Eigenvalues and Eigenvectors.

Definition.

Let V be a vector space over $\mathbb{K}$ and let $T: V \to V$ be a linear transformation. A scalar $\lambda \in \mathbb{K}$ is said to be an eigenvalue of T if there exists a non-zero vector $v \in V$ such that

$$Tv = \lambda v.$$

The non-zero vector v is called an eigenvector of T associated with the eigenvalue λ. The null space

$$(E_\lambda(T) \equiv) N(T - \lambda I) = \{v \in V : Tv = \lambda v\}$$

is called the eigenspace of T associate with the eigenvalue λ. The dimension of the eigenspace $E_\lambda(T)$ is called the geometric multiplicity of λ.

Theorem

Let T be a linear operator on a finite dimensional vector space V over $\mathbb{K}$ and $\mathcal{B}$ is a basis for V. Then $\lambda \in \mathbb{K}$ is an eigenvalue of T iff $\det [T - \lambda I]_{\mathcal{B}} = 0$.

Definition.

Let A be an $n \times n$ matrix over $\mathbb{K}$. Then a scalar $\lambda \in \mathbb{K}$ is an eigenvalue of A if there exists a non-zero vector $v \in \mathbb{K}^{n \times 1}$ such that

$$Av = \lambda v.$$

The non-zero vector v is called an eigenvector of A associated with an eigenvalue ~~of~~ λ

Theorem:

λ is an eigenvalue of A iff $\det(A-\lambda I)=0$.

Definition (characteristic polynomial)

Let A be an $n\times n$ matrix over $\mathbb{K}$. Then the polynomial

$$p(\lambda) := \det(\lambda I - A)$$

is called the characteristic polynomial of A.

$p(\lambda)=0$ is called the characteristic eqn of A

Properties.

(i) $p(\lambda) = \lambda^n - (\operatorname{tr} A)\lambda^{n-1} + \dots \quad + (-1)^n \det A$.

In particular, when $n=2$

$$p(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A .$$

(b) When $n=3$,

$$p(\lambda) = \lambda^3 - (\operatorname{tr} A)\lambda^2 + s_1\lambda - \det A,$$

where $s_1 =$ sum of the minors of of the main diagonal elements

Definition (Algebraic multiplicity)

An eigenvalue λ_0 of A is said to have the algebraic multiplicity m if $p(\lambda_0) = p'(\lambda_0) = \dots\ p^{(m)}(\lambda_0) = 0$ but $p^{(m+1)}(\lambda_0) \neq 0$.

Definition (Similar matrices)

Let A and B be two $n\times n$ matrices over $\mathbb{K}$. Then We say that A is similar to B if there exists an invertible matrix P s.t $B = P^{-1}AP$.

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Exercise. Prove that similarity is an equivalence relation on $M_n(\mathbb{K})$

Theorem. Let T be a linear operator on a finite dimensional vector space over $\mathbb{K}$. If $\mathcal{B}$ and $\mathcal{B}'$ are bases for V, then $[T]_{\mathcal{B}}$ is similar to $[T]_{\mathcal{B}'}$.

Lemma. Similar matrices have the same characteristic polynomial.

Proof. If $B = P^{-1}AP$, then

$$\det(\lambda I - B) = \det(\lambda I - P^{-1}AP)$$

$$= \det(\lambda P^{-1}P - P^{-1}AP)$$

$$= \det(P^{-1}\cancel{\lambda}P - P^{-1}AP)$$

$$= \det(P^{-1}(\lambda I - A)P)$$

$$= \det P^{-1} \cdot \det(\lambda I - A)\det P$$

$$= \det(\lambda I - A)$$

Definition. Let T be a linear operator on a finite dimensional vector space over $\mathbb{K}$. Then the characteristic polynomial of T is defined as

$$p(\lambda) = \det([\lambda I - T]_{\mathcal{B}}), \text{ for some basis } \mathcal{B} \text{ of } V$$

Example 1

Consider the linear transformation $T : \mathbb{C}^{2\times 1} \to \mathbb{C}^{2\times 1}$ by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Take $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. Then

$$A = [T]_{\mathcal{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then $P_A(\lambda) = \det(\lambda I - A)$

$$\overset{\shortparallel}{P_T(\lambda)} = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1.$$

$\therefore$ The eigenvalues of T (or A) are $\pm i$.

Example

Take $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x, y) = (-y, x).$$

Then $[T]_{\mathcal{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, where $\mathcal{B} = \{(1,0), (0,1)\}$.

$P_T(\lambda) = \lambda^2 + 1 \longrightarrow$ char. polynomial.

T has no eigenvalues (in $\mathbb{R}$).

