Tutorial Sheet 1

II B. Tech. (Common)
Mathematics-II
(MCI102)

Notations:

$M_{m\times n}(\mathbb{F})$: The set of all matrices of order $m \times n$ with entries from $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
$P_n(\mathbb{R})$ or $\mathbb{R}_n[x]$: The set of all polynomials of degree at most n in one variable with real coefficients.
$P(\mathbb{R})$ or $\mathbb{R}[x]$: The set of all polynomials in one variable with real coefficients.
$C(\mathbb{R})$: The set of all real-valued continous functions
$\mathbb{R}^\infty$: The set of all real-valued sequences.
$C[0,1]$: The set of all real-valued continous functions defined on $[0,1]$.
$D[0,1]$: The set of all real-valued differentiable functions defined on $[0,1]$.

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1. Find the rank of the matrix $A=\begin{bmatrix} a & -1 & 1 \\ -1 & a & -1 \\ -1 & -1 & a \\ 1 & 1 & 1 \end{bmatrix}$ if (a) $a \neq -1$ (b) $a = -1$.

2. Find x such that the rank of the matrix $A=\begin{bmatrix} 1 & 3 & -3 & x \\ 2 & 2 & x & -4 \\ 1 & 1-x & 2x+1 & -5-3x \end{bmatrix}$ is 2.

3. Using the row reduced echelon form of the matrix, find the inverse of the following matrices.
(a) $\begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$ Answers: (a) $\begin{bmatrix} 1/8 & -5/8 & 3/4 \\ -1/4 & 3/4 & -1/2 \\ 38 & -3/8 & 1/4 \end{bmatrix}$ (b) $\begin{bmatrix} 8 & -1/2 & -2 \\ -1 & 1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

4. Solve the following systems of linear equations (if possible).
(a) $x_1 + x_2 = 4$, $x_2 - x_3 = 1$, $2x_1 + x_2 + 4x_3 = 7$ (b) $x_1 + 3x_2 + x_3 = 0$, $2x_1 - x_2 + x_3 = 0$
(c) $x_1 + 2x_2 - x_3 = 10$, $-x_1 + x_2 + 2x_3 = 2$, $2x_1 + x_2 - 3x_3 = 2$
(d) $x + y + z - 3w = 1$, $2x + 4y + 3z + w = 3$, $3x + 6y + 4z - 2w = 4$
(e) $x_1 + 2x_2 - x_3 = 10$, $-x_1 + x_2 + 2x_3 = 2$, $2x_1 + x_2 - 3x_3 = 8$
Answers: (a) (3,1,0) (b) $c(-4/7, -1/7, 1)$, $c \in \mathbb{R}$, (c) inconsistent
(d) $(0,0,1,0) + c(-2,1,0,0) + d(10,0,-7,1), c, d \in \mathbb{R}$ (e) $(5c/3 + 2, -c/3 + 4, c), c \in \mathbb{R}$

5. Determine the conditions for which the following system $x + y + z = 1$, $x + 2y - z = b$, $5x + 7y + az = b^2$ admits (a) only one solution, (b) no solution (c) many solutions.
Answers: (a) $a \neq 1$ (b) $a = 1, b \neq -1, 3$ (c) $a = 1, b = -1$ or $b = 3$.

