Linear equations with constant coefficients.

An n^{th} order linear ODE with constant coefficients is of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = b(x), \longrightarrow (*)$$

where $a_0 \neq 0, a_1, a_2, \dots a_n \in \mathbb{R}$. b is a real valued function defined on an interval I.

Without loss of generality, we assume that $a_0 = 1$

Then $(*)$ can be written as

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = b(x) \longrightarrow (1)$$

If we denote $L \equiv \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \dots + a_{n-1} \frac{d}{dx} + a_n$, then (1) can be written as

$$L(y) = b(x) \longrightarrow (2).$$

The second order linear homogeneous eqn with constant coefficient.

Consider the ODE

$$(3) \longleftarrow \quad y'' + a_1 y' + a_2 y = 0 \quad (\text{or}) \quad L(y) = 0 \text{ with}$$

$$L \equiv \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_2$$

Suppose that $y(x) = e^{mx}$ is a solution of (3). Then

$$L(e^{mx}) = 0$$

ie $\left(\frac{d^2}{dx^2} + a_1 \frac{d}{dy} + a_2 \right) e^{mx} = 0$

$$\Rightarrow m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = 0$$

$$\Rightarrow (m^2 + a_1 m + a_2) e^{mx} = 0$$

$$\Rightarrow m^2 + a_1 m + a_2 = 0 \qquad (\because e^{mx} \neq 0)$$

The polynomial

$$p(m) = m^2 + a_1 m + a_2$$

is called the characteristic polynomial of L.

Let m_1, m_2 be two roots of $p(m)$.

Case (i) When $m_1 \neq m_2$, $m_1, m_2 \in \mathbb{R}$, then

$\phi_1(x) = e^{m_1 x}$ and $\phi_2(x) = e^{m_2 x}$ are solns of (3).

Case ii) When $m_1 = m_2$, then

$\phi_1(x) = e^{m_1 x}$ and $x e^{m_1 x}$ are solns of (3).

Case (iii) When $m_1 \neq m_2$ and $m_1 = a + ib$, $m_2 = a - ib$, then

$\phi_1(x) = e^{ax} \cos bx$ and $\phi_2(x) = e^{ax} \sin bx$ are solns of (3).

Notice that $\{\phi_1(x), \phi_2(x)\}$ is linearly independent on $\mathcal{F}(\mathbb{R})$,

where

$\mathcal{F}(\mathbb{R})$ = The space of real valued functions defined on $\mathbb{R}$

The n^{th} order linear homogeneous eqn with constant coefficients.

Consider the n^{th} order linear homogeneous eqn

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n = 0, \longrightarrow (1)$$

where $a_1, a_2, \dots a_n \in \mathbb{R}$.

The polynomial $p(m) = m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n$ is called the characteristic polynomial of (1).

Theorem. Assume that all the roots of $p(m)$ are real.
Let $m_1, m_2, \dots, m_s$ be the distinct roots of the characteristic polynomial $p(m)$. Suppose m_i has multiplicity d_i (Thus $d_1 + d_2 + \dots + d_s = n$). Then the 'n' functions

$$e^{m_1 x}, x e^{m_1 x}, \dots, x^{d_1 - 1} e^{m_1 x}$$

$$e^{m_2 x}, x e^{m_2 x}, \dots, x^{d_2 - 1} e^{m_2 x}$$

$$\vdots$$

$$e^{m_s x}, x e^{m_s x}, \dots, x^{d_s - 1} e^{m_s x}$$

are real solutions of $L(y) = 0$. The above collection is a linearly independent set on any interval I.

Theorem

Consider the ODE

$$(*) \leftarrow L(y) \equiv y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0, \quad a_1, a_2, \dots a_n \in \mathbb{R}$$

Let $r_1, \bar{r}_1, r_2, \bar{r}_2, \dots, r_j, \bar{r}_j, r_{2j+1}, \dots r_s$ be the distinct roots of the characteristic polynomial

$$p(r) = r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n,$$

where $r_k = \sigma_k + i\tau_k$ $(k = 1, 2, \dots j)$ σ_k, τ_k are real and $\tau_k \neq 0$ and $r_{2j+1}, \dots r_s$ are real.

Suppose that r_k has multiplicity m_k. (Then $2(m_1 + \dots + m_j) + m_{2j+1} + \dots + m_s = n$)

Then

$$\left.\begin{array}{llll}
e^{\sigma_1 x}\cos\tau_1 x, & x e^{\sigma_1 x}\cos\tau_1 x, & \dots & x^{m_1-1} e^{\sigma_1 x}\cos\tau_1 x, \\
e^{\sigma_1 x}\sin\tau_1 x, & x e^{\sigma_1 x}\sin\tau_1 x, & \dots & x^{m_1-1} e^{\sigma_1 x}\sin\tau_1 x, \\
 & \vdots & & \\
e^{\sigma_j x}\cos\tau_j x, & x e^{\sigma_j x}\cos\tau_j x, & \dots & x^{m_j-1} e^{\sigma_j x}\cos\tau_j x \\
e^{\sigma_j x}\sin\tau_j x, & x e^{\sigma_j x}\sin\tau_j x, & \dots & x^{m_j-1} e^{\sigma_j x}\sin\tau_j x \\
e^{r_{2j+1} x}, & x e^{r_{2j+1} x}, & \dots & x^{m_{2j+1}-1} e^{r_{2j+1} x} \\
 & \vdots & & \\
e^{r_s x}, & x e^{r_s x} & \dots & x^{m_s-1} e^{r_s x}
\end{array}\right\} (**) \text{ fundamental soln}$$

are solns of ①. The collection given (*) is linearly independent on

any interval. Further, every soln of ⊛ can be written as linear combination of (⊛⊛).
(with real coeff.)

