MAPM312 Project - Parabolic PDEs

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1 Introduction

1.1 Problem Statement

The following set of coupled partial differential equations will be analysed at the hand of the Crank-Nicolson model:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= D_u \frac{\partial^2 u}{\partial x^2} + u(1-u) - \frac{auv}{1+\lambda u} \\
\frac{\partial v}{\partial t} &= D_v \frac{\partial^2 v}{\partial x^2} - \frac{v}{ab} + \frac{auv}{b(1+\lambda u)}
\end{aligned}
$$

$$
\begin{aligned}
&a \leq x \leq b, \quad 0 \leq t \\
&u_x(a,t) = 0, \quad u_x(b,t) = 0
\end{aligned}
$$

2 Derivation

From the problem statement we have the following system of PDEs.

$$u_t = D_u u_{xx} + f(u,v) \tag{1a}$$
$$v_t = D_v v_{xx} + g(u,v) \tag{1b}$$

where

$$f(u,v) = u(1-u) - \frac{auv}{1+\lambda u}$$
$$g(u,v) = -\frac{v}{ab} + \frac{auv}{b(1+\lambda u)}$$

We use the following finite difference approximations, derived from the Taylor series expansion, in place of the u_t, u_{xx}, v_t, and v_{xx} in equations (1).

$$u_t = \frac{u(x,t+k) - u(x,t)}{k} - \frac{k}{2}u_{tt}(x,\tau) \tag{3a}$$
$$v_t = \frac{v(x,t+k) - v(x,t)}{k} - \frac{k}{2}v_{tt}(x,\sigma) \tag{3b}$$
$$u_{xx} = \frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} - \frac{h^2}{12}u_{xxxx}(\xi,t) \tag{3c}$$
$$v_{xx} = \frac{v(x-h,t) - 2v(x,t) + v(x+h,t)}{h^2} - \frac{h^2}{12}v_{xxxx}(\epsilon,t) \tag{3d}$$

By discarding the truncation terms from equations (3) and letting $U_{i,j}$ and $V_{i,j}$ approximate the true solutions $u(x_i,t_j)$ and $v(x_i,t_j)$ respectively we can replace equations (1) with the following system of equations. We write $f(u_i^j, v_i^j)) = f_i^j$ and $g(u_i^j, v_i^j)) = g_i^j$.

$$\frac{U_i^{j+1} - U_i^j}{k} = \frac{D_u}{2}\left\{\left[\frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{h^2}\right] + \left[\frac{U_{i-1}^{j} - 2U_i^{j} + U_{i+1}^{j}}{h^2}\right]\right\} + f_i^j \tag{4a}$$
$$\frac{V_i^{j+1} - V_i^j}{k} = \frac{D_v}{2}\left\{\left[\frac{V_{i-1}^{j+1} - 2V_i^{j+1} + V_{i+1}^{j+1}}{h^2}\right] + \left[\frac{V_{i-1}^{j} - 2V_i^{j} + V_{i+1}^{j}}{h^2}\right]\right\} + g_i^j \tag{4b}$$

Multiplying equations (4) throughout by $2k$ and rearranging the terms yields the following with $r = D_u k/h^2$ and $s = D_v k/h^2$.

$$(2+2r)U_i^{j+1} - rU_{i-1}^{j+1} - rU_{i+1}^{j+1} = (2-2r)U_i^j + rU_{i-1}^j + rU_{i+1}^j + 2kf_i^j \tag{5a}$$
$$(2+2s)V_i^{j+1} - sV_{i-1}^{j+1} - sV_{i+1}^{j+1} = (2-2s)V_i^j + sV_{i-1}^j + sV_{i+1}^j + 2kg_i^j \tag{5b}$$

We now investigate the boundary conditions. When $i = 0$ we have may use the central difference formula for the first derivatives as follows:

$$\frac{\partial U_0^{j+1}}{\partial x} \approx \frac{{U_1}^{j+1} - U_{-1}^{j+1}}{2h} \tag{6a}$$
$$\frac{\partial V_0^{j+1}}{\partial x} \approx \frac{{V_1}^{j+1} - V_{-1}^{j+1}}{2h} \tag{6b}$$

Since the derivatives at the boundaries are given as 0 this implies that:

$$U_1^{j+1} = U_{-1}^{j+1} \tag{7a}$$
$$V_1^{j+1} = V_{-1}^{j+1} \tag{7b}$$

The same argument follows for the right endpoint where $i = N$ and we have:

$$U_{N-1}^{j+1} = U_{N-1}^{j+1} \tag{8a}$$
$$V_{N+1}^{j+1} = V_{N-1}^{j+1} \tag{8b}$$

