# MAPM312 Project - Parabolic PDEs

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# 1 Introduction

## 1.1 Problem Statement

The following set of coupled partial differential equations will be analysed at the hand of the Crank-Nicolson model:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= D_u \frac{\partial^2 u}{\partial x^2} + u(1-u) - \frac{auv}{1+\lambda u} \\
\frac{\partial v}{\partial t} &= D_v \frac{\partial^2 v}{\partial x^2} - \frac{v}{ab} + \frac{auv}{b(1+\lambda u)}
\end{aligned}
$$

$$
\begin{aligned}
&a \leq x \leq b, \quad 0 \leq t \\
&u_x(a,t) = 0, \quad u_x(b,t) = 0
\end{aligned}
$$

## 2 Derivation

From the problem statement we have the following system of PDEs.

$$u_t = D_u u_{xx} + u\left(1 - u - \frac{av}{1+\lambda u}\right) \tag{1a}$$

$$v_t = D_v v_{xx} - v\left(\frac{1}{ab} - \frac{au}{b(1+\lambda u)}\right) \tag{1b}$$

$$a \leq x \leq b, \quad 0 \leq t \leq T$$

We use the following finite difference approximations, derived from the Taylor series expansion, in place of the $u_t$, $u_{xx}$, $v_t$, and $v_{xx}$ in equations (1). The spacial and temporal domains are descretized using $N$ and $M$ points respectively and $h$ and $k$ are the related step lengths.

$$u_t = \frac{u(x,t+k) - u(x,t)}{k} - \frac{k}{2}u_{tt}(x,\tau) \tag{2a}$$

$$v_t = \frac{v(x,t+k) - v(x,t)}{k} - \frac{k}{2}v_{tt}(x,\sigma) \tag{2b}$$

$$u_{xx} = \frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} - \frac{h^2}{12}u_{xxxx}(\xi,t) \tag{2c}$$

$$v_{xx} = \frac{v(x-h,t) - 2v(x,t) + v(x+h,t)}{h^2} - \frac{h^2}{12}v_{xxxx}(\epsilon,t) \tag{2d}$$

Let $n = 1 \dots N$ and $m = 1 \dots M$. Discard the truncation terms from equations (2) and let $U_n^m$ and $V_n^m$ approximate the true solutions $u(x_n, t_m)$ and $v(x_n, t_m)$. The initial $u$ and $v$ in the source terms from (1) are replaced with the central differences:

$$u = \frac{U_n^{m+1} + U_n^m}{2} \tag{3a}$$

$$v = \frac{V_n^{m+1} + V_n^m}{2} \tag{3b}$$

We can then replace equations (1) with the following system of equations.

$$\begin{aligned}\frac{U_n^{m+1} - U_n^m}{k} = \frac{D_u}{2}\left\{\left[\frac{U_{n-1}^{m+1} - 2U_n^{m+1} + U_{n+1}^{m+1}}{h^2}\right] + \left[\frac{U_{n-1}^m - 2U_n^m + U_{n+1}^m}{h^2}\right]\right\} \\ + \left[\frac{U_n^{m+1} + U_n^m}{2}\right]\left[1 - U_n^m - \frac{aV_n^m}{1+\lambda U_n^m}\right]\end{aligned} \tag{4a}$$

$$\begin{aligned}\frac{V_n^{m+1} - V_n^m}{k} = \frac{D_v}{2}\left\{\left[\frac{V_{n-1}^{m+1} - 2V_n^{m+1} + V_{n+1}^{m+1}}{h^2}\right] + \left[\frac{V_{n-1}^m - 2V_n^m + V_{n+1}^m}{h^2}\right]\right\} \\ - \left[\frac{V_n^{m+1} + V_n^m}{2}\right]\left[\frac{1}{ab} - \frac{aU_n^m}{b(1+\lambda U_n^m)}\right]\end{aligned} \tag{4b}$$

