The auxiliary $(D+3)^2 = 0$

$\Rightarrow r = -3$ (twice)

$\therefore$ $C.F = Ae^{-3t} + Bte^{-3t}$

$= (A+Bt)e^{-3t}$

$P.I = \frac{1}{(D+3)^2}(-2t)$

$= \frac{1}{9}\left(\frac{1}{(1+D/3)^2}(-2t)\right)$

$= -\frac{2}{9}(1+D/3)^{-2}(t)$

$= -\frac{2}{9}\left\{1 - \frac{2D}{3}(t) + \cdots\right\}(t)$

$= -\frac{2}{9}\left(t - \frac{2}{3}\right)$

$= \frac{-2t}{9} + \frac{4}{27}$

$\therefore y(t) = (A+Bt)e^{-3t} - \frac{2t}{9} + \frac{4}{27} \longrightarrow (*)$

To find $x(t)$:

From $(*)$

$(Dy)(t) = (A+Bt)e^{-3t}.(-3) + Be^{-3t} - \frac{2}{9}$.

$= -3(A+Bt)e^{-3t} + Be^{-3t} - \frac{2}{9} \longrightarrow (**)$

Substitute $(*)$ and $(**)$ in (2)

$$x = -\frac{1}{2}(Dy+y)$$

$$= -\frac{1}{2}\left[\left(A-\frac{1}{2}B\right)+Bt\right]e^{-3t} + \frac{t}{9} + \frac{1}{27}.$$

when $x = y = 0$, at $t=0$.

$$\left.\begin{aligned} 0 &= A + \frac{4}{27} \\ \text{and} \quad 0 &= A - \frac{B}{2} + \frac{1}{27} \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -4/27 \\ B &= -2/9. \end{aligned}$$

∴ The desired soln are

$$x = -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{27}(1+3t)$$

$$y = -\frac{2}{27}(2+3t)e^{-3t} + \frac{2}{27}(2-3t).$$

Example

Solve

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$$

Soln $D = \frac{d}{dt}$

(143)

$$Dx + (D-2)y = 2\cos t - 7\sin t \longrightarrow (1)$$

$$(D+2)(x) - Dy = 4\cos t - 3\sin t \longrightarrow (2)$$

Eliminate y

$$(1) \times D \Rightarrow \quad D^2x + \cancel{D(D-2)y} = -2\sin t - 7\cos t$$

$$(2) \times (D-2) \Rightarrow \quad (D-2)(D+2)x - \cancel{(D-2)Dy} = -4\sin t - 3\cos t$$

$$-8\cos t + 6\sin t$$

$$D^2x + (D-2)(D+2)x = -18\cos t$$

$$\Rightarrow \quad 2(D^2-2)x = -18\cos t$$

$$\Rightarrow (D^2-2)x = -9\cos t$$

The char. $p(r) = r^2 - 2 = 0$

$$\Rightarrow r = \pm\sqrt{2}.$$

$$\therefore C.F = Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x}$$

$$P.I = \frac{1}{D^2-2}(-9\cos t)$$

$D^2 = -2$,

$$= -9 \frac{1}{D^2-2}\cos t$$

$$= -9.\left(\frac{1}{-1-2}\cos t\right)$$

$$= 3\cos t.$$

$$\therefore \; x(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} \; 3\cos t$$

Substitute in (2), we get

$$y'(t) = (2+\sqrt{2}) A e^{\sqrt{2}t} + (2-\sqrt{2}) B e^{-\sqrt{2}t} + 2\cos t$$

$$y(t) = (\sqrt{2}+1) A e^{\sqrt{2}t} - (\sqrt{2}-1) B e^{-\sqrt{2}t} + 2\sin t + C.$$

System of first order linear DE.

Consider the system

$$(*)\quad \begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)\,x_1 + a_{12}(t)\,x_2 + \dots + a_{1n}(t)\,x_n + f_1(t) \\ \quad\vdots \\ \dfrac{dx_n}{dt} = a_{n1}(t)\,x_1 + a_{n2}(t)\,x_2 + \dots + a_{nn}(t)\,x_n + f_n(t) \end{cases}$$

In matrix form

$$X' = AX + F. \longrightarrow (*)$$

where $X = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$; $A = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & \dots & a_{2n}(t) \\ & \vdots & \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}_{n\times n}$

$$F = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

$F = 0$, then $(*)$ is homogeneous.