6. Which of the following are vector spaces?
(a) $V = C[a,b]$ over $\mathbb{R}$ with $(f+g)(x) = f(x) + g(x)$ and $(\lambda \cdot f)(x) = \lambda f(x)$ for all $\lambda \in \mathbb{R}, f, g \in C[a,b]$.
(b) $V = \{$ all $n \times n$ Hermitian matrices$\}$ over $\mathbb{C}$ with usual addition and scalar multiplication.
(c) $V = \mathbb{R}[x]$ over $\mathbb{R}$ with usual addition and scalar multiplication of polynomials.
(d) $V = \mathbb{R}^\infty$ over $\mathbb{R}$ with $a + b = \{a_n + b_n\}_{n=1}^\infty$ and $\lambda a = \{\lambda a_n\}_{n=1}^\infty$ for all $a = \{a_n\}_{n=1}^\infty$, $b = \{b_n\}_{n=1}^\infty$ $\in \mathbb{R}^\infty$ and $\lambda \in \mathbb{R}$.
(e) $V = \mathbb{R}^+$ over $\mathbb{R}$ with $x + y = xy$ and $\lambda x = x^\lambda$ for all $x, y \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}$.
(f) V is the set of all real-valued continous functions defined on an open interval I which have at most finite number of points of discontinuity over $\mathbb{R}$ with pointwise addition and scalar multiplication of functions.
(g) $V = \{t_\alpha : \mathbb{R} \to \mathbb{R} \mid t_\alpha(x) = x + \alpha, \alpha \in \mathbb{R}\}$ over $\mathbb{R}$ with composition of mappings and $\lambda t_\alpha = t_{\alpha\lambda}$ for all $t_\alpha \in V$ and $\lambda \in \mathbb{R}$.
(h) $V = \mathbb{R}^2$ over $\mathbb{R}$ with componentwise addition and $\lambda(x,y) = (3\lambda x, y)$ for all $(x,y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.
(i) $V = \{$ all real polynomials of degree 4 or 6$\}$ over $\mathbb{R}$ with usual addition and scalar multiplication.
(j) $V = \{$all $n \times n$ skew-Hermitian matrices$\}$ over $\mathbb{C}$ with usual matrix addition and scalar multiplication.
Answers: (a) Yes (b) No (c) Yes (d) Yes (e) Yes (f) Yes (g) Yes (h) No (i) No (j) No

7. Which of the following are subspace of $\mathbb{R}[x]$?

   (a) $S = \mathbb{R}_n[x]$ (b) $S = \{f(x) \in \mathbb{R}[x] : f(x) = f(1-x)\ \forall\ x\}$ (c) $S = \{f(x) \in \mathbb{R}[x] : f(x) = f(-x)\ \forall\ x\}$
   (d) $S = \{f(x) \in \mathbb{R}[x] : f(1) \geq 0\}$ (e) $S = \{f(x) \in \mathbb{R}[x] : f'(0) + f(0) = 0\}$(f) $S = \{f(x) \in \mathbb{R}[x] : f(x)$ has a root in $[-1, 1]\}$

8. Which of the following are subspaces of the vector space $\mathbb{R}^n$?

   (a) $S = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\colon x_n = 0\}$ (b) $S = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\colon x_1 + x_2 + \cdots + x_n = 0\}$
   (c) $S = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\colon x_1^2 + x_2^2 + \cdots + x_n^2 \geq 1\}$ (d) $S = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\colon x_i = x_{n+i-1}\ \forall\ i = 1, 2, \ldots, n\}$

9. Which of the following are subspaces of the vector space $M_{2\times2}(\mathbb{R})$?

   (a) $S = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R})\colon a + b = 0\right\}$ (b) $S = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R})\colon a + b + c + d = 0\right\}$
   (c) $S = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R})\colon \ \mathbf{det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0\right\}$ (d) $S = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R})\colon b = c = 0\right\}$
   (e) $S = \{A \in M_{2\times2}(\mathbb{R})\colon A = A^T\}$ (f) $S = \{A \in M_{2\times2}(\mathbb{R})\colon A = -A^T\}$
   (g) $S = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R})\colon c = 0\right\}$ (h) $S = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R})\colon b = 0\right\}$

   Answers: (a) Yes (b) Yes (c) No (d) Yes (e) Yes (f) Yes (g) Yes (h) Yes

10. Which of the following are subspaces of the vector space $C[0, 1]$?

    (a) $S = \{f \in C[0, 1]\colon f(0) = 0\}$ (b) $S = \{f \in C[0, 1]\colon f(0) = 0, f(1) = 0\}$ (c) $S = D[0, 1]$

    Answers: (a) Yes (b) Yes (c) Yes

11. Find all subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$.

12. Write TRUE/FALSE with proper justifications.

    (a) Any set containing the zero vector is linearly dependent

    (b) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S

    (c) Subsets of linearly independent sets are linearly independent

    (d) Subsets of linearly dependent sets are linearly dependent

    Answers: (a) TRUE (b) FALSE (c) TRUE (d) FALSE

13. Determine the linear independence/dependence of the following sets in the corresponding vector spaces.

    (a) $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $P_3(\mathbb{R})$ (b) $\{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$ in $\mathbb{R}^3$
    (c) $\left\{\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} -2 & 6 \\ 4 & -8 \end{bmatrix}\right\}$ in $M_{2\times2}(\mathbb{R})$