1. Find the characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

S_1 = sum of main diagonal elements

= 1+2=3

S_2 = Det (A) =|A|

$= \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$

S_2 = 2-0 = 2

The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$.

2. Find the characteristic equation of $\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = sum of the main diagonal elements

= 2+1-4 = -1

S_2 = sum of minor of main diagonal elements

$= \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$

= (-4-6)+(-8+5)+(2+9) = -10+(-3)+11 = -2

S_3 = Det (A) =|A|

$= \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix}$

= 2(-4-6)-(-3)(-12+15)+1(6+5)

= 2(-10) +3(3) +1(11) = -20+9+11= 0

The characteristic equation is $\lambda^3 + \lambda^2 - 2\lambda = 0$

3. Find the Eigenvalues of $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$S_1 = 1+2+1=4$$

$$S_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (2\text{-}1)+(1\text{-}0)+(2\text{-}1) = 3$$

$$S_3 = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(2\text{-}1)+(\text{-}1\text{-}0)+0 = 0$$

Therefore the characteristic equation is $\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$

To find the Eigenvalues

$$\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$$

$$\lambda(\lambda^2 \text{-}4\lambda\text{+}3) = 0$$

$$\lambda = 0,\ (\lambda^2 \text{-}4\lambda\text{+}3) = 0$$

$$(\lambda-1)(\lambda-3) = 0$$

$$\lambda = 1, 3$$

The Eigen values are 1, 3, and 0.

Properties of Eigenvalues.

i. The sum of the Eigenvalues of a matrix is the sum of the elements of main diagonal
ii. The product of the Eigenvalues is equal to the determinant of the matrix.
iii. The Eigen values of the triangular matrix are just the diagonal element of the matrix
iv. If λ is an Eigenvalue of a matrix A, then $1/\lambda$, $(\lambda \neq 0)$ is Eigen value of A^{-1} if inverse exists
v. If $\lambda_1, \lambda_2, \ldots\ldots\lambda_n$ are the Eigenvalues of a matrix A, then A^m has a Eigenvalues $\lambda_1^m, \lambda_2^m, \ldots\ldots\lambda_n^m$

1. Find the sum & product of the Eigenvalues of the matrix A= $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & -6 \end{bmatrix}$

Solution:

Sum of the Eigen values = sum of the main diagonal elements

= 2+3-6=-1

Product of the Eigen value = |A|

$$= 2\begin{vmatrix} 3 & 1 \\ 1 & -6 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -6 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

= 2(-18-1)-1(-6-2)+2(1-6) = - 40

Sum = -1 and Product = - 40

2. The product of two Eigenvalues of the matrix A= $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ **is 16. Find the third Eigenvalue of A.**

Solution:

Given: The product of two Eigen values of A is 16

(i.e) $\lambda_1\lambda_2=16$

By property, Product of Eigen values = |A|

$$\lambda_1\lambda_2\lambda_3 = |A|$$

$$16\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1)+2(-6+2)+2(2-6)= 32$$

$$\lambda_3 = \frac{32}{16} = 2$$

The third Eigen value is 2.

3. Two Eigenvalues of the matrix A = $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ **are 3 and 0. What is the third eigenvalue?**

What is the product of the eigenvalues of A?

Solution:

Given: If $\lambda_1 = 3$, $\lambda_2 = 0$, and $\lambda_3 =?$

By property, Sum of the Eigenvalues = sum of the main diagonals.

$$\lambda_1+\lambda_2+\lambda_3 = 8+7+3= 18$$

$$3+0+\lambda_3 = 18$$

$$\lambda_3 = 18 - 3 = 15$$

By property, Product of the Eigen values = |A|

$$(3)(0)(15) = |A|$$

$$|A| = 0$$

The third eigenvalue is 15, The product of the eigenvalues of A is 0.

4. If 3 and 15 are two Eigenvalues of the matrix A = $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ **then find the third eigenvalue and hence** = |A|

Solution:

Given: If $\lambda_1 = 3$, $\lambda_2 = 15$, and $\lambda_3 =?$

By property, Sum of the Eigenvalues = sum of the main diagonals.

$$\lambda_1+\lambda_2+\lambda_3 = 8+7+3= 18$$

$$3+15+\lambda_3 = 18$$

$$\lambda_3 = 18 - 18 = 0$$

By property, Product of the Eigenvalues = |A|

$$(3)(15)(0) = |A|$$

$$|A| = 0$$

The third eigenvalue is 0, The product of the eigenvalues of A is 0.

