Endsem: Probability and Statistics (100 marks)

Cheat Sheet:

- The probability density function (PDF) of a Gaussian random variable $X \sim N(\mu, \sigma^2)$ is given by:
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}.$$
- The probability mass function (PMF) of a Poisson random variable $X \sim \text{Poisson}(\lambda)$ is:
$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$
- The PDF of a Gamma random variable $X \sim \text{Gamma}(\alpha, \beta)$ is:
$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0,$$
where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$ is the Gamma function.

Each question: 5 marks

1. MGF: Let $Z = X_1 + X_2 + \ldots X_N$ where X_i are iid and N is a positive discrete random variable. Prove that $M_Z(t) = M_N(logM_X(t))$.
2. **Log-normal distribution:** Let $Y = e^X$ where $X \sim \mathcal{N}(\mu, \sigma^2)$. Obtain the pdf of Y.

Each question: 8 marks

1. Let $\mathcal{D} = \{x_1, \ldots x_n\}$ denote i.i.d samples from a Poisson random variable with unknown parameter γ. Find an MLE estimate for the unknown parameter γ. (5mks) What is its Mean Squared Error (MSE) (3mks)?
2. MLE: Consider a Gaussian random variable X with known mean μ but an unknown variance σ^2. Suppose you observe k iid samples from this random variable which is denoted by $\mathcal{D} = \{x_1, x_2, \ldots, x_k\}$. Find the maximum likelihood estimate for the unknown variance σ^2. (5mks) Is the MLE estimate biased ? justify why. (3 mks)
3. Consider a sequence $\{X_n, n = 1, 2, 3, \ldots\}$ such that
$$X_n = \begin{cases} n & \text{with probability } \frac{1}{n^2}, \\ 0 & \text{with probability } 1 - \frac{1}{n^2}. \end{cases}$$
   (a) Show that $X_n \xrightarrow{P} 0$ (Convergence in probability to 0). (4mks)
   (b) Show that $X_n \xrightarrow{a.s.} 0$ (Almost sure convergence to 0). (4mks)
4. Given a Markov coin with the following transition probability matrix P and initial distribution $\mu = [0.1, 0.9]$, Use the following 4 independent $Uniform[0, 1]$ samples $\{0.3, 0.7, 0.23, 0.97\}$ to obtain/generate 4 successive toss outcomes of the Markov coin. (Hint: The first toss is to be sampled from the initial distribution). $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix}$
5. Let X and Y be independent random variables with common distribution function F. Obtain the pdfs of (i) $Z_1 = max(X, Y)$ (ii) $Z_2 = min(X, Y)$.

Each question: 10 marks

1. Let $D = \{x_1, \ldots, x_n\}$ denote i.i.d samples from a uniform random variable $U[a, b]$ where a and b are unknown. Find an MLE estimate for the unknown parameters a and b.

2. Bayesian Inference/ Conjugate prior problem: Suppose $D = \{x_1, \ldots, x_n\}$ is a data set consisting of independent samples of a Poisson random variable with unknown parameter λ^*. Now assume a prior model $\Lambda \sim Gamma(\alpha, \beta)$ on the unknown parameter λ^* (see hint below for gamma distribution). Obtain an expression for the posterior distribution on λ^*. (7mks). What is the MAP estimate for λ^*? (3mks)

   Hint: Use Prior belief: $\Lambda \sim$ Gamma(α, β),

$$f_\Lambda(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}, \quad \lambda > 0.$$

   and Likelihood of observing x given $\Lambda = \lambda$:

$$f_{X|\Lambda}(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}.$$

3. (i) (4mks) For a Markov chain, let F_{ii} denote the probability of the chain ever returning to state i having started in state i and let f_{ii}^n denote the probability of visiting state i for the first time in exactly n steps, having started in state i. Show that $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^n$.
   (ii) (6mks) For a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrices given below, use the above equality to find F_{ii} for $i = 1, 2, 3$. From the values of F_{ii}, deduce which states are transient and recurrent. $P = \begin{bmatrix} p & 1-p & 0 \\ p & 1-2p & p \\ 0 & 0 & 1 \end{bmatrix}$

4. Gaussian: Suppose $\mathbf{X} = A\mathbf{Z} + \mu$ where A is an $n \times n$ matrix and $\mathbf{Z}$ is a standard normal vector of length n. Derive the expression for mean $E[\mathbf{X}]$ and covariance matrix $C_{\mathbf{X}}$. (5 mks) Also derive the expression for the pdf of $\mathbf{X}$. (5mks)

5. **Shifted Exponential:** Let X be a random variable following a shifted exponential distribution with rate parameter $\lambda > 0$ and shift parameter μ. The probability density function (PDF) of X is given by:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda(x-\mu)}, & \text{if } x \geq \mu, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Derive the MGF of X, state its region of convergence (5 mks)
   (b) Using the MGF, obtain the first and second moments of X. What is the variance of X ? (5 mks)