Mathematics - II

System of linear equations

Terminology:

$\mathbb{N}$ - The set of all natural numbers

$\mathbb{Z}$ - The set of all integers

$\mathbb{Q}$ - The set of all rational numbers

$\mathbb{Q}^c$ - The set of all irrational numbers

$\mathbb{R}$ - The set of all real numbers

$\mathbb{C}$ - The set of all complex numbers

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

(The elements of $\mathbb{K}$ are called scalars).

$\mathbb{R}^n$ - The n-dimensional Euclidean space.

$\mathbb{C}^n$ - The n-dimensional Unitary space

Problem. To find the 'n' scalars $x_1, x_2, \ldots x_n$ which satisfy the conditions

$$\left.\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= y_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= y_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= y_m \end{aligned}\right\} \longrightarrow (*)$$

where $y_1, y_2, \ldots y_m$ and A_{ij}, $1 \le i \le m$, $1 \le j \le n$ are scalars.

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We call (*) the system of 'm' linear equations with 'n' unknowns.

Any n-tuple $(x_1, x_2, \ldots x_n) \in \mathbb{K}^n$ which satisfies (*) is called a solution of the system.

If $y_1 = y_2 = \ldots = y_m = 0$, we say that the system is homogeneous otherwise it is called non-homogeneous.

<u>Example</u> (Elimination method)

Solve

$$\left.\begin{aligned} 2x_1 - x_2 + x_3 &= 0 \\ x_1 + 3x_2 + 4x_3 &= 0 \end{aligned}\right\} \rightarrow (*)$$

<u>Soln.</u> Multiply the second equation in (*) by 2. Then

$$\begin{array}{rl} & 2x_1 - x_2 + x_3 = 0 \\ (-) & 2x_1 + 6x_2 + 8x_3 = 0 \\ \hline & 0 - 7x_2 - 7x_3 = 0 \end{array}$$

$$\Rightarrow x_2 = -x_3$$

Multiply the first eqn in (*) by 3. Then

$$\begin{array}{rl} & 6x_1 - 3x_2 + 3x_3 = 0 \\ (+) & x_1 + 3x_2 + 4x_3 = 0 \\ \hline & 7x_1 + 7x_3 = 0 \end{array}$$

$$\Rightarrow x_1 = -x_3.$$

$\therefore$ If (x_1, x_2, x_3) is a soln of (*), then

$$x_1 = x_2 = -x_3.$$

$\therefore$ The solution set of (*) is

$$\{(x_1, x_2, x_3) \in \mathbb{K}^3 : x_1 = x_2 = -x_3\} = \{(a, a, -a) : a \in \mathbb{K}\}.$$

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Matrices and elementary row operations.

We can write the system (*) symbolically by

$$AX = Y.$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ & & \vdots & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} ; \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} ; \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

A is called the matrix of the coefficients.

Definition (of matrix)

An $m \times n$ matrix over $\mathbb{K}$ is a function A from the set of pair of integers $1 \le i \le m$, $1 \le j \le n$ into $\mathbb{K}$, i.e.,

$$A : \{(i,j) : 1 \le i \le m, 1 \le j \le n\} \longrightarrow \mathbb{K}.$$

The entries of the matrix A are the scalars $A(i,j) = A_{ij}$.

The set of all $m \times n$ matrices over $\mathbb{K}$ is defined by $M_{m \times n}(\mathbb{K})$.

If $m = n$, we will write simply $M_n(\mathbb{K})$ or $\mathbb{K}^{m \times n}$.

We now consider the three elementary row operations on $m \times n$ matrix A over $\mathbb{K}$:

(i) multiplication of one row of A by a non-zero scalars (scale)

(ii) replacement of the rth row of A by row r plus c times row s, where $c \in \mathbb{K}$ and $r \neq s$. (Replacement)

(iii) interchange of two rows of A (swap)

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Matrix addition

Let A, B be two $m \times n$ matrices over $\mathbb{K}$. Then the addition $A+B$ is the $m \times n$ matrix C whose $(i,j)^{th}$ entry is

$$C_{ij} = (A+B)_{ij} = A_{ij} + B_{ij}, \quad 1 \le i \le m, \; 1 \le j \le n.$$

- An $n \times n$ matrix A is called identity matrix if

$$A_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

The $n \times n$ identity matrix is denoted by I_n (or simply I).

