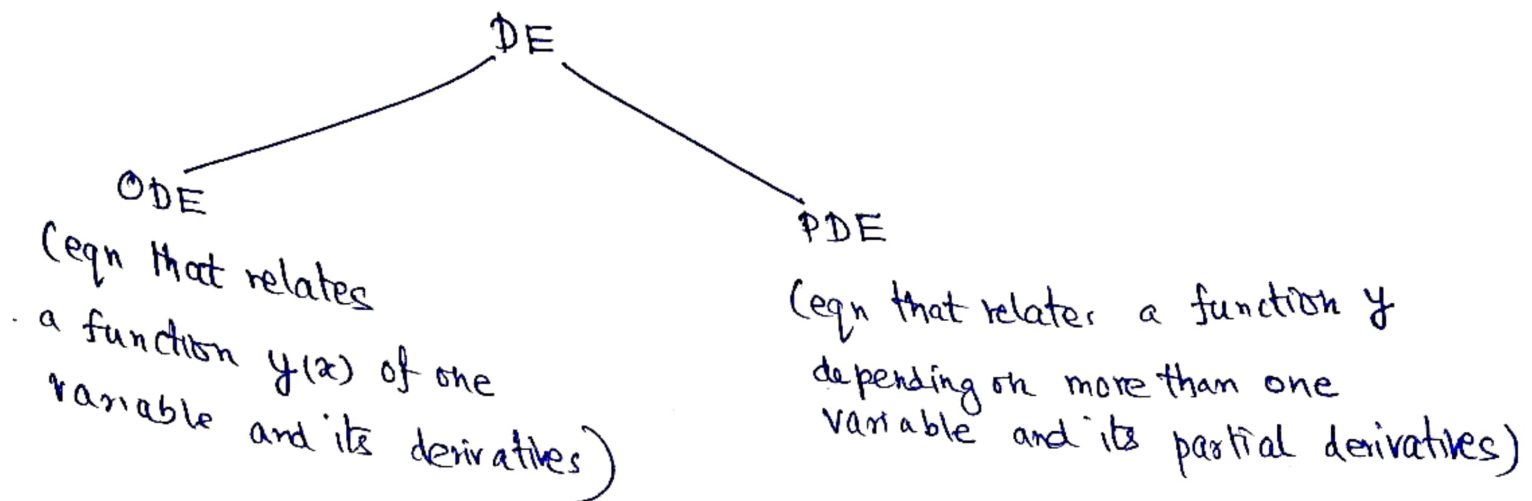


## Differential Equations.

A differential equation is an equation that relates one (or more) functions and their derivatives.



A general ODE can be written as

$$\textcircled{1} \leftarrow F(x, y, y', \dots, y^{(n)}) = 0, \quad \text{where } y = y(x), \quad y^{(k)}(x) = \frac{d^k y}{dx^k}.$$

The order of the ODE  $\textcircled{1}$  is defined as the order of the highest derivative of  $y$  that occurs in the equation.

For example,

$$x^2 + 2x y'' + 2x y' = 0$$

The order of the above eqn is 2.

The degree of the ODE  $\textcircled{1}$  is defined as greatest power of the highest order derivative occurring in the eqn after the equation has been made free of radicals and fractions in its derivatives.

For example

$$k \frac{d^2 y}{dx^2} = f(x) \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{3/2}, k > 0$$

The degree of the above eqn is 2.

Consider the  $n^{\text{th}}$  order ODE

$$F(x, y, y', \dots, y^{(n)}) \equiv 0 \longrightarrow (1)$$

(or)

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \longrightarrow (2)$$

The ODE (1) is said to be linear if it can be written

$$(3) \leftarrow a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_{n-1}(x) y' + a_n(x) y = f(x)$$

with  $a_0(x) \neq 0$ . We assume that the functions  $a_0(x), \dots, a_n(x), f(x)$

are defined over some interval  $a \leq x \leq b$ . The functions

$a_0(x), a_1(x), \dots, a_n(x)$  are called coefficients of the equation and  $f(x)$  is called a forcing function.

