

THE RIEMANN HYPOTHESIS IS FALSE

TATENDA KUBALALIKA

ABSTRACT. Let Θ be the supremum of the real parts of the zeros of the Riemann zeta function. We demonstrate that $\Theta \geq \frac{3}{4}$. This disproves the Riemann Hypothesis, which asserts that $\Theta = \frac{1}{2}$.

Keywords and phrases: Riemann zeta function; zeros; Riemann Hypothesis; disproof.

2020 Mathematics Subject Classifications: 11M26, 11M06.

The Riemann zeta function is a function of the complex variable s , defined in the half-plane $\Re(s) > 1$ by $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ and in the whole complex plane by analytic continuation. Euler noticed that for $\Re(s) > 1$, $\zeta(s)$ can be expressed as a product $\prod_p (1 - p^{-s})^{-1}$ over the entire set of primes, which entails that $\zeta(s) \neq 0$ for $\Re(s) > 1$. It can be shown that $\zeta(s)$ extends to \mathbb{C} as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the functional equation $\xi(s) = \xi(1 - s)$, where $\xi(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(\frac{1}{2}z)\zeta(z)$ and $\Gamma(w) = \int_0^{\infty} e^{-x}x^{w-1}dx$. Define ρ to be a complex (non-real) zero of ξ , hence a complex zero of ζ . Let $\Lambda(n)$ denote the von Mangoldt function, which is equal to $\log p$ if $n = p^r$ for some prime p and $r \in \mathbb{N}$, and 0 otherwise. The importance of the ρ 's in the distribution of primes can be clearly seen from the Riemann explicit formula $\psi(x) := \sum_{n \leq x} \Lambda(n) = x - \sum_{|\rho| \leq x} \frac{x^{\rho}}{\rho} + O(\log^2 x)$. In the literature, ψ is sometimes referred to as the Chebyshev ψ function after P.L. Chebyshev, who pioneered its study. It can be shown that $\psi(x) - x \ll x^b(\log x)^2$ if $\zeta(s) \neq 0$ for $\Re(s) > b$. In particular, the Riemann Hypothesis (RH) is equivalent to the statement that $b = \frac{1}{2}$. Let χ be a Dirichlet character and $L(s, \chi)$ be the corresponding L-function. Most of the analytic properties of ζ are also known to hold for $L(s, \chi)$ (see e.g. Chapter 10 of [2]). Let Θ_{χ} be the supremum of the real parts of the zeros of $L(s, \chi)$. The Generalised Riemann Hypothesis (GRH) is equivalent to the statement that $\Theta_{\chi} = \frac{1}{2}$ for each χ , and the Generalised Prime Number Theorem is equivalent to the fact that $L(s, \chi) \neq 0$ at $\Re(s) = 1$ for each χ . For a far more thorough discussion of the RH and GRH, the interested reader is kindly referred to the nearly exhaustive book of Borwein *et al* [3].

MAIN RESULTS

Lemma 1 (Plancherel's identity, [2, Theorem 5.4]). *Suppose that*

$$\nu(s) = \sum_{n=1}^{\infty} v_n n^{-s}$$

is a Dirichlet series whose abscissa of convergence is $c > 0$. Let $V(x) = \sum_{n \leq x} v_n$. Then for $\sigma = \Re(s) > c$, one has

$$2\pi \int_0^{\infty} |V(x)|^2 x^{-2\sigma-1} dx = \int_{\mathbb{R}} \left| \frac{\nu(\sigma + it)}{\sigma + it} \right|^2 dt.$$

Definitions. Let: Λ be the von Mangoldt function, μ be the Mobius function and p be a prime. For $x \geq 0$, define $\theta(x) := \sum_{p \leq x} \log p$, $\psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{r=1}^{\infty} \theta(x^{1/r})$. Let $\gamma = 0.57721 \dots$ be the Euler-Mascheroni constant and $\sigma = \Re(s)$. From now on, let $\sigma > 1$ unless specified otherwise. Define $k_1(s) := \frac{1}{s-1} - \gamma$,

$$K_1(\sigma) := \int_{\mathbb{R}} \left| \frac{k_1(\sigma + it)}{\sigma + it} \right|^2 dt = \frac{\pi((2\sigma^2 - 3\sigma + 1)\gamma^2 + (2 - 2\sigma)\gamma + 1)}{\sigma(2\sigma - 1)(\sigma - 1)}, \quad (1)$$

