Vector spaces.

Definition

A vector space over $\mathbb{K}$ is a non-empty set V together with two operations, addition carrying $V \times V$ into V and scalar multiplication carrying $\mathbb{K} \times V$ into V with the following properties:

(I). The operation of addition, written '+' satisfies

(i) $v + w = w + v$, $\forall\, v, w \in V$.

(ii) $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$, $\forall\, v_1, v_2, v_3 \in V$.

(iii) there exists a unique element $0 \in V$ such that

$$0 + v = 0, \ \forall\, v \in V$$

(The element 0 is called zero vector in V).

(iv) For each $v \in V$, there exists a unique element w such that $v + w = 0$.

(The element 'w' is called the additive inverse element of v and it is denoted by $-v$).

(II) The operation of scalar multiplication, written without a sign satisfies

(v) $1v = v$, $\forall\, v \in V$

(vi) $(\alpha\beta)v = \alpha(\beta v)$, $\forall\, \alpha, \beta \in \mathbb{K}$.

(III) The two operations are related by the distributive laws:

(vii) $\alpha(v+w) = \alpha v + \alpha w, \quad \forall\, \alpha \in \mathbb{K}, \quad v, w \in V.$

(viii) $(\alpha+\beta)v = \alpha v + \beta v, \quad \forall\, \alpha, \beta \in \mathbb{K}, v \in V.$

- A vector space over $\mathbb{R}$ is called a real vector space.
- A vector space over $\mathbb{C}$ is called a complex vector space.

Example 1. (The n-tuple space, $\mathbb{K}^n$).

$$\mathbb{K}^n = \{ (x_1, x_2, \dots x_n) : x_1, x_2, \dots x_n \in \mathbb{K} \}$$

For $x = (x_1, x_2, \dots x_n)$ and $y = (y_1, y_2, \dots y_n) \in \mathbb{K}^n$, define

$$x + y := (x_1+y_1, x_2+y_2, \dots x_n+y_n)$$

$$cx = (cx_1, cx_2, \dots, cx_n)$$

Then $\mathbb{K}^n$ is a vector space over $\mathbb{K}$ with respect to the above operations.

The zero element in $\mathbb{K}^n$ is $(0, 0, \dots 0)$.

The additive inverse of $(x_1, x_2 \dots x_n)$ is $(-x_1, -x_2, \dots -x_n)$.

- The space $\mathbb{R}^n$ is called the n-dimensional Euclidean space.
- The space $\mathbb{C}^n$ is called the n-dimensional Unitary space.

Example 2. The space of $m\times n$ matrices over $\mathbb{K} \equiv M_{m\times n}(\mathbb{K})$ (or) $\mathbb{K}^{m\times n}$.

For $A, B \in M_{m\times n}(\mathbb{K})$, we define $A+B$ by

$$(A+B)_{ij} := A_{ij} + B_{ij} \quad , \begin{array}{l} 1\le i\le m \\ 1\le j\le n \end{array}$$

and

$$(cA)_{ij} = cA_{ij}. \quad c\in\mathbb{K}.$$

Then $M_{m\times n}(\mathbb{K})$ is a vector space over $\mathbb{K}$ w.r.t the above operations.

The zero element in $M_{m\times n}(\mathbb{K})$ is the zero matrix.

The additive inverse of the matrix A is $-A$.

- When $m=n$, we write $M_{n\times n}(\mathbb{K})$ as $M_n(\mathbb{K})$.

Example 3.

The space of polynomials functions over $\mathbb{K} \equiv \mathbb{K}[x]$

$p\in\mathbb{K}[x]$ means $p:\mathbb{K}\to\mathbb{K}$ such that

$$p(x) = a_0 + a_1 x + \dots + a_n x^n, \quad n\in\mathbb{N}\cup\{0\},$$
$$a_0, a_1, \dots a_n \in \mathbb{K}.$$

For $p, q\in\mathbb{K}[x]$, define $p+q$ by

$$(p+q)(x) := p(x)+q(x) \ , \ x\in\mathbb{K}$$

Define cp by

$$(cp)(x) = c\,p(x) \ , \ x\in\mathbb{K}$$

Then $\mathbb{K}[x]$ is a vector space over $\mathbb{K}$. The zero vector in $\mathbb{K}[x]$ is the zero polynomial and the additive inverse of the polynomial p is the polynomial $-p$.

Example 4 The space of functions from a set S to $\mathbb{K}$.

