Theorem

Let V be a vector space which is spanned by a finite number of vectors, $v_1, v_2, .. v_m$. Then any indepedent set of vectors in V is finite and its cardinality is less than or equal to m

Proof.

It is enough to prove that every subset S of V containing more than m vectors is linearly dependent.

Let $S = \{u_1, u_2, .. , u_n\}$ be a such set, $n > m$.

Since $\{v_1, v_2, .. v_m\}$ spans V, $\exists$ scalars $A_{ij} \in \mathbb{K}$ such that

$$u_j = \sum_{i=1}^{m} A_{ij} v_i .$$

For any scalars $c_1, c_2, .. c_n$, we have

$$c_1 u_1 + c_2 u_2 + .. + c_n u_n = \sum_{j=1}^{n} c_j u_j$$

$$= \sum_{j=1}^{n} c_j \sum_{i=1}^{m} A_{ij} v_i$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} c_j \right) v_i$$

Since $m < n$, $AC = 0$ has at least one non-zero soln,

say that $(c_1, c_2 .. c_n) = c$ is not zero-vector.

For this non-zero $c = (c_1, c_2 .. c_n)$, we have

$$c_1 v_1 + c_2 v_2 + . \quad + c_m v_n = 0.$$

Corollary.

If V is a finite-dimensional vector space, then any two bases of V have the same (finite) number of elements.

Proof.

Since V is finite-dimensional, it has a finite basis.

Let $\{v_1, v_2, \dots v_n\}$ and $\{u_1, u_2, \dots u_m\}$ be two bases for V.

Since $\{v_1, v_2, \dots v_n\}$ spans V and $\{u_1, u_2, \dots u_m\}$ is linearly independent,

$$m \leq n \longrightarrow (1) \quad \text{(by previous thm)}$$

IIIly $\{u_1, u_2, \dots u_m\}$ spans V and $\{v_1, v_2, \dots v_n\}$ is linearly independent,

$$n \leq m \longrightarrow (2)$$

From (1) & (2) $m = n$.

Definition

$\dim V :=$ the number of elements in a basis.

Corollary.

Let V be a finite dimensional vector space of dimension n. Then

(i) any subset of V which contains more than n elements is linearly dependent.

(ii) Any subset of V which contains fewer than n elements cannot span V.

Subspaces.

Let V be a vector space over $\mathbb{K}$. A subset W of V is said to be a subspace of V if W is a vector space over $\mathbb{K}$ with respect to vector addition and scalar multiplication of V.

Examples.

(i) For any vector space V over $\mathbb{K}$, V is a subspace of V

(ii) $W = \{0\}$ is a subspace of V; it is called zero subspace.

(iii) Take $V = M_n(\mathbb{R})$ and W = The set of all $n\times n$ symmetric matrices.

W is a subspace of $M_n(\mathbb{R})$.

(iv) Take $V = M_n(\mathbb{C})$ and W = The set of all $n\times n$ self-adjoint matrices.

Then W is a subspace of $M_n(\mathbb{C})$

(v) Take $V = \mathbb{K}[x]$ and $W = \mathbb{K}_n[x]$. Then

$\mathbb{K}_n[x]$ is a subspace of $\mathbb{K}[x]$

(vi) Take $V = M_n(\mathbb{C})$ and W = The set of all $n\times n$ lower triangular matrices.

W is a subspace of $M_n(\mathbb{C})$.

Theorem.

A non-empty subset W of V is a subspace of V iff for every $v, w \in W$ and $c \in \mathbb{K}$, $cv + w \in W$.

Theorem

Let V be a vector space over $\mathbb{K}$. Then arbitrary intersection of subspaces of V is a subspace of V.

Proof.

Let $W = \bigcap_{\alpha} W_\alpha$, W_α's are subspaces of V.

Clearly $0 \in W_\alpha, \forall \alpha \Rightarrow 0 \in W$. $\therefore W \neq \phi$.

Let $u, v \in W$ and $c \in \mathbb{K}$. Then

$$u \in W_\alpha, v \in W_\alpha, \forall \alpha$$

Since W_α's are subspaces of V,

$$cu + v \in W_\alpha, \text{ for every } \alpha$$

$$\Rightarrow cu + v \in W.$$

$\therefore W$ is a subspace of V

Definition: Let S be a subset of a vector space V over $\mathbb{K}$. Then the intersection of all subspaces of V containing S is called the subspace generated by S. It is denoted by $\text{gen}(S)$.

Properties.

(i) $gen(\phi) = \{0\}$.

(ii) If S is a non-empty set, then

$$span(S) = gen(S)$$

(iii) If S is a subspace then $span(S) = S$

(iv) $span(S)$ is the smallest subspace containing S.

Sum of subsets in V.

Let $S_1, S_2, \dots S_k$ be subsets of V. Then the set $S_1 + S_2 + \dots + S_k$ is defined as

$$S_1 + S_2 + \dots + S_k := \{ v_1 + v_2 + \dots + v_k : v_j \in S_j,\ j = 1, 2, \dots k \}$$

Lemma. If $W_1, W_2, \dots W_k$ are subspaces of V, then $W_1 + W_2 + \dots + W_k$ is a subspace of V.

Proof. Since $0 \in W_j$, $j = \overline{1,k}$, $0 \in W_1 + W_2 + \dots + W_k$.

$$\therefore W_1 + W_2 + \dots + W_k \neq \phi.$$

Let $u, v \in W_1 + W_2 + \dots + W_k$ and $c \in \mathbb{K}$. Then

$$u = w_1 + w_2 + \dots + w_k$$
$$v = w_1' + w_2' + \dots + w_k'$$
, $w_j, w_j' \in W_j$, $j = \overline{1,k}$

Now

$$cu + v = (cw_1 + w_1') + \dots + (cw_k + w_k') \in W_1 + \dots + W_k$$

($\because$ W_k's are subspaces)

$\therefore W_1 + W_2 + \dots + W_k$ is a subspace of V

Determinants.

Let A be an $n\times n$ matrix over $\mathbb{K}$. A matrix which is obtained from A by deleting one column and row is called a minor of a matrix A. We denote M^{ij} is the minor of the matrix by deleting i^{th} row and j^{th} column of A.

The determinant of the matrix A is defined by the recurrence relation

$$\det A := \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(M^{ij}) \quad \text{(Expansion using row)}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(M^{ij}) \quad \text{(Expansion using column)}$$

When $A = (a)_{1\times 1}$, $\det A = a$. The terms $(-1)^{i+j}\det(M^{ij})$ are called cofactors.

Properties.

i) $\det 0 = 0$; $\det I = 1$

ii) $\det A^t = \det A$

iii) $\det(cA) = c^n \det A$, $c \in \mathbb{K}$

iv) $\det(AB) = \det A . \det B$

v) $\det A = 0$ if two rows are identical

vi) $\det(e_{swap}(A)) = -\det A$

vii) A is invertible iff $\det A \neq 0$.

Definition

Let A be an m×n matrix over $\mathbb{K}$. Then the subspace spanned by the rows of the matrix A is called the row space of A and it is denoted by rowspace(A).

Ill^ly we can define column space of A

Properties,

(i) The dimension of rowspace(A) is equal to the rank of the matrix A

(ii) The non-zero rows of a row-reduced echelon matrix of A is a basis for rowspace of A.

(iii) rank(A) = the size (order) of the highest non-vanishing minor of the matrix A.

(iv) If A is row-equivalent to B, then

rowspace(A) = rowspace(B)