Theorem (Existence & uniqueness)

Consider the IVP

$$x'(t) = A(t)\,x(t) + F(t) \;;\; x(t_0) = x_0. \longrightarrow (1)$$

Assume that $A(t)$ and $F(t)$ are continuous on a interval I containing t_0. Then (1) has unique soln in I.

~~Wronskian~~

Theorem

$$= \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$$

Let $x_1, x_2, \ldots x_n$ be n soln vectors of the homogeneous system

$$x'(t) = A(t)\,x(t).$$

In the interval I. Then $\{x_1, \ldots x_n\}$ is l. independent on I iff

$$W(x_1, x_2, \ldots x_n)(t) = \begin{vmatrix} x_{11}(t) & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ x_{31} & x_{32} & \cdots & x_{3n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \neq 0, \; \forall t \in I$$

Example

$$x_1 = \begin{pmatrix} \cos t \\ -\tfrac{1}{2}\cos t + \tfrac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix} ; \quad x_2 = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

$$x_3 = \begin{pmatrix} \sin t \\ -\tfrac{1}{2}\sin t - \tfrac{1}{2}\cos t \\ -\sin t + \cos t \end{pmatrix} ; \quad W(x_1, x_2, x_3) = e^t \neq 0.$$

Theorem (Homogeneous linear system with constant coefficients)

Consider the system

$$(*) \leftarrow x' = AX \quad , \quad A \text{ is an } n\times n \text{ constant matrix}$$

If $\lambda_1, \lambda_2, \ldots \lambda_n$ are 'n' distinct real eigenvalues of A, and $K_1, K_2, \ldots K_n$ are the corresponding eigenvectors, then the general soln of ① on $(-\infty, \infty)$ is given by

$$X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + \ldots + C_n K_n e^{\lambda_n t}.$$

Example

Solve

$$\frac{dx}{dt} = -4x + y + z$$

$$\frac{dy}{dt} = x + 5y - z$$

$$\frac{dz}{dt} = y - 3z.$$

Soln

$$x' = A\cancel{X}x, \quad \text{where} \quad A = \begin{pmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{pmatrix}$$

The eigenvalues of A are $-3, -4, 5$.

For $\lambda_1 = -3$,

$$(A|b) \simeq \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 8 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \quad \begin{matrix} x = \\ y = \\ z = \end{matrix} \begin{pmatrix} 1 \\ 0 \\ +1 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ; \quad X_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-3t}$$

$\lambda_2 = -4$

$$K_2 = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} e^{-4t}$$

$\lambda_3 = 5$

$$K_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} e^{5t}$$

$\therefore$ The g. soln is

$$X(t) = Ae^{-3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + Be^{-4t}\begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} + Ce^{5t}\begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}$$

Theorem

Consider the system $X' = AX$ ①, A is an $n \times n$ diagonalizable matrix, whose eigenvalues are real.

Let $\lambda_1, \lambda_2, \ldots \lambda_k$ be the all distinct real eigenvalues of A of multiplicity $m_1, \ldots m_k$ respectively.

Let $K_{1j}, K_{2j}, \ldots K_{m_j j}$ be the m_k-linearly independent eigenvectors corresponding to λ_j, $j = 1, 2, \ldots k$. Then the g. soln is ① is given by

$X(t) =$ $\cancel{c_1 K e^{\lambda_1}}$ $c_{11} K_{11} e^{\lambda_1 t} + \cancel{c_{21}} K$.

$$x(t) = C_{11}K_{11}e^{\lambda_1 t} + C_{12}K_{12}e^{\lambda_1 t} + \dots + C_{1,m_1}K_{1,m_1}e^{\lambda_1 t}$$
$$+$$
$$C_{21}K_{21}e^{\lambda_2 t} + C_{22}K_{22}e^{\lambda_2 t} + \dots + C_{2,m_2}K_{2,m_2}e^{\lambda_2 t}$$
$$+$$
$$\vdots$$
$$C_{k_1}K_{k_1}e^{\lambda_k t} + C_{k_2}K_{k_2}e^{\lambda_k t} + \dots + C_{k,m_k}K_{2,m_k}e^{\lambda_k t}$$