Non-Symmetric Matrix With Non-Repeated Eigenvalues

1. Find all the Eigenvalues and Eigenvectors of the matrix $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

Solution:

Given: A= $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

To find the characteristic equation of A

Formula: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1= sum of main diagonal

=1+2-1=2

S_2 = sum of minor of main diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

= (-2+1)+(-1-8)+(2+3) = -1-9+5 = -5

$S_3 = \text{Det}(A) = |A|$

$$= \begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = 1(-2+1)+1(-3+2)+4(3-4)$$

$$=1(-1)+1(-1)+4(-1) = -1-1-4 = -6$$

Hence the characteristic equation is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

To solve the characteristic equation:

If $\lambda=1$ By synthetic division

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & 0 & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

Therefore the $\lambda=1$ and other roots are given by $\lambda^2 - \lambda - 6 = 0$

$$(\lambda+2)(\lambda-3) = 0$$

$$\lambda = -2, 3$$

Therefore Eigenvalues are 1,-2, 3

To find the Eigenvectors:

To get the Eigenvectors solve: $(A-\lambda I)X=0$

$$\left[\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\left[\begin{pmatrix}1-\lambda & -1 & 4\\ 3 & 2-\lambda & -1\\ 2 & 1 & -1-\lambda\end{pmatrix}\right]\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\left.\begin{aligned}(1-\lambda)x_1-x_2+4x_3=0\\ 3x_1+(2-\lambda)x_2-x_3=0\\ 2x_1+x_2+(-1-\lambda)x_3=0\end{aligned}\right\} \qquad \ldots.(1)$$

Case 1: Substitute $\lambda=1$ in, (1) we get

$$0x_1-x_2+4x_3=0 \quad \ldots. \quad (2)$$
$$3x_1+x_2-x_3=0 \quad \ldots. \quad (3)$$
$$2x_1+x_2-2x_3=0 \quad \ldots. \quad (4)$$

Solving (2) and (3) by cross multiplication rule, we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & & 4 & & 0 & & -1\\ 1 & & -1 & & 3 & & 1\end{array}$$

$$\frac{x_1}{1-4} = \frac{x_2}{12-0} = \frac{x_3}{0+3}$$

$$\Rightarrow \frac{x_1}{-3} = \frac{x_2}{12} = \frac{x_3}{3}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{4} = \frac{x_3}{1}$$

$$\text{Therefore } X_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

eigenspace for eigenvalue 1=span{X_1}

Case 2: Substitute λ= -2 in (1), we get

$3x_1 - x_2 + 4x_3 = 0$ ....(5)

$3x_1 + 4x_2 - x_3 = 0$ ....(6)

$2x_1 + x_2 + x_3 = 0$ ....(7)

Solving (5) and (6) by cross multiplication rule we get

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & & 4 & & 3 & & -1 \\ 4 & & -1 & & 3 & & 4 \end{array}$$

$$\frac{x_1}{1-16} = \frac{x_2}{12+3} = \frac{x_3}{12+3}$$

$$\Rightarrow \frac{x_1}{-15} = \frac{x_2}{15} = \frac{x_3}{15}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\text{Therefore } X_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

eigen space of eigenvalue -2=span{X_2}

Case 3: Substitute λ=3 in (1) we get

$$-2x_1-x_2+4x_3=0 \quad \ldots. \ (8)$$
$$3x_1-x_2-x_3=0 \quad \ldots. \ (9)$$
$$2x_1+x_2-4x_3=0 \quad \ldots. \ (10)$$

Solving (8) and (9) by cross multiplication rule we get

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 4 & -2 & -1 \\ -1 & -1 & 3 & -1 \end{array}$$

$$\frac{x_1}{1+4} = \frac{x_2}{12-2} = \frac{x_3}{2+3}$$

$$\Rightarrow \frac{x_1}{5} = \frac{x_2}{10} = \frac{x_3}{5}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

Therefore $X_3=\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ eigenspace for 3=span{X_3}

Result: The Eigen values of A are 1,-2, 3 and the Eigenvectors are $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Non-Symmetric Matrix With Repeated Eigenvalues

1. Find all the Eigenvalues and Eigenvectors of the matrix $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

Solution:

Given: A= $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

To find the characteristic equation of A

Formula: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = sum of main diagonal

= -2+1+0=-1

S_2 = sum of minor of main diagonal elements

$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$

= (0-12)+(0-3)+(-2-4) = -12-3-6=-21

S_3 = Det (A)=|A|

$= \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ = -2(0-12)-2(0-6)-3(-4+1) =45

Hence the characteristic equation is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

if λ=1; 1+1-21-45 ≠0
if λ=-1; -1+1-21-45 ≠0
if λ=2; 8+4-42-45 ≠0
if λ=-2; -8+4+42-45≠0
if λ=3; 27+9-63-45≠0
if λ=-3; -27+9+63-45≠0

Therefore λ=-3 is a root

By synthetic division

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

Therefore the λ= -3 and other roots are given by $\lambda^2 - 2\lambda - 15 = 0$

(λ-5)(λ+3) = 0

λ= 5,-3,-3

Therefore Eigenvalues are 5, -3,-3 and Here the Eigenvalues are repeated.

