Theorem (Rank - Nullity theorem)

Let $T: V \to W$ be a linear transformation from V into W. If V is finite-dimensional, then

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

Proof.

Let $\{v_1, v_2, \ldots v_k\}$ be a basis for $N(T)$.

Choose $v_{k+1}, v_{k+2}, \ldots v_n$ such that $\{v_1, v_2, \ldots v_k, v_{k+1}, \ldots v_n\}$ is a basis for V.

Claim $S = \{Tv_{k+1}, Tv_{k+2}, \ldots Tv_n\}$ is a basis for $R(T)$.

S is linearly independent:

Let $c_{k+1}Tv_{k+1} + \ldots + c_nTv_n = 0$. Then

$$T(c_{k+1}v_{k+1} + \ldots + c_nv_n) = 0.$$

$$\Rightarrow c_{k+1}v_{k+1} + \ldots + c_nv_n \in N(T).$$

$\therefore$ there exists ~~$v \in N(T)$ s.t~~ scalar $c_1, c_2, \ldots c_k$ such that

$$c_{k+1}v_{k+1} + \ldots + c_nv_n = c_1v_1 + c_2v_2 + \ldots + c_kv_k.$$

$$\Rightarrow c_1v_1 + c_2v_2 + \ldots + c_nv_n - c_{k+1}v_{k+1} - \ldots - c_nv_n = 0$$

Since $\{v_1, v_2 \ldots v_n\}$ is a basis for V,

$$c_1 = c_2 = \ldots = c_k = c_{k+1} = \ldots = c_n = 0.$$

In particular, $c_{k+1} = \ldots = c_n = 0$.

S spans $R(T)$: (ie, $span(S) = R(T)$.

Let $w \in span(S)$. Then

$$w = d_{k+1} T v_{k+1} + \dots + d_n T(v_n)$$

$$= T(d_{k+1} v_{k+1} + \dots + d_n v_n) \in R(T).$$

$\therefore$ $span(S) \subseteq R(T) \longrightarrow (1)$

Let $w_1 \in R(T)$. Then

$w_1 = T(v)$, for some $v \in V$.

$$\Rightarrow w_1 = T(c_1 v_1 + \dots + c_n v_n)$$

$$\Rightarrow w_1 = T(c_1 v_1 + \dots + c_k v_k) + T(c_{k+1} v_{k+1} + \dots + c_n v_n)$$

$$= c_1 T(v_1) + \dots + c_k T(v_k) + c_{k+1} T v_{k+1} + \dots + c_n T v_n$$

$$= c_{k+1} T(v_{k+1}) + \dots + c_n T v_n \quad (\because v_1, v_2 \dots v_k \in N(T))$$

$\in span(S)$.

$\therefore$ $R(T) \subseteq span(S) \longrightarrow (2)$

From (1) & (2), $R(T) = span(S)$.

$\therefore$ $rank(T) = \dim(R(T))$
$= n - k$

$\Rightarrow$ $rank(T) + nullity(T) = \dim V$.

Theorem.

Let V and W be finite dimensional vector spaces over $\mathbb{K}$ such that $\dim V = \dim W$. If T is a linear transformation from V into W, then TFAE.

(i) T is invertible.

(ii) T is one-one.

(iii) T is onto

Proof. Use rank-nullity theorem.

Representation of ~~matrices~~ linear transformation by matrices.

Definition (Ordered basis)

If V is a finite dimensional vector space, an ordered basis for V is a finite sequence of vectors which is a basis for V.

Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be an ~~basis~~ ordered basis for V.

Let $v \in V$. Then there exist unique scalars $c_1, c_2, \ldots, c_n$ such that

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

We call c_i the ith coordinate of v with respect to the ordered basis $\mathcal{B}$. The matrix

$$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$ is called the coordinate matrix of v w.r. to $\mathcal{B}$.

Let V be an n-dimensional vector space over $\mathbb{K}$ and let W be an m-dimensional vector space over $\mathbb{K}$.

Let $\mathcal{B} = \{v_1, v_2, \ldots v_n\}$ be an ordered basis for V and let $\mathcal{B}' = \{w_1, w_2, \ldots w_m\}$ be an ordered basis for W.