- An $m \times n$ matrix A is called zero matrix if

$$A_{ij} = 0 \quad \text{for all } 1 \le i \le m, \; 1 \le j \le n.$$

The $m \times n$ zero matrix is denoted by $0_{m \times n}$ (or simply 0)

Properties (of addition)

(i) $A+B = B+A$ (Commutative)

(ii) $(A+B)+C = A+(B+C)$ (Associative)

(iii) $A+0 = 0+A = A$.

(iv) For each $m \times n$ matrix A, there exists a unique matrix B such that

$$A+B = B+A = 0$$

The matrix B is called the additive inverse of A and it is denoted by $-A$.

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Matrix multiplication.

Let A be an $m \times n$ matrix over $\mathbb{K}$ and B be an $n \times p$ matrix over $\mathbb{K}$. Then the product AB is the $m \times p$ matrix C over $\mathbb{K}$ whose $(i,j)^{th}$ entry is

$$(AB)_{ij} = C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}, \quad \begin{array}{l} 1 \le i \le m \\ 1 \le j \le p. \end{array}$$

Properties. (of multiplication)

(i) $(AB)C = A(BC)$

(ii) $A(B+C) = AB + AC$

(iii) $(A+B)C = AC + BC$

In general, $AB \neq BA$ even if $m = n = p$. For example

if $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then

$$BC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } CB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore BC \neq CA.$

Scalar multiplication.

Let A be an $m \times n$ matrix and c be a scalar. Then the scalar multiplication cA is defined as

$$(cA)_{ij} = c A_{ij}, \quad \begin{array}{l} 1 \le i \le m \\ 1 \le j \le n \end{array}$$

Property. $c(AB) = (cA)B = A(cB)$, where $c \in \mathbb{K}$

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Invertible matrices.

Let A be a $m \times n$ matrix over $\mathbb{K}$. Then A is said to have left (right) inverse if there exists an $n \times m$ matrix B (C) over $\mathbb{K}$ such that

$$BA = I_{n \times n}$$
$$(AC = I_{m \times m})$$

The matrix B (C) is called a left (right) inverse of A

Remark.

(i) In general, left (or right) inverse may not exist.

(ii) If a matrix has a left inverse, it is not necessarily unique. For example,

take $A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Choose $B = (1 \ 0)$ and $C = (2 \ 1)$.

Then $BA = (1 \ 0)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1) = I_1$

$CA = (2, 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1) = I_1$.

Definition An $n \times n$ matrix A is said to be invertible if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

The matrix B is unique and it is denoted by A^{-1}.

Theorem. A matrix A is invertible iff it has left and right inverse.

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Definition. (Transpose of a matrix)

Let A be an $m \times n$ matrix over $\mathbb{K}$. Then the transpose of A is the $n \times m$ matrix A^t defined by

$$(A^t)_{ij} = A_{ji}, \quad 1 \le i \le n, \quad 1 \le j \le m.$$

Properties.

(i) $(A+B)^t = A^t + B^t$.

(ii) $(AB)^t = B^t A^t$.

(iii) $(A^t)^t = A$

(iv) $(cA)^t = cA^t$, $c \in \mathbb{K}$

(v) $(A^t)^{-1} = (A^{-1})^t$ if A is invertible

Definition (Adjoint of a matrix)

Let A be an $m \times n$ matrix over $\mathbb{K}$. Then the adjoint of A is the $n \times m$ matrix A^* defined by

$$(A^*)_{ij} = \overline{A_{ji}}, \quad 1 \le i \le n, \quad 1 \le j \le m.$$

Properties.

(i) $A^* = \overline{(A^t)} = (\overline{A})^t$ (where $(\overline{A})_{ij} = \overline{A_{ij}}$)

(ii) $(A+B)^* = A^* + B^*$

(iii) $(AB)^* = B^* A^*$

(iv) $(A^*)^* = A$

(v) $(cA)^* = \bar{c} A^*$, $c \in \mathbb{K}$.