    Answers: (a) LI (b) LI (c) LD

14. Let u and v be distinct vectors in any vector space V over F. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

15. Let $\{u, v, w\}$ be linearly independent in a real vector space V. Show that $\{\lambda u, \lambda v, \lambda w\}$, $\{u + \lambda v, v, w\}$, $\{u+v, u+w, v+w\}$, $\{u+v+w, v+w, w\}$ are also linearly independent in V and $\{u+\lambda v, v+\lambda w, w+\lambda u\}$ may not be linearly independent in V, where $\lambda \in \mathbb{R}$.

16. For $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 0 & 7 \\ -1 & 4 & 3 \end{bmatrix}$, examine whether $(1, 1, 1)$ and $(1, -1, 1)$ are in (a) the row space of A; (b) the coloumn space of A. Answers: (a) no, yes (b) yes, no

17. Write true/false with proper justifications:

    (a) Every vector space has a finite basis. (b) A vector space can not have finite basis.

    (c) If a vector space has a finite basis, then the number of vectors in every basis is same.

    (d) If S generates/spans V, then every vector can be written as a linear combination of vectors in S uniquely.

    (e) Suppose V is finite dimensional. If S_1 is a linearly independent subset of V and S_2 is a subset of V that spans V, then S_1 cannot contain more vectors than S_2.

    Answers: (a) False (b) False (c) True (d) False (e) True

18. Find a basis and the dimension of the following vector spaces.

    (a) $\mathbb{R}^n$ over $\mathbb{R}$, (b) $\mathbb{C}$ over $\mathbb{R}$ (c) $P(\mathbb{R})$ over $\mathbb{R}$ (d) $M_{m\times n}(\mathbb{R})$ over $\mathbb{R}$

19. Find a basis and the dimension of the following subspaces W of the corresponding vector spaces.

    (a) $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ (b) $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0, 2x + y + z = 0\}$

    (c) $W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : b = c = 0\right\}$ (d) $W = \left\{A \in M_{2\times 2}(\mathbb{R}) : A = A^T\right\}$

    (e) $W = \left\{A \in M_{2\times 2}(\mathbb{R}) : A = -A^T\right\}$ (f) $W = \{A \in M_{n\times n}(\mathbb{R}) : \textbf{trace}(A) = 0\}$

    (g) Fix $a \in \mathbb{R}$. $W = \{f(x) \in P_n(\mathbb{R}) : f(a) = 0\}$ (h) $W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : a + b = 0\right\}$

20. Give three different basis for $M_{2\times 2}(\mathbb{R})$ and $\mathbb{R}^2$.

21. For what real values of k, does the the set $\{(k, 0, 1), (1, k + 1, 1), (1, 1, 1)\}$ form a basis of $\mathbb{R}^3$? [**Ans.** $k \neq 0, 1$]

22. Using the row echelon form of a matrix, find the row rank, the column rank, the rank and the nullity of the following matrices. Also find a basis for the row space, the column space, and the null space of the matrices.

    (a) $A=\begin{bmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 6 & 9 \\ 1 & 1 & 2 & 6 \end{bmatrix}$ (b) $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$ (c) $A=\begin{bmatrix} -2 & 0 & 0 & 3 \\ 1 & 5 & 3 & 0 \\ 3 & 2 & 1 & 6 \\ 3 & 5 & 3 & -3 \end{bmatrix}$ Answers: (a) 2 (b) 2 (c) 3

23. Examine whether T is a linear transformation. If T is linear, find $Ker(T)$, $Im(T)$ and verify rank-nullity theorem for (a) – (s).