Let V be the set of all functions from a set S to $\mathbb{K}$.
i.e.,

$$V = \{ f : S \to \mathbb{K} \}.$$

Define addition and scalar multiplication on V by

$$(f+g)(x) := f(x) + g(x), \quad \forall x \in S$$

$$(cf)(x) := cf(x), \quad \forall x \in S, \quad c \in \mathbb{K}$$

Then V is a vector space over $\mathbb{K}$ w.r.to the above operations.

Example 5.

Take $V = \{ (x_1, x_2, \dots x_n) : x_1, x_2, \dots x_n \in \mathbb{C} \}$ and $\mathbb{K} = \mathbb{R}$.

Then Define

$$x+y := (x_1+y_1, x_2+y_2, \dots x_n+y_n), \text{ for } x = (x_1, \dots x_n)$$
$$y = (y_1, \dots y_n)$$

$$cx = (cx_1, cx_2, \dots cx_n), \quad c \in \mathbb{R}$$

Then V is a vector space over $\mathbb{R}$.

Note: The above vector space V is different from $\mathbb{C}^n$.

Example 6. (Not a vector space)

Take $V = \{ (x_1, x_2 \dots x_n) : x_1, x_2 \dots x_n \in \mathbb{R} \}$ and $\mathbb{K} = \mathbb{C}$.

Then define $x+y = (x_1+y_1, \dots, x_n+y_n)$; $cx = (cx_1, cx_2 \dots cx_n)$

Then V is not a vector space over $\mathbb{C}$ (because $i(1, 2, \dots n) = (i, 2i, \dots in) \notin V$)

Definition (Linear combination)

A vector $w \in V$ is said to be a linear combination of $v_1, v_2, \dots v_n$ if there exists scalars $c_1, c_2, \dots c_n$ such that

$$w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n \equiv \sum_{i=1}^{n} c_i v_i$$

Definition (Linearly independent and dependent)

Let V be a vector space over $\mathbb{K}$. A non-empty subset S of V is said to be linearly dependent if there exists distinct vectors $v_1, v_2, \dots v_n$ in S and scalars $c_1, c_2, \dots c_n$ not all of which are zero, such that

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0.$$

A set which is not linearly dependent is called linearly independent.

Theorem. A subset E of V is linearly independent iff for each finite subset $\{u_1, u_2, \dots u_n\}$ of E,

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0 \Rightarrow c_1 = c_2 = \dots = c_n = 0$$

Theorem A finite set $\{v_1, v_2, \dots v_n\}$ in V is linearly independent iff whenever $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$ implies $c_1 = c_2 = \dots = c_n = 0$.

Note

- $\{0\}$ is linearly dependent subset in V.
- $\{v\}$, $v \neq 0$ is linearly independent subset in V.
- Any subset containing zero vector is linearly dependent.
- Every subset of a linearly independent set is linearly independent.
- Every superset of a linearly dependent set is linearly dependent.
- For convenience, empty set is linearly independent.

Definition (Spanning set)

Let S be a non-empty subset of a vector space V over $\mathbb{K}$. The set of all finite linear combination of elements of S is called the span of S and it is denoted by span(S).

- $v \in span(S)$ iff there exists $v_1, v_2 .. v_m$ in S and scalars $c_1, c_2 .. c_m$ such that

$$v = c_1 v_1 + c_2 v_2 + .. + c_m v_m.$$

- For convenience, we define $span(\phi) := \{0\}$.

- $$span(S) = \{ c_1 v_1 + ... + c_m v_n : n \in \mathbb{N}, v_1, v_2 .. v_n \in S \}.$$

Definition (Basis)

Let V be a vector space over $\mathbb{K}$. A set S is said to be a basis for V if

(i) S is a linearly independent set in V

(ii) $\text{span}(S) = V$.

V is said to be finite dimensional if it has a finite basis.

Theorem

- Every non-zero vector space has a basis.

Theorem

If we assume, empty set is finite and its cardinality is zero, then the trivial vector space $V = \{0\}$ has a basis.

[Because $S = \phi$ is linearly independent & $\text{span}(\phi) = \{0\}$].

Example 1.

Take $V = \mathbb{K}^n$.

Define $e_1 = (1, 0, 0 \dots)$, $e_2 = (0, 1, 0 \dots 0)$, $\dots$ $e_n = (0, 0, \dots 0, 1)$

Then $S = \{e_1, e_2, \dots e_n\}$ is a basis for $\mathbb{K}^n$.