(vi) $(A^{-1})^* = (A^*)^{-1}$.

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Row-equivalent matrices.

Definition.

Let A and B be two $m \times n$ matrices over $\mathbb{K}$. We say that A is row-equivalent to B if B can be obtained from A by a finite number of elementary row operations.

Theorem (without proof).

If A is row-equivalent to B, then the homogeneous systems of linear equations

$$Ax = 0 \quad \text{and} \quad Bx = 0$$

have exactly the same solutions.

Definition (Row-reduced matrix)

An $m \times n$ matrix R is said to be row-reduced if

(i) the first non-zero entry in each non-zero row of R is equal to one.

(ii) each column of R which contains the leading non-zero entry of some row has all its other entries '0'.

Definition. (Row-echelon matrix).

An $m \times n$ matrix R is said to be row-echelon if

(i) all the zero rows should appear bottom of the matrix.

(ii) if rows $1, 2, \ldots r$ are non-zero rows of R and if the leading non-zero entry of row i occurs in column k_i, $i = 1, 2, \ldots r$ then $k_1 < k_2 < \ldots < k_r$.

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Definition

If the matrix R is row-echelon, then the first non-zero entry of each rows is called a pivot and the column in which pivot appears are called pivot columns.

Definition (Row-reduced echelon matrix)

An $m \times n$ matrix R is said to be row-reduced echelon matrix if

(i) it is row-reduced,

(ii) it is row-echelon.

Examples.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ — row-reduced echelon matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$ — not row-reduced matrix

$$C = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$ — not row-reduced

$$D = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ — not row-reduced but row-echelon

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Theorem

i) Every $m \times n$ matrix A is row-equivalent to a row-reduced matrix.

ii) Every $m \times n$ matrix A is row-equivalent to a row-echelon matrix. (Gauss elimination)

(iii) Every $m \times n$ matrix A is row-equivalent to a <u>unique</u> row-reduced echelon matrix. (Gauss-Jordan elimination)

Definition (Rank of a matrix)

Let A be an $m \times n$ matrix over $\mathbb{K}$. Then the rank of the matrix A is defined as

$$\text{rank}(A) := \text{the number of non-zero rows in the row-reduced echelon matrix } R \text{ of } A$$

$$\parallel$$

$$\text{the no. of pivots of } R.$$

$$\parallel$$

$$\text{the no. of pivot columns of } R.$$

Properties.

Let A be an $m \times n$ matrix over $\mathbb{K}$. Then

(i) $0 \leq \text{rank}(A) \leq \min\{m, n\}$.

(ii) $\text{rank}(A) = \text{rank}(A^t) = \text{rank}(AA^t) = \text{rank}(A^tA)$
$= \text{rank}(\bar{A}) = \text{rank}(A^*) = \text{rank}(AA^*) = \text{rank}(A^*A)$.

(iii) $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

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(iv) $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$

(v) <u>Sylvester inequality</u>:

$$\text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - n,$$

where n is the number of columns of A.

<u>Theorem</u>

(i) An $m \times n$ matrix A has right inverse iff $\text{rank}(A) = m$

(ii) An $m \times n$ matrix A has left inverse iff $\text{rank}(A) = n$.

(iii) An $n \times n$ matrix A is invertible iff $\text{rank}(A) = n$ iff A is row-equivalent to identity matrix

<u>Theorem.</u> (To find the inverse of a matrix).

If A is an invertible $n \times n$ matrix and if a sequence of elementary row operations reduces A to identity I, then the same sequence elementary row operations when applied to I yields A^{-1}.

<u>Example</u>

Find the row-reduced echelon matrix of the matrix

$$A = \begin{pmatrix} 2 & -2 & 4 & -2 \\ 2 & 1 & 10 & 7 \\ -4 & 4 & -8 & 4 \\ 4 & -1 & 14 & 6 \end{pmatrix}.$$ Find $\text{rank}(A)$.