Let $T : V \to W$ be a linear transformation from V into W. For each $v_j \in V$, $T(v_j)$ is uniquely written as

$$Tv_j = \sum_{i=1}^{m} A_{ij} w_i, \quad \text{where } A_{ij} \in \mathbb{K}.$$

The $m \times n$ matrix $A = (A_{ij})$ is called the matrix of T with respect to the ordered bases $\mathcal{B}$ and $\mathcal{B}'$. We denote A by $[T]_{\mathcal{B},\mathcal{B}'}$. If $V = W$ and $\mathcal{B} = \mathcal{B}'$, we write simply $[T]_{\mathcal{B}}$ instead of $[T]_{\mathcal{B},\mathcal{B}'}$.

<u>Example 1</u> Let $T : \mathbb{K}^2 \to \mathbb{K}^2$ be a linear transformation defined by $T(v_1, v_2) = (v_1, 0)$.

Find the matrix of T w.r. to the standard basis $\{e_1, e_2\}$.

$e_1 = (1,0)$, $e_2 = (0,1)$

<u>Soln</u>

(i) Verify that T is linear.

Now $T(1,0) = (1,0) = 1e_1 + 0e_2$

$T(0,1) = (0,0) = 0e_1 + 0e_2$

$$\therefore [T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Example 2.

Take $V = P_3(\mathbb{R})$, $W = P_3(\mathbb{R})$

$$\mathcal{B} = \{ \underset{f_0}{1}, \underset{f_1}{x}, \underset{f_2}{x^2}, \underset{f_3}{x^3} \}, \quad \mathcal{B}' = \{1, x, x^2, x^3\}$$

Define $D : P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ by

$$Df = f'$$

ie, $(Df)(x) = f'(x)$, $x \in \mathbb{R}$

Then D is linear.

$$Df_0 = 0 = 0f_0 + 0f_1 + 0f_2 + 0f_3$$

$$Df_1 = f_0 = 1f_0 + 0f_1 + 1f_2 + 1f_3$$

$$Df_2 = 2f_1 = 0f_0 + 2f_1 + 0f_2 + 0f_3$$

$$Df_3 = 3f_2 = 0f_0 + 0f_1 + 3f_2 + 0f_3$$

$$\therefore \quad [D]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem. Let V be a vector space over $\mathbb{K}$. Let $\mathcal{B} = \{v_1, v_2 .. v_n\}$ and $\mathcal{B}' = \{v_1', v_2', .. v_n'\}$ be two ordered basis for V. Suppose T is a linear transformation from V into V. If $P = [P_1, P_2, .. P_n]$ is the $n \times n$ matrix with columns

$$P_j = [v_j']_{\mathcal{B}'}$$

Then

$$[T]_{\mathcal{B}'} = P^{-1} [T]_{\mathcal{B}} P.$$

Example

Take Example 1 (Page 41).

Choose $\mathcal{B} = \{(1,0),(0,1)\}$ and $\mathcal{B}' = \{(1,1),(2,1)\}$.

Prove that $\mathcal{B}'$ is a basis for $\mathbb{K}^2$.

$$(1,1) = 1e_1 + 1e_2 \Rightarrow P_1 = [(1,1)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(2,1) = 2e_1 + 1e_2 \Rightarrow P_2 = [(2,1)]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad \left(\text{Show that } P^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\right)$$

$$T(1,1) = (1,0) = \alpha_1(1,1) + \alpha_2(2,1)$$
$$= 1(1,1) + -1(2,1)$$

To find α_1, α_2,

$$\alpha_1 + 2\alpha_2 = 1$$
$$\alpha_1 + \alpha_2 = 0$$
$$\Rightarrow \alpha_2 = 1, \ \alpha_1 = -1$$

$$T(2,1) = (2,0)$$
$$= \alpha_1(1,1) + \alpha_2(2,1)$$
$$= -2(1,1) + 2(2,1)$$

$$\therefore [T]_{\mathcal{B}'} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$$

Verify that

$$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}} P.$$