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Superhedging in Incomplete Markets

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Abstract

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This work conducts research in the area of incomplete markets, regarding the use of the superhedging theory and also develops a literature review that addresses the mathematical, statistical and financial prerequisites for the use of such tool. Based in these theoretical concepts, distinct incomplete markets are analyzed, providing the superhedging price for specific contracts and the economic intuition for the results obtained. Additionally, the impact of regulation on incomplete markets is discussed considering the context in which decisions are supported by utility curves.

Keywords: *Superhedging*; Incomplete Markets; Asset Pricing; Regulation.

Summary

1 Introduction

Asset pricing is a common problem in many markets. It occurs when the fair price of an object needs to be determined, when a new product is created or when the price of an indirectly traded asset needs to be computed. Not only this topic is relevant for the success of companies and financial institutions, but it also is important when determining the utilities of individuals. Therefore, it affects offer and demand for goods, work, currency and others. Because prices influence how economic agents allocate their resources, its exact determination can cause good or bad investments, impacting the development of a country.

In the cases where the asset that needs to be priced is available on the market or is a new product to be sold, like the initial public offering of the stock of a company, there are already many different methodologies to determine its price. According to Duffie (2003, p.642), in the context of complete markets with no arbitrage, these types of problem can be reduced to the computation of *state prices*. This concept was originally developed by Arrow (1953), and it states that the value of an asset is the weighted sum of its future cash flows discounted by the marginal rate of substitution between investment opportunities and present consumption, at each period of time. Consequently, the subsequent discussions about asset pricing in this context are about different techniques to estimate cash flows and discount rates.

However, in the cases where the asset is not directly available on the market, a new theory is needed. To exemplify this problem, consider the health insurance market: many people have a history of diabetes in their families, being prone to develop the same condition throughout their lives, so they would be willing to buy a health plan that specifically covers this illness. However, that is not possible. There's no such particular plan; there are only plans that cover many illnesses at the same time. Thus, the health plan market is incomplete since it is not possible to perfectly hedge against some contingencies. As a consequence, the theory of superhedging (protection against risks for the least possible price) has to be used to estimate prices in this context.

The theory of superhedging, according to Föllmer & Schied (2004), is intrinsic to incomplete markets, because it aims to capture the consequences of its imperfections when determining prices.

Henderson & Hobson (2004) define that a portfolio is protected by a superhedge (or that a portfolio has been super replicated) when the portfolio that provides the protection always generates a result equal or greater than the result of the protected portfolio. They also affirm that the price of the hedge is the one that minimizes the expenses needed to provide the protection. Another example of superhedging is the one applied to the derivatives market, cited by Argesanu (2004). In this case, the superhedging portfolio is the one that demands the least amount of capital, and also eliminates all the risks associated with the derivative. As an illustration, suppose that the derivative in question is the short sale of a dollar futures contract. Hence, the most evident hedge is the purchase of the same contract (long position). However, imagine a scenario where the liquidity for this contract suddenly and sharply decreases, so that buying it becomes impossible. Now the portfolio that can protect the short position is the purchase of dollars, an operation that certainly has a higher cost than just purchasing a futures contract.

This problem can also be studied from the point of view of the utility theory. Carasuss & Rásonyi (2005) explain that the superhedging cost is the least amount of initial wealth necessary to cover against oscillations in the utility. That is, the superhedging price is the cost that the agent is willing to incur in order to maintain the same utility level that he has when protected as that when he is unprotected (but without the costs).

In short, the superhedging problem consists in pricing a contract that guarantees the hedge (risk elimination) at the lowest possible cost. The solution, in general, will imply in a contract that covers additional contingencies (because of the incompleteness of the market). Moreover, as a consequence of this excessive coverage, the superhedging price will be higher than the price of the hedge in a situation with complete markets.

It is noteworthy to mention that this theory can be applied in diverse areas of finance, from fixed and variable income to exotic options. Whenever there are an incomplete market and the necessity of protection against risk, a superhedging will be imperative.

In conclusion, the goal of this work is the review of the theoretical tools that underlies the theory of superhedging, and the application of the theory to the health insurance market and the options market, in a two time period context with a finite number of states of nature. This theory has gained more ground in recent years, because it offers potential impact to the economic literature by shedding light on financial regulation issues in incomplete markets (like policies turning hedge operations extremely costly), and by explaining the hedge dynamics in different insurance markets.

In section 2, the linear algebra literature (subsection 2.1) and some elements of probability (subsection 2.2) are reviewed. The motivation for the study of the first subject is the fact that asset markets (the ones considered in this work) can be represented by price and quantity vectors, and, their relationship pervades linear transformations and the dual space. Still, from primitive assets, one can create various portfolios and other types of assets, which leads to the study of concepts such as vector spaces, vector subspaces, basis and linear combinations. The motivation for the second topic is based on the economic model used in this work. The model works in a two-time periods context, including future prices and different scenarios, thus, involving the calculation of probabilities and expectations. Also, section 2 presents the concept of Choquet integral (subsection 2.3); that is applied in a particular case (and is recent in the literature) of superhedging.

In section 3, the methodology used for the computation of superhedging prices is presented. This section presents the model with two-time periods, various financial assets and their market prices, and the possible scenarios of states of nature (subsection 3.1). Then, the principle of no-arbitrage (subsection 3.2) is studied, followed by the risk-neutral measure (subsection 3.3). These are essential tools for the development of the model, and they are present in the fundamental theorem of asset pricing (subsection 3.4). Next, prices in different time periods are compared (subsection 3.5). In subsection 3.6, the concept of replicating portfolios is developed, their expected return calculated and the idea of redundant assets is introduced. Subsection 3.7 introduces various derivative instruments that can be used to price insurance contracts. Finally, the superhedging price is explained as a new asset is added to the model (subsection 3.8), and then the concept is exemplified by applying it to options (subsection 3.9).

In section 4, the tools developed before are applied to different situations in the context of incomplete markets. Initially, an example of pricing a car insurance contract is examined and then related to a decision problem of a rational agent (subsection 4.1). Next, is a case where the superhedging toolset is used to price a non-trivial asset portfolio (subsection 4.2). In subsection 4.3 two other examples are considered, in these there is a regulator that must decide whether or not to create a new contract. In the first one, the regulator is indifferent between the possible choices. However, in the second one, the regulator's decision is obtained through utility modeling. Lastly, subsection 4.4 analyses the situation in which the health insurance market is subject to regulatory changes, discussing the potential impacts of different types of customers.

In section 5, the main findings of the superhedging model are presented, along with critics and suggestion for future works and other applications.

2 Literature Review

This section presents the mathematical tools that underpin the financial concepts needed to develop the superhedging model. The study of this framework is motivated by the fact that a financial market in discrete time, and with a finite number of states of nature, can be modeled as a vector subspace, where the price of assets are given by a linear functional. Moreover, the study of hedge operations depends on concepts such as linear combinations and generated spaces, and the study of complete markets is closely related to the set of risk neutral measures.

2.1 Linear Algebra

The definitions and demonstrations of this subsection are based on Callioli *et al* (1989, p. 42–151).

2.1.1 Vector Spaces and Vector Subspaces

The concept of vector spaces comes from the study of two sets: geometry vectors and the set of real matrices of m lines and n columns. At first, they do not appear to be related to each other. However, by analyzing the structure of both of these sets, it is possible to realize that there are many similarities regarding several critical operations. Thus, the definition of vector spaces intends to capture all those sets that have these operations, so that they can be studied in aggregate. These sets also have an intimate relationship with the financial market, since a stock portfolio, for example, also has the properties of a vector space, as will be exemplified below.

Definition. A non-empty set V is a vector space over ℝ if, and only if, the following properties are well defined:

1. The addition of two elements of V (u and v, for example) generates a new element in V (element u + v), that satisfies the:
    a. Commutative Property: $u+v=v+u, \forall u,v \in V$
    b. Associative Property: $u+(v+w)=(u+v)+w, \forall u,v,w \in V$
    c. Neutral Element Existence: $\exists\ o \in V\ |u+o=u, \forall u \in V$
    d. Additive Inverse Existence: $\forall v \in V, \exists(-v) \in V \mid v+(-v)=0$
2. The multiplication of a real number (α, for example) by an element of the vector space (u, for example) is well defined and satisfies:
    a. $\propto (\beta u)=(\alpha\beta)u \mid \alpha,\beta \in \mathbb{R}\ e\ u \in V$

b. $(\alpha + \beta)u = \alpha u + \beta u \mid \alpha, \beta \in \mathbb{R}\ e\ u \in V$
c. $\alpha(u + v) = \alpha u + \alpha v \mid \alpha \in \mathbb{R}\ e\ u, v \in V$
d. $1 * u = u \mid u \in V$

Note that a portfolio is composed of elements, in this case they are stocks. Moreover, the purchase of a stock A and a stock B is no different from the purchase of stock B and stock A (commutative property). Furthermore, a stock can be bought or sold (additive inverse), and doubling the quantity of stocks in a portfolio, doubles the quantity of each stock (property 2.c). It becomes apparent that the definition of vector spaces is essential for the study of financial portfolios.

Now vector subspaces, which are sets contained in vector spaces and with akin properties, are defined.

Definition. Let V be a vector space over ℝ. A vector subspace of V is a subset $W \subset V$, such that the following properties are valid:

1. There is a neutral element in W: $0 \in W$;
2. $u + v \in W \mid \forall u, v \in W$;
3. $(\alpha u) \in W \mid \forall \alpha \in \mathbb{R}\ e\ \forall u \in W$.

From this definition, it can be shown that if W is a vector subspace of V, and V is a vector subspace of ℝ, then W is also a subspace of ℝ. In fact, one can understand a portfolio of stocks contained in the portfolio of a bank as a vector subspace of the latter.

2.1.2 Finitely Generated Vector Spaces, Linear Dependence, Basis, and Dimension

Several spaces can be generated from linear combinations of some elements, leading to the definition of linear combinations and generated space.

Definition. Let V be a vector space over ℝ and $S = \{u_1, \dots, u_n\}$ a subset of V. Another set $[S] \subset V$ can be constructed from S by taking $[S] = \{\propto_1 u_1 + \cdots + \propto_n u_n \mid \propto_1, \dots, \propto_n \in \mathbb{R}\}$. It is possible to prove that $[S]$ is a vector subspace of V, and the name 'linear combination' is given to the operation used to create the elements of $[S]$. Also, $[S]$ is the generated subspace by S.

Definition. A vector space V is said to be finitely generated if a finite subset $S \subset V$ exists and satisfies $V = [S]$.

One should realize that an investment portfolio is nothing more than a linear combination of stocks and their quantities. In addition, a set is said to be replicable when it is formed by a linear combination of several other assets. Moreover, for a portfolio (or an asset) to be unique, it must be linearly independent, which raises the next definition:

Definition. Given a vector space V and a set $L = \{u_1, \dots, u_n\} \subset V$ of elements from this space, L is said to be linearly independent if, and only if, the equation $\propto_1 u_1 + \cdots + \propto_n u_n = 0$ with $\propto_i \in \mathbb{R}$ is only possible for $\propto_1 = \cdots = \propto_n = 0$. L is said to be linearly dependent if the equation is valid for some $\propto_i$ nonzero.

A unique set of stocks (linearly independent) forms a basis for a portfolio, because the combination of purchases and sales of these stocks (linear combination) creates the portfolio (finitely generated vector space). Also, note that the definition above suggests that a market is non-redundant when there are no replicable assets, that is, the stock portfolio is linearly independent.

The mathematical definition for the basis of a finitely generated vector space is a linear independent subset that can generate the former through linear combinations.

Definition. The basis of a vector space V is a finite subset $B \subset V$ such that:

1. $[B] = V$;
2. $B\ is\ linearly\ independent.$

The definition of dimension is related to the idea that a space can be generated by different basis, but with the same size. However, it is important to study the basis invariance theorem, which proof is in Callioli et al. (1989, p. 99-101), before defining the concept of dimension.

Invariance Theorem. Any two basis of a finitely generated vector space V contains the same number of vectors.

Definition. The dimension of a finitely generated vector space V is defined as the number of vectors in any of its basis.