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Soln

$$A \sim \begin{pmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & 10 & 7 \\ -4 & 4 & -8 & 4 \\ 4 & -1 & 14 & 6 \end{pmatrix} \qquad R_1 \to \frac{1}{2}R_1$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 10 \end{pmatrix} \qquad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 4R_1 \\ R_4 \to R_4 - 4R_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 3 & 6 & 9 \\ 0 & 3 & 6 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_3 \leftrightarrow R_4$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 6 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_2 \to \frac{1}{2}R_2.$$

$$\sim \begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 - 3R_2. \end{array}$$

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$$\sim \begin{pmatrix} \underline{1} & 0 & 4 & 0 \\ 0 & \underline{1} & 2 & 0 \\ 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 - 3R_3. \end{array}$$

$\hookrightarrow$ row-reduced echelon matrix

$\therefore \text{rank}(A) = 3.$

<u>Example.</u>

Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 \\ \frac{1}{2} & 1/3 & 1/4 \\ \frac{1}{3} & 1/4 & 1/5 \end{pmatrix}$$ if it exists.

<u>Soln</u>

$$(A; I) = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} ; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & \frac{1}{12} & 4/45 \end{pmatrix} ; \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

$R_2 \to R_2 - \frac{1}{2}R_1$

$R_3 \to R_3 - \frac{1}{3}R_1$

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$$\sim \begin{pmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 1/12 & 4/45 \end{pmatrix} : \begin{pmatrix} 1 & 0 & 0 \\ -6 & 12 & 0 \\ -1/3 & 0 & 1 \end{pmatrix} \qquad R_2 \to 12R_2.$$

$$\sim \begin{pmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 1 \\ 0 & 0 & 1/180 \end{pmatrix} : \begin{pmatrix} 4 & -6 & 0 \\ -6 & 12 & 0 \\ 1/6 & -1 & 1 \end{pmatrix} \qquad \begin{matrix} R_1 \to R_1 - \frac{1}{2}R_2 \\ R_3 \to R_3 - \frac{1}{12}R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 4 & -6 & 0 \\ -6 & 12 & 0 \\ 30 & -180 & 180 \end{pmatrix} \qquad R_3 \to \frac{180}{1} R_3.$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \qquad \begin{matrix} R_1 \to R_1 + \frac{1}{6}R_3 \\ R_2 \to R_2 - R_3. \end{matrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix}.$$

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Solution of the linear system

Consider the linear system

① $\leftarrow AX = Y$,

where A is an $m \times n$ matrix, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$.

We now form the augmented matrix A' of the system ①.

It is an $m \times (n+1)$ matrix whose first n columns are the columns of A and whose last column is Y. It is denoted by $(A|Y)$.

Definition

The linear system $AX = Y$ is said to be consistent if it has a solution. otherwise the system is inconsistent.

Theorem

Consider the linear system $AX = Y$. $\rightarrow (*)$ Then

(i) $(*)$ is consistent iff $\text{rank}(A) = \text{rank}(A|Y)$.

(ii) $(*)$ is inconsistent iff $\text{rank}(A) < \text{rank}(A|Y)$.

In the first case,

(i) $r(A) = r(A|Y) =$ the number of unknowns iff $(*)$ has a unique solution.

(ii) $r(A) = r(A|Y) <$ the number of unknowns iff $(*)$ has infinitely many solutions.

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Example.

Solve the system

$$\left.\begin{aligned} 2x+2y+2z &= 4 \\ x+y+2z &= 4 \\ x+y+2z &= 4 \end{aligned}\right\} \rightarrow (*)$$

Soln

$$(A|B) = \begin{pmatrix} 2 & 2 & 2 & 4 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 2 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 2 & 4 \end{pmatrix} \qquad R_1 \rightarrow \tfrac{1}{2} R_1$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad \begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} R_1 \rightarrow R_1 - R_2 \\ R_3 \rightarrow R_3 - R_2 \end{matrix}$$

$\therefore \quad x+y=0 \Rightarrow x=-y$

$z=2$

$\therefore$ The solution set of $(*)$ is

$$\{(x, -x, 2) : x \in \mathbb{K}\}.$$

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