

# MAPM312 Project - Parabolic PDEs

Stolk, Martin - 212295187  
Koen, Salomon - 212288350  
Hatherly, Michael - 212240382

April 20, 2014

**Contents**

|          |                             |          |
|----------|-----------------------------|----------|
| <b>1</b> | <b>Introduction</b>         | <b>2</b> |
| 1.1      | Problem Statement . . . . . | 2        |
| <b>2</b> | <b>Derivation</b>           | <b>3</b> |
| <b>3</b> | <b>Stability Analysis</b>   | <b>5</b> |
| <b>4</b> | <b>Results</b>              | <b>6</b> |
| <b>5</b> | <b>Discussion</b>           | <b>7</b> |
| <b>6</b> | <b>Conclusion</b>           | <b>8</b> |

# 1 Introduction

## 1.1 Problem Statement

The following set of coupled partial differential equations will be analysed at the hand of the Crank-Nicolson model:

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u \frac{\partial^2 u}{\partial x^2} + u(1 - u) - \frac{auv}{1 + \lambda u} \\ \frac{\partial v}{\partial t} &= D_v \frac{\partial^2 v}{\partial x^2} - \frac{v}{ab} + \frac{auv}{b(1 + \lambda u)}\end{aligned}$$

$$\begin{aligned}a \leq x \leq b, \quad 0 \leq t \\ u_x(a, t) = 0, \quad u_x(b, t) = 0\end{aligned}$$

## 2 Derivation

From the problem statement we have the following system of PDEs.

$$u_t = D_u u_{xx} + f(u, v) \quad (1a)$$

$$v_t = D_v v_{xx} + g(u, v) \quad (1b)$$

where

$$f(u, v) = u(1 - u) - \frac{auv}{1 + \lambda u}$$

$$g(u, v) = -\frac{v}{ab} + \frac{auv}{b(1 + \lambda u)}$$

We use the following finite difference approximations, derived from the Taylor series expansion, in place of the  $u_t$ ,  $u_{xx}$ ,  $v_t$ , and  $v_{xx}$  in equations (1).

$$u_t = \frac{u(x, t + k) - u(x, t)}{k} - \frac{k}{2} u_{tt}(x, \tau) \quad (3a)$$

$$v_t = \frac{v(x, t + k) - v(x, t)}{k} - \frac{k}{2} v_{tt}(x, \sigma) \quad (3b)$$

$$u_{xx} = \frac{u(x - h, t) - 2u(x, t) + u(x + h, t)}{h^2} - \frac{h^2}{12} u_{xxx}(\xi, t) \quad (3c)$$

$$v_{xx} = \frac{v(x - h, t) - 2v(x, t) + v(x + h, t)}{h^2} - \frac{h^2}{12} v_{xxx}(\epsilon, t) \quad (3d)$$

By discarding the truncation terms from equations (3) and letting  $U_{i,j}$  and  $V_{i,j}$  approximate the true solutions  $u(x_i, t_j)$  and  $v(x_i, t_j)$  respectively we can replace equations (1) with the following system of equations. We write  $f(u_i^j, v_i^j) = f_i^j$  and  $g(u_i^j, v_i^j) = g_i^j$ .

$$\frac{U_i^{j+1} - U_i^j}{k} = \frac{D_u}{2} \left\{ \left[ \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{h^2} \right] + \left[ \frac{U_{i-1}^j - 2U_i^j + U_{i+1}^j}{h^2} \right] \right\} + f_i^j \quad (4a)$$

$$\frac{V_i^{j+1} - V_i^j}{k} = \frac{D_v}{2} \left\{ \left[ \frac{V_{i-1}^{j+1} - 2V_i^{j+1} + V_{i+1}^{j+1}}{h^2} \right] + \left[ \frac{V_{i-1}^j - 2V_i^j + V_{i+1}^j}{h^2} \right] \right\} + g_i^j \quad (4b)$$

Multiplying equations (4) throughout by  $2k$  and rearranging the terms yields the following with  $r = D_u k/h^2$  and  $s = D_v k/h^2$ .