This definition ensures that a portfolio can be constructed from a set of primitive assets (basis). Therefore, the dimension of a portfolio defines the quantity of different assets that need to be used to form it. That is, the higher the dimension of a portfolio, the greater the amount of primitive assets that are needed.

2.1.3 Linear Transformations and Inner Product

Analogous to vector spaces, linear transformations are functions from a vector space to another, that preserve the properties of vector addition and scalar multiplication.

Definition. Let U and V be vector spaces over $\mathbb{R}$. A function $F: U \to V$ is a linear transformation from U to V if, and only if, the following properties are valid:

1. $F(u_1 + u_2) = F(u_1) + F(u_2),\ \forall\, u_1, u_2 \in U$
2. $F(\alpha u) = \alpha F(u),\ \forall\, \alpha \in \mathbb{R}\ e\ u \in U$

Definition. Given $x, y \in \mathbb{R}^n$, the inner product is denoted by $< x, y >$ and is defined as $< x, y >= \sum_{i=0}^{n} x_i y_i$.

The concept of inner product raises an important example of linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$, because, for any $y \in \mathbb{R}^n$, the function $\begin{matrix} \varphi_y: \mathbb{R}^n \to \mathbb{R} \\ x \longmapsto < x, y > \end{matrix}$ is a linear transformation. This notion will be used to determine the price of a portfolio since the inner product of a vector of quantities and a vector of prices will result in a unique price for it. The notation $x \cdot y$ can also represent an inner product.

2.1.4 Dual Space

The definition of a dual space pervades the concepts presented earlier and unites them in a set of linear combinations of vector spaces.

Definition. Given a vector space U over $\mathbb{R}$, it is possible to show that the collection of linear transformations of U in $\mathbb{R}$ is also a vector space over $\mathbb{R}$. Thus, $L(U, \mathbb{R})$ denotes all the linear combinations of U in $\mathbb{R}$ and define it as the dual space of U. Moreover, each element of the dual space of U is said to be a linear functional over $\mathbb{R}$.

An important result follows this definition. Given a linear functional $F: \mathbb{R}^n \longrightarrow \mathbb{R}$, it's possible to find $y \in \mathbb{R}^n$ such that $F(x) =< x, y >,\ \forall x \in \mathbb{R}^n$. That is, every linear transformation can be represented by an inner product.

2.1.5 Upper and Lower Bounds, Supremum and Infimum

A set $S \subset \mathbb{R}$ has an upper bound when there's a $\pi_{Upper} \in \mathbb{R}$ such that $s^i \leq \pi_{upper}$, $\forall s^i \in S$. In this case, π_{Upper} is said to be an upper bound for S. A set $S \subset \mathbb{R}$ has a lower bound when there's a $\pi_{Lower} \in \mathbb{R}$ such that $s^i \geq \pi_{Lower}$, $\forall s^i \in S$. In this case, π_{Lower} is said to be a lower bound for S. If S has an upper and lower bound, then S is a limited set, that is, $\exists k > 0: s^i \in S \Longrightarrow |s^i| \leq k$.

Let $S \subset \mathbb{R}$ be a non-empty set with an upper bound. A number $\pi_{Sup} \in \mathbb{R}$ is said to be the supremum of the set S when it is the lowest of the upper bounds of S:

a. For each $s^i \in S$, $s^i \leq \pi_{Sup}, \pi_{Sup} \in \mathbb{R}$;

b. If $\pi_{Upper} \in \mathbb{R}$ is such that $s^i \leq \pi_{Upper}$ for all $s^i \in S$, then $\pi_{Sup} \leq \pi_{Upper}$.

So it is written $\pi_{Sup} = \sup S$.

Let $S \subset \mathbb{R}$ be a non-empty set with an inferior bound. A number $\pi_{Inf} \in \mathbb{R}$ is said to be the infimum of the set S when it is the largest of the lower bounds of S:

a. For all $s^i \in S$, $s^i \geq \pi_{Inf}, \pi_{Inf} \in \mathbb{R}$;

b. If $\pi_{Inferior} \in \mathbb{R}$ is such that $s^i \geq \pi_{Lower}$ for every $s^i \in S$, then $\pi_{Inf} \geq \pi_{Lower}$.

Moreover, $\pi_{Inf} = \inf S$.

The definitions above are necessary to the understanding of the computation of arbitrage-free prices of financial assets, the identification of the price intervals where they occur, and the calculation of the superhedging price.

2.2 Elements of Probability

This subsection presents the fundamentals of probability necessary to the formulation of the superhedging model and to the exploration of certain financial topics under the probabilistic perspective. It is based on Fernandez (2009, pg. 25–30 e 87).

2.2.1 Probability Measures

First, let's discuss what a sample space is. Consider a probabilistic experiment; a set Ω is the sample space of this experiment if each possible outcome of the experiment is associated to a value in Ω. Take the flipping of a coin as an example, if heads relates to 1, and tails to 0, then $\Omega = \{0,1\}$ is the sample space of this experiment. Furthermore, an event is defined as any subset of Ω, like $\{0\}$, $\{1\}$, $\{0,1\}$ and $\emptyset$ in the previous example. Also, an event that has only one element is said to be an elementary event.

In addition, it is necessary to study the frequency and the relative frequency in which the events occur. In order to accomplish this, a function that associates each event to its occurrence probability is used. This function receives the name of probability measure. However, not every sample space has a probability for all of its events. Thus, urging the definition of the concept of sigma-algebra.

Definition. A class of events $\mathcal{F}$ in Ω is a sigma-algebra when:

1. $\Omega \in \mathcal{F}$;
2. If $A \in \mathcal{F}$, then $A^C \in \mathcal{F}$;
3. If $\{A_1, \ldots\} \subset \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega;\ \mathcal{F})$ is called a measurable space, because it refers to the set of events that can have a probability measure, which is defined below:

Definition. A function P is said to be a probability measure for the sigma-algebra $\mathcal{F}(\Omega)$ if the following properties hold:

1. $P: \mathcal{F} \rightarrow [0; 1]$ and $P(\Omega) = 1$;
2. If $\{A_1, \dots, A_i, \dots\} \subset \mathcal{F}$ and $A_i \cap A_j, i \neq j$, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$. This property is also known as the sigma-additive probability.

Finally, the triple $(\Omega, \mathcal{F}, \mathrm{P})$ receives the name probability space. This definition is fundamental to the study of finance, since it allows the association of different probabilities to events, and it can be used to access the expectations of the agents.

2.2.2 Mathematical Expectation

The expected value of a random variable is given by its average. In other words, it is the value that one expects to observe when realizing an experiment. Mathematically:

Definition. Given a random variable $X: \Omega \rightarrow \mathbb{R}$ that is $\mathcal{F} - measurable$ and limited, its expectation with respect to a probability measure P is defined as:

$$E(X) = \int X dP \coloneqq \int_{-\infty}^{0} (P(\{w \in \Omega: \mathrm{X(w)} \geq \lambda\}) - 1) \mathrm{d}\lambda + \int_{0}^{+\infty} P(\{w \in \Omega: \mathrm{X(w)} \geq \lambda\}) \mathrm{d}\lambda$$

For the finite case:

$$E(X) = \sum_{i=1}^{n} X(\omega_i) P(\omega_i)$$

2.3 Choquet Integral

Based on Castro & Faro (2005), the concept of Choquet integral is presented below. This notion is an expansion of the usual integral, because it incorporates discrete functions and uses the idea of capacity (related to probability).

Definition. Given the set $\Omega = \{1, \dots, K\}$ and the family of subsets $2^{\Omega} \in S$, a capacity v is a function $v: 2^{\Omega} \rightarrow [0; 1]$ that satisfies:

1. $v(0) = 0 \; and \; v(\Omega) = 1$;
2. $\forall \; E, F \in 2^{\Omega}: E \subset F \Longrightarrow v(E) \leq v(F)$.

Observe that, unlike the case of probabilities, a capacity does not necessarily satisfies the additive property, i.e., there are cases where $v(E + F) \neq v(E) + v(F)$. In fact, this possibility is what allows the study of problems arising from market failures, such as incompleteness and transaction costs. For example, purchasing some goods together may cost more than the total cost of purchasing them separately.

Definition. Given a function a, and supposing its image is $Im(a) = \{\alpha_1, \dots, \alpha_N\}$, in such a way that $\alpha_1 \geq \cdots \geq \alpha_N$, and also defining $E_j = a^{-1}(\propto_i)$, the Choquet integral is:

$$I_v(a) = \sum_{i=1}^{N} (\alpha_i - \alpha_{i+1}) * v\left(\bigcup_{j=1}^{i} (E_j)\right)$$

Note that when v is a probability measure, the integral above is simply the classical case of mathematical expectation presented previously.

A particular case of capacity that occurs in the context of incomplete markets is given by:

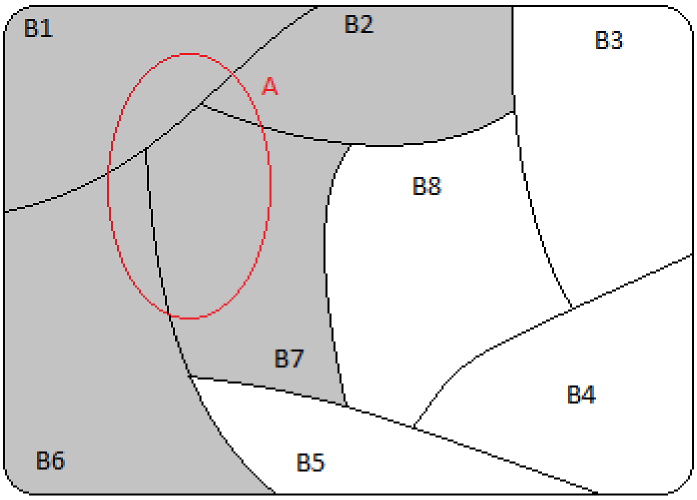
$$v: 2^{\Omega} \to [0,1]$$

Where there's a partition $\{B_k\}_{k=1}^{k}$ of Ω, i.e., $B_k \cap B_j = \emptyset, \forall k \neq j$ and $\bigcup_{k=1}^{k} B_k = \Omega$, and a probability measure P such that for every event $A \subseteq \Omega$:

$$v(A) = \sum_{k: B_k \cap A \neq \emptyset} P(B_k)$$

This measure is exemplified in the picture below:

Figure 1: Partition Example



Source: Prepared by the author

Where $v(A) = P(B_1) + P(B_2) + P(B_6) + P(B_7)$. In the example, to buy the protection A against the contingencies B_1, B_2, B_6 and B_7, it is necessary to spend the sum of the price of each contingency.

In this case,

$$I_v(a) = \sum_{k=1}^{k} \frac{\max a(w)}{w \in B_k} * P(B_k)$$

3 Methodology

This section presents the financial model that allows the computation of the superhedging price for market assets, and the concepts necessary to the understanding of the analysis. The following discussion is based on Föllmer & Schied (2004). For an approach incorporating various periods or continuous time, see Schachermayer (2003).

3.1 Modeling Scenarios, Financial Assets and Prices

Financial assets, prices and scenarios can be modeled using the definitions of linear algebra and the elements of probability presented in the preceding sections. Now, suppose that a market works in two periods (no loss of generality). The present time (time $t = 0$) – where the market prices are known, and the future (time $t = 1$) – where prices are random. Also, assume that there are $d + 1$ assets in the market, from the asset 0 (in this case, the risk-free asset) to the asset d. Thus, the set $\bar{\pi} = \{\pi^0, \pi^1, \ldots, \pi^d\} = \{\pi^0, \pi\} \in \mathbb{R}^{d+1}$ describes the price of the various assets π^i at time $t = 0$. The future prices, however, are associated with a measure of uncertainty. They are described by a probability space $(\Omega; \Sigma; \mathrm{P})$ in such a way that there is a function S that defines future prices according to a scenario $\omega \in \Omega$, where Ω is the set of states of nature (there are a finite number of states, and each state $\omega \in \Omega$ has a positive physical probability $P(\omega)$ of occurrence). Furthermore, $S^i(\omega)$ is the price (or payoff) of the asset i when the scenario ω materializes, and, generically, $\bar{S} = \{S^0, S^1, \ldots, S^d\} = \{S^0, S\} \in \mathbb{R}^{d+1}$ is the set containing the asset prices at time $t = 1$. For the case of the risk-free asset, $\pi^0 = 1$ (normalization), and $S^o = 1 + r,\ \forall \omega \in \Omega$, where r represents the risk-free interest rate. Finally, given the prices, an investor must choose which assets to buy or sell, and at what quantity. This choice is given by $\bar{\xi} = \{\xi^0, \xi^1, \ldots, \xi^d\} = \{\xi^0, \xi\} \in \mathbb{R}^{d+1}$.

