Theorem

Let $M(x,y)\,dx + N(x,y)\,dy = 0$ is a homogeneous ODE.

(i) If $xM + yN \neq 0$, then

$$\frac{1}{xM+yN}$$ is an integrating factor

(ii) If $xM + yN = 0$, then

$$\frac{1}{xy}, \text{ or } \frac{1}{x^2}, \frac{1}{y^2}$$ is an integrating factor.

First order linear ODE.

Theorem

Consider the ODE

$$\frac{dy}{dx} + p(x)y = q(x) \quad \longrightarrow (1)$$

Then I.F of (1) is

$\pm e^{\int p(x)\,dx}$, where either $+$ or $-$ sign maybe chosen.

The solns of (1) are given by

$$y\, e^{\int p\,dx} = \int q(x)\, e^{\int p(x)\,dx}\, dx + C.$$

Bernoulli's equation

Consider the ODE

$$\frac{dy}{dx} + p(x)\,y = q(x)\,y^n, \quad n \in \mathbb{R}.$$

$\longrightarrow (2)$

- When $n = 0$ or 1, then ① is linear

- When $n > 1$, choose $v(x) = [y(x)]^{1-n}$. Then

$$\frac{dv}{dx} = (1-n)\, y(x)^{-n} \cdot \frac{dy}{dx}$$

$$= (1-n)\, y^{-n} \cdot \left(q(x)\, y^{n} - p(x)\, y \right)$$

$$= (1-n) \left[q(x) - p(x)\, y^{1-n} \right]$$

$$= (1-n) \left[q(x) - p(x)\, v(x) \right]$$

$$\Rightarrow \boxed{\frac{dv}{dx} + (1-n)\, p(x)\, v(x) = (1-n)\, q(x)}$$ — First order linear ODE.

Example

$$(x-a)\frac{dy}{dx} + 3y = 12(x-a)^3, \quad x \neq a. \quad \rightarrow (*)$$

Soln

$$\frac{dy}{dx} + \frac{3}{x-a}\, y = 12(x-a)^2.$$

$$p(x) = \frac{3}{x-a}\ ; \quad q(x) = 12(x-a)^2$$

$$\text{I.F} = \pm e^{\int p(x)\,dx}$$

$$= \pm e^{\int \frac{3}{x-a}\,dx} = \pm e^{3 \ln|x-a|}$$

$$= \pm |x-a|^3$$

If we choose + sign for $x > a$ and − sign for $x < a$, the

$(x-a)^3$ is an I.F of ⊛.

∴ The general soln is

$$y(x)\,(x-a)^3 = \int 12(x-a)^2 (x-a)^3\, dx + c \quad , x \neq a$$

$$= \cancel{12}^{2} \frac{(x-a)^6}{\cancel{6}} + c$$

$$\Rightarrow \boxed{y(x) = 2(x-a)^3 + c\,(x-a)^{-3}} \;, x \neq a$$

<u>Example</u>

$$y' + 4xy = -xy^3.$$

<u>Soln</u> It is a Bernoulli's eqn.

Here $p(x) = 4x$, $q(x) = -x$, $n = 3$.

Take $v(x) = y^{-2}$. Then

$$\frac{dv}{dx} - 8x\, v(x) = 2x \longrightarrow ⊛.$$

An I.F of (⊛) is

$$I.F = \pm \exp\left(\int p\, dx\right) = \pm e^{\int -8x\, dx} = \pm e^{-4x^2}$$

Choose $I.F = e^{-4x^2}$, $x \in \mathbb{R}$. The g. soln of (⊛) is

$$v(x)\, e^{-4x^2} = \int 2x\, e^{-4x^2}\, dx + c$$

$$v(x)\, e^{-4x^2} = -\frac{1}{4} e^{-4x^2} + c$$

$$\Rightarrow v(x) = -\frac{1}{4} + c e^{4x^2}$$

$$\Rightarrow y(x) = \left(c e^{4x^2} - 1/4\right)^{-1/2}.$$

Orthogonal trajecteries.

Consider the family of curves

$$f(x, y, c) = 0 \longrightarrow (*)$$

where c is an arbitrary parameter.

To find the eqn of family of curves orthogonal to $(*)$.

Recall that two curves are orthogonal if their tangent lines are perpendicular at every point of intersection.

Differenting $(*)$ w.r.to x, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0 \longrightarrow (**)$$

After eliminating c from $*)$ & $(**)$, let the ODE be

$$F\left(x, y, \frac{dy}{dx}\right) = 0.$$

The orthogonal family of curves cuts the family of curves $(*)$ at right angles. $\therefore$ The desired family of curve is given by $F\left(x, y, -\frac{dx}{dy}\right) = 0. \longrightarrow (1)$

This is the first order ODE. Its general soln is of the form

$$\phi(x, y, k) = 0 \longrightarrow ③$$

where k is an arbitrary parameter. The family of curves determined by ③ are called orthogonal trajectories of (*).

In polar coordinates.

If the eqn of a family of curve is given in polar coordinates

$$f(r, \theta, c) = 0 \longrightarrow (*)$$

then the orthogonal trajectories of (*) are obtained by replacing $r \frac{d\theta}{dr}$ by $-\frac{1}{r}\frac{dr}{d\theta}$ in the D.E of (*).

Example

Find the family of curves orthogonal to the the family of circles

$$x^2 + y^2 - cx = 0 \rightarrow ①$$

Soln

Differentiating ① w.r.to x

$$2x + 2y\frac{dy}{dx} - c = 0 \longrightarrow ②$$

$$\Rightarrow 2x^2 + 2xy\frac{dy}{dx} - cx = 0 \longrightarrow ③$$

$$\Rightarrow 2x^2 + 2xy\frac{dy}{dx} - x^2 - y^2 = 0 \quad \text{from ①}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

(By substituting $y = vx$)

$$\frac{y}{x^2+y^2} = k$$

$$\Rightarrow x^2 + y^2 - cy = 0$$