$$(2 + 2r)U_i^{j+1} - rU_{i-1}^{j+1} - rU_{i+1}^{j+1} = (2 - 2r)U_i^j + rU_{i-1}^j + rU_{i+1}^j + 2kf_i^j \quad (5a)$$

$$(2 + 2s)V_i^{j+1} - sV_{i-1}^{j+1} - sV_{i+1}^{j+1} = (2 - 2s)V_i^j + sV_{i-1}^j + sV_{i+1}^j + 2kg_i^j \quad (5b)$$

We now investigate the boundary conditions. When  $i = 0$  we have may use the central difference formula for the first derivatives as follows:

$$\frac{\partial U_0^{j+1}}{\partial x} \approx \frac{U_1^{j+1} - U_{-1}^{j+1}}{2h} \quad (6a)$$

$$\frac{\partial V_0^{j+1}}{\partial x} \approx \frac{V_1^{j+1} - V_{-1}^{j+1}}{2h} \quad (6b)$$

Since the derivatives at the boundaries are given as 0 this implies that:

$$U_1^{j+1} = U_{-1}^{j+1} \quad (7a)$$

$$V_1^{j+1} = V_{-1}^{j+1} \quad (7b)$$

The same argument follows for the right endpoint where  $i = N$  and we have:

$$U_{N-1}^{j+1} = U_{N-1}^{j+1} \quad (8a)$$

$$V_{N+1}^{j+1} = V_{N+1}^{j+1} \quad (8b)$$

We may then substitute these into their respective equations so that:

$$(2 + 2r)U_0^{j+1} - 2rU_1^{j+1} = (2 - 2r)U_0^j + 2rU_1^j + 2kf_i^j \quad (9a)$$

$$(2 + 2s)V_0^{j+1} - 2sV_1^{j+1} = (2 - 2s)V_0^j + 2sV_1^j + 2kg_i^j \quad (9b)$$

$$(2 + 2r)U_N^{j+1} - 2rU_{N-1}^{j+1} = (2 - 2r)U_N^j + 2rU_{N-1}^j + 2kf_i^j \quad (9c)$$

$$(2 + 2s)V_N^{j+1} - 2sV_{N-1}^{j+1} = (2 - 2s)V_N^j + 2sV_{N-1}^j + 2kg_i^j \quad (9d)$$

These systems may be rewritten into the following matrix equations:

$$\begin{bmatrix} 2+2r & -2r & \dots & 0 \\ -r & 2+2r & \dots & 0 \\ 0 & -r & \ddots & \vdots \\ \vdots & \vdots & \ddots & -r \\ 0 & 0 & -2r & 2+2r \end{bmatrix} \begin{bmatrix} f_1^{j+1} \\ f_2^{j+1} \\ f_3^{j+1} \\ \vdots \\ f_n^{j+1} \end{bmatrix} = \begin{bmatrix} 2-2r & 2r & \dots & 0 \\ r & 2-2r & \dots & 0 \\ 0 & r & \ddots & \vdots \\ \vdots & \vdots & \ddots & r \\ 0 & 0 & 2r & 2-2r \end{bmatrix} \begin{bmatrix} f_1^j \\ f_2^j \\ f_3^j \\ \vdots \\ f_n^j \end{bmatrix} \quad (10)$$

$$\begin{bmatrix} 2+2s & -2s & \dots & 0 \\ -s & 2+2s & \dots & 0 \\ 0 & -s & \ddots & \vdots \\ \vdots & \vdots & \ddots & -s \\ 0 & 0 & -2s & 2+2s \end{bmatrix} \begin{bmatrix} g_1^{j+1} \\ g_2^{j+1} \\ g_3^{j+1} \\ \vdots \\ g_n^{j+1} \end{bmatrix} = \begin{bmatrix} 2-2s & 2s & \dots & 0 \\ s & 2-2s & \dots & 0 \\ 0 & s & \ddots & \vdots \\ \vdots & \vdots & \ddots & s \\ 0 & 0 & 2s & 2-2s \end{bmatrix} \begin{bmatrix} g_1^j \\ g_2^j \\ g_3^j \\ \vdots \\ g_n^j \end{bmatrix} \quad (11)$$

### 3 Stability Analysis

## 4 Results

## 5 Discussion



## 6 Conclusion