3.2 No-Arbitrage Principle

Given the costs and payoffs of a portfolio, a certain market is free of arbitrage when it is not possible to obtain profits without risk and investments.

Definition. Similarly to Araujo *et al* (2011), a portfolio $\bar{\xi} \in \mathbb{R}^{d+1}$ is an arbitrage portfolio if, and only if, $\bar{\pi} \cdot \bar{\xi} \leq 0$, $\bar{\xi} \cdot \bar{S} \geq 0$ with $P\big(\{\bar{\xi} \cdot \bar{S} \geq 0\}\big) = 1$ and $P\big(\{\bar{\xi} \cdot \bar{S} > 0\}\big) > 0$. In addition, it said that there's no arbitrage when $\bar{\pi} \cdot \bar{\xi} = 0 \leftrightarrow \bar{\xi} \cdot \bar{S} = 0$ or $\bar{\xi} \cdot \bar{S} > 0 \leftrightarrow \bar{\pi} \cdot \bar{\xi} > 0$.

The idea of arbitrage-free market is an extremely relevant hypothesis in a wide range of financial models. It is justified by the argument that whenever an arbitrage opportunity appears, economic agents will make use of it, hence changing the market prices until it is gone.

3.3 Risk-Neutral Measure

A fundamental concept for the development of the superhedging problem is present below.

Definition. A probability measure P^* is risk-neutral when:

$$\pi^i = E_{P^*}\left(\frac{S^i}{1+r}\right), i = 0,1, \ldots, d$$

This definition clarifies that, in a risk-neutral and arbitrage-free context, the price of an asset is given by the expected value of its discounted payoffs with respect to the risk-neutral probability measure. Momentarily disregarding the interest rate ($r = 0$), it is true that:

$$E_{P^*}(S) = \sum_{w \in \Omega} S(w) P^*(w)$$

However, there is not just one probability measure defined on a probability space $(\Omega; \mathcal{F}; \mathrm{P})$. Besides the various risk-neutral measures, there is also the physical (or real) probability measure (not risk neutral). Thus, Δ is defined as the set of all the probability measures over $(\Omega; \mathcal{F})$. In addition, it is also essential to define the set that only contains risk neutral measures that are equivalent to the physical measure:

Definition. A probability measure P^* defined over $(\Omega; \mathcal{F})$ is equivalent to a probability measure P, defined over the same measurable space, if, and only if, they agree on the same events. That is, they associate zero probabilities to the same events, and positive probabilities (but potentially different in magnitude) to the others. Formally:

$$P^* \approx P \Leftrightarrow [\forall A \in \mathcal{F}, P^*(A) = 0 \Leftrightarrow P(A) = 0]$$

Since $P(\omega) > 0, \forall \omega \in \Omega$, it follows that:

$$P^* \approx P \Leftrightarrow P^*(\omega) > 0, \forall \omega \in \Omega$$

Moreover, it is written $P^* \gg 0$.

With that, it is desired to identify those risk-neutral measures that have an equivalent non-risk-neutral measure, in such a way that the choice between them does not affect the superhedging price.

Definition. The set of equivalent risk-neutral measures (also known as the set of equivalent martingale measures) is defined as:

$$\mathcal{P} \coloneqq \{P^* \in \Delta: P^* \; is\; a\; risk - neutral\; measure\; and\; P^* \approx P\}$$

3.4 Fundamental Theorem of Asset Pricing

The set defined in the previous subsection is of big importance to the study of financial markets and is even present in the fundamental theorem of asset pricing. Therefore, the next subsections will detail other elements that are relevant to the better comprehension of this theorem.

Fundamental Theorem of Asset Pricing. A financial market $M = \{(S^0, S^1, \dots, S^d); (1, \pi^1, \dots, \pi^d)\}$ if arbitrage-free if, and only if, $\mathcal{P} \neq \emptyset$.

Thus, for a market to be free of arbitrage opportunities, there must be at least one risk-neutral measure, but this does not imply that there can't be other risk-free measures.

3.5 Change of Basis

When dealing with financial markets, it is common to convert prices to a common basis. So that the comparison between today's prices and future prices is possible and unbiased (by the inflation, for example). In order to do that, define $\tilde{\pi} \coloneqq \frac{\pi^i}{\pi^1}$ and $\tilde{S} \coloneqq \frac{S^i}{S^1}$. It is possible to show that if the financial market $M = \{S^i; \pi^i\}_{i=0}^{d}$ is free of arbitrage, then the financial market $\widetilde{M} = \{\tilde{S}^i; \tilde{\pi}^i\}_{i=0}^{d}$ also is, independently of the chosen basis (in this case π^1, S^1).

3.6 Replicable Portfolios, Redundant Assets and Return

Consider $\mathrm{N} \coloneqq \{\bar{\xi} \cdot \bar{S}: \bar{\xi} \in \mathbb{R}^{d+1}\} = span\{S^0, S^1, \dots, S^d\}$, which represents the vector subspace of all payoffs that can be obtained by a portfolio. Any element $\nu \in \mathrm{N}$ is a replicable payoff, since it represents a linear combination of the basis assets. Also, it is known that when there is no arbitrage opportunity, the price of a replicable payoff is well-defined, that is, $\forall \nu \in \mathrm{N}, \pi(\nu) \coloneqq \bar{\pi} \cdot \bar{\xi}$ if $\nu = \bar{\xi} \cdot \bar{S}$. Observe that $\forall P^* \in \mathcal{P}$ and $\exists \bar{\xi} \in \mathbb{R}^{d+1}\; such\; that\; \nu = \bar{\xi} \cdot \bar{S}$, so that:

$$\pi(\nu) = \pi(\bar{\xi} \cdot \bar{S}) = \sum_{i=0}^{d} \xi^i \pi(S^i) = \sum_{i=0}^{d} \xi^i E_{P^*}\left(\frac{S^i}{1+r}\right) = E_{P^*}\left(\frac{\sum_{i=0}^{d} \xi^i S^i}{1+r}\right) = E_{P^*}\left(\frac{\nu}{1+r}\right)$$

for any $\nu \in \mathrm{N}$.

Notice that when there is no arbitrage, the condition $\bar{\xi} \cdot \bar{S} = 0\ P.q.c.$ (which means $P\left(\{\bar{\xi} \cdot \bar{S} = 0\}\right) = 1$) implies that $\bar{\pi} \cdot \bar{\xi} = 0$ is valid. Without any loss of generality, we can assume that $\bar{\xi} \cdot \bar{S} = 0 \Longrightarrow \bar{\xi} = 0$. Because if that were not true, there would be $\xi^i \neq 0$ for some $i \in \{0,1,\dots,d\}$ and the asset i could be written as the linear combination of the other assets $\pi^i = \frac{1}{\xi^i}\sum_{j\neq i}\pi^j\xi^j$ and $S^i = \frac{1}{\xi^i}\sum_{j\neq i}S^j\xi^j$. Hence, $S^i = \frac{1}{\xi^i}\sum_{j\neq i}S^j\xi^j$ would be a redundant asset since it would be formed by the combination of the other assets in the market. So now it is desirable to define when a financial market is or isn't redundant.

Definition. A financial market $M = \left\{S^i; \pi^i\right\}_{i=0}^{d}$ is non-redundant if $\bar{\xi} \cdot \bar{S} = 0 \Longrightarrow \bar{\xi} = 0$. Furthermore, the condition is also valid for the vector of discounted net gains, i.e., $\bar{\xi} \cdot \bar{Y} = 0 \Longrightarrow \bar{\xi} = 0$.

Additionally, note that the definition above can be rewritten in terms of linear dependence, in other words, a financial market is non-redundant if the asset portfolio is linearly independent.

It is also possible define the return of a replicable payoff.

Definition. Consider an arbitrage-free market M and a replicable payoff $\nu \in N$ such that $\pi(\nu) \neq 0$. Define the return of ν as $R(\nu) \coloneqq \frac{\nu - \pi(\nu)}{\pi(\nu)}$. Notice that $R(\nu)$ is a random variable, because $R(\nu)\colon \Omega \to \mathbb{R}$. In the case of the risk-free asset, $R(S^0)$ is a constant function $R(S^0) = \frac{S^0(w) - \pi^0}{\pi^0} = r$.

Next, the return of a replicable portfolio is calculated.

Proposition. Consider an arbitrage-free market M and a replicable payoff $\nu \in \mathrm{N}$ such that $\pi(\nu) \neq 0$. Then, it is possible to prove that:

a. Taking any risk-neutral measure P^*, the expected value of the return of ν is equal to the return of the risk-free asset: $E_{P^*}\big(R(\nu)\big) = r$;

b. Considering any $P^* \approx P$ such that $E_P\left(\left|S^i\right|\right) < \infty, \forall i$, the expected return of ν is given by $E_P\big(R(\nu)\big) = r - Cov_{P^*}\left(\frac{dP^*}{dP}; R(\nu)\right)$.

In fact, since the model considers economic agents to be risk-neutral, it is expected that they will require no extraordinary returns. In other words, the remuneration from a replicable portfolio is equal to the risk-free interest rate.

3.7 Derivative Contracts

Another relevant aspect of financial markets is related to derivatives. Derivatives are contracts that depend on the value of other assets through a linear relation. Such contracts have an exercise price, denoted by K, that represents the price paid for the underlying asset in the future (when the contract reaches maturity). The payoff of a derivative will be denoted by $C^i_{Derivative}$. Below some types of derivative contracts are presented.

Forward Contract. In this type of contract, the agent agrees to sell an asset at time $t = 1$ for a given price K, determined at $t = 0$. The seller of this contract profits when the future market price of the underlying asset S^i is lower than the exercise price K. Because, in this way, the asset would sell for a price higher than what is available in the market. Similarly, the buyer of this contract gains when the price of the underlying asset in the future is higher than the exercise price agreed upon. Formally, the profit of a long position in a forward contract is given by the function $C^i_{FC}\colon \begin{matrix} \Omega \to \mathbb{R} \\ \text{w} \longmapsto C^i_{FC}(w) \coloneqq S^i - K \end{matrix}$.

Call Option. A call option gives its buyer the right (but not the obligation) to buy the underlying asset i at time $t = 1$ for a fixed price K. Formally, the buyer's profit with a call option over S^i is given by $C^i_{Call}\colon \begin{matrix} \Omega \to \mathbb{R} \\ \text{w} \longmapsto C^i_{Call}(w) \end{matrix}$, where $C^i_{Call}(w) \coloneqq \text{máx}\{0; S^i(w) - K\} = \left(S^i(w) - K\right)^+$.

Put Option. A put option gives its buyer the right (but not the obligation) to sell the underlying asset i at time $t = 1$ for a fixed price K. Formally, the payoff of a put option for its buyer is given by $C^i_{Put}\colon \begin{matrix} \Omega \to \mathbb{R} \\ w \longmapsto C^i_{Put}(w) \end{matrix}$, where $C^i_{Put}(w) \coloneqq \text{máx}\{0; K - S^i\} = \left(K - S^i(w)\right)^+$.

Put-Call Parity. The combination of buying a call option and selling a put option for the same exercise price K generates the payoff of a forward contract. Because if the future price of the underlying asset is higher than K, the payoff is equal to $S^i(w) - K$. However, if it is lower, the payoff is $-\left(S^i(w) - K\right)$. Therefore:

$$C^i_{Call} - C^i_{Put} = S^i(w) - K$$

Thus, if the price of a call option $\pi\left(C^i_{Call}\right)$ is a known value, it is possible to recover the price of the corresponding put option $\pi\left(C^i_{Put}\right)$ by using the previous relation. Moreover, it is possible to further develop it to obtain:

$$\pi\left(C^i_{Call}\right) - \pi\left(C^i_{Put}\right) = \pi^i - \frac{K}{1 + r}$$

It is important to note that derivative contracts are not necessarily tied to specific assets. That is, a portfolio or even an index can be used as an underlying 'asset' for a derivative. For example, an option on the value $\nu = \bar{\xi} \cdot \bar{S}$ of a portfolio is denominated as a basket or index option.