$q = \sum_{n=2}^{\infty} \mu(n) \frac{\zeta'}{\zeta}(n)$, $k_2(s) := k_1(s) - q$ and

$$K_2(\sigma) := \int_{\mathbb{R}} \left| \frac{k_2(\sigma + it)}{\sigma + it} \right|^2 dt = \frac{\pi((2\sigma^2 - 3\sigma + 1)(\gamma + q)^2 + (2 - 2\sigma)(\gamma + q) + 1)}{\sigma(2\sigma - 1)(\sigma - 1)}. \quad (2)$$

From (1) and (2), note that the function $K_2(\sigma) - K_1(\sigma)$ has a real-analytic continuation to $\sigma > \frac{1}{2}$. Define

$$\alpha(s) = -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \quad (3)$$

and

$$\beta(s) = -\sum_{n=1}^{\infty} \mu(n) \frac{\zeta'}{\zeta}(ns) = \sum_p \frac{\log p}{p^s} \quad (4)$$

[2, p.28]. Let

$$f(\sigma) = 2\pi \int_1^{\infty} \psi^2(x) x^{-2\sigma-1} dx - K_1(\sigma), \quad (5)$$

$$g(\sigma) = 2\pi \int_1^{\infty} \theta^2(x) x^{-2\sigma-1} dx - K_2(\sigma) \quad (6)$$

and

$$h(\sigma) = f(\sigma) - g(\sigma) = 2\pi \int_1^{\infty} (\psi^2(x) - \theta^2(x)) x^{-2\sigma-1} dx + K_2(\sigma) - K_1(\sigma). \quad (7)$$

Theorem 1. *The function $h(\sigma)$ has a simple pole at $\sigma = \frac{3}{4}$.*

Proof. By the Prime Number Theorem [2, p.179], we know that there exists some constant $d > 0$ such that $\psi(y) = y(1 + O(e^{-d\sqrt{\log y}}))$ uniformly for $y \geq 1$. Note that $\psi(y) - \theta(y) \ll \sqrt{y}$ [2, p.49]. Hence $\psi(x) - \theta(x) = \theta(\sqrt{x}) + O(x^{1/3}) = \sqrt{x}(1 + O(e^{-d\sqrt{\log x}}))$ uniformly for $x \geq 1$. Thus

$$\psi^2(x) - \theta^2(x) = (\psi(x) - \theta(x))(\psi(x) + \theta(x)) = 2x^{3/2}(1 + O(e^{-d\sqrt{\log x}})) \quad (8)$$

uniformly for $x \geq 1$. Inserting (8) into the integral on the extreme right-hand side of (7) yields

$$h(\sigma) = 2\pi \left(\sigma - \frac{3}{4} \right)^{-1} + (K_2(\sigma) - K_1(\sigma)) + O_{\sigma}(1), \quad (9)$$

where the implicit constant is uniform for $\sigma \geq \frac{3}{4}$. Recall that the function $K_2(\sigma) - K_1(\sigma)$ has a real-analytic continuation to $\sigma > \frac{1}{2}$. Thus it follows from (9) that $h(\sigma)$ has a simple pole at $\sigma = \frac{3}{4}$, as claimed. \square

Theorem 2. *Let $\Theta \in [\frac{1}{2}, 1]$ be the supremum of the real parts of the zeros of ζ . Then $h(\sigma)$ has a real-analytic continuation to $\sigma > \Theta$.*

Proof. Let ρ denote a complex (non-real) zero of ζ . By Theorem 9.6(A) of [1], we know that for $\sigma > \Theta$ and $s \neq 1$, one has

$$\alpha(s) = \sum_{|\Im(s) - \Im(\rho)| \leq 1} \frac{1}{s - \rho} + O(\log |2s|). \quad (10)$$

Let $T > 0$. Define $N(T)$ to be the number of those ρ with $|\Im(\rho)| \leq T$. By Theorem 9.2 of [1], we know that $N(T+1) - N(T) \ll \log T$. Hence for fixed $\sigma > \Theta$, one has

$$\sum_{|\Im(s) - \Im(\rho)| \leq 1} \frac{1}{s - \rho} \ll \sum_{|\Im(s) - \Im(\rho)| \leq 1} 1 \ll \log |2s|. \quad (11)$$