To find the Eigenvectors:

To get the Eigenvectors solve (A-λI)X=0

$$\left[\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{r}(-2-\lambda)x_1+2x_2-3x_3=0\\ 2x_1+(1-\lambda)x_2-6x_3=0\\ -x_1-2x_2+-\lambda x_3=0\end{array}\right\}\ \ldots \qquad (1)$$

Case 1: Substitute $\lambda=5$ in (1) we get

$$-7x_1+2x_2-3x_3=0 \qquad \ldots.(2)$$
$$2x_1-4x_2-6x_3=0 \qquad \ldots.(3)$$
$$-x_1-2x_2-5x_3=0 \qquad \ldots.(4)$$

Solving (3)and (4) by cross multiplication rule we get

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -4 & -6 & 2 & -4 \\ -2 & -5 & -1 & -2 \end{array}$$

$$\frac{x_1}{20-12}=\frac{x_2}{6+10}=\frac{x_3}{-4-4}$$

$$\Rightarrow \frac{x_1}{8}=\frac{x_2}{16}=\frac{x_3}{-8} \quad \Rightarrow \frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{-1}$$

Therefore $X_1=\begin{pmatrix}1\\2\\-1\end{pmatrix}$

Case 2: Substitute $\lambda=-3$ in (1), we get

$$x_1+2x_2-3x_3=0 \qquad \ldots.\ (5)$$
$$2x_1+4x_2-6x_3=0 \qquad \ldots.\ (6)$$
$$x_1+2x_2-3x_3=0 \qquad \ldots.\ (7)$$

Since (5),(6),(7) are all same, So we considered only one equation

$$x_1+2x_2-3x_3=0$$

Put $x_1=0$

$$2x_2-3x_3=0$$

$$\Rightarrow 2x_2=3x_3$$

$$\frac{x_2}{3}=\frac{x_3}{2}$$

Therefore Eigenvector is $X_2=\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

Put $x_2=0$

$$x_1-3x_3=0$$

$$\Rightarrow x_1=3x_3$$

$$\frac{x_1}{3}=\frac{x_3}{1}$$

eigensapce of -3=span{X_2,X_3}

Therefore Eigenvector is $X_3=\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

Result: The Eigenvalues are-3,-3,5 and Eigenvectors are $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

Symmetric Matrix With Non-Repeated Eigenvalues

1. Find all the Eigen values and Eigen vectors of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Solution:

Given: A= $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

To find the characteristic equation of A

Formula: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = sum of main diagonal

=1+5+1=7

S_2 = sum of minor of main diagonal elements

$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$ = (5-1)+(1-9)+(5-1)=0

$S_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$ =1(5-1)-1(1-3)+3(1-15) = -36

Hence the characteristic equation is $\lambda^3 - 7\lambda^2 + 0\lambda - 36 = 0$

if λ=1; 1-7+0+36 $\neq$0

if λ=-1; -1-7+0+36 $\neq$0

if λ=2; 8-24+0+36 $\neq$0

if λ=-2; -8-24+0+36=0

λ= -2 is a root

To solve the characteristic equation:

if λ=-2 By synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

Therefore the λ=-2 and other roots are given by $\lambda^2 - 9\lambda + 18 = 0$

$$(\lambda-6)(\lambda-3) = 0$$

$$\lambda = 3,6$$

Therefore Eigenvalues are-2, 3, 6

To find the Eigenvectors:

To get the Eigenvectors solve (A-λI)X=0

$$\left[\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{r} (1-\lambda)x_1+x_2+3x_3=0 \\ x_1+(5-\lambda)x_2+x_3=0 \\ 3x_1+x_2+(1-\lambda)x_3=0 \end{array}\right\} \quad \ldots(1)$$

Case 1: Substitute λ=-2 in (1), we get

$$3x_1+x_2+3x_3=0 \ \ldots(2)$$

$$x_1+7x_2+x_3=0 \ \ldots(3)$$

$$3x_1+x_2+3x_3=0 \ \ldots(4)$$

Since (2) and (4) are same we consider, solving (2)and (3) by cross multiplication rule we get

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & 3 & & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\Rightarrow \frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{-10} = \frac{x_2}{0} = \frac{x_3}{10}$$