    (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (a_1, -a_2)$ (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (a_1, 0)$

    (c) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + 2y, 2x + y, x + 2)$ (d) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x, -y, 2z)$

    (e) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (yz, zx, xy)$ (f) $T : M_{m\times n}(\mathbb{R}) \to M_{n\times m}(\mathbb{R})$ defined by $T(A) = A^t$

    (g) $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ defined by $T(A) = \frac{1}{2}(A+A^t)$ (h) $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by $T(f(x)) = f'(x)$

    (i) $T : C(\mathbb{R}) \to \mathbb{R}$ defined by $T(f(x)) = \int_a^b f(t)dt$, where $a, b \in \mathbb{R}$ and $a < b$

    (j) $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by $T(a_0 + a_1x + a_2x^2) = (a_0, a_1, a_2)$

    (k) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt$

    (l) $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(f(x)) = \begin{bmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{bmatrix}$

    (m) $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \textbf{trace}(A)$

    (n) $T : P(\mathbb{R}) \to P(\mathbb{R})$ defined by $T(f(x)) = \int_0^x f(t)dt$

    (o) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (1, a_2)$ (p) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (a_1, a_1^2)$

    (q) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (\sin a_1, 0)$ (r) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (|a_1|, a_2)$

    (s) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (a_1 + 1, a_2)$

Answers: (a) Yes, $Ker(T) = \{(0,0)\}$, $Im(T) = \mathbb{R}^2$ (b) Yes, $Ker(T) = Y-$ axis, $Im(T) = X-$ axis (c) Yes, $Ker(T) = \{(0,0)\}$, $Im(T) = \mathbb{R}^2$ (d) Yes, $Ker(T) = L\{(1,1,0)\}$, $Im(T) = \mathbb{R}^2$ (e) No (f) Yes, $Ker(T) = 0_{m\times n}$, $Im(T) = M_{n\times m}(\mathbb{R})$ (g) Yes, $Ker(T) = \{A \in M_{n\times n}(\mathbb{R})\colon A = A^t\}$, $Im(T) = \{A \in M_{n\times n}(\mathbb{R})\colon A = -A^t\}$ (h) Yes, $Ker(T) = \{\text{all constant polynomials in} P_n(\mathbb{R})\}$, $Im(T) = P_{n-1}(\mathbb{R})$ (i) Yes (j) Yes (k) Yes (l) Yes (m) Yes (n) Yes (o) No (p) No (q) No (r) No (s) No

24. (a) Is there a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(1,0) = (1,4)$ and $T(1,1) = (2,5)$? If yes, what is $T(2,3)$?

    (b) Is there a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$? If yes, find T and what is $T(8,11)$?

    (c) Is there a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?

    (d) Determine the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1), (1,0,1), (1,1,0)$ of $\mathbb{R}^3$ to $(1,1,1), (1,1,1), (1,1,1)$ respectively. Verify rank-nullity theorem after finding $Ker(T), Im(T)$.

    Answers: (a) Yes, $(5,11)$ (b) Yes, $(5,-3,16)$ (c) No (d) $T(x,y,z) = \left(\dfrac{x+y+z}{2}, \dfrac{x+y+z}{2}, \dfrac{x+y+z}{2}\right)$

25. Find all linear transformations $T\colon \mathbb{F} \to \mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

26. Find all linear transformations $T\colon \mathbb{F}^2 \to \mathbb{F}^2$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

27. A linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x_1,x_2,x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3)$. Find the matrix of T relative to the ordered bases

    (a) $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

    (b) $\{(0,1,0), (1,0,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(0,1), (1,0)\}$ of $\mathbb{R}^2$.

    (c) $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

    Answers: (a) $\begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 1 & -2 \\ -2 & 3 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 4 & 1 \\ -5 & -1 & -2 \end{bmatrix}$

28. The matrix of a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the ordered basis $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ is given by $\begin{bmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{bmatrix}$. Find T and also find the matrix of T with ordered basis $\{(2,1,1), (1,2,1), (1,1,2$ of $\mathbb{R}^3$.

    Answers: $T(x,y,z) = (-x+y+3z, x+y+z, x-3y+5z)$, $m(T) = \begin{bmatrix} -1/2 & 2 & 3/2 \\ 3/2 & 2 & -1/2 \\ 3/2 & -2 & 7/2 \end{bmatrix}$