The equation (3) is often written as

$$L(y) = f(x), \longrightarrow (3b)$$

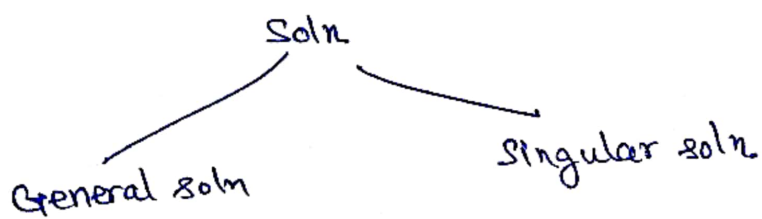
$$\text{where } L \equiv a_0(x) \frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{d}{dx} + a_n(x)$$

If  $f(x) \equiv 0$  on  $a \leq x \leq b$ , then (3) is called a homogeneous linear ODE.

- When one or more coefficients<sup>fn's</sup> of (3) depends on  $x$ , then (3) is called variable coefficient equation.
- When all the coefficient functions are constants, then (3) is called constant coefficient equation.

### Solutions of ODE.

- A function  $\phi(x)$  is said to be a solution of (1) if it satisfies the equation (1) for  $y = \phi(x)$ . In this case,  $\phi$  is called an integral curve.
- A solution of  $n^{\text{th}}$  order ODE<sup>(1)</sup> that contains 'n' arbitrary constants is called the general solution of (1). If the arbitrary constant are assigned specific value, then it is called a particular solution of (1).
- A solution that cannot be obtained from the general solution by assigning suitable value for arbitrary constants is called a singular solution of (1).



Theorem An  $n^{\text{th}}$  order linear ODE has no singular solutions.

### Lemma

If  $y_1, y_2, \dots, y_n$  are solutions of the linear homogeneous DE

$$L(y) = 0, \longrightarrow \textcircled{1}$$

then  $c_1 y_1 + c_2 y_2 + \dots + c_n y_n$  is also soln of  $\textcircled{1}$ .

Proof.

$$\begin{aligned} L(c_1 y_1 + \dots + c_n y_n) &= c_1 L(y_1) + c_2 L(y_2) + \dots + c_n L(y_n) \\ &= 0 \end{aligned}$$

Consider the non-homogeneous linear DE

$$L(y) = f(x), \longrightarrow \textcircled{2}$$

The soln of  $\textcircled{1}$ , i.e.  $c_1 y_1 + \dots + c_n y_n$ , is called the complementary solution (function) of  $\textcircled{2}$ .  
 $\downarrow$   
 $y_c(x)$

A solution  $y_p(x)$  of  $\textcircled{2}$  which does not contain any arbitrary constants is called a particular integral of  $\textcircled{2}$ .

The complete solution of  $\textcircled{2}$  is given by

$$y(x) = y_c(x) + y_p(x)$$

First order ODE:

Consider the first order ODE

$$F(x, y, y') = 0 \longrightarrow \textcircled{*}$$

or  $y' = f(x, y)$



# Separable equations.

Suppose that  $f(x,y) = g(x)h(y)$  in  $(*)$ .

Then  $(1)$  can be written as

$$y'(x) \equiv \frac{dy}{dx} = g(x)h(y)$$

$$\Rightarrow \frac{y'(x)}{h(y(x))} = g(x)$$

Integrating both sides, we get

$$\int \frac{y'(x)}{h(y(x))} dx = \int g(x) dx + C$$

$$\text{Let } u = y(x), \left. \begin{array}{l} \frac{du}{dx} = y'(x) \end{array} \right\} \Rightarrow \int \frac{1}{h(u)} du = \int g(x) dx + C$$

Symbolically, we write

$$\frac{1}{h(u)} du = g(x) dx$$

Example

$$y' - 2y + a = 0, \quad a \text{ is constant.}$$

Soln

$$y' = 2y - a$$

$$\frac{dy}{dx} = 2y - a$$

$$\frac{dy}{2y - a} = dx$$

Integrating both sides, we get

$$\frac{1}{2} \int \frac{2 dy}{2y - a} = \int dx + C$$

$$\Rightarrow \frac{1}{2} \ln|2y - a| = x + C,$$

if  $y \neq a/2$ .

$$\ln|2y - a| = 2x + C \quad \rightarrow \text{general const}$$

$$\Rightarrow |2y - a| = k e^{2x}, \quad k > 0.$$

$$\Rightarrow 2y - a = \pm k e^{2x}$$

$$\boxed{y = \pm \frac{1}{2} k e^{2x} + \frac{a}{2}}$$