Basket Call. A basket call has the same payoff relation as a call option; however, the underlying instrument is a portfolio, instead of a single asset. Formally, given a portfolio $\nu \in$ N, the payoff of a basket call is $C_{Call}^{\nu} \coloneqq (\nu - K)^{+}$.

Straddle: A straddle is the combination of the purchases of call and put options for the same exercise price K, that must be equal to the price of the underlying asset (or portfolio) at $t = 0$. In other words, it is a combination of call and put options at the money. Formally $C_{Straddle}^{\nu}: \begin{matrix} \Omega \to \mathbb{R} \\ w \longmapsto C_{Straddle}^{\nu}(w) \end{matrix}$, where:

$$C_{Straddle}^{\nu} \coloneqq (\nu - K)^{+} + (K - \nu)^{+} = |\nu - K|$$

Because the options are *at the money*:

$$C_{Straddle}^{\nu} \coloneqq [\nu(w) - \pi(w)]^{+} + [\pi(w) - \nu(w)]^{+} = |\nu(w) - \pi(w)|$$

Notice that the payoff of a straddle increases as the price of the underlying security deviates from its current price, regardless of the direction of the change.

Butterfly Spread. A butterfly spread is a derivative that increases in value as the price of the underlying portfolio at $t = 1$ gets closer to its present value (price in $t = 0$). In other words, the bet is that the price of the portfolio will not change from $t = 0$ to $t = 1$. Formally, $C_{BS}^{\nu}: \begin{matrix} \Omega \to \mathbb{R} \\ w \mapsto C_{BS}^{\nu}(w) \end{matrix}$, where $C_{BS}^{\nu} \coloneqq (K - |\nu(w) - \pi(w)|)^{+}$.

Discount Certificate. A discount certificate of a portfolio can be defined as $C_{CD}^{\nu}: \begin{matrix} \Omega \to \mathbb{R} \\ w \mapsto C_{CD}^{\nu}(w) \end{matrix}$, where $C_{CD}^{\nu} \coloneqq \min\{\nu(w); K\} = \nu - (\nu - K)^{+}$. This derivative receives that name because its buyer receives the portfolio discounted a value that depends on a rate K. Also, it is equivalent to the purchase of the portfolio and the sale of a call option with the same portfolio as the underlying security. With this, the following price relationship is valid:

$$\pi(C_{CD}^{\nu}) = \pi(\nu) - \pi(C_{Call}^{\nu})$$

After the preceding discussion, it is evident that derivatives are highly flexible and can be adapted to many different situations, and even simulate a wide range of contracts, from fixed-income contracts (through the purchase and sale of two call options and two put options) to insurance contracts (like the option to sell a hit car to the insurer).

3.8 Introducing and Pricing a New Asset and the Superhedging Price

This section discusses the implications of the introduction of a new asset to the market, how to calculate its price at the time of entry and the related theorems. For that, the concept of contingent claim is explored. Informally, a contingent claim is a derivative negotiated at time $t = 0$, and that pays a value at $t = 1$ that depends on the state of nature, and, by assumption, this work only focuses on the contracts that generate non-negative payoffs.

Definition. A *contingent claim* is a random variable in a probability space $(\Omega, \mathcal{F}, P)$, where $0 \leq C < \infty \; P.q.c.$. It is also true that a *contingent claim* is a derivative of the primitive assets $S^0, \ldots, S^d$ if C is measurable with respect to $\mathcal{F}(S^0, \ldots, S^d)$, in other words, if $\exists f: \mathbb{R}^{d+1} \longrightarrow \mathbb{R}$ measurable such that $C = f(S^0, \ldots, S^d)$. Moreover, denote $\pi^{d+1} := \pi^C$ and $S^{d+1} := S^C$.

With the introduction of a new asset in the market, it is necessary to calculate its price so that there is no opportunity for arbitrage.

Definition. A number π^C is called an arbitrage-free price for C if the extended market $M' = \{(S^0, \ldots, S^d, S^{d+1}); (\pi^0, \ldots, \pi^d, \pi^{d+1})\}$ is free from arbitrage.

The notation $\pi(C)$ identifies the set of arbitrage-free prices for C. Moreover, an additional assumption is that the introduction of a new asset will not affect the price of the existing assets. This hypothesis is reasonable when the trading volume of the new asset is small, but, in general, this assumption is not valid. Now, it is possible to characterize the arbitrage-free prices for C.

Theorem. Suppose that the set $\mathcal{P}$ of equivalent risk-neutral measures for the initial market is non-empty, that is, there is no arbitrage in the initial market. Then, the set of arbitrage-free prices for a contingent claim C is non-empty and is given by:

$$\pi(C) = \left\{E_{P^*}\left(\frac{C}{1+r}\right) : P^* \in \mathcal{P}\right\}$$

Next, is the discussion of a dual characterization for lower and upper bounds of $\pi(C)$. For that, consider the notation:

$$\pi_{inf}(C) := \inf[\pi(C)] \;\; e \; \pi_{sup}(C) := \sup[\pi(C)]$$

Theorem. In an arbitrage-free market M, the arbitrage bounds for a contingent claim C are given by:

$$\pi_{inf}(C) = \inf_{P^* \in \mathcal{P}} E_{P^*}\left(\frac{C}{1+r}\right) = \max\left\{m \geq 0 : \exists \xi \in \mathbb{R}^d \; with \; m + \bar{\xi} \cdot Y \leq \frac{C}{1+r} \; P - q.c.\right\}$$

$$\pi_{sup}(C) = \sup_{P^* \in \mathcal{P}} E_{P^*}\left(\frac{C}{1+r}\right) = \min\left\{m \geq 0 : \exists \xi \in \mathbb{R}^d \; with \; m + \bar{\xi} \cdot Y \geq \frac{C}{1+r} \; P - q.c.\right\}$$

The result of the theorem is that $\pi_{Sup}(C)$ is the lowest possible price for a portfolio $\bar{\xi} \cdot \bar{S} \geq C \; P.q.c.$. This portfolio receives the name of superhedging strategy for C, and the identities presented above are called the superhedging duality relations.

Having identified what the *superhedging price* is, it is now necessary to understand when this phenomenon occurs, i.e., when there is indeed a range of arbitrage-free prices. First, the definition of a replicable contingent claim is presented:

Definition. C is replicable if $C = \bar{\xi} \cdot \bar{S}\ P.q.c.$ for some $\bar{\xi} \in \mathbb{R}^{d+1}$. This portfolio $\bar{\xi}$ is named the portfolio that replicates C.

This definition leads to the following corollary:

Corollary. Take an arbitrage-free market and let C be a contingent claim. Then:

a. C is replicable if, and only if, C admits a unique arbitrage-free price;

b. If C is non-replicable, then $\pi_{Inf}(C) < \pi_{Sup}(C)$ and $\pi(C) = \left(\pi_{Inf}(C); \pi_{Sup}(C)\right)$.

Hence, whenever the contingent claim is non-replicable, the superhedging price exists, and, consequently, so will the range of arbitrage-free prices. This fact motivates the next definition.

Definition. An arbitrage-free market is complete if every contingent claim is replicable.

So, the market is incomplete if there is at least one contingent claim that is non-replicable. Additionally, this concept relates to the set of risk-neutral measures.

Theorem. An arbitrage-free model is complete if, and only if, there's only one risk-neutral measure, that is, $\#\mathcal{P} = 1$.

When there is more than one risk-neutral measure, not only the market is free from arbitrage, but it also is incomplete. As a consequence, some contingent claim is not replicable, implying that there is a range of prices for this asset that are free from arbitrage and $\pi_{Sup}(C)$ is its superhedging price. In the next subsection, the superhedging theory is applied to the case of options.

3.9 Superhedging for Options

In this section, the previously explored concepts are applied to the case of options. Specifically by finding the range of arbitrage-free prices for call and put options. Consider an arbitrage-free market M with a call option $C_{Call}^i \coloneqq \left(S^i - K\right)^+$. Since $C_{Call}^i \leq S^i$, it is true that $\forall P^* \in \mathcal{P}\ E_{P^*}\left(\frac{C_{Call}^i}{1+r}\right) \leq E_{P^*}\left(\frac{S^i}{1+r}\right) = \pi^i$. Notice that $x \longmapsto x^+ = \begin{cases} x, x > 0 \\ 0, x \leq 0 \end{cases}$ is a convex function, i.e., the payoff of a call option is also a convex function. Hence, by Jensen's inequality for convex functions, as in Jensen (1906), $\pi^i \geq E_{P^*}\left(\frac{C_{Call}^i}{1+r}\right) = E_{P^*}\left(\frac{\left(S^i - K\right)^+}{1+r}\right) \geq \left[E_{P^*}\left(\frac{S^i - K}{1+r}\right)\right]^+ = \left[\pi^i - \frac{K}{1+r}\right]^+, \forall P^* \in \mathcal{P}$.

Therefore, the following universal bounds for an arbitrage-free market are valid:

$$\left[\pi^i - \frac{K}{1+r}\right]^+ \leq \pi_{Inf}(C) \leq \pi_{Sup}(C) \leq \pi^i$$

For the equivalent put option:

$$\left[\frac{K}{1+r} - \pi^i\right]^+ \leq \pi_{Inf}(C) \leq \pi_{Sup}(C) \leq \frac{K}{1+r}$$

Observe that if the interest rate is non-negative ($r \geq 0$), then $\pi_{Inf}\left(C_{Call}^i\right) \geq \left(\pi^i - K\right)^+$. In other words, the price of the right to buy an asset i at $t = 0$ for the price K is strictly less than any price for C_{Call}^i that is free from arbitrage. In conclusion, there is a time value to the call option, and $\left(\pi^i - K\right)^+$ is the intrinsic value of a call option.

4 Applications of the Superhedging Toolset

This section presents applications of the superhedging theory to the automobile insurance market and to the health insurance market. It discusses the pricing of different contracts and the problem of hedge for the consumers of insurances. In addition, this section explores other methods for calculating the price of superhedging strategies and presents an example where the solution of this problem generates a non-trivial result. Finally, it discusses the impacts of regulations on incomplete markets and what is the decision process behind them.

4.1 Superhedging Application to the Car Insurance Market

In this subsection, the superhedging model is used to analyze the case of a car insurance market that has a finite number of contracts. Suppose that, at time $t = 0$, it is possible to buy four different types of insurance contract. The first one protects its buyers against car crashes, the second safeguards against mechanical failures and fires, the third shields against the theft of the car or any electronic device used inside it, and the fourth type secures against all the eventualities mentioned before. Therefore, the set Ω of states of nature is $\Omega = \left\{\begin{matrix}\text{crashes; mechanical failures; fires;} \\ \text{car theft; theft of electronic devices}\end{matrix}\right\}$, and the sigma-algebra $\mathcal{F}$ is composed of all the possible combinations of every state of nature, disregarding repetitions. Additionally, let P be a probability over $\mathcal{F}$.

