system of equations. We also write $f(u_n^m, v_n^m)) = f_n^m$ and $g(u_n^m, v_n^m)) = g_n^m$.

$$\frac{U_n^{m+1} - U_n^m}{k} = \frac{D_u}{2} \left\{ \left[\frac{U_{n-1}^{m+1} - 2U_n^{m+1} + U_{n+1}^{m+1}}{h^2} \right] + \left[\frac{U_{n-1}^m - 2U_n^m + U_{n+1}^m}{h^2} \right] \right\} + f_n^m \tag{4a}$$

$$\frac{V_n^{m+1} - V_n^m}{k} = \frac{D_v}{2} \left\{ \left[\frac{V_{n-1}^{m+1} - 2V_n^{m+1} + V_{n+1}^{m+1}}{h^2} \right] + \left[\frac{V_{n-1}^m - 2V_n^m + V_{n+1}^m}{h^2} \right] \right\} + g_n^m \tag{4b}$$

Multiplying equations (4) throughout by $2k$ and rearranging the terms yields the following with $r = D_u k / h^2$ and $s = D_v k / h^2$.

$$(2 + 2r)U_n^{m+1} - rU_{n-1}^{m+1} - rU_{n+1}^{m+1} = (2 - 2r)U_n^m + rU_{n-1}^m + rU_{n+1}^m + 2k f_n^m \tag{5a}$$

$$(2 + 2s)V_n^{m+1} - sV_{n-1}^{m+1} - sV_{n+1}^{m+1} = (2 - 2s)V_n^m + sV_{n-1}^m + sV_{n+1}^m + 2k g_n^m \tag{5b}$$

We now investigate the boundary conditions. When $n = 1$ we have may use the central difference formula for the first derivatives as follows:

$$U_x(a, t) = 0 = \frac{U_2 - U_0}{2h} \tag{6a}$$

$$V_x(a, t) = 0 = \frac{V_2 - V_0}{2h} \tag{6b}$$

Since the derivatives at the boundaries are given as 0 this implies that:

$$U_2 = U_0 \tag{7a}$$

$$V_2 = V_0 \tag{7b}$$

The same argument follows for the right endpoint where $n = N$ and we have:

$$U_x(b, t) = 0 = \frac{U_{N+1} - U_{N-1}}{2h} \tag{8a}$$

$$V_x(b, t) = 0 = \frac{V_{N+1} - V_{N-1}}{2h} \tag{8b}$$

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hence

$$U_{N+1} = U_{N-1} \tag{9a}$$

$$V_{N+1} = V_{N-1} \tag{9b}$$

We may then substitute (7) and (9) into their respective equations from (5) so that:

$$(2+2r)U_1^{m+1} - 2rU_2^{m+1} = (2-2r)U_1^m + 2rU_2^m + 2kf_1^m \tag{10a}$$

$$(2+2s)V_1^{m+1} - 2sV_2^{m+1} = (2-2s)V_1^m + 2sV_2^m + 2kg_1^m \tag{10b}$$

$$(2+2r)U_N^{m+1} - 2rU_{N-1}^{m+1} = (2-2r)U_N^m + 2rU_{N-1}^m + 2kf_N^m \tag{10c}$$

$$(2+2s)V_N^{m+1} - 2sV_{N-1}^{m+1} = (2-2s)V_N^m + 2sV_{N-1}^m + 2kg_N^m \tag{10d}$$

rewritten into the following matrix equations:

$$
\begin{bmatrix}
2+2r & -2r & \dots & 0 \\
-r & 2+2r & \dots & 0 \\
0 & -r & \ddots & \vdots \\
\vdots & \vdots & \ddots & -r \\
0 & 0 & -2r & 2+2r
\end{bmatrix}
\begin{bmatrix}
U_1^{m+1} \\
U_2^{m+1} \\
U_3^{m+1} \\
\vdots \\
U_n^{m+1}
\end{bmatrix}
=
\begin{bmatrix}
2-2r & 2r & \dots & 0 \\
r & 2-2r & \dots & 0 \\
0 & r & \ddots & \vdots \\
\vdots & \vdots & \ddots & r \\
0 & 0 & 2r & 2-2r
\end{bmatrix}
\begin{bmatrix}
U_1^m \\
U_2^m \\
U_3^m \\
\vdots \\
U_n^m
\end{bmatrix}
+ 2k
\begin{bmatrix}
f_1^m \\
f_2^m \\
f_3^m \\
\vdots \\
f_n^m
\end{bmatrix}
$$

$$
\begin{bmatrix}
2+2s & -2s & \dots & 0 \\
-s & 2+2s & \dots & 0 \\
0 & -s & \ddots & \vdots \\
\vdots & \vdots & \ddots & -s \\
0 & 0 & -2s & 2+2s
\end{bmatrix}
\begin{bmatrix}
V_1^{m+1} \\
V_2^{m+1} \\
V_3^{j+1} \\
\vdots \\
V_n^{m+1}
\end{bmatrix}
=
\begin{bmatrix}
2-2s & 2s & \dots & 0 \\
s & 2-2s & \dots & 0 \\
0 & s & \ddots & \vdots \\
\vdots & \vdots & \ddots & s \\
0 & 0 & 2s & 2-2s
\end{bmatrix}
\begin{bmatrix}
V_1^m \\
V_2^m \\
V_3^m \\
\vdots \\
V_n^m
\end{bmatrix}
+ 2k
\begin{bmatrix}
g_1^m \\
g_2^m \\
g_3^m \\
\vdots \\
g_n^m
\end{bmatrix}
$$

3 Stability Analysis

We investigate the stabiliy of the Crank-Nicolson method using the equation from (1a). A similar argument may be used to show the stability of the Crank-Nicolson method using the equation from (1b).

$$\frac{U_n^{m+1} - U_n^m}{k} = \frac{D_u}{2} \left(\frac{U_{n-1}^{m+1} - 2U_n^{m+1} + U_{n+1}^{m+1}}{h^2} + \frac{U_{n-1}^m - 2U_n^m + U_{n+1}^m}{h^2} \right) \tag{11}$$

We then substitute $U_n^m = \xi^m e^{iknh}$ and rearrange the terms in (11) such that:

$$\begin{aligned}
\xi^{m+1} e^{iknh} - \xi^m e^{iknh} = \frac{D_u k}{2h^2} \Big(& \xi^{m+1} e^{ik(n+1)h} + \xi^m e^{ik(n+1)h} \\
& -2\xi^{m+1} e^{iknh} - 2\xi^m e^{iknh} \\
& +\xi^{m+1} e^{ik(n-1)h} + \xi^m e^{ik(n-1)h} \Big)
\end{aligned} \tag{12}$$

We may then divide throughout by $\xi^m e^{iknh}$ which gives the following:

$$\xi - 1 = \frac{D_u k}{2h^2} \left(\xi e^{ikh} + e^{ikh} - 2\xi - 2 + \xi e^{-ikh} + e^{-ikh} \right) \tag{13}$$

Rearranging (13) and replacing the complex exponential gives:

$$\xi = 1 - \frac{D_u \tau}{2h^2} (\xi + 1)[1 - \cos(kh)] \tag{14}$$

Taking the absolute value of ξ gives:

$$|\xi| = \left| \frac{1 - D_u \tau / h^2 [1 - \cos(kh)]}{1 + D_u \tau / h^2 [1 - \cos(kh)]} \right| \leq 1 \tag{15}$$

Since (15) is true for any value of k the Crank-Nicolson model is unconditionally stable.

4   Results

5   Discussion

6 Conclusion