<u>Example</u> Find the general soln of

$$y^{(iv)}+y=0 \longrightarrow (1)$$

<u>Soln</u> The char. polynomial of (1) is

$$p(r)=r^4+1$$

The roots of $p(r)$ are

$$\frac{1}{\sqrt{2}}(1+i),\ \frac{1}{\sqrt{2}}(1-i),\ \frac{1}{\sqrt{2}}(-1+i),\ \frac{1}{\sqrt{2}}(-1-i).$$

The fundamental solns are

$$\phi_1(x)=e^{\frac{x}{\sqrt{2}}}\cos\left(\frac{x}{\sqrt{2}}\right)$$

$$\phi_2(x)=e^{\frac{x}{\sqrt{2}}}\sin\left(\frac{x}{\sqrt{2}}\right)$$

$$\phi_3(x)=e^{-\frac{x}{\sqrt{2}}}\cos\left(x/\sqrt{2}\right)$$

$$\phi_4(x)=e^{-x/\sqrt{2}}\sin\left(x/\sqrt{2}\right)$$

The general soln of (1) is

$$y(x)=c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+c_4\phi_4(x)$$

Linear independence and dependence

Theorem (Existence and Uniqueness)

Consider the ODE

$$(1) \leftarrow L(y) \equiv y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_{n-1}(x)\, y' + a_n y = 0$$ on an interval

I satisfying

$$\phi(x_0) = \alpha_0,\ \phi'(x_0) = \alpha_1,\ \dots,\ \phi^{(n-1)}(x_0) = \alpha_{n-1}.$$

where $a_1(x), \dots a_n(x) \in C(I)$.

Then (1) has a unique solution

Theorem. Let $\phi_1, \phi_2, \dots \phi_n$ are 'n' solutions of ODE

$$y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_{n-1}(x)\, y' + a_n y = 0 \text{ on } I,$$

where $a_1(x), a_2(x) \dots, a_n(x) \in C(I)$. Then they are linearly independent on I iff

$$W(\phi_1, \phi_2, \dots \phi_n)(x) := \begin{vmatrix} \phi_1(x) & \phi_2(x) \dots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) \dots & \phi_n'(x) \\ \vdots & & \\ \phi_1^{(n-1)}(x), & \phi_2^{(n-1)}(x) \dots & \phi_n^{(n-1)}(x) \end{vmatrix} \neq 0$$

(Wronskian of $\phi_1, \phi_2 \dots \phi_n$)

for all $x \in I$.

Proof. We prove the result for $n=2$. We can easily modify the result for an arbitrary n.

$\Leftarrow$: Assume that $W(\varphi_1, \varphi_2)(x) \neq 0, \ \forall x \in I$.

Let $c_1, c_2 \in \mathbb{R}$ s.t

$$c_1\varphi_1 + c_2\varphi_2 = 0.$$

ie, $$c_1\varphi_1(x) + c_2\varphi_2(x) = 0, \quad \forall x \in I \quad \text{①}$$

Differentiating ① w.r.to x,

$$c_1\varphi_1'(x) + c_2\varphi_2'(x) = 0, \quad \forall x \in I \quad \text{②}$$

For each fixed $x \in I$, ① & ② are linear homogeneous eqns.

Since $W(\varphi_1, \varphi_2)(x) \neq 0, \forall x \in I$, the above system has a unique soln.

$$\therefore c_1 = c_2 = 0.$$

$\Rightarrow$: Assume that $\{\varphi_1, \varphi_2\}$ is linearly independent on $\mathcal{F}(I)$ (or I)

If possible, there exists $x_0 \in I$ such that

$$W(\varphi_1, \varphi_2)(x_0) = 0.$$

Then the system

$$c_1\varphi_1(x_0) + c_2\varphi_2(x_0) = 0$$

$$c_1\varphi_1'(x_0) + c_2\varphi_2'(x_0) = 0$$

has a soln c_1, c_2 such that at least one of c_1, c_2 is not zero.

Let c_1, c_2 be such soln and consider the function

$$\varphi(x) = c_1\varphi_1(x) + c_2\varphi_2(x).$$

Since φ_1, φ_2 are soln of $L(y)=0$,

$$L(\varphi)=0 \text{ and } \varphi(x_0)=\varphi'(x_0)=0$$

From the previous theorem (Uniqueness)

$$\varphi(x)=0, \forall x \in I \longrightarrow \Leftarrow \text{ to linearly independence of } \{\varphi_1, \varphi_2\}.$$

Example

$$\varphi_1(x)=x^2; \quad \varphi_2(x)=x|x|, \quad x \in (-\infty, \infty)$$

Find the Wronskian of φ_1, φ_2.

Soln

$$\varphi_1'(x)=2x, \qquad \varphi_2'(x)=\begin{cases} 2x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -2x & \text{if } x<0 \end{cases}$$

$$= |2x| = 2|x|$$

$$W(\varphi_1, \varphi_2)(x) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix}$$

$$= 2x^2|x| - 2x^2|x|$$

$$=0.$$

Example $\varphi_1(x)=x, \quad \varphi_2(x)=x^2$

$W(\varphi_1, \varphi_2)(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0$ for any interval that does not contain '0'.