The cost of each contract is given by:

$$\pi = \{\pi^1, \pi^2, \pi^3, \pi^4\}$$

As a simplification, it is assumed that the insurer pays the contractor the value of the car (in monetary units) whenever one of the states of nature that is specified in the contract happens. That is, for the first contract a payment is issued if there is a car crash, and for the second when a mechanical failure or a fire takes place, and so on. The payoffs can then be written as:

$$S^1 = \{1; 0; 0; 0; 0\}$$
$$S^2 = \{0; 1; 1; 0; 0\}$$
$$S^3 = \{0; 0; 0; 1; 1\}$$
$$S^4 = \{1; 1; 1; 1; 1\}$$

Next, it is verified if there is any opportunity for arbitrage in this market, by calculating how many risk-neutral measures exist in this model. For this, recall that a probability measure P is risk-neutral when:

$$\pi^i = E_P\left(\frac{S^i}{1+r}\right), i = 1,2,3,4$$

Let $P = \{(\alpha; \beta; \gamma; \varepsilon; 1 - \alpha - \beta - \gamma - \varepsilon) : \alpha, \beta, \gamma, \varepsilon \in (0,1)\}$, and, without any loss of generality, assume that $r = 0$. Then P must satisfy:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \varepsilon \\ 1 - \alpha - \beta - \gamma - \varepsilon \end{pmatrix} = \begin{pmatrix} \pi^1 \\ \pi^2 \\ \pi^3 \\ \pi^4 \end{pmatrix}$$

Which can be rewritten as the linear system below:

$$\begin{cases} \alpha = \pi^1 \\ \beta + \gamma = \pi^2 \\ \varepsilon + 1 - \alpha - \beta - \gamma - \varepsilon = \pi^3 \\ \alpha + \beta + \gamma + \delta + \epsilon + 1 - \alpha - \beta - \gamma - \varepsilon = \pi^4 \end{cases} \Rightarrow \begin{cases} \alpha = \pi^1 \\ \gamma = \pi^2 - \beta \\ 1 - \alpha - \beta - \gamma = \pi^3 \\ 1 = \pi^4 \end{cases}$$

$$\Rightarrow 1 - \pi^1 - \beta - \gamma + \beta + \gamma = \pi^3 + \pi^2$$

$$\Rightarrow \pi^4 = \pi^1 + \pi^2 + \pi^3 = 1$$

Because P is a probability measure, $\{\alpha; \beta; \gamma; \varepsilon; 1 - \alpha - \beta - \gamma - \varepsilon\}$ must also be probabilities. Hence:

$$\begin{cases} 0 \leq \alpha = \pi^1 \leq 1 \\ 0 \leq \gamma = \pi^2 - \beta \leq 1 \\ 0 \leq 1 - \alpha - \beta - \gamma - \varepsilon \leq 1 \end{cases} \Rightarrow \begin{matrix} \beta \leq \pi^2 \\ 0 \leq 1 - \pi^1 - \beta - \pi^2 + \beta - \varepsilon \leq 1 \end{matrix} \Rightarrow \varepsilon \leq 1 - \pi^1 - \pi^2$$

Thus, $\alpha \in (0;1), \beta \in (0;\pi^2), \varepsilon \in (0;1-\pi^1-\pi^2)$, and assuming that $\pi^1 = 0{,}2$, $\pi^2 = 0{,}4$, $\pi^3 = 0{,}4$ and $\pi^4 = 1$, it is possible to see that there are many risk-neutral measures. This is true because, given α and $\varepsilon, \beta \in (0; 0{,}4)$, which is an interval that contains an infinite amount of numbers. Therefore, this market is free of arbitrage. However, it is incomplete since it has more than on risk-neutral measure. As a consequence, if an agent wants to buy a contract that cannot be replicated, like the protection against only the theft of electronic devices inside the vehicle, it will not be possible. For this reason, it is necessary to price the superhedging strategies for this market. The computation of the price for each contract that insures against only one state of nature is exemplified below, starting with the one against the robbery of electronics.

As the market is incomplete, it is known that there is a range of arbitrage-free prices for the contract and that the superhedging price is given by:

$$\pi_{sup}(theft\ of\ electronics) = \pi_{sup}\big((0{,}0{,}0{,}0{,}1)\big) = \sup_{P \in \mathcal{P}} E_P\big((theft\ of\ electronics)\big)$$

Taking the expectation with respect to the risk-neutral measure:

$$\begin{aligned} \pi_{sup}(theft\ of\ electronics) &= \sup_{P \in \mathcal{P}} (1 - \alpha - \beta - \gamma - \varepsilon) * 1 \\ &= \sup_{\beta \in (0;\pi^2), \varepsilon \in (0; 1-\pi^1-\pi^2)} (1 - \pi^1 - \beta - \pi^2 + \beta - \varepsilon) \\ &= \sup_{\varepsilon \in (0; 0{,}4)} (0{,}4 - \varepsilon) \end{aligned}$$

Now, it is necessary to find the smallest upper bound of $(0{,}4 - \varepsilon)$, where $\varepsilon \in (0; 0{,}4)$. It is possible to state that 0,4 is an upper bound, because $\forall \varepsilon \in (0; 0{,}4)$, $0{,}4 \geq 0{,}4 - \varepsilon$. Next, comes the proof that 0,4 is the smallest upper bound. For this, let $k \in (0, \infty)$ and it is valid that $0{,}4 - k < 0{,}4 - \varepsilon, \forall \varepsilon \in (0; 0{,}4) e\ \forall k \in (0, \infty)$. So, 0,4 is the samallest upper bound:

$$\pi_{sup}(theft\ of\ electronics) = 0{,}4 = \pi^3$$

In fact, if one wants the protection against only theft of electronics, then, given the constraints of the market, the optimal outcome is to buy the third contract. Because this contract has an additional coverage against car robbery, its buyer pays a higher price (premium for the additional protection) than what he would have paid in a complete market for the protection he wants. Moreover, even though the fourth contract also protects electronics, the premium for the extra insurances that are not needed is much higher, seeing that it covers all states of nature.

Now, the superhedging price for the contract that shields against automobile robbery is calculated.

$$\pi_{sup}(car\ theft) = \pi_{sup}\big((0,0,0,1,0)\big) = \sup_{P\,\in\,\mathcal{P}} \varepsilon = \sup_{\varepsilon\,\in\,(0;\,1-\pi^1-\pi^2)} \varepsilon = \sup_{\varepsilon\,\in\,(0;\,0{,}4)} \varepsilon$$
$$= 0{,}4 = \pi^3$$

The cheapest contract that covers car theft is the third one, because the premium for the additional protection is lower than that in the fourth contract.

Next, the superhedging price for the insurance against fires.

$$\pi_{sup}(fire) = \pi_{sup}\big((0,0,1,0,0)\big) = \sup_{P\,\in\,\mathcal{P}} \gamma = \sup_{\beta\,\in\,(0;\,\pi^2)} \pi^2 - \beta = \sup_{\beta\,\in\,(0;\,0{,}4)} 0{,}4 - \beta$$

First, it is imperative to prove that 0,4 is an upper bound for $0{,}4 - \beta, \beta \in (0; 0{,}4)$. I fact, it is valid that $0{,}4 \geq 0{,}4 - \beta, \forall \beta \in (0; 0{,}4)$, thus 0,4 is an upper bound. Then it is proved that 0,4 is the smallest upper bound. Taking $k \in (0, \infty)$, $0{,}4 - k < 0{,}4 - \beta, \beta \in (0; 0{,}4)$. So, 0,4 is the least upper bound, and:

$$\pi_{sup}(fire) = 0{,}4 = \pi^2$$

Hence, the second contract is the one that guarantees the hedge for the least possible price.

Then, the superhedging price for the insurance against mechanical failures only is determined:

$$\pi_{sup}(mechanical\ failures) = \pi_{sup}\big((0,1,0,0,0)\big) = \sup_{P\,\in\,\mathcal{P}} \beta = \sup_{\beta\,\in\,(0;\,\pi^2)} \beta$$
$$= \sup_{\beta\,\in\,(0;\,0{,}4)} \beta = 0{,}4 = \pi^2$$

Consequently, the second contract is the one that insures against mechanical failures for the smallest price.

Finally, the superhedging price for the contract that only covers crashes is calculated. However, this contract already exists in the market, implying that its price is unique and equal to the superhedging price:

$$\pi_{sup}(crash) = \pi_{sup}\big((1,0,0,0,0)\big) = \sup_{P\,\in\,\mathcal{P}} \alpha = \sup_{P\,\in\,\mathcal{P}} \pi^1 = \pi^1 = 0{,}2$$

With that, the equality between those prices has been verified.

Now, consider that in this financial market there is an economical agent interested in protecting himself against possible damages to his vehicle. Because he is a rational agent, it is fair to suppose that the solution to his problem is given by the minimization of the price of protection subject to a minimum level of utility, like stated below:

$$\min \pi_{sup}(S)$$
$$s.a.: U(S) = S^i(w_j) \geq \bar{U}$$

Where S represents the portfolio of contracts and Ū is the minimum level of security required by the individual. Notice that the objective function is not differentiable, so, to simplify the analysis, the problem is restricted to the discrete case.

Suppose that the minimum security level required by the agent does not depend on the state of nature, that is, a security level can be satisfied with any contract that protects the agent against at least one of the damages. Thus, define $\bar{U} \in \{0; 1; 2; 3; 4; 5\}$, which implies that agent doesn't want to be protected against 'half event'.

Observe that when the agent defines the minimum safety level to be null ($\bar{U} = \{0\}$), he is accepting *ex-ante* that his utility in the next day, for example, can be extremely reduced. If $\bar{U} = \{1\}$, then the solution for his problem is given by choosing the cheapest contract that safeguards against at least one state of nature. Recalling that $\pi^1 = 0{,}2$, $\pi^2 = 0{,}4$, $\pi^3 = 0{,}4$ and $\pi^4 = 1$, the solution would be $S = S^1$.

For the case where $\bar{U} = \{2\}$, $S \in \{S^1; S^2\}$. Furthermore, when $\bar{U} = \{3\}$, the solution is given by $S \in \{(S^1; S^2); (S^1; S^3)\}$. If $\bar{U} = \{4\}$, then $S \in \{S^2; S^3\}$, and when $\bar{U} = \{5\}$, $S \in \{S^4\}$.

Notice that many strategies above can be replicated via the combination of the last contract with the others. For example, when the minimum insurance level is four, an equivalent solution is to sell the first contract and buy the fourth one.

With this example, it is clear that the application of the superhedging model can solve the problem of choosing between different insurance contracts.

4.2 Superhedging with Nontrivial Results

This subsection presents a model for a financial market in which the superhedging price is determined like in the previous section, but the portfolio that is used as the hedge to a non-replicable contract cannot be determined in an immediate way. Unlike in the preceding subsection, the superhedging strategy will be determined as a linear combination of all available contracts in such a way that the cost of protection is minimized.

Consider a financial market in which three contracts are available – contracts A, B and C. Define the set of states of nature as $\Omega \coloneqq \{w^1, w^2, w^3, w^4\}$ and the sigma-algebra $\mathcal{F}$ as $\mathcal{F} = 2^{\Omega}$. Let the payoffs of each contract be: $S^A \equiv S^1 = \{S^1(w^1) = 1; S^1(w^2) = 1; S^1(w^3) = 1; S^1(w^4) = 1\} = \{1; 1; 1; 1\}$, $S^B \equiv S^2 = \{S^2(w^1) = 1; S^2(w^2) = 1; S^2(w^3) = 1; S^2(w^4) = 0\} = \{1; 1; 1; 0\}$ and $S^C \equiv S^3 = \{S^3(w^1) = 0; S^3(w^2) = 1; S^3(w^3) =$

$1; S^3(w^4) = 1\} = \{0; 1; 1; 1\}$. And their prices are $\pi^A \equiv \pi^1 = 1$, $\pi^B \equiv \pi^2 = \frac{3}{4}$ and $\pi^C \equiv \pi^3 = \frac{3}{5}$.