Example

Solve $\quad x' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} X.$

Soln

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = -1$ (twice)

$\lambda_2 = 5.$

For $\lambda_1 = -1$,

$$(A+I \mid 0) = \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x - y + z = 0$

$\Rightarrow x = y - z.$

Choose $k_2 = 1,\ k_3 = 0$

$$K_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad X_1 = e^{-t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Choose: $k_2 = 1,\ k_3 = 1$

$$K_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad X_2 = e^{-t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

at $\lambda = 5$

$$(A - 5I \mid 0) = \left(\begin{array}{ccc|c} -4 & -2 & 2 & 0 \\ -2 & 4 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{aligned} y &= -z \\ x &= z \end{aligned}$$

Choose $z = 1,\quad x = 1,\quad y = -1$

$$K_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; \quad X_3 = e^{5t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore X(t) = A e^{-t}\begin{pmatrix}1\\1\\0\end{pmatrix} + B e^{-t}\begin{pmatrix}0\\1\\1\end{pmatrix} + C e^{5t}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Eigenvalue of multiplicity ~~two~~

Special case: For 3x3 matrix (or) 2x2 system

Let λ_1 be an eigenvalue of multiplicity two and there is only one l.independent eigenvector. Then the second soln can be found of the form

eigenvector — g.e.vect of order 1

$$X_2 = K t e^{\lambda_1 t} + P e^{\lambda_1 t}$$

$$(AK - \lambda_1 K) t e^{\lambda_1 t} + (AP - \lambda_1 P - K) e^{\lambda_1 t} = 0$$

$$\Rightarrow (A - \lambda_1 I) K = 0$$

$$\Rightarrow (A - \lambda_1 I) P = K$$

Example $X' = \begin{pmatrix}3 & -18\\ 2 & -9\end{pmatrix} X$

$\lambda = -3$ (twice) $\qquad \lambda = -3$

$$X_1 = \begin{pmatrix}3\\1\end{pmatrix} e^{-t}$$

$$(A + 3I) P = \begin{pmatrix}3\\1\end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix}1/2\\0\end{pmatrix}$$

$$X(t) = C_1 X_1(t) + C_2 X(t)$$

rank$(A - \lambda I) = 1$

$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$2x_1 - 6x_2 = 0$

$x_2 = 1$

$\begin{pmatrix}3 & -8\\2 & -9\end{pmatrix}$

$6P_1 - 18P_2 = 3$

$2P_1 - 6P_2 = 1$

$\Rightarrow$

$6P_1 = 18P_2 + 3$

$= 3$

$6P_1 = 3$

$P_1 = 1/2$

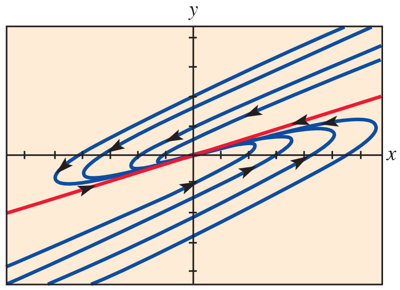

FIGURE 10.2.3 A phase portrait of system (10)

By assigning various values to c_1 and c_2 in the solution in Example 4, we can plot trajectories of the system in (10). A phase portrait of (10) is given in **FIGURE 10.2.3**. The solutions $\mathbf{X}_1$ and $-\mathbf{X}_1$ determine two half-lines $y = \frac{1}{3}x$, $x > 0$, and $y = \frac{1}{3}x$, $x < 0$, respectively, that are shown in red in Figure 10.2.3. Because the single eigenvalue is negative and $e^{-3t} \to 0$ as $t \to \infty$ on *every* trajectory, we have $(x(t), y(t)) \to (0, 0)$ as $t \to \infty$. This is why the arrowheads in Figure 10.2.3 indicate that a particle on any trajectory would move toward the origin as time increases and why the origin is an attractor in this case. Moreover, a moving particle on a trajectory $x = 3c_1e^{-3t} + c_2(te^{-3t} + \frac{1}{2}e^{-3t})$, $y = c_1e^{-3t} + c_2te^{-3t}$, $c_2 \neq 0$, approaches $(0, 0)$ tangentially to one of the half-lines as $t \to \infty$. In contrast, when the repeated eigenvalue is positive the situation is reversed and the origin is a repeller. See Problem 23 in Exercises 10.2. Analogous to Figure 10.2.2, Figure 10.2.3 is typical of all 2×2 homogeneous linear systems $\mathbf{X}' = \mathbf{AX}$ that have repeated negative eigenvalues. See Problem 34 in Exercises 10.2.