Therefore $X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Case 2: Substitute $\lambda=3$ in (1), we get

$$-2x_1+x_2+3x_3=0 \quad \ldots (5)$$
$$x_1+2x_2+x_3=0 \quad \ldots (6)$$
$$3x_1+x_2-2x_3=0 \quad \ldots (7)$$

Solving (5) and (6) by cross multiplication rule we get

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & -2 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{2+3} = \frac{x_3}{-4-1}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

Therefore $X_2=\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

Case 3: Substitute $\lambda=6$ in (1), we get

$$-5x_1+x_2+3x_3=0 \quad \dots (8)$$
$$x_1-x_2+x_3=0 \quad \dots (9)$$
$$3x_1+x_2-5x_3=0 \quad \dots (10)$$

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & -5 & & 1 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{1+3}=\frac{x_2}{3+5}=\frac{x_3}{5-1}$$

$$\Rightarrow \frac{x_1}{4}=\frac{x_2}{8}=\frac{x_3}{4} \qquad \Rightarrow \frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{1}$$

Therefore $X_3=\begin{pmatrix}1\\2\\1\end{pmatrix}$

Therefore the Eigenvalues of A are 6,-2, 3

Result: The Eigenvalues of A are 6,-2, 3 and the Eigenvectors are $\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}$

Symmetric Matrix With Repeated Eigenvalues

1. Find all the Eigenvalues and Eigenvectors of $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

Solution:

Given: A= $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

To find the characteristic equation of A

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1= sum of main diagonal

=6+3+3=12

S_2 = sum of minor of main diagonal elements

$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

= (9-1)+(18-4)+(18-4) = 8+14+14 = 36

S_3 = Det (A)=|A|

$= \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$

Hence the characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

if $\lambda=1$; $1-12+36-32 \neq 0$

if $\lambda=-1$; $-1-12-36-32 \neq 0$

if $\lambda=2$; $8-42+72-32=0$

By synthetic division

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

Therefore the $\lambda=2$ is a root

and other roots are given by $\lambda^2 - 10\lambda + 16 = 0$

$$(\lambda-8)(\lambda-2) = 0$$

$$\lambda = 8, 2$$

Therefore Eigenvalues are 8, 2, 2.

To find the Eigenvectors:

To get the Eigenvectors solve $(A-\lambda I)\,X=0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \ldots\ldots \text{(A)}$$

Case (1): If $\lambda = 8$, then the equation (A) becomes

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e) $-2x_1-2x_2+2x_3=0$ .... (1)

$-2x_1-5x_2-x_3=0$ ..... (2)

$2x_1-x_2-5x_3=0$ ..... (3)

Solving (1) and (2) by rule of cross multiplication, we get

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -2 & 2 & -2 & -2 \\ -5 & -1 & -2 & -5 \end{array}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\Rightarrow \frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence the corresponding Eigenvector is $X_1=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case (2): If $\lambda = 2$ then the equation (A) becomes

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e) $4x_1-2x_2+2x_3=0$ .... (4)

$-2x_1+x_2-x_3=0$ .... (5)

$2x_1-x_2+x_3=0$ .... (6)

Here (4), (5), (6) represents the same equation,

$$2x_1 - x_2 + x_3 = 0$$

If $x_1=0$ we get $-x_2+x_3=0$

$$-x_2 = -x_3$$

$$x_2 = x_3$$

$$\text{(i.e)}\ \frac{x_2}{1} = \frac{x_3}{1}$$

Hence the corresponding eigenvector is $X_2=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Let $X_3=\begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as x_3 is orthogonal to x_1 and x_2 since the given matrix is symmetric

$$[2\ -1\ 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \ \text{ or } 2l-m+n=0 \qquad \ldots. \qquad (7)$$

$$[0\ 1\ 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \ \text{ or } 0l+m+n=0 \qquad \ldots. \qquad (8)$$

Solving (7) and (8) by rule of cross multiplication, we get

$$\begin{array}{cccc} & l & m & n \\ -1 & 1 & 2 & -1 \\ 1 & 1 & 0 & 1 \end{array}$$

$$\frac{l}{-1-1}=\frac{m}{0-2}=\frac{n}{2-0}$$

$$\Rightarrow \frac{l}{-2}=\frac{m}{-2}=\frac{n}{2}$$

Hence the corresponding Eigenvector is $X_3=\begin{bmatrix}1\\1\\-1\end{bmatrix}$

Result: The Eigenvalues are 8, 2, 2 and the Eigenvectors are $\begin{bmatrix}2\\-1\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix}$

For finding orthogonal matrix P, normalize all the above eigenvectors