Example

Solve $(2xy+1)\,dx + (x^2+4y)\,dy = 0$, $\longrightarrow$ ①

Soln.

$P(x,y) = 2xy+1$ | $Q(x,y) = x^2+4y$.

$\frac{\partial P}{\partial y} = 2x$ | $\frac{\partial Q}{\partial x} = 2x$.

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$\therefore$ ① is exact.

To find the soln $F(x,y)=c$.

We have $\frac{\partial F}{\partial x} = P(x,y)$

$$\Rightarrow \frac{\partial F}{\partial x} = 2xy+1.$$

Integrating w.r.to x we get

$$F(x,y) = \int (2xy+1)\,dx + \phi(y)$$

$$\Rightarrow F(x,y) = x^2y + x + \phi(y) \longrightarrow ②$$

Differentiating ② w.r.to y partially, we get

$$\frac{\partial F}{\partial y} = x^2 + \phi'(y)$$

$$\Rightarrow x^2+4y = x^2 + \phi'(y)$$

$$\Rightarrow \phi'(y) = 4y$$

$$\Rightarrow \phi(y) = 2y^2 + c$$

$$\therefore F(x,y) = x^2y + x + 2y^2 + c$$

$\boxed{x^2y + x + 2y^2 = K}$, $K \in \mathbb{R}$.

Integrating factors.

Consider the ODE

$$M(x,y)\,dx + N(x,y)\,dy = 0 \longrightarrow (1).$$

A function $\mu(x,y)$ is said to be an integrating factor of (1) if

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0$$

is an exact ODE.

Suppose that (1) has a solution

$F(x,y) = c$, where $\cancel{F \in C^1(\Omega)}$ (2)
F is a differentiable function

Differentiating (2) w.r.to x, we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0 \longrightarrow (2)$$

We rewrite (1) as

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \longrightarrow (3)$$

$$(2) \times N(x,y) \Rightarrow \quad N(x,y)\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot N(x,y)\frac{dy}{dx} = 0$$

$$(3) \times \frac{\partial F}{\partial y} \Rightarrow \quad M(x,y)\frac{\partial F}{\partial y} + \frac{\partial F}{\partial y}N(x,y)\frac{dy}{dx} = 0$$

$$\underline{\quad - \qquad\qquad - \qquad\qquad\qquad - \quad}$$

$$\Rightarrow N(x,y)\frac{\partial F}{\partial x} - M(x,y)\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow N(x,y)\frac{\partial F}{\partial x} = M(x,y)\frac{\partial F}{\partial y}$$

$$\Rightarrow \frac{\partial F/\partial x}{M(x,y)} = \frac{\partial F/\partial y}{N(x,y)} =: \mu(x,y) \text{ (say)}$$

$$\Rightarrow \frac{\partial F}{\partial x} = \mu(x,y)M(x,y) =: P(x,y)$$

$$\frac{\partial F}{\partial y} = \mu(x,y)N(x,y) =: Q(x,y)$$

Then $P\,dx + Q\,dy = 0$ is an exact ODE.

Lemma 1

(i) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} =: f(x)$, then

$\mu(x,y) = e^{\int f(x)\,dx}$. is an I.F of ①

(ii) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} =: g(y)$, then

$\mu(x,y) = e^{-\int g(y)\,dy}$ is an I.F of ①.

Example $(y^2 - x)\,dx + 2y\,dy = 0. \longrightarrow (*)$

Soln $M(x,y) = y^2 - x \quad | \quad N(x,y) = 2y$

$\dfrac{\partial M}{\partial y} = 2y \quad | \quad \dfrac{\partial N}{\partial x} = 0.$

$(*)$ is not exact.

$$g(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - 0}{2y} = 1$$

$\therefore$ I.F $\mu(x,y) = e^{x}$.

$\therefore\ e^x(y^2-x)\,dx + e^x\,2y\,dy = 0$ is an exact ODE.

$$\frac{\partial F}{\partial x} = P(x,y) = e^x(y^2-x) \quad ; \quad Q(x,y) = \frac{\partial F}{\partial y} = e^x\,2y$$

Integrating w.r.to y, we get

$$F(x,y) = \int e^x\,2y\,dy + \varphi(x)$$

$$\Rightarrow F(x,y) = \cancel{2}e^x \cdot \frac{y^2}{\cancel{2}} + \varphi(x)$$

$$F(x,y) = e^x y^2 + \varphi(x) \longrightarrow (**)$$

Differentiating (**) w.r.to x partially,

$$P(x,y) = e^x y^2 + \varphi'(x)$$

$$\Rightarrow e^x(y^2-x) = e^x y^2 + \varphi'(x)$$

$$\Rightarrow -xe^x = \varphi'(x)$$

$$\Rightarrow \varphi(x) = -\int xe^x\,dx + C$$

$$\Rightarrow \varphi(x) = -xe^x + e^x + C$$

$\therefore$ $F(x,y) = C$

$$\Rightarrow \boxed{e^x y^2 - xe^x + e^x = K}$$