Equations reducible to separable form

Consider the first order ODE of the form

$$y' = f(ax+by+c), \quad b \neq 0 \longrightarrow ①$$

Let $t = ax+by+c$. Then Differentiating w.r.to x,

$$\frac{dt}{dx} = a + b\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$$

Now ① becomes

$$\frac{1}{b}\left(\frac{dt}{dx} - a\right) = f(t)$$

$$\Rightarrow \frac{dt}{dx} - a = bf(t)$$

$$\Rightarrow \frac{dt}{dx} = a + bf(t)$$

$$\Rightarrow \frac{dt}{a+bf(t)} = dx$$

Integrating, we obtain

$$\boxed{\int \frac{dt}{a+bf(t)} = x + c.} \quad \text{with } t = ax+by+c$$

Homogeneous first order ODE.

A function $f(x,y)$ is said to be homogeneous of degree n in a region Ω if for every $\lambda > 0$

$$f(\lambda x, \lambda y) = \lambda^n f(x,y), \quad \forall \ (x,y) \in \Omega.$$

Consider the ODE

$$P(x,y)\,dx + Q(x,y)\,dy = 0, \longrightarrow (1)$$

where P and Q are homogeneous polynomials of same degree.

We rewrite (1) as

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} =: \phi(x,y) \longrightarrow (2)$$

Notice that $\phi(x,y)$ is homogeneous of degree 0. ie,

$$\phi(\lambda x, \lambda y) = \phi(x,y)$$

Take $\lambda = 1/x$. Then

$$\phi(x,y) = \phi(1, y/x) \longrightarrow (3)$$

The above eqn (3) suggests making the substitution

$$y/x = v \quad \text{or} \quad y = vx$$

If $y = vx$, then

$$\frac{dy}{dx} = v + x.\frac{dv}{dx}$$

Subsitituting this value in (2), we get

$$v + x\frac{dv}{dx} = \phi(1,v)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \phi(1,v) - v$$

$$\Rightarrow \quad \frac{dv}{\phi(1,v)-v} = \frac{1}{x}\,dx$$

Integrating, $$\boxed{\int \frac{dv}{\phi(1,v)-v} = \ln|x| + C}$$ with $y = vx$

Example.

Solve $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

Soln

$$y^2 + (x^2 - xy)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2}{x^2-xy}$$

Take

$$\phi(x,y) = -\frac{y^2}{x^2-xy}.$$

Then ϕ is a homogeneous function of degree 0.

$$\phi(1,v) = -\frac{v^2}{1-v}$$

$$\phi(1,v) - v = \frac{-v^2}{1-v} - v$$

$$= \frac{-\cancel{v^2} - v + \cancel{v^2}}{1-v}$$

$$= \frac{v}{v-1}$$

Now

$$\int \frac{dv}{\phi(1,v)-v} = \ln|x| + c$$

$$\Rightarrow \cancel{\int \frac{1-v}{v^2}\,dv = \ln|x| + c}$$

$$\Rightarrow \int \frac{v-1}{v}\,dv = \ln|x| + c \quad , x \neq 0$$

$$\Rightarrow \int \left(1 - \frac{1}{v}\right)dv = \ln|x| + c$$

$$\Rightarrow v - \ln|v| = \ln|x| + c$$

$$\Rightarrow v = \ln|xv| + c$$

$$\Rightarrow \frac{y}{x} = \ln|y| + c$$

$$\Rightarrow \boxed{\ln|y| - \frac{y}{x} = c}$$

Exact ODE

An expression $P(x,y)\,dx + Q(x,y)\,dy$ is said to be exact in a region Ω if it coincides with the differential

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy,$$

for some function $F(x,y)$. ie,

$$P\,dx + Q\,dy = dF \longrightarrow (1)$$

If $P(x,y)\,dx + Q(x,y)\,dy = 0$, then

$$dF = 0$$

$$\Rightarrow F(x,y) = C$$

From (1), we get

$$\frac{\partial F}{\partial x} = P(x,y) \quad \& \quad \frac{\partial F}{\partial y} = Q(x,y)$$

If $F(x,y) \in C^2(\Omega)$, then

$$\frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial Q}{\partial x}.$$

$$\therefore \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial y}}$$

<u>Theorem</u>. Let Ω be a simply connected region. Assume that $P(x,y)$ and $Q(x,y) \in C^1(\Omega)$. Then $P\,dx + Q\,dy$ is exact in Ω iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in Ω. In this case, $\exists\, F(x,y) \in C^2(\Omega)$ s.t

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = P\,dx + Q\,dy.$$

To find $F(x,y)$.

$$\frac{\partial F}{\partial x} = P(x,y) \longrightarrow ①$$

Integrating ① w.r.to x, we get

$$F(x,y) = \int P(x,y)\, dx + \phi(y), \longrightarrow ②$$

where ϕ is some arbitrary function of y.

Differenting ② w.r.to y, we get

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int P(x,y)\, dx\right] + \phi'(y)$$

$$\Rightarrow Q(x,y) = \frac{\partial}{\partial y}\left[\int P(x,y)\, dx\right] + \phi'(y)$$

$$\Rightarrow \phi'(y) = -\frac{\partial}{\partial y}\left[\int P(x,y)\, dx\right] + Q(x,y)$$

$$\Rightarrow \phi(y) = \int \left[Q(x,y) - \frac{\partial}{\partial y}\left(\int P(x,y)\, dx\right)\right] dy$$

$\therefore$ The solution of an exact ODE is

$$\int P(x,y)\, dx + \int\left[Q(x,y) - \frac{\partial}{\partial y}\left(\int P(x,y)\, dx\right)\right] dy = c.$$