Eigenvalue of Multiplicity Three

When the coefficient matrix $\mathbf{A}$ has only one eigenvector associated with an eigenvalue λ_1 of multiplicity three, we can find a solution of the form (12) and a third solution of the form

$$\mathbf{X}_3 = \mathbf{K}\frac{t^2}{2}e^{\lambda_1 t} + \mathbf{P}te^{\lambda_1 t} + \mathbf{Q}e^{\lambda_1 t}, \tag{15}$$

where
$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}, \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}.$$

By substituting (15) into the system $\mathbf{X}' = \mathbf{AX}$, we find that the column vectors $\mathbf{K}$, $\mathbf{P}$, and $\mathbf{Q}$ must satisfy

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{K} = \mathbf{0} \tag{16}$$

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K} \tag{17}$$

and
$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{Q} = \mathbf{P}. \tag{18}$$

Of course, the solutions of (16) and (17) can be used in forming the solutions $\mathbf{X}_1$ and $\mathbf{X}_2$.

EXAMPLE 5 Repeated Eigenvalues

Solve $\mathbf{X}' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{X}$.

SOLUTION The characteristic equation $(\lambda - 2)^3 = 0$ shows that $\lambda_1 = 2$ is an eigenvalue of multiplicity three. By solving $(\mathbf{A} - 2\mathbf{I})\mathbf{K} = \mathbf{0}$ we find the single eigenvector

$$\mathbf{K} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We next solve the systems $(\mathbf{A} - 2\mathbf{I})\mathbf{P} = \mathbf{K}$ and $(\mathbf{A} - 2\mathbf{I})\mathbf{Q} = \mathbf{P}$ in succession and find that

$$\mathbf{P} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix}.$$

Using (12) and (15), we see that the general solution of the system is

$$\mathbf{X} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^{2t}\right] + c_3\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\frac{t^2}{2}e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix}e^{2t}\right]. \equiv$$

REMARKS

When an eigenvalue λ_1 has multiplicity m, then we can either find m linearly independent eigenvectors or the number of corresponding eigenvectors is less than m. Hence the two cases listed on pages 602 and 603 are not all the possibilities under which a repeated eigenvalue can occur. It could happen, say, that a 5×5 matrix has an eigenvalue of multiplicity 5 and there exist three corresponding linearly independent eigenvectors. See Problems 33 and 53 in Exercises 10.2.

10.2.3 Complex Eigenvalues

If $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, $\beta > 0$, $i^2 = -1$, are complex eigenvalues of the coefficient matrix $\mathbf{A}$, we can then certainly expect their corresponding eigenvectors to also have complex entries.*

For example, the characteristic equation of the system

$$\begin{aligned}\frac{dx}{dt} &= 6x - y\\ \frac{dy}{dt} &= 5x + 4y\end{aligned} \tag{19}$$

is $$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0.$$

From the quadratic formula we find $\lambda_1 = 5 + 2i$, $\lambda_2 = 5 - 2i$.

Now for $\lambda_1 = 5 + 2i$ we must solve

$$\begin{aligned}(1-2i)k_1 - \quad\quad k_2 &= 0\\ 5k_1 - (1+2i)k_2 &= 0.\end{aligned}$$

Since $k_2 = (1 - 2i)k_1$,[†] the choice $k_1 = 1$ gives the following eigenvector and a solution vector:

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}, \quad \mathbf{X}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t}.$$

In like manner, for $\lambda_2 = 5 - 2i$ we find

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(5-2i)t}.$$

We can verify by means of the Wronskian that these solution vectors are linearly independent, and so the general solution of (19) is

$$\mathbf{X} = c_1\begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t} + c_2\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(5-2i)t}. \tag{20}$$

Note that the entries in $\mathbf{K}_2$ corresponding to λ_2 are the conjugates of the entries in $\mathbf{K}_1$ corresponding to λ_1. The conjugate of λ_1 is, of course, λ_2. We write this as $\lambda_2 = \bar{\lambda}_1$ and $\mathbf{K}_2 = \bar{\mathbf{K}}_1$. We have illustrated the following general result.