Multiplying equations (4) throughout by $2k$ and rearranging the terms such that all the $U^{m+1}$

and $V^{m+1}$ terms are taken to the LHS yields the following with $r = D_u k/h^2$ and $s = D_v k/h^2$:

$$-rU_{n-1}^{m+1} + \left(2 + 2r - k + kU_n^m + \frac{akV_n^m}{1+\lambda U_n^m}\right) U_n^{m+1} - rU_{n+1}^{m+1}$$
$$= rU_{n-1}^m + \left(2 - 2r + k - kU_n^m - \frac{akV_n^m}{1+\lambda U_n^m}\right) U_n^m + rU_{n+1}^m \tag{5a}$$

$$-sV_{n-1}^{m+1} + \left(2 + 2s + \frac{k}{ab} - \frac{akU_n^m}{b(1+\lambda U_n^m)}\right) V_n^{m+1} - sV_{n+1}^{m+1}$$
$$= sV_{n-1}^m + \left(2 - 2s - \frac{k}{ab} + \frac{akU_n^m}{b(1+\lambda U_n^m)}\right) V_n^m + sV_{n+1}^m \tag{5b}$$

We now investigate the boundary conditions. When $n = 1$ we have may use the central difference formula for the first derivatives as follows:

$$U_x(a,t) = 0 = \frac{U_2 - U_0}{2h} \tag{6a}$$
$$V_x(a,t) = 0 = \frac{V_2 - V_0}{2h} \tag{6b}$$

Since the derivatives at the boundaries are given as 0 this implies that:

$$U_2 = U_0 \tag{7a}$$
$$V_2 = V_0 \tag{7b}$$

The same argument follows for the right endpoint where $n = N$ and we have:

$$U_x(b,t) = 0 = \frac{U_{N+1} - U_{N-1}}{2h} \tag{8a}$$
$$V_x(b,t) = 0 = \frac{V_{N+1} - V_{N-1}}{2h} \tag{8b}$$

hence we find that

$$U_{N+1} = U_{N-1} \tag{9a}$$
$$V_{N+1} = V_{N-1} \tag{9b}$$

We may then substitute (7) and (9) into their respective equations from (5) so that:

$$\left(2 + 2r - k + kU_1^m + \frac{akV_1^m}{1+\lambda U_1^m}\right) U_1^{m+1} - 2rU_2^{m+1} = \left(2 - 2r + k - kU_1^m - \frac{akV_1^m}{1+\lambda U_1^m}\right) U_1^m + 2rU_2^m \tag{10a}$$

$$\left(2 + 2s + \frac{k}{ab} - \frac{akU_1^m}{b(1+\lambda U_1^m)}\right) V_1^{m+1} - 2sV_2^{m+1} = \left(2 - 2s - \frac{k}{ab} + \frac{akU_1^m}{b(1+\lambda U_1^m)}\right) V_1^m + 2sV_2^m \tag{10b}$$

$$\left(2 + 2r - k + kU_N^m + \frac{akV_N^m}{1+\lambda U_N^m}\right) U_N^{m+1} - 2rU_{N-1}^{m+1} = \left(2 - 2r + k - kU_N^m - \frac{akV_N^m}{1+\lambda U_N^m}\right) U_N^m + 2rU_{N-1}^m \tag{10c}$$

$$\left(2 + 2s + \frac{k}{ab} - \frac{akU_N^m}{b(1+\lambda U_N^m)}\right) V_N^{m+1} - 2sV_{N-1}^{m+1} = \left(2 - 2s - \frac{k}{ab} + \frac{akU_N^m}{b(1+\lambda U_N^m)}\right) V_N^m + 2sV_{N-1}^m \tag{10d}$$

rewritten into the following matrix equations where:

$$
\begin{aligned}
L_u &= 2 + 2r - k + kU_n^m + \frac{akV_n^m}{1+\lambda U_n^m} \\
R_u &= 2 - 2r + k - kU_n^m - \frac{akV_n^m}{1+\lambda U_n^m} \\
L_v &= 2 + 2s + \frac{k}{ab} - \frac{akU_n^m}{b(1+\lambda U_n^m)} \\
R_v &= 2 - 2s - \frac{k}{ab} + \frac{akU_n^m}{b(1+\lambda U_n^m)}
\end{aligned}
$$