Now, it must be verified if this market is free of arbitrage and incomplete. Utilizing the probability space $(\Omega, \mathcal{F}, P)$, where $P = \{\alpha, \beta, \gamma, 1 - \alpha - \beta - \gamma\}$, it is possible to determine the risk-neutral probability measures. Thus, it is valid that:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 1 - \alpha - \beta - \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 3/4 \\ 3/5 \end{pmatrix}$$

Which can be rewritten as:

$$\begin{cases} \alpha + \beta + \gamma + 1 - \alpha - \beta - \gamma = 1 \\ \alpha + \beta + \gamma = \dfrac{3}{4} \\ \beta + \gamma + 1 - \alpha - \beta - \gamma = \dfrac{3}{5} \end{cases} \Rightarrow \begin{cases} \alpha + \beta + \gamma = \dfrac{3}{4} \\ 1 - \alpha = \dfrac{3}{5} \end{cases} \Rightarrow \begin{cases} \alpha + \beta + \gamma = \dfrac{3}{4} \\ \alpha = \dfrac{2}{5} \end{cases} \Rightarrow \left\{ \beta + \gamma = \frac{7}{20} \right.$$

$$\Rightarrow \gamma = \frac{7}{20} - \beta$$

Because P is a probability measure over a measurable space $(\Omega, \mathcal{F})$, the inequality below is satisfied:

$$0 \leq \gamma = \frac{7}{20} - \beta \leq 1 \Rightarrow 0 \leq \frac{7}{20} - \beta \Rightarrow \beta \leq \frac{7}{20}$$

So, the set of risk-neutral measures is given by $\mathcal{P} = \left\{ \alpha = \frac{2}{5}; \beta; \frac{7}{20} - \beta; 1 - \alpha - \beta - \left(\frac{7}{20} - \beta\right)\right\} : \beta \in \left(0; \frac{7}{20}\right) \Rightarrow \mathcal{P} = \left\{ \frac{2}{5}; \beta; \frac{7}{20} - \beta; \frac{1}{4}\right\} : \beta \in \left(0; \frac{7}{20}\right)$. It is clear that there are infinite risk-neutral measures. Therefore, this financial market is arbitrage-free and incomplete. So now, it is of interest to calculate the superhedging price for a contract D defined as $D \coloneqq \{0; 1; 0; 0\}$. Assuming that (without loss of generality) $r = 0$:

$$\pi_{sup}(D) = \sup_{P \in \mathcal{P}} E_P(D) = \sup_{P \in \mathcal{P}} \beta = \sup_{\beta \in \left(0; \frac{7}{20}\right)} \beta = \frac{7}{20}$$

Notice that the superhedging price for the contract D is not directly given by any other contract available in the market. That is, this hedge strategy was created from a linear combination of the replicable assets of this market. To find the optimal choice for the quantity of each asset, an optimization problem must be solved.

Let θ_A, θ_B and θ_C be the weight of each asset A, B and C in the portfolio created to super-replicate D. The values of θ_A, θ_B and θ_C must be chosen in such a way that it minimizes the cost of the portfolio, at the same time it satisfies $D = \{0; 1; 0; 0\}$. This results in the following problem:

$$\min_{\theta_A, \theta_B, \theta_C} \left\{1 * \theta_A + \frac{3}{4}\theta_B + \frac{3}{5}\theta_C : (\theta_A + \theta_B; \theta_A + \theta_B + \theta_C; \theta_A + \theta_B + \theta_C; \theta_A + \theta_C) \geq (0; 1; 0; 0)\right\}$$

The minimization problem above represents the choice of buying θ_A contracts of type A, θ_B contracts of type B, and θ_C contracts of type C, at the least possible price. However, the final combination must create a contract $D = \{0; 1; 0; 0\}$, to put it another way, the payoff of the combination of the contracts A,B and C have to be equal to the payoff of contract D.

As an example, let the payoff of D be $S^D(w^1) = 0$ when the state of nature w^1 occurs. Then, buying A,B and C will generate a payoff of $1 * \theta_A + 1 * \theta_B + 0 * \theta_C$ if w^1 materializes since $S^A(w^1) = S^B(w^1) = 1$ and $S^C(w^1) = 0$. Applying the same reasoning, minimization problem for this example is:

$$\min_{\theta_A, \theta_B, \theta_C} \left\{1 * \theta_A + \frac{3}{4}\theta_B + \frac{3}{5}\theta_C : \begin{pmatrix} \theta_A + \theta_B \geq 0 \\ \theta_A + \theta_B + \theta_C \geq 1 \\ \theta_A + \theta_B + \theta_C \geq 0 \\ \theta_A + \theta_C \geq 0 \end{pmatrix}\right\}$$

The second inequality restriction makes the third unnecessary, because $\theta_A + \theta_B + \theta_C \geq 1 \geq 0$. Subtracting the first inequality from the second results in $\theta_C \geq 1$. Subtracting the fourth inequality from the second leads to $\theta_B \geq 1$. In addition, subtracting the first and the fourth from the second proves that $-\theta_A \geq 1 \Rightarrow \theta_A \leq -1$.

Since this is a minimization problem with relation to θ_A, θ_B and θ_C, where they all are multiplied by positive numbers in the objective function, the solution to the problem will be given by the lowest possible values for the variables. Therefore, $\theta_A = -1$, $\theta_B = 1$ and $\theta_C = 1$. These numbers satisfy the restrictions and result in a price of $-1 + \frac{3}{4} + \frac{3}{5} = \frac{7}{20}$ for the asset D, via the selling of a contract A and the purchase of contracs B and C. Another possible way of solving this minimization problem would be through the use of the Karush-Kuhn-Tucker conditions, as demonstrated in Karush (1939) and Kuhn & Tucker (1951).

In conclusion, the solution of the superhedging problem in incomplete markets can generate nontrivial strategies.

4.3 *Superhedging*, Regulation and Utility Curves

Incomplete markets are related to the impossibility of replicating some types of contracts, which causes a loss of utility to those agents that have to execute a superhedging strategy since they have to expend capital for unnecessary protections. Based in these ideas, the next two subsections presents examples of incomplete markets that have a central regulator, which will intervene in the economy in order to increase the utility of consumers. In the first example, the regulator is indifferent between two different policies, while in the second one, these policies generate contrasting results so that the final choice of the authority depends on the problem of maximizing its utility.

4.3.1 Regulation in an Indifference Context

Consider a financial market where three contracts are available – namely, contracts A, B and C. Define the set of states of nature as $\Omega \coloneqq \{w^1, w^2, w^3, w^4\}$, and the sigma-algebra $\mathcal{F}$ as $\mathcal{F} = 2^{\Omega}$. The payoffs of the contracts are given by: $S^A = \{S^A(w^1) = k; S^A(w^2) = k; S^A(w^3) = k; S^A(w^4) = k\} = \{k; k; k; k\}$, $S^B = \{S^B(w^1) = 0; S^B(w^2) = k; S^B(w^3) = k; S^B(w^4) = k\} = \{0; k; k; k\}$ and $S^C = \{S^C(w^1) = k; S^C(w^2) = k; S^C(w^3) = k; S^C(w^4) = 0\} = \{k; k; k; 0\}$. Furthermore, their respective prices are π^A, π^B and π^C.

Using the probability space $(\Omega, \mathcal{F}, P)$, where $P = \{\alpha, \beta, \gamma, 1 - \alpha - \beta - \gamma\}$, it is possible to calculate the risk-neutral probability measures. First, the relations below are determined:

$$\begin{pmatrix} k & k & k & k \\ 0 & k & k & k \\ k & k & k & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 1 - \alpha - \beta - \gamma \end{pmatrix} = \begin{pmatrix} \pi^A \\ \pi^B \\ \pi^C \end{pmatrix} \Rightarrow \begin{cases} k(\alpha + \beta + \gamma + 1 - \alpha - \beta - \gamma) = \pi^A \\ k(\beta + \gamma + 1 - \alpha - \beta - \gamma) = \pi^B \\ k(\alpha + \beta + \gamma) = \pi^C \end{cases}$$

$$\Rightarrow \begin{cases} \pi^A = k \\ \alpha = \dfrac{k - \pi^B}{k} \\ \gamma = \dfrac{\pi^C - k + \pi^B - k\beta}{k} \end{cases}$$

Next, the set of risk-neutral measures is computed:

$$0 \le \gamma \le 1 \Rightarrow \beta \le \frac{\pi^C - k + \pi^B}{k}$$

$$\mathcal{P} = \left\{ \left(\frac{k - \pi^B}{k}; \beta; \frac{\pi^C - k + \pi^B - k\beta}{k}; \frac{k - \pi^C}{k} \right) : \beta \in \left(0; \frac{\pi^C - k + \pi^B}{k} \right) \right\}$$

Thus, the market is incomplete and arbitrage-free. Define $e_j \coloneqq \{0, \ldots, 0, k, 0, \ldots, 0\}$, where k is located in the j-th column of the vector. Notice that e_j represents each of the assets that forms the basis of this financial market. Now, calculate the superhedging price for these assets:

$$\pi_{sup}(B_1) = \sup_{P \in \mathcal{P}} E_P(B_1) = \sup_{P \in \mathcal{P}} \mathrm{k}\left(\frac{k - \pi^B}{k}\right) = k - \pi^B$$

$$\pi_{sup}(B_2) = \sup_{P \in \mathcal{P}} E_P(B_2) = \sup_{P \in \mathcal{P}} \mathrm{k}\beta = \pi^C - k + \pi^B$$

$$\pi_{sup}(B_3) = \sup_{P \in \mathcal{P}} E_P(B_3) = \sup_{P \in \mathcal{P}} \mathrm{k}\left(\frac{\pi^C - k + \pi^B - k\beta}{k}\right) = \pi^C - k + \pi^B$$

$$\pi_{sup}(B_4) = \sup_{P \in \mathcal{P}} E_P(B_4) = \sup_{P \in \mathcal{P}} \mathrm{k}\left(\frac{k - \pi^C}{k}\right) = k - \pi^C$$

Suppose that there is a central regulator in this market and that he must choose a new contract to be created and traded in this market. There are two possible choices, contract $D = \{0; k; 0; 0\}$ or contract $E = \{k; k; 0; 0\}$.

The analysis must be based on the incompleteness of the market, that is, the chosen contract must be the one that, when implemented in the market, drives the superhedging prices down by the greatest quantity possible. To determine that, there is only need to evaluate the primitive assets, because the superhedging price of any other asset is given by a linear combination of the prices of the primitive assets. By studying the superhedging prices before and after the introduction of the new contract, we can figure out which is the most beneficial contract.

First, imagine that the regulator chooses to implement the contract D. In order to do that, its price must first be determined. Because no opportunity of arbitrage should be created in this process, the price of the contract must be in the interval of arbitrage-free prices. So, the interval is determined:

$$\pi_{sup}(D) = \sup_{P \in \mathcal{P}} E_P(D) = \sup_{P \in \mathcal{P}} \mathrm{k}\beta = \pi^C - k + \pi^B$$

$$\pi_{inf}(D) = \inf_{P \in \mathcal{P}} E_P(D) = \inf_{P \in \mathcal{P}} \mathrm{k}\beta = 0$$

Notice that the direction in which the superhedging price of the primitive assets will move depends on the chosen price of the new contract. For simplicity's sake, it is chosen that the price of the contract D is equal to the average of the extreme prices of the interval, i.e., $\pi^D := \frac{\pi_{sup}(D)+\pi_{inf}(D)}{2} = \frac{\pi^C - k + \pi^B}{2}$. Calculating the new set of risk-neutral measures:

$$\begin{pmatrix} k & k & k & k \\ 0 & k & k & k \\ k & k & k & 0 \\ 0 & k & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 1-\alpha-\beta-\gamma \end{pmatrix} = \begin{pmatrix} \pi^A \\ \pi^B \\ \pi^C \\ \pi^D \end{pmatrix} \Rightarrow \begin{cases} \alpha = \dfrac{k-\pi^B}{k} \\ \beta = \dfrac{\pi^C - k + \pi^B}{2k} \\ \gamma = \dfrac{\pi^C - k + \pi^B}{2k} \end{cases} \Rightarrow$$

$$\mathcal{P} = \left\{\left(\frac{k-\pi^B}{k}; \frac{\pi^C - k + \pi^B}{2k}; \frac{\pi^C - k + \pi^B}{2k}; \frac{k-\pi^C}{k}\right)\right\}$$

Observe that $\mathcal{P}$ has a unitary cardinality, which means that there is only one risk-neutral measure. Hence, by making the contract D available in the market, the regulator has made the market complete, implying that the reduction in prices was the highest possible, given D's price. Because the market is now complete, the creation of that contract was the optimal choice, as it decreased market inefficiencies.

If the ruler had chosen to create the contract E instead, the results would have been identical. This is true due to the fact that the new contract would make it possible to obtain all the primitive assets (it would also complete the market) through linear combination of the contracts A, B, C and D, or A, B, C and E.