Claim: S spans $\mathbb{K}^n$.

Let $x \in \mathbb{K}^n$. Then $x = (x_1, x_2, \dots x_n)$, $x_1, x_2 \dots x_n \in \mathbb{K}$.

$$= x_1(1, 0, \dots 0) + x_2(0, 1, \dots 0) + \dots + x_n(0, \dots 0, 1)$$
$$= x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

$\therefore$ span$(S) = \mathbb{K}^n$.

<u>Claim</u>: S is linearly independent

Let $c_1e_1 + c_2e_2 + \dots + c_ne_n = 0$

$\Rightarrow c_1(1,0,\dots 0) + c_2(0,1,\dots 0) + \dots + c_n(0,0,\dots 0,1) = (0,0\dots 0)$

$\Rightarrow (c_1, c_2, \dots c_n) = (0, 0, \dots 0)$

$\Rightarrow c_1 = c_2 = \dots = c_n = 0.$

$\therefore$ S is linearly independent.

Thus S is a basis for $\mathbb{K}^n$.

<u>Example 2</u>

$V = \mathbb{K}^{m\times n} \equiv M_{m\times n}(\mathbb{K})$.

For each $1 \le p \le m, 1 \le q \le n$, consider the matrix E^{pq} defined by

$$(E^{pq})_{ij} = \begin{cases} 1 & \text{if } (i,j) = (p,q) \\ 0 & \text{otherwise} \end{cases}$$

Then the set $S = \{ E^{pq} : 1 \le p \le m \;\; 1 \le q \le n \}$ is a basis for $M_{m\times n}(\mathbb{K})$.

<u>Claim.</u> S spans $M_{m\times n}(\mathbb{K})$.

Let $A \in M_{m\times n}(\mathbb{K})$. Then $A = [A_{ij}]$.

Clearly $A = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{pq}$ $\quad \therefore$ span$(S) = M_{m\times n}(\mathbb{K})$

<u>Claim</u> S is linearly independent

Let $\sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{pq} = 0$. Then

$$A = [A_{ij}] = 0$$

$$\Rightarrow A_{ij} = 0, \ \forall \ \begin{matrix} 1 \le i \le m \\ 1 \le j \le n. \end{matrix}$$

$\therefore$ S is linearly independent.

$\therefore$ S is a basis for $M_{m\times n}(\mathbb{K})$.

<u>Example.3</u>

Take $V = \mathbb{K}[x]$. Define $f_k(x) = x^k$, $k = 0, 1, 2, 3 \ldots$

Then

$$S = \{1, f_0, f_1, f_2, \ldots\} \text{ is a basis for } \mathbb{K}[x].$$

<u>claim.</u> S spans $\mathbb{K}[x]$.

Let $f \in \mathbb{K}[x]$. Then

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n, \quad a_0, a_1, \ldots a_n \in \mathbb{K}. \quad , x \in \mathbb{K}$$

$$= a_0 \cancel{f_0} + a_1 f_1(x) + \ldots + a_n f_n(x)$$

$$\Rightarrow f = a_0 f_0 + a_1 f_1 + \ldots + a_n f_n.$$

$\therefore$ span$(S) = \mathbb{K}[x]$.

<u>claim.</u> S is linearly independent.

To prove S is l. independent, it is enough to prove that for each n, the set $\{f_0, f_1, \ldots f_n\}$ is linearly independent.

Let

$$c_0 f_0 + c_1 f_1 + \dots + c_n f_n = 0. \text{ Then}$$

$$c_0 f_0(x) + c_1 f_1(x) + \dots + c_n f_n(x) = 0, \quad \forall\, x \in \mathbb{K}$$

$$\Rightarrow c_0 + c_1 x + \dots + c_n x^n = 0, \quad \forall\, x \in \mathbb{K}$$

The polynomial $p(x) = c_0 + c_1 x + \dots + c_n x^n$ has more than n roots/zeros. Therefore by Fundamental theorem of algebra,

$$p(x) = 0, \ \forall\, x \in \mathbb{K}.$$

$$\Rightarrow c_0 = c_1 = \dots = c_n = 0.$$

$\therefore$ S is linearly independent.

$\therefore$ S is a basis for $\mathbb{K}[x]$.

Rank of a matrix.

Theorem Let A be an $m \times n$ matrix over $\mathbb{K}$. Then

rank(A) = the maximum number of linearly independent row vectors of A.

Theorem. Row equivalent matrices have the same rank.