Tutorial Sheet 5

II B. Tech. (Common)
Mathematics-II
(MCI102)

1. Form partial differential equations by eliminating arbitrary constants a and b from the following relations:
   (a) $z = ax + by + a^2 + b^2$ **Ans:** $z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$
   (b) $z = (x-a)^2 + (y-b)^2$ **Ans:** $4z = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$

2. Eliminate the arbitrary functions and hence obtain the partial differential equations:
   (a) $y = f(x-at) + F(x+at)$ **Ans:** $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$
   (b) $z = f\left(\frac{y}{x}\right)$ **Ans:** $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = 0$
   (c) $z = x^n f\left(\frac{y}{x}\right)$ **Ans:** $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = nz$
   (d) $lx + my + nz = \phi\left(x^2 + y^2 + z^2\right)$ **Ans:** $(ny - mz)\left(\frac{\partial z}{\partial x}\right) + (lz - nx)\left(\frac{\partial z}{\partial y}\right) = (mx - ly)$

3. Solve the following partial differential equations:
   (a) $p\tan x + q\tan y = \tan z$ **Ans:** $\frac{\sin x}{\sin y} = f\left(\frac{\sin y}{\sin z}\right)$
   (b) $y^2 p - xyq = x(z - 2y)$ **Ans:** $f(x^2 + y^2,\ zy - y^2) = 0$

4. Solve the following partial differential equations:
   (a) $(D^3 - 4D^2D' + 4DD'^2)z = 0$ **Ans:** $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$
   (b) $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$ **Ans:** $z = \phi_1(y-x) + \phi_2(y+x) + x\phi_3(y+x) + x^2\phi_4(y+x)$
   (c) $(2D^2 - 5DD' + 2D'^2)z = 24(y-x)$ **Ans:** $z = \phi_1(2y+x) + \phi_2(y+2x) + \frac{(y-x)^3}{5}$
   (d) $q = (z+px)^2$ **Ans:** $z = \phi_1(y-x) + x\phi_2(y-x) + \frac{(e^{2x+3y})}{25}$

5. Find the complete integrals of the following equations:
   (a) $q = 3p^2$ **Ans:** $z = ax + 3a^2y + b$
   (b) $p^2 - y^2q = y^2 - x^2$ **Ans:** $z = \left(\frac{x}{2}\right)(a^2 - x^2)^{\frac{1}{2}} + \left(\frac{a^2}{2}\right)\sin^{-1}\left(\frac{x}{a}\right) - \left(\frac{a^2}{y}\right) - y + b$
   (c) $z^2(z^2p^2 + q^2) = 1$ **Ans:** $9a^4(ax + y + b)^2 = (a^2z^2 + 1)^3$
   (d) $z = \phi_1(y-x) + x\phi_2(y-x) + \frac{e^{(2x+3y)}}{25}$ **Ans:** $xz = ay + 2\sqrt{ax} + b$

6. Show that a family of spheres $x^2 + y^2 + (z-c)^2 = r^2$ satisfies the first-order linear partial differential equation $yp - xq = 0$.

7. Show that a family of spheres $(x-a)^2 + (y-b)^2 + z^2 = r^2$ satisfies the first-order nonlinear partial differential equation $z^2(p^2 + q^2 + 1) = r^2$.

8. Find the general solution of the first-order linear partial differential equations:
   (a) $xu_x + yu_y = u$ **Ans:** $u(x,y) = x^n g\left(\frac{y}{x}\right)$
   (b) $x^2u_x + y^2u_y = (x+y)u$, where $u = u(x,y)$ **Ans:** $u(x,y) = xyh\left(\frac{x-y}{xy}\right)$

9. Obtain the solutions of the equations:
   (a) $(y-u)u_x + (u-x)u_y = x - y$ with the condition $u = 0$ on $xy = 1$.
   (b) $x(y^2+u)u_x - y(x^2+u)u_y = (x^2 - y^2)u$, with the data $x + y = 0,\ \ u = 1$.
   Ans:(a) $u(x,y) = \frac{1-xy}{x+y}$
   (b) $2xyu + x^2 + y^2 - 2u + 2 = 0$.

10. Use the separation of variables $u(x,y) = f(x) + g(y)$ to solve the equations :
    (a) $u_x^2 + u_y^2 = 1$, **Ans:** $u(x,y) = \lambda x + y\sqrt{1-\lambda^2} + C$
    (b) $u_x^2 + u_y + x^2 = 0$. **Ans:** $u(x,y) = \frac{1}{2}\lambda^2\sin^{-1}\left(\frac{x}{\lambda}\right) + \frac{x}{2}\sqrt{\lambda^2 - x^2} - \lambda^2 y + C$

11. Deteremine the region in which the given equation is hyperbolic, parabolic, or elliptic, and transform the equation in the respective region to canonical form.
    (a) $y^2u_{xx} - x^2u_{yy} = 0$,

(b) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$.
Ans: (a) Hyperbolic everywhere except on the coordinate axes $x = 0$ and $y = 0$;
$u_{\xi\eta} = \frac{\eta}{2(\xi^2-\eta^2)}u_\xi - \frac{\xi}{2(\xi^2-\eta^2)}u_\eta$,
(b)Parabolic everywhere; $u_{\eta\eta} = 0$ for $y \neq 0$.

12. Obtain the general solution of the following equations:
(a) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$ **Ans:** $u(x, y) = yf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$
(b) $4u_{xx} + 5u_{xy} + u_{yy}u_x + u_y = 2$ **Ans:** $u(x, y) = \frac{8}{3}\left(y - \frac{1}{4}\right) + \frac{1}{3}g\left(y - \frac{x}{4}\right)e^{\frac{1}{3}(y-x)} + f(y - x)$

13. Find the characteristic equations and characteristics, and then reduce the equations $u_{xx} \mp (sech^4x)u_{yy} = 0$ to the canonical forms.
Ans: Canonical forms: $u_{\xi\eta} = \frac{(\eta-\xi)}{[4-(\xi-\eta)^2]}(u_\xi - u_\eta)$ and $u_{\alpha\alpha} + u_{\beta\beta} = \frac{2\beta}{1-\beta^2}u_\beta$, $|\beta| < 1$.