4.3.2 Regulation with Different Methods of Choice

Consider a financial market where two contracts are available – namely, contracts A and B. Define the set of states of nature as $\Omega := \{w^1, w^2, w^3, w^4\}$ and the related sigma-algebra $\mathcal{F}$ as $\mathcal{F} = 2^\Omega$. Then, the payoffs of each contract is given by: $S^A = \{S^A(w^1) = 1; S^A(w^2) = 1; S^A(w^3) = 0; S^A(w^4) = 0\} = \{1;1;0;0\}$ and $S^B = \{S^B(w^1) = 0; S^B(w^2) = 0; S^B(w^3) = 0; S^B(w^4) = 0\} = \{0;0;1;1\}$. Moreover, their prices are π^A and π^B. Also, the probability space $(\Omega, \mathcal{F}, P)$ with $P = \{(p_1;p_2;p_3;p_4): p_1 + p_2 = \alpha, p_3 + p_4 = (1-\alpha), p_i > 0, \forall i \ e \ \sum_{i=1}^4 p_i = 1\}$ is used.

Now, consider the partition definition as presented in Araujo *et al* (Preprint, 2010).

Definition. Consider the set $\Omega = \{w^1, \dots, w^n\}$. A partition of Ω is a family of subsets $\{B_j\}_{j=1}^J$ of Ω such that $\bigcup_{j=1}^J B_j = \Omega$ and $\forall i,j, i \neq j, B_i \cap B_j = \emptyset$.

When a financial market is formed by a partition, the set of arbitrage-free prices can be calculated through the relations presented in Araujo *et al* (Preprint, 2010, Teorema 17):

$$\pi_{sup}(C) = \sum_{j=1}^{J} P(B^j) * \max{}_{w \in B^j} C(w), \forall C \in \mathbb{R}^n$$

$$\pi_{inf}(C) = \sum_{j=1}^{J} P(B^j) * \min{}_{w \in B^j} C(w), \forall C \in \mathbb{R}^n$$

Notice that the relations above capture the case where a Choquet integral is needed and that they are equivalent to what was exhibited in the previous sections, but offer a more direct way to calculate the prices.

It is noted that the market $M = \{A; B\}$ can be written as a partition by using two sets - $B^1 = \{w^1, w^2\}$ and $B^2 = \{w^3, w^4\}$, since the union of both results in Ω, while their intersection is empty. Therefore, the equations above can be used to calculate the superhedging price of a portfolio $C = \{c^1; c^2; c^3; c^4\}$, where $c^i = S(w^i)$.

$$\pi_{sup}(C) = P(\{w^1, w^2\}) * \max\{c^1; c^2\} + P(\{w^3, w^4\}) * \max\{c^3; c^4\}$$
$$\pi_{inf}(C) = P(\{w^1, w^2\}) * \min\{c^1; c^2\} + P(\{w^3, w^4\}) * \min\{c^3; c^4\}$$

Next, the superhedging prices for Arrow's assets are calculated. Those assets are defined by $e_j \coloneqq \{0, \dots, 0, 1, 0, \dots, 0\}$, where 1 is located in j-th column of the vector. Moreover, in this case, they can be written as $e_1 = \{1; 0; 0; 0\}, e_2 = \{0; 1; 0; 0\}, e_3 = \{0; 0; k; 0\}$ and $e_4 = \{0; 0; 0; k\}$. So that:

$$\pi_{sup}(e_1) = (\alpha) * \max(1; 0) + (1 - \alpha) * \max(0; 0) = \alpha$$
$$\pi_{sup}(e_2) = (\alpha) * \max(0; 1) + (1 - \alpha) * \max(0; 0) = \alpha$$
$$\pi_{sup}(e_3) = (\alpha) * \max(0; 0) + (1 - \alpha) * \max(1; 0) = (1 - \alpha)$$
$$\pi_{sup}(e_4) = (\alpha) * \max(0; 0) + (1 - \alpha) * \max(0; 1) = (1 - \alpha)$$
$$\pi_{inf}(e_1) = (\alpha) * \min(1; 0) + (1 - \alpha) * \min(0; 0) = 0$$
$$\pi_{inf}(e_4) = (\alpha) * \min(0; 0) + (1 - \alpha) * \min(0; 1) = 0$$

Now, suppose that there is a regulator in this market who will intervene in the economy in order to introduce a new contract, with the objective of reducing the incompleteness of the market. However, there are only two possible choices of contracts, contract $D = \{1; 0; 0; 0\}$ and contract $E = \{0; 0; 0; 1\}$. Using the same comparative statics applied in the previous subsection, it is possible to determine the superhedging prices for the primitive assets.

Initially, consider that the contract D is chosen by the regulator, creating a new market $M' = \{A; B; D\}$. As previously done, let the price of the new contract be given by the average of the extremes of the arbitrage-free price interval.

$$\pi^D = \frac{\pi_{sup}(D) + \pi_{inf}(D)}{2} = \frac{\alpha}{2}$$

If the partition is altered to $B^1 = \{w^1\}$, $B^2 = \{w^2\}$ and $B^3 = \{w^3, w^4\}$, the following superhedging prices for the primitive assets are obtained:

$$\pi_{sup}(e_1) = \left(\frac{\alpha}{2}\right) * 1 + \left(\frac{\alpha}{2}\right) * 0 + (1-\alpha) * \max(0;0) = \frac{\alpha}{2}$$

$$\pi_{sup}(e_2) = \left(\frac{\alpha}{2}\right) * 0 + \left(\frac{\alpha}{2}\right) * 1 + (1-\alpha) * \max(0;0) = \frac{\alpha}{2}$$

$$\pi_{sup}(e_3) = \left(\frac{\alpha}{2}\right) * 0 + \left(\frac{\alpha}{2}\right) * 0 + (1-\alpha) * \max(1;0) = (1-\alpha)$$

$$\pi_{sup}(e_4) = \left(\frac{\alpha}{2}\right) * 0 + \left(\frac{\alpha}{2}\right) * 0 + (1-\alpha) * \max(0;1) = (1-\alpha)$$

Next, consider that the contract E was chosen by the regulator instead of the contract D, so that the new financial market is given by $M'' = \{A; B; E\}$. Define its price as:

$$\pi^E = \frac{\pi_{sup}(E) + \pi_{inf}(E)}{2} = \frac{(1-\alpha)}{2}$$

If the partition is changed one more time to $B^1 = \{w^1, w^2\}$, $B^2 = \{w^3\}$ e $B^3 = \{w^4\}$, the new superhedging prices would be:

$$\pi_{sup}(e_1) = (\alpha) * \max(1;0) + \frac{(1-\alpha)}{2} * 0 + \frac{(1-\alpha)}{2} * 0 = \alpha$$

$$\pi_{sup}(e_2) = (\alpha) * \max(0;1) + \frac{(1-\alpha)}{2} * 0 + \frac{(1-\alpha)}{2} * 0 = \alpha$$

$$\pi_{sup}(e_3) = (\alpha) * \max(0;0) + \frac{(1-\alpha)}{2} * 1 + \frac{(1-\alpha)}{2} * 0 = \frac{(1-\alpha)}{2}$$

$$\pi_{sup}(e_4) = (\alpha) * \max(0;0) + \frac{(1-\alpha)}{2} * 0 + \frac{(1-\alpha)}{2} * 1 = \frac{(1-\alpha)}{2}$$

The results are summarized in the table below:

Chart 1: Summarized Results

Superhedging Price	Financial Market		
	Original	M'	M''
$\pi_{sup}(A_1)$	α	$\frac{\alpha}{2}$	α
$\pi_{sup}(A_2)$	α	$\frac{\alpha}{2}$	α
$\pi_{sup}(A_3)$	$(1-\alpha)$	$(1-\alpha)$	$\frac{(1-\alpha)}{2}$
$\pi_{sup}(A_4)$	$(1-\alpha)$	$(1-\alpha)$	$\frac{(1-\alpha)}{2}$

Cardinality of $\mathcal{P}$	Higher than 1	Higher than 1	Higher than 1

Source: Prepared by the author

In sum, even with the creation of new contracts by the regulator, the market is still incomplete. However, observe that the superhedging price of each primitive asset lowers after the creation of either D or E. Opposed to what was presented in the last subsection, where both contracts completed the market, the decision between each contract is nontrivial, since none completes the market, and, thus, their impact in the economy differs. Consequently, the solution to this problem pervades the maximization of the regulator's utility curve.

The remainder of this subsection will exhibit different choice criteria for the regulator's utility curve and explain the logic behind them. As a simplifying assumption, consider that the objective of the regulator is to maximize the consumers' utility, disregarding the utility of the firms and other agents.

First, let the government be a utilitarian, that is, every contract is equally important to its objective function. Thus, the optimal decision is that in which the price of every contract is reduced by the maximum amount. The economic intuition behind this analysis is that there is no social class that is more important than the others, implying that the government's decision is not influenced by the voting majority, but only by the total benefit generated by the policy. Said utility can be written as:

$$U_{Gov}(\pi_{sup}) = -\sum_{i=1}^{4} \pi_{sup}(e_i)$$

The solution to this problem is given by maximizing the ruler's utility with relation to the superhedging prices for Arrow's assets in the different markets:

$$U_{Gov,Original\ Market} = -\big(\alpha + \alpha + (1-\alpha) + (1-\alpha)\big) = -2$$

$$U_{Gov,Market\ M'} = -\left(\frac{\alpha}{2} + \frac{\alpha}{2} + (1-\alpha) + (1-\alpha)\right) = \alpha - 2$$

$$U_{Gov,Market\ M''} = -\left(\alpha + \alpha + \frac{(1-\alpha)}{2} + \frac{(1-\alpha)}{2}\right) = -1 - \alpha$$

Consequently, if $\alpha > 0{,}5$, the best choice is the contract D. Though, if $\alpha < 0{,}5$, contract E is the best option. In both cases, the markets M' and M'' are already better than the original financial market due to them being less incomplete.

Second, consider that there are two economic classes involved in the analysis, and that the government's utility depends only on the utility of the class in the worst situation. Moreover, let each of the basis assets represent a consumption contract for different incomes, in other words, the cheapest basis asset is what the class with the lower purchasing power can acquire, while the most expensive asset is what the class with the higher purchasing power can buy. Seeing that the government wants to improve the situation of the poor, its utility curve is said to be Walrasian and it is written as:

$$U_{Gov} = -\min\{\pi_{sup}(e_i)\}_{i=1}^{4}$$

Observe that the resolution of the regulator's maximization problem is given by the contract that generates the greatest reduction in price to the cheapest asset:

$$U_{Gov,Original\ Market} = \begin{cases} -\alpha, if\ \alpha < 0{,}5 \\ -(1-\alpha), if\ \alpha > 0{,}5 \end{cases}$$

$$U_{Gov,Market\ M'} = \begin{cases} -\dfrac{\alpha}{2}, if\ \alpha < 0{,}5 \\ -(1-\alpha), if\ \alpha > 0{,}5 \end{cases}$$

$$U_{Gov,Market\ M''} = \begin{cases} -\alpha, if\ \alpha < 0{,}5 \\ -\dfrac{(1-\alpha)}{2}, if\ \alpha > 0{,}5 \end{cases}$$

The maximization of the government M' and M'' elucidates that, when $\alpha < 0{,}5$, the optimal decision is to create contract D, but when $\alpha > 0{,}5$, the best choice is certainly contract E.

Third, the government may want to give preference to the contract with the highest price. In the context of the health services sector, this situation can be justified for the case of the treatment of rare diseases, because the cost for this type of event is usually high. Thus, the government would be giving support to the reduction of the superhedging price for this type of insurance, lowering the survival cost for the ones afflicted by rare diseases. A utility function that takes this in consideration is:

$$U_{Gov} = -\max\{\pi_{sup}(e_i)\}_{i=1}^{4}$$

The solution to the maximization problem is given by the contract that generates the highest reduction in price to the most expensive asset:

$$U_{Gov,Original\ Market} = \begin{cases} -\alpha, if\ \alpha > 0{,}5 \\ -(1-\alpha), if\ \alpha < 0{,}5 \end{cases}$$

$$U_{Gov,Market\ M'} = \begin{cases} -\dfrac{\alpha}{2}, if\ \alpha > 0{,}5 \\ -(1-\alpha), if\ \alpha < 0{,}5 \end{cases}$$

$$U_{Gov,Market\ M''} = \begin{cases} -\alpha, if\ \alpha > 0{,}5 \\ -\dfrac{(1-\alpha)}{2}, if\ \alpha < 0{,}5 \end{cases}$$

The maximization of the government's utility with respect to M' and M'' shows that, when $\alpha > 0{,}5$, the optimal decision is to create the contract D, but when $\alpha < 0{,}5$, the best choice is the contract E.