We may then substitute these into their respective equations so that:

$$(2+2r)U_0^{j+1} - 2rU_1^{j+1} = (2-2r)U_0^j + 2rU_1^j + 2kf_i^j \tag{9a}$$
$$(2+2s)V_0^{j+1} - 2sV_1^{j+1} = (2-2s)V_0^j + 2sV_1^j + 2kg_i^j \tag{9b}$$
$$(2+2r)U_N^{j+1} - 2rU_{N-1}^{j+1} = (2-2r)U_N^j + 2rU_{N-1}^j + 2kf_i^j \tag{9c}$$
$$(2+2s)V_N^{j+1} - 2sV_{N-1}^{j+1} = (2-2s)V_N^j + 2sV_{N-1}^j + 2kg_i^j \tag{9d}$$

These systems may be rewritten into the following matrix equations:

$$\begin{bmatrix} 2+2r & -2r & \dots & 0 \\ -r & 2+2r & \dots & 0 \\ 0 & -r & \ddots & \vdots \\ \vdots & \vdots & \ddots & -r \\ 0 & 0 & -2r & 2+2r \end{bmatrix} \begin{bmatrix} U_1^{j+1} \\ U_2^{j+1} \\ U_3^{j+1} \\ \vdots \\ U_n^{j+1} \end{bmatrix} = \begin{bmatrix} 2-2r & 2r & \dots & 0 \\ r & 2-2r & \dots & 0 \\ 0 & r & \ddots & \vdots \\ \vdots & \vdots & \ddots & r \\ 0 & 0 & 2r & 2-2r \end{bmatrix} \begin{bmatrix} U_1^j \\ U_2^j \\ U_3^j \\ \vdots \\ U_n^j \end{bmatrix} + 2k \begin{bmatrix} f_1^j \\ f_2^j \\ f_3^j \\ \vdots \\ f_n^j \end{bmatrix}$$

$$\begin{bmatrix} 2+2s & -2s & \dots & 0 \\ -s & 2+2s & \dots & 0 \\ 0 & -s & \ddots & \vdots \\ \vdots & \vdots & \ddots & -s \\ 0 & 0 & -2s & 2+2s \end{bmatrix} \begin{bmatrix} V_1^{j+1} \\ V_2^{j+1} \\ V_3^{j+1} \\ \vdots \\ V_n^{j+1} \end{bmatrix} = \begin{bmatrix} 2-2s & 2s & \dots & 0 \\ s & 2-2s & \dots & 0 \\ 0 & s & \ddots & \vdots \\ \vdots & \vdots & \ddots & s \\ 0 & 0 & 2s & 2-2s \end{bmatrix} \begin{bmatrix} V_1^j \\ V_2^j \\ V_3^j \\ \vdots \\ V_n^j \end{bmatrix} + 2k \begin{bmatrix} g_1^j \\ g_2^j \\ g_3^j \\ \vdots \\ g_n^j \end{bmatrix}$$

3 Stability Analysis

We investigate the stabiliy of the Crank-Nicolson method using the equation from (1a). A similar argument may be used to show the stability of the Crank-Nicolson method using the equation from (1b).

$$\frac{U_m^{n+1} - U_m^n}{\tau} = \frac{D_u}{2}\left(\frac{U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}}{h^2} + \frac{U_{m-1}^n - 2U_m^n + U_{m+1}^n}{h^2}\right) \tag{10}$$

We then substitute $U_m^n = \xi^n e^{ikmh}$ and rearrange the terms in (10) such that:

$$\begin{aligned} \xi^{n+1}e^{ikmh} - \xi^n e^{ikmh} = \frac{D_u\tau}{2h^2}\Big(&\xi^{n+1}e^{ik(m+1)h} + \xi^n e^{ik(m+1)h} \\ &-2\xi^{n+1}e^{ikmh} - 2\xi^n e^{ikmh} \\ &+\xi^{n+1}e^{ik(m-1)h} + \xi^n e^{ik(m-1)h}\Big) \end{aligned} \tag{11}$$

We may then divide throughout by $\xi^n e^{ikmh}$ which gives the following:

$$\xi - 1 = \frac{D_u\tau}{2h^2}\left(\xi e^{ikh} + e^{ikh} - 2\xi - 2 + \xi e^{-ikh} + e^{-ikh}\right) \tag{12}$$

Rearranging (12) and replacing the complex exponential with the trigonometric identity gives us:

$$\xi = 1 - \frac{D_u\tau}{2h^2}(\xi + 1)[1 - \cos(kh)] \tag{13}$$

When we take the absolute value of (13) to following inequality holds $\forall \tau \in R$.

$$|\xi| = \left|\frac{1 - D_u\tau/h^2[1 - \cos(kh)]}{1 + D_u\tau/h^2[1 - \cos(kh)]}\right| \leq 1 \tag{14}$$

Since (14) is true for any τ the Crank-Nicolson model is unconditionally stable.

4 Results

5 Discussion

6 Conclusion