14. Solve $x(z + 2a)p + (xz + 2yz + 2ay)q = z(z + a)$. **Ans:** $\phi\left\{\frac{(x+y)}{z^2}, \frac{x(z+a)}{z^2}\right\} = 0$

15. Find the solution of $2x(y + z^2)p + y(2y + z^2)q = z^3$. **Ans:** $\phi\left\{\frac{x}{y^2}, \frac{z}{y} - \frac{2}{z}\right\} = 0$

16. Solve $(x + y + z)(p - q) + a(px - qy + x - y) = 0$. **Ans:** $\phi\{u + z, av^2 + 4uz - au^2\} = 0$

17. Find the surface whose tangent planes cut off an intercept of constant length k from the axis of z.
Ans: $\phi\left\{\frac{y}{x}, \frac{z-k}{x}\right\} = 0$

18. Find the complete integral of $p^2 + q^2 = (x^2 + y^2)z$.
Ans: $4z^{\frac{1}{2}} = x(x^2 + a^2)^{\frac{1}{2}} + a^2sinh^{-1}\left(\frac{x}{a}\right) + y(y^2 - a^2)^{\frac{1}{2}} - a^2cosh^{-1}\left(\frac{y}{a}\right) + b$

19. Solve $(D + D')^2z = 2\cos y - x\sin y$. **Ans:** $z = \phi_1(y - x) + x\phi_2(y - x) + x\sin y$

20. Find the solution of $(D^3 + D^2D' - DD'^2 - D'^3)z = e^y\cos 2x$
Ans: $z = \phi_1(y + x) + \phi_2(y - x) + x\phi_3(y - x) - \frac{1}{25}e^y(\cos 2x + 2\sin 2x)$

21. Find the solution of $(D^2 + DD' - 6D'^2)z = x^2\sin(x + y)$
Ans: $z = \phi_1(y - 3x) + \phi_2(y + 2x) + \left[\frac{x^2}{4} - \frac{13}{32}\right]\sin(x + y) - 3\frac{x}{8}\cos(x + y)$

22. Reduce the equation to canonical form $u_{xx} + (2\csc y)u_{xy} + (\csc^2 y)u_{yy} = 0$. **Ans:** $u_{\eta\eta} = (\sin^2\eta\cos\eta)u_\xi$

23. Use $u = f(\xi)$, $\xi = \frac{x}{\sqrt{y\kappa t}}$ to solve the parabolic system

$$u_t = \kappa u_{xx}, \quad -\infty < x < \infty, \quad t > 0u(x, 0) = 0, \quad x < 0; \quad u(x, 0) = u_0, \quad x > 0,$$

where κ and u_0 are constants.
Ans: $u(x, t) = u_0\left[\frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4\kappa t}}} e^{-\alpha^2}\,d\alpha + \frac{1}{2}\right]$

24. Find the general solution of the wave equation $u_{tt} - c^2u_{xx} = 0$, where c is constant.
Ans: $u(x, t) = \phi(x + ct) + \psi(x - ct)$, provided ϕ and ψ are arbitrary but twice differentiable functions.

25. Use separation of variables $u(x, y) = X(x)Y(y) \neq 0$, Solve the initial value problem

$$u_x + 2u_y = 0, u(0, y) = 4e^{-2y}$$

. **Ans:** $u(x, y) = 4e^{4x-2y}$.

26. Use separation of variables $u(x, y) = f(x)g(y) \neq 0$, give the general solution of the equation $y^2u_x^2 + x^2u_y^2 = (xyu)^2$. **Ans:** $u(x, y) = c\exp\frac{\lambda}{2}x^2 + \frac{1}{2}y^2\sqrt{1 - \lambda^2}$, where c is an arbitrary constant.

27. Reduce the following equations

$$u_x - u_y = u,$$

$$yu_x + u_y = x,$$

to canonical form, and obtain the general solution.
Ans: $u(x, y) = f(x + y)e^{-y}$, and $u(x, y) = xy - \frac{1}{3}y^3 + f\left(x - \frac{y^2}{2}\right)$, where f is an arbitrary function.

28. Use $v = \ln u$ and $v = f(x) + g(y)$ to solve the equation $x^2 u_x^2 + y^2 u_y^2 = u^2$.
Ans: $u(x, y) = e^v = C x^\lambda y^{\sqrt{1-\lambda^2}}$, where C is an integrating constant.

29. Find the integral surface of the equation $u u_x + u_y = 1$, so that the surface passes through an initial curve represented parametrically by $x = x_0(s), \quad y = y_0(s), \quad u = u_0(s)$, where s is a parameter.
Ans: $F(x, y, s) = 2x - (y - 2s)^2 - 4s^2 = 0$.

30. Find the solution of the characteristic initial-value problem

$$y^3 u_{xx} - y u_{yy} + u_y = 0, u(x, y) = f(x) \text{ on } x + \frac{y^2}{2} = 4 \text{ for } 2 \leq x \leq 4, u(x, y) = g(x) \text{ on } x - \frac{y^2}{2} = 4 \text{ for } 0 \leq x \leq 2,$$

with $f(2) = g(2)$. **Ans:** $u(x, y) = f\left(\frac{x}{2} - \frac{y^2}{4} + 2\right) + g\left(\frac{x}{2} + \frac{y^2}{4}\right) - f(2)$