MAPM312 Project - Parabolic PDEs

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1 Introduction

1.1 Problem Statement

The following set of coupled partial differential equations will be analysed at the hand of the FTCS model:

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u \frac{\partial^2 u}{\partial x^2} + u(1-u) - \frac{auv}{1+\lambda u} \\ \frac{\partial v}{\partial t} &= D_v \frac{\partial^2 v}{\partial x^2} - \frac{v}{ab} + \frac{auv}{b(1+\lambda u)}\end{aligned}$$

$$\begin{aligned}&a \leq x \leq b, \quad 0 \leq t \\ &u_x(a,t) = 0, \quad u_x(b,t) = 0\end{aligned}$$

2 Derivation

From the problem statement we have the following system of PDEs.

$$u_t = D_u u_{xx} + u(1-u) - \frac{auv}{1+\lambda u} \tag{1a}$$
$$v_t = D_v v_{xx} - \frac{v}{ab} + \frac{auv}{b(1+\lambda u)} \tag{1b}$$

We use the following finite difference approximations, derived from the Taylor series expansion, in place of the u_t, u_{xx}, v_t, and v_{xx} in equations (1).

$$u_t = \frac{u(x,t+k) - u(x,t)}{k} - \frac{k}{2}u_{tt}(x,\tau) \tag{2a}$$
$$v_t = \frac{v(x,t+k) - v(x,t)}{k} - \frac{k}{2}v_{tt}(x,\sigma) \tag{2b}$$
$$u_{xx} = \frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} - \frac{h^2}{12}u_{xxxx}(\xi,t) \tag{2c}$$
$$v_{xx} = \frac{v(x-h,t) - 2v(x,t) + v(x+h,t)}{h^2} - \frac{h^2}{12}v_{xxxx}(\epsilon,t) \tag{2d}$$

By discarding the truncation terms from equations (2) and letting $\overline{u}_{i,j}$ and $\overline{v}_{i,j}$ approximate the true solutions $u(x_i,t_j)$ and $v(x_i,t_j)$ respectively we can replace equations (1) with the following system of equations.

$$\frac{\overline{u}_{i,j+1} - \overline{u}_{i,j}}{k} = D_u\left(\frac{\overline{u}_{i-1,j} - 2\overline{u}_{i,j} + \overline{u}i+1,j}{h^2}\right) + \overline{u}_{i,j}(1-\overline{u}_{i,j-1}) - \frac{a\overline{u}_{i,j}\overline{v}_{i,j}}{1+\lambda\overline{u}_{i,j-1}} \tag{3a}$$
$$\frac{\overline{v}_{i,j+1} - \overline{v}_{i,j}}{k} = D_v\left(\frac{\overline{v}_{i-1,j} - 2\overline{v}_{i,j} + \overline{v}i+1,j}{h^2}\right) - \frac{\overline{v}_{i,j}}{ab} + \frac{a\overline{u}_{i,j}\overline{v}_{i,j}}{1+\lambda\overline{u}_{i,j-1}} \tag{3b}$$

Multiplying equations (3) throughout by k and rearranging the terms yields:

$$\overline{u}_{i,j+1} = r_u\overline{u}_{i-1,j} + (1-2r_u)\overline{u}_{i,j} + r_u\overline{u}_{i+1,j} + k\overline{u}_{i,j} - k\overline{u}_{i,j-1} - \frac{ka\overline{u}_{i,j}\overline{v}_{i,j}}{1+\lambda\overline{u}_{i,j-1}} \tag{4a}$$
$$\overline{v}_{i,j+1} = r_v\overline{v}_{i-1,j} + (1-2r_v)\overline{v}_{i,j} + r_v\overline{v}_{i+1,j} - \frac{k\overline{v}_{i,j}}{ab} + \frac{ka\overline{u}_{i,j}\overline{v}_{i,j}}{b(1+\lambda\overline{u}_{i,j})} \tag{4b}$$

where $r_u = D_u k/h^2$ and $r_v = D_v k/h^2$.

3 Stability Analysis

4 Results

5 Discussion

6 Conclusion