Therefore, the characteristic equation is $\lambda^2 - 6\lambda + 5 = 0$

By Cayley-Hamilton theorem, $A^2 - 6A + 5I = 0$

i.e., $A^2 = 6A - 5I$

**3. Verify Cayley-Hamilton theorem, find $A^4$ and $A^{-1}$ when A =** $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

**Solution**: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\ of\ the\ minirs\ of\ the\ main\ diagonal\ elements = 3 + 2 + 3 = 8$

$S_3 = |A| = 2(4-1) + 1(-2+1) + 2(1-2) = 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

**To prove that:** $A^3 - 6A^2 + 8A - 3I = 0$---------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$A^3 - 6A^2 + 8A - 3I$$
$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

**To find $A^4$:**

(1) $\Rightarrow A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^3 = 6A^2 - 8A + 3I$ ---------- (2)

Multiply by A on both sides, $A^4 = 6A^3 - 8A^2 + 3A = 6(6A^2 - 8A + 3I) - 8A^2 + 3A$

Therefore, $A^4 = 36A^2 - 48A + 18I - 8A^2 + 3A = 28A^2 - 45A + 18I$

Hence, $A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

**To find $A^{-1}$:**

Multiplying (1) by $A^{-1}$, $A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

4. **Verify that A = $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ satisfies its own characteristic equation and hence find $A^4$**

Solution:Given A =$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 =$ Sum of the main diagonal elements $= 0$

$S_2 = |A| = -1 - 4 = -5$

Therefore, the characteristic equation is $\lambda^2 - 0\lambda - 5 = 0$ i.e., $\lambda^2 - 5 = 0$

To prove: $A^2 - 5I = 0$---------- (1)

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

**To find $A^4$:**

From (1), we get, $A^2 - 5I = 0 \Rightarrow A^2 = 5I$

Multiplying by $A^2$ on both sides, we get, $A^4 = A^2(5I) = 5A^2 = 5\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$

5. **Find $A^{-1}$ if A = $\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, using Cayley-Hamilton theorem**

## Definition

Let $V$ be a vector space over $\mathbb{K}$. A function $\langle\, , \rangle : V \times V \to \mathbb{K}$ is said to be an inner product on $V$ if it satisfies the following properties.

(i) $\langle v+w, u\rangle = \langle v,u\rangle + \langle w,u\rangle, \quad \forall\, u,v,w \in V$

(ii) $\langle cv, w\rangle = c\langle v,w\rangle, \quad \forall\, c \in \mathbb{K}$ and $v,w \in V$

(iii) $\langle v,w\rangle = \overline{\langle w,v\rangle}, \quad \forall\, v,w \in V$

(iv) $\langle v,v\rangle \geq 0, \forall\, v \in V$ & $\langle v,v\rangle = 0$ iff $v=0$

A vector space together with an inner product is called an inner product space.

**Example. 1**

$V = \mathbb{R}^n$, standard inner product on $\mathbb{R}^n$.

Define $\langle x,y\rangle = \sum_{i=1}^{n} x_i y_j$, $\quad x = (x_1, \ldots x_n)$, $y = (y_1, \ldots y_n)$

Then $\langle x,y\rangle$ is an inner product on $\mathbb{R}^n$.

**Example 2**

$V = \mathbb{C}^n$, standard inner product on $\mathbb{C}^n$

Define $\langle z,w\rangle = \sum_{j=1}^{n} z_j \overline{w_j}$ $\quad z = (z_1, \ldots z_n)$, $w = (w_1, \ldots w_n)$

Then $\langle \cdot \rangle$ is an inner product on $\mathbb{C}^n$.

## Definition

Let $v, w \in V$ (IPS). Then $v$ is said to be orthogonal to $w$ if $\langle v, w \rangle = 0$.

Let $S \subseteq V$. We say that $S$ is an orthogonal set if all pairs of distinct vectors in $S$ are orthogonal.

We say that $S$ is an orthonormal set if

(i) $S$ is an orthogonal set

(ii) $\|v\| = 1, \forall v \in S$.

Note: $\|v\| = +\sqrt{\langle v, v \rangle} > 0$.

$\|v\| = 0$ iff $v = 0$.

## Lemma

An orthogonal set of non-zero vectors is linearly independent.

**Proof** Exercise (See class notes).

- zero vector is orthogonal to any vectors.

  ie $\langle 0, v \rangle = 0, \forall v \in V$.

- $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in V$.

# Orthogonal diagonalization

**Definition.**

Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Then $A$ is said to be orthogonally diagonalizable if there exists an orthogonal matrix $P$ such that

$$P^{-1}AP = D \qquad (\text{Note that } P^{-1} = P^t).$$

**Theorem.**

Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Then TFAE

(i) $P$ is orthogonal (ie $PP^t = P^tP = I$).

(ii) The rows of $P$ are ~~orthogonal~~. orthonormal.

(iii) The columns of $P$ are orthonormal.

**Theorem** (Spectral theorem) or (Principal Axes theorem)

Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Then TFAE

(i) $A$ has an orthonormal basis

(ii) $A$ is orthogonally diagonalizable

(iii) $A$ is symmetric

**Lemma** If $A$ is an $n \times n$ symmetric matrix, then eigenvectors of $A$ corresponding to distinct eigenvalues are orthogonal.

**Proof** Let $Ax = \lambda x$ & $Ay = \mu y$, $\lambda \neq \mu$

$$\langle x, Ay \rangle = \langle x, \mu y \rangle \Rightarrow \langle A^t x, y \rangle = \mu \langle x, y \rangle$$

$$\Rightarrow \langle Ax, y \rangle = \mu \langle x, y \rangle \quad (\because A^t = A)$$

$$\Rightarrow (\lambda - \mu)\langle x, y \rangle = 0 \Rightarrow \langle x, y \rangle = 0.$$

**Theorem** (Gram-Schmidt process)

Let $V$ be an inner product space and $\{v_1, v_2 \ldots v_n\}$ be a linearly independent set in $V$. Define

$$w_1 = v_1$$

$$\cancel{w_{m+1} = \sum}$$

$$w_{m+1} = v_{m+1} - \sum_{k=1}^{m} \frac{\langle v_{m+1}, w_k \rangle}{\|w_k\|^2} w_k, \quad k = \overline{1, n-1}.$$

Then $\{w_1, w_2 \ldots w_n\}$ is an orthogonal set.

**Example**

Take $v_1 = (3, 0, 4)$

$v_2 = (-1, 0, 7) \quad \in \mathbb{R}^n$

$v_3 = (2, 9, 11)$

Then

$$w_1 = (3, 0, 4)$$

$$w_2 = (-1, 0, 7) - \frac{\langle (-1,0,7), (3,0,4) \rangle}{25} (3, 0, 4)$$

$$= (-4, 0, 3)$$

$$w_3 = (2, 9, 11) - \frac{\langle (2,9,11), (3,0,4) \rangle}{25} (3, 0, 4)$$

$$- \frac{\langle (2,9,11), (-4,0,3) \rangle}{25} (-4, 0, 3)$$

$$= (0, 9, 0).$$