Lastly, consider a utility curve with generic weights. Let $\theta = \{\theta^1; \theta^2; \theta^3; \theta^4\}$ be a weight vector for the prices of the contracts, where $\sum_{i=1}^{4} \theta^i = 1$ and $\theta^i > 0$. Then, the generic utility is given by:

$$U_{Gov} = \theta^1 \pi_{sup}(e_1) + \theta^2 \pi_{sup}(e_2) + \theta^3 \pi_{sup}(e_3) + \theta^4 \pi_{sup}(e_4)$$

A result similar to the first example is obtained whenever $\theta^i = 1, i \in \{1; 2; 3; 4\}$. This definition of utility is more generic since it considers a linear combination of the assets' superhedging prices in its computation.

4.4 Comments Concerning Regulation in Incomplete Markets

In this subsection, another method of regulating incomplete markets is discussed, specifically the idea the regulator can modify existing contracts, instead of just creating new ones. Based on the results of the superhedging problem presented in the previous subsections, comments about the impacts of regulations in incomplete markets are exhibited in the following paragraphs. Consider an incomplete market, like the health insurance market. This market can be classified as incomplete as there are numerous contracts that cannot be replicated, for example, an insurance contract that covers only specific diseases is not directly available for purchase.

The market that we analyze is composed of two different insurance plans: plan A, which grants its buyer access to a minimum level of services, and plan B', that extends the services (additional services) offered by plan A. Define the set of states of nature as $\Omega \coloneqq \{w^1, w^2, w^3, w^4\}$ and the related sigma-algebra $\mathcal{F}$ as $\mathcal{F} = 2^\Omega$. The payoffs of the plans are given by: $S^A = \{S^A(w^1) = 1; S^A(w^2) = 1; S^A(w^3) = 0; S^A(w^4) = 0\} = \{1; 1; 0; 0\}$ and $S^{B'} = \{S^{B'}(w^1) = 0; S^{B'}(w^2) = 0; S^{B'}(w^3) = 0; S^{B'}(w^4) = 0\} = \{1; 1; 1; 1\}$. Note that $S^{B'} = \{1; 1; 1; 1\}$ can be substituted by a plan $S^B = \{0; 0; 1; 1\}$ since $S^{B'}$ can be formed by a linear combination of S^A and S^B. Additionally, the prices of the relevant contracts are π^A and π^B. Finally, we use the probability space $(\Omega, \mathcal{F}, P)$, with $P = \{(p_1; p_2; p_3; p_4): p_1 + p_2 = \alpha, p_3 + p_4 = (1 - \alpha), p_i > 0, \forall i \ e \ \sum_{i=1}^{4} p_i = 1\}$.

It is possible to write the market $M = \{A, B\}$ as a partition by using $B^1 = \{w^1, w^2\}$ and $B^2 = \{w^3, w^4\}$. Hence, the relations presented in the subsection 4.3.2 are used in order to calculate the superhedging price of some contracts:

$$\pi_{sup}(A) = (\alpha) * \max(1;1) + (1-\alpha) * \max(0;0) = \alpha$$

$$\pi_{sup}(B) = (\alpha) * \max(0;0) + (1-\alpha) * \max(1;1) = (1-\alpha)$$

$$\pi_{sup}(\{1;1;1;0\}) = (\alpha) * \max(1;1) + (1-\alpha) * \max(1;0) = \alpha + (1-\alpha) = 1$$

$$\pi_{inf}(\{1;1;1;0\}) = (\alpha) * \min(1;1) + (1-\alpha) * \min(1;0) = \alpha$$

$$\pi_{sup}(\{0;0;0;1\}) = (\alpha) * \max(0;0) + (1-\alpha) * \max(0;1) = (1-\alpha)$$

$$\pi_{inf}(\{0;0;0;1\}) = (\alpha) * \min(0;0) + (1-\alpha) * \min(0;1) = 0$$

Consider now that the ‘Agência Nacional de Saúde Suplementar’ (Brazil’s regulator body that is responsible for the health plans market) decides to expand the minimal insurance plan A, so that it offers a greater quantity of services, increasing the quality of life of those that use it. Suppose that the payoff of this new plan is given by $\bar{A} = \{1;1;1;0\}$. As a consequence of that, plan B is altered to $\bar{B} = \{0;0;0;1\}$.

The new market $M' = \{\bar{A}, \bar{B}\}$ can also be partitioned in $B^1 = \{w^1, w^2, w^3\}$ and $B^2 = \{w^4\}$ and, using the following price rule $\pi^i = \frac{\pi_{sup}(i)+\pi_{inf}(i)}{2}$, define $\pi^{\bar{A}} = \frac{1+\alpha}{2}$ and $\pi^{\bar{B}} = \frac{1-\alpha}{2}$. Calculating the superhedging price for the new contracts:

$$\pi_{sup}(A) = \left(\frac{1+\alpha}{2}\right) * \max(1;1;0) + \left(\frac{1-\alpha}{2}\right) * 0 = \left(\frac{1+\alpha}{2}\right)$$

$$\pi_{sup}(B) = \left(\frac{1+\alpha}{2}\right) * \max(0;0;1) + \left(\frac{1-\alpha}{2}\right) * 1 = \left(\frac{1+\alpha}{2}\right) + \left(\frac{1-\alpha}{2}\right) = 1$$

It is possible to conclude important facts about the impacts of regulation on prices of different health plans. First, the price of the minimum health plan (plan A in the original market, and $\bar{A}$ in the altered market) increased with the change and, therefore, those that could buy A before, might not be able to buy it after the new policy, which means that a portion of society lost access to health plans. Furthermore, those that did not need the protection offered by the extra services (those that only needed the original plan A) will have to spend more to buy the new plan $\bar{A}$, since plan A became a non-replicable contract in the market M'.

However, it is valid to assume that the change occurred because there were some individuals who needed the extra insurance ($\{w^3\}$), but could not afford plan B in the original market. Hence, the political change benefits these individuals, because it reduced the price of contract $\bar{A}$, since in the original market it cost $\pi_{sup}(\{1;1;1;0\}) = 1$, while in the new market its’ price is $\pi_{sup}(\{1;1;1;0\}) = \frac{1+\alpha}{2} < 1$.

Another effect of the policy appears in the price of the contract B. Individuals that only had to buy plan B in the original market were spending $(1-\alpha)$, but with the expansion of plan A, plan B became non-replicable and its superhedging cost went up to 1.

The effects of such policy on the prices of different health plans are now evident. In addition, it is arguable that this regulation is only positive when the government considers that the benefits for the individuals that will use the new contract $\bar{A}$ outweighs all the utility loss by the other consumers.

To grant more validity to this analysis, it is essential to consider the case in which the health plan B is not reduced to $\bar{B}$, in other words, the altered market is $M' = \{\bar{A}, B\}$. However, it is important to notice that, in this new market, the asset $B' = \{1; 1; 1; 1\}$ that was redundant in the previous example, is not anymore. As a consequence, it must be included in the study (contract B is directly available). Furthermore, the contract A can be replicated by purchasing B' and selling B, implying that $\pi_A = \alpha$ and $\pi_B = 1 - \alpha$.

If both contracts $\bar{A}$ and B are purchased, a protection superior to that generated by B' is obtained, so their purchase price must be higher than B''s. So:

$$\pi_{\bar{A}} + \pi_B \geq \pi_{\bar{A}+B} = \pi_{\{1;1;2;1\}} \geq \pi_{\{1;1;1;1\}} = 1 = \pi_{B'}$$

It is also true that:

$$\pi_{\bar{A}} + \pi_B \geq 1 \Rightarrow \pi_{\bar{A}} + (1 - \alpha) \geq 1 \Rightarrow \pi_{\bar{A}} \geq \alpha$$

Since $B' = \{1; 1; 1; 1\}$ and $\pi_{B'} = 1$, it is correct that:

$$\pi_{\bar{A}} \leq 1$$

Thus, $1 \geq \pi_{\bar{A}} \geq \alpha$.

In case the regulator decides to forcefully fix the price of the health plan ($\pi_{\bar{A}} = \alpha$), foreseeing the exit of many agents from the market, two main effects would be observed. The first one is that, even though this measure could greatly increase the benefits of the plan extension policy for its consumers, the operational margin for the health plan providers would be extremely decreased, making the practical applicability of this regulation remarkably questionable since the financial stability of the companies responsible for the plans would be compromised and there would be strong pressure from lobbyists to reverse the regulator's decision. The second effect is that if $\pi_{\bar{A}} = \alpha$, then it is possible to buy contract $\bar{A}$ with the revenue from the sale of contract A, resulting in a contract $\bar{A} - A = \{0; 0; 1; 0\}$ that has no cost, which means that the expansion of the minimum health plan, while maintaining the original price, results in arbitrage opportunities, that pressures the government to unpin the price. As a conclusion, if $\bar{A}$ is introduced in the market and there are no arbitrage opportunities, then $\pi_{\bar{A}} > \alpha$.

It is relevant to explain that this analysis is only valid when the contract B' is available in the market and, in the case it is not, another toolset is required in order to evaluate the movements of the market. Moreover, it is imperative to assume that there are no transaction costs that prevent the execution of arbitrage opportunities in the health plan market.

5 Conclusion

The study of the superhedging problem allowed the identification of market inefficiencies that result from the existence of non-replicable contracts. In the situations previously analyzed, the inefficiencies arose when economic agents had to spend capital in order to create safeguards against particular non-replicable events, leading to the purchase of unnecessary protections. In fact, as the quantity of contracts that are available for the protection against unfavorable states of nature decreases, the amount of money spent on a hedge operation increases. The reciprocal of that implies in reduction of costs because there would be more hedge options available. Thus, the incompleteness of the market creates inefficiencies, as it prevents the allocation of the demanded contracts to those that demand them, and also raises the protection costs when compared to those of a complete market.

The analysis of the models that had a regulator intervening in the economy explained the positive and negative effects of the political decisions. Subsection 4.3 explored the case where a new contract was created, expanding the number of available assets in the market. In consequence of that, the market came closer to being complete, reducing the inefficiencies from the impossibility of replicating contracts. In the first of those examples, the introduction of any contract completed the market, reducing the superhedging price of all primitive assets. In the second example, the conclusion still pointed towards the same result: The introduction of a non-replicable asset reduces the price of the primitive assets. As a result, it reduces the inefficiencies of an incomplete market. Therefore, the intervention in financial markets is justified by the analysis of consumer prices.

However, in the subsection 4.4, it became clear that regulations in incomplete markets also have adverse effects. The analysis of the model demonstrated that expanding the coverage of health plans implies in the loss of utility to a broad group of people, as they have to perform super-replications of contracts that were previously available in the market. Even if the benefits to the population that earns with the expansion of the basic health plan is more relevant to the regulator than the loss of utility of the others, justifying the intervention, there are still other factors to be questioned. Because the model only considers the benefits to consumers, but not to firms or even the government (such as gains from taxation), it is important to question if the introduction of a new contract is possible from an operational standpoint. In other words, the market could be incomplete because of the high price required to make an extended health plan economically viable, or because the requirement for the availability of the extended plan drove many of the service providers out of the market.

It can be concluded that interventions in incomplete markets do not have precise results. In order to have a better understanding of the consequences of interventions, a thorough analysis of the market structure and the type of proposed intervention is imperative.

In summary, the use of the superhedging toolset is relevant for the analysis of incomplete markets, because it allows the resolution of nontrivial cost minimization problems and the understanding of the dynamics of interventionist policies. Although the constructed model leads to surprising theoretical results, a study that uses real market data is of fundamental importance to the validation of the results seen previously, as it can, for instance, clarify which markets are incomplete and what are the arbitrage-free price intervals in practice. In addition to a statistical study, adding transaction costs to the model is an interesting expansion of this work, as it would allow the analysis of different types of market inefficiencies and interventionist policies, such as the increase in tax rates and the worsening of the judicial system.

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