$$
\begin{bmatrix} L_u & -2r & \dots & \dots & 0 \\ -r & L_u & -r & \ddots & \vdots \\ 0 & \ddots & L_u & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -r \\ 0 & \dots & \dots & -2r & L_u \end{bmatrix}
\begin{bmatrix} U_1^{m+1} \\ U_2^{m+1} \\ U_3^{m+1} \\ \vdots \\ U_n^{m+1} \end{bmatrix}
=
\begin{bmatrix} R_u & 2r & \dots & \dots & 0 \\ r & R_u & r & \ddots & \vdots \\ 0 & \ddots & R_u & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & r \\ 0 & \dots & \dots & 2r & R_u \end{bmatrix}
\begin{bmatrix} U_1^m \\ U_2^m \\ U_3^m \\ \vdots \\ U_N^m \end{bmatrix}
$$

$$
\begin{bmatrix} L_v & -2s & \dots & \dots & 0 \\ -s & L_v & -s & \ddots & \vdots \\ 0 & \ddots & L_v & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -s \\ 0 & \dots & \dots & -2s & L_v \end{bmatrix}
\begin{bmatrix} V_1^{m+1} \\ V_2^{m+1} \\ V_3^{j+1} \\ \vdots \\ V_n^{m+1} \end{bmatrix}
=
\begin{bmatrix} R_v & 2s & \dots & \dots & 0 \\ s & R_v & s & \dots & 0 \\ 0 & \ddots & R_v & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & s \\ 0 & \dots & \dots & 2s & R_v \end{bmatrix}
\begin{bmatrix} V_1^m \\ V_2^m \\ V_3^m \\ \vdots \\ V_n^m \end{bmatrix}
$$

## 3 Stability Analysis

We investigate the stabiliy of the Crank-Nicolson method using the equation from (1a). A similar argument may be used to show the stability of the Crank-Nicolson method using the equation from (1b).

$$\frac{U_n^{m+1} - U_n^m}{k} = \frac{D_u}{2}\left(\frac{U_{n-1}^{m+1} - 2U_n^{m+1} + U_{n+1}^{m+1}}{h^2} + \frac{U_{n-1}^{m} - 2U_n^{m} + U_{n+1}^{m}}{h^2}\right) \tag{12}$$

We then substitute $U_n^m = \xi^m e^{iknh}$ and rearrange the terms in (12) such that:

$$\begin{aligned}\xi^{m+1}e^{iknh} - \xi^m e^{iknh} = \frac{D_u k}{2h^2}\Big(&\xi^{m+1}e^{ik(n+1)h} + \xi^m e^{ik(n+1)h} \\ &-2\xi^{m+1}e^{iknh} - 2\xi^m e^{iknh} \\ &+\xi^{m+1}e^{ik(n-1)h} + \xi^m e^{ik(n-1)h}\Big)\end{aligned} \tag{13}$$

We may then divide throughout by $\xi^m e^{iknh}$ which gives the following:

$$\xi - 1 = \frac{D_u k}{2h^2}\left(\xi e^{ikh} + e^{ikh} - 2\xi - 2 + \xi e^{-ikh} + e^{-ikh}\right) \tag{14}$$

Rearranging (14) and replacing the complex exponential gives:

$$\xi = 1 - \frac{D_u \tau}{2h^2}(\xi + 1)[1 - \cos(kh)] \tag{15}$$

Taking the absolute value of $\xi$ gives:

$$|\xi| = \left|\frac{1 - D_u\tau/h^2[1-\cos(kh)]}{1 + D_u\tau/h^2[1-\cos(kh)]}\right| \leq 1 \tag{16}$$

Since (16) is true for any value of $k$ the Crank-Nicolson model is unconditionally stable.

## 4 Results

## 5 Discussion

# 6 Conclusion