Theorem 10.2.2 Solutions Corresponding to a Complex Eigenvalue

Let $\mathbf{A}$ be the coefficient matrix having real entries of the homogeneous system (2), and let $\mathbf{K}_1$ be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$, α and β real. Then

$$\mathbf{K}_1 e^{\lambda_1 t} \quad \text{and} \quad \bar{\mathbf{K}}_1 e^{\bar{\lambda}_1 t}$$

are solutions of (2).

*When the characteristic equation has real coefficients, complex eigenvalues always appear in conjugate pairs.

[†]Note that the second equation in the system is simply $1 + 2i$ times the first equation.

It is desirable and relatively easy to rewrite a solution such as (20) in terms of real functions. To this end we first use Euler's formula to write

$$e^{(5+2i)t} = e^{5t}e^{2ti} = e^{5t}(\cos 2t + i \sin 2t)$$

$$e^{(5-2i)t} = e^{5t}e^{-2ti} = e^{5t}(\cos 2t - i \sin 2t).$$

Then, after we multiply complex numbers, collect terms, and replace $c_1 + c_2$ by C_1 and $(c_1 - c_2)i$ by C_2, (20) becomes

$$\mathbf{X} = C_1\mathbf{X}_1 + C_2\mathbf{X}_2, \tag{21}$$

where $$\mathbf{X}_1 = \left[\begin{pmatrix}1\\1\end{pmatrix}\cos 2t - \begin{pmatrix}0\\-2\end{pmatrix}\sin 2t\right]e^{5t} = \begin{pmatrix}\cos 2t\\ \cos 2t + 2\sin 2t\end{pmatrix}e^{5t}$$

and $$\mathbf{X}_2 = \left[\begin{pmatrix}0\\-2\end{pmatrix}\cos 2t + \begin{pmatrix}1\\1\end{pmatrix}\sin 2t\right]e^{5t} = \begin{pmatrix}\sin 2t\\ -2\cos 2t + \sin 2t\end{pmatrix}e^{5t}.$$

It is now important to realize that the two vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ in (21) are themselves linearly independent *real* solutions of the original system. Consequently, we are justified in ignoring the relationship between C_1, C_2 and c_1, c_2, and we can regard C_1 and C_2 as completely arbitrary and real. In other words, the linear combination (21) is an alternative general solution of (19).

The foregoing process can be generalized. Let $\mathbf{K}_1$ be an eigenvector of the coefficient matrix $\mathbf{A}$ (with real entries) corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$. Then the two solution vectors in Theorem 10.2.2 can be written as

$$\mathbf{K}_1e^{\lambda_1 t} = \mathbf{K}_1e^{\alpha t}e^{i\beta t} = \mathbf{K}_1e^{\alpha t}(\cos \beta t + i \sin \beta t)$$

$$\overline{\mathbf{K}}_1e^{\overline{\lambda}_1 t} = \overline{\mathbf{K}}_1e^{\alpha t}e^{-i\beta t} = \overline{\mathbf{K}}_1e^{\alpha t}(\cos \beta t - i \sin \beta t).$$

By the superposition principle, Theorem 10.1.2, the following vectors are also solutions:

$$\mathbf{X}_1 = \frac{1}{2}(\mathbf{K}_1e^{\lambda_1 t} + \overline{\mathbf{K}}_1e^{\overline{\lambda}_1 t}) = \frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1)\,e^{\alpha t}\cos \beta t - \frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1)e^{\alpha t}\sin \beta t$$

$$\mathbf{X}_2 = \frac{i}{2}(-\mathbf{K}_1e^{\lambda_1 t} + \overline{\mathbf{K}}_1e^{\overline{\lambda}_1 t}) = \frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1)e^{\alpha t}\cos \beta t + \frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1)e^{\alpha t}\sin \beta t.$$

For *any* complex number $z = a + ib$, both $\frac{1}{2}(z + \bar{z}) = a$ and $\frac{i}{2}(-z + \bar{z}) = b$ are *real* numbers. Therefore, the entries in the column vectors $\frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1)$ and $\frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1)$ are real numbers. By defining

$$\mathbf{B}_1 = \frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1) \quad \text{and} \quad \mathbf{B}_2 = \frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1), \tag{22}$$

we are led to the following theorem.