For $\sigma > \frac{1}{2}$ and $n \geq 2$, note that

$$\left| \mu(n) \frac{\zeta'}{\zeta}(ns) \right| \leq \left| \frac{\zeta'}{\zeta}(ns) \right| \leq \sum_{m=1}^{\infty} \Lambda(m) m^{-n\sigma} \ll 2^{-n\sigma} \quad (12)$$

as $n \rightarrow \infty$. Combining (12) with (11) and (10) reveals that both $\alpha(s)$ and $\beta(s)$ are $\ll \log |2s|$ for $\sigma > \Theta$ and $s \neq 1$. Note that $\psi(y) = 0 = \theta(y)$ for every $y \in [0, 1]$. Thus by Lemma 1, we also have

$$f(\sigma) = 2\pi \int_1^{\infty} \psi^2(x) x^{-2\sigma-1} dx - K_1(\sigma) = \int_{\mathbb{R}} \frac{|\alpha(\sigma + it)|^2 - |k_1(\sigma + it)|^2}{|\sigma + it|^2} dt \quad (13)$$

and

$$g(\sigma) = 2\pi \int_1^{\infty} \theta^2(x) x^{-2\sigma-1} dx - K_2(\sigma) = \int_{\mathbb{R}} \frac{|\beta(\sigma + it)|^2 - |k_2(\sigma + it)|^2}{|\sigma + it|^2} dt \quad (14)$$

for $\sigma > 1$. It is known [1, p.20] that for σ sufficiently near 1^+ , one has

$$\alpha(\sigma) = -\frac{\zeta'}{\zeta}(\sigma) = (\sigma - 1)^{-1} - \gamma + O(\sigma - 1) = k_1(\sigma) + O(\sigma - 1) \quad (15)$$

hence

$$\beta(\sigma) = -\frac{\zeta'}{\zeta}(\sigma) - \sum_{n=2}^{\infty} \mu(n) \frac{\zeta'}{\zeta}(n\sigma) = k_1(\sigma) - q + O(\sigma - 1) = k_2(\sigma) + O(\sigma - 1). \quad (16)$$

From (15) and (16), we deduce that both $\alpha_0(s) := |\alpha(s)|^2 - |k_1(s)|^2$ and $\beta_0(s) := |\beta(s)|^2 - |k_2(s)|^2$ are regular at $s = 1$. As a consequence, they both have real-analytic continuations to $\sigma > \Theta$ whether or not $\Theta < 1$. By combining this with the fact that both $\alpha_0(s)$ and $\beta_0(s)$ are $\ll \log^2 |2s|$ for $\sigma > \Theta$, we deduce that the integrals on the respective extreme right-hand sides of (13) and (14) are absolutely convergent for $\sigma > \Theta$ and have real-analytic continuations there. That is, both $f(\sigma)$ and $g(\sigma)$ have real-analytic continuations to $\sigma > \Theta$ hence so must $h(\sigma) = f(\sigma) - g(\sigma)$. This completes the proof. \square

Corollary 1. *Let Θ be as defined in Theorem 2. Then $\Theta \geq \frac{3}{4}$.*

Proof. Suppose that $\Theta < \frac{3}{4}$ and let $\varepsilon \in (0, \frac{3}{4} - \Theta)$. Then by Theorem 2, $h(\sigma)$ must have a real-analytic continuation to $\sigma > \frac{3}{4} - \varepsilon$, contradicting Theorem 1. We therefore deduce that our supposition must be false, so we are done. \square

This disproves the Riemann hypothesis, which asserts that $\Theta = \frac{1}{2}$.

REFERENCES

- [1] E. C. Titchmarsh, *The theory of the Riemann zeta function*, second ed., The Clarendon Press Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown.
- [2] H. L. Montgomery, R. C. Vaughan, *Multiplicative number theory. I. Classical theory. Cambridge Studies in Advanced Mathematics*, **97**, Cambridge University Press, Cambridge, 2007.
- [3] P.B. Borwein, S. Choi, B. Rooney and A. Weirathmueller, *The Riemann Hypothesis: A resource for the aficionado and virtuoso alike*, (2008), Springer-Berlin.

Email address: `tatendakubalalika@yahoo.com`, `tkubalalika@gmail.com`.