Sec A - Q3.1

Pigeonhole Principle

1 Pigeonhole Principle

Contrapositive statement:
If there are no pigeonholes with at least 2 pigeons, then less than $N + 1$ pigeons are put in N pigeonholes.

Proof:
We have that, out of N pigeonholes, there are no pigeonholes with at least 2 pigeons.
$\implies$ All pigeonholes have less than 2 pigeons.
$\implies$ There is at most 1 pigeon in each pigeonhole. Since there are N pigeonholes, there can be $N * 1 = N$ pigeons at most
Therefore, total no. of pigeons $\leq N$
$\implies$ Less than $N + 1$ pigeons are put in the N pigeonholes.
Hence proved.

2 General Pigeonhole Principle

Contrapositive statement:
If there are no pigeonholes with at least $k + 1$ pigeons, then less than $Nk + 1$ pigeons are put in N pigeonholes.

Proof:
We have that, out of N pigeonholes, there are no pigeonholes with at least $k + 1$ pigeons.
$\implies$ All pigeonholes have less than $k + 1$ pigeons.
$\implies$ There is at most k pigeons in each pigeonhole. Since there are N pigeonholes, there can be $N * k = Nk$ pigeons at most
Therefore, total no. of pigeons $\leq Nk$
$\implies$ Less than $Nk + 1$ pigeons are put in the N pigeonholes.
Hence proved.

3 12 integers, difference of 2 is divisible by 11

Required to prove:
When we select any 12 random integers, the difference of at least 2 of them will be divisible by 11.

Proof:
For any integer, when divided by 11, it can give the following as remainder: 0, 1, 2,..., 10. Hence, there are 11 possible remainders. Let's assume each remainder to be a pigeonhole. Therefore, we have 11 pigeonholes such that the first pigeonhole contains numbers with remainder 0, second pigeonhole contains numbers with remainder 1,..., eleventh pigeonhole contains numbers with remainder 10.

Now, we have 12 numbers (12 pigeons). Using pigeonhole principle, we can show that at least 2 of them lie in the same pigeonhole (as number of pigeonholes $(N) = 11$ and number of pigeons $(N + 1) = 12$).

The pigeons lying in the same pigeonhole give the same remainder when divided by 11 (from our definition of the pigeonholes in this problem). Therefore, at least 2 numbers give the same remainder when divided by 11. Let these numbers be of the form:

$$a = 11q_1 + r \text{ and } b = 11q_2 + r$$
$$\implies a - b = 11(q_1 - q_2)$$

Therefore, the difference of numbers with the same remainder when divided by 11 is divisible by 11 and there are at least 2 such numbers.
Hence proved.

4 Important points to consider for grading

- Last part of the question is of 1 mark, the first two parts hold 2 marks in total.
- If only the general pigeonhole principle is proved, 2 marks are awarded. However, if only the basic pigeonhole principle is proved, only one mark is awarded.
- For only writing the contrapositive statement correctly, half a mark is awarded.
- Since the question clearly mentions that the constraposition should be used to prove the pigeonhole principle, if some other method is used, no marks may be given for the answer.
- In the last part, it is important to mention what the pigeonholes and pigeons are in order to get full marks for the provided answer.