Theorem 10.2.3 Real Solutions Corresponding to a Complex Eigenvalue

Let $\lambda_1 = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix $\mathbf{A}$ in the homogeneous system (2), and let $\mathbf{B}_1$ and $\mathbf{B}_2$ denote the column vectors defined in (22). Then

$$\begin{aligned} \mathbf{X}_1 &= [\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t]e^{\alpha t} \\ \mathbf{X}_2 &= [\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t]e^{\alpha t} \end{aligned} \tag{23}$$

are linearly independent solutions of (2) on $(-\infty, \infty)$.

The matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ in (22) are often denoted by

$$\mathbf{B}_1 = \text{Re}(\mathbf{K}_1) \quad \text{and} \quad \mathbf{B}_2 = \text{Im}(\mathbf{K}_1) \tag{24}$$

since these vectors are, respectively, the *real* and *imaginary* parts of the eigenvector $\mathbf{K}_1$. For example, (21) follows from (23) with

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\mathbf{B}_1 = \text{Re}(\mathbf{K}_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_2 = \text{Im}(\mathbf{K}_1) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

EXAMPLE 6 Complex Eigenvalues

Solve the initial-value problem

$$\mathbf{X}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}\mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{25}$$

SOLUTION First we obtain the eigenvalues from

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2 - \lambda & 8 \\ -1 & -2 - \lambda \end{vmatrix} = \lambda^2 + 4 = 0.$$

The eigenvalues are $\lambda_1 = 2i$ and $\lambda_2 = \overline{\lambda_1} = -2i$. For λ_1 the system

$$\begin{aligned} (2 - 2i)k_1 + \qquad\quad 8k_2 &= 0 \\ -k_1 + (-2 - 2i)k_2 &= 0 \end{aligned}$$

gives $k_1 = -(2 + 2i)k_2$. By choosing $k_2 = -1$ we get

$$\mathbf{K}_1 = \begin{pmatrix} 2 + 2i \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Now from (24) we form

$$\mathbf{B}_1 = \text{Re}(\mathbf{K}_1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_2 = \text{Im}(\mathbf{K}_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Since $\alpha = 0$, it follows from (23) that the general solution of the system is

$$\begin{aligned} \mathbf{X} &= c_1\left[\begin{pmatrix} 2 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin 2t\right] + c_2\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\sin 2t\right] \\ &= c_1\begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix} + c_2\begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}. \end{aligned} \tag{26}$$

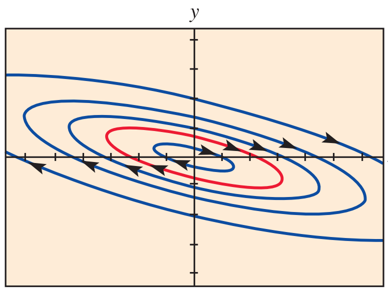

FIGURE 10.2.4 A phase portrait of system (25) in Example 6

Some graphs of the curves or trajectories defined by the solution (26) of the system are illustrated in the phase portrait in **FIGURE 10.2.4**. Now the initial condition $\mathbf{X}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, or equivalently $x(0) = 2$, and $y(0) = -1$, yields the algebraic system $2c_1 + 2c_2 = 2, -c_1 = -1$ whose solution is $c_1 = 1, c_2 = 0$. Thus the solution to the problem is $\mathbf{X} = \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix}$. The specific trajectory defined parametrically by the particular solution $x = 2\cos 2t - 2\sin 2t, y = -\cos 2t$ is the red curve in Figure 10.2.4. Note that this curve passes through $(2, -1)$. ≡