Linear transformations.

Definition.

Let V and W be two vector spaces over K. A function $T : V \to W$ is said to be a linear transformation from V into W if

(i) $T(x+y) = T(x+y)$, $\forall\, x, y \in V$

(ii) $T(cx) = cT(x)$, $\forall\, x \in V$ and $c \in K$

(or)

$T(cx+y) = cT(x)+T(y)$, $\forall\, x, y \in V$ & $c \in K$

Properties.

(i) $T(0) = 0$

(ii) $T(-v) = -T(v)$, $\forall\, v \in V$

1. Show that in a vector space V, if $x+y = x+z$, then $y=z$. (Cancellation law)

Example 1.

Let $T : V \to V$ by $T(v) = v$, $\forall\, v \in V$. Then

$$T(cv+w) = cv+w$$

$$= cTv + T(w).$$

$\therefore$ T is a linear transformation. T is called the identity transformation and it is denoted by I.

Example 2.

Let $T : V \to W$ by $T(v) = 0$, $\forall\, v \in V$.

Then T is a linear transformation and it is called the zero transformation '0'.

Example 3.

$$V = \mathbb{K}[x], \quad W = \mathbb{K}[x]$$

Define $D : V \to W$ by $Df = f'$, ie $(Df)(x) = f'(x)$.

Then

$$D(\alpha f + g)(x) = (\alpha f + g)'(x)$$
$$= \alpha f'(x) + g'(x)$$
$$= \alpha (Df)(x) + (Dg)(x)$$

ie, $D(\alpha f + g) = \alpha Df + Dg$.

$\therefore$ D is a linear transformation from $\mathbb{K}[x]$ into $\mathbb{K}[x]$.

Example 4.

Let A be a fixed $m \times n$ matrix with entries in $\mathbb{K}$.

Define $T : \mathbb{K}^{n\times 1} \to \mathbb{K}^{m \times 1}$ by

$$T(X) = AX,$$

ie,
$$T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ & \vdots & & \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Then

$$T(\alpha X + Y) = A(\alpha X + Y)$$
$$= A(\alpha X) + AY$$
$$= \alpha AX + AY$$
$$= \alpha TX + TY$$

$\therefore$ T is linear.

Example 5.

$V = C(\mathbb{R})$ = The space of real continuous functions defined on $\mathbb{R}$

Define $T : V \to V$ by

$$(Tf)(x) = \int_0^x f(t)\,dt$$

Then

$$T(\alpha f+g)(x) = \int_0^x (\alpha f+g)(t)\,dt$$

$$= \alpha \int_0^x f(t)\,dt + \int_0^x g(t)\,dt$$

$$= \alpha (Tf)(x) + (Tg)(x)$$

i.e, $T(\alpha f+g) = \alpha Tf + Tg$

Range space and Null space

Let $T : V \to W$ be a linear transformation.

Define

$$\ker(T) \equiv N(T) := \{ v \in V : Tv = 0 \}$$

and

$$R(T) \equiv Im(T) := \{ w \in W : Tv = w \text{ for some } v \in V \}$$

Lemma

(i) $N(T)$ is a subspace of V

(ii) $R(T)$ is a subspace of W.

Proof. Since $T(0) = 0$, $N(T) \neq \phi$.

Let $c \in \mathbb{K}$ and $v_1, v_2 \in N(T)$. Then

$$T(cv_1+v_2) = cTv_1 + Tv_2$$

$$= c0 + 0$$

$$= 0$$

$$\Rightarrow cv_1+v_2 \in N(T).$$

$\therefore$ $N(T)$ is a subspace of V.

(ii) Since $T(0)=0$, $R(T) \neq \phi$.

Let $c \in \mathbb{K}$ and $w_1, w_2 \in R(T)$. Then $\exists$ $v_1, v_2 \in V$ such that

$$T(v_1) = w_1 \quad \text{and} \quad T(v_2) = w_2$$

Now Take $v = cv_1+v_2 \in V$ Then

$$T(cv_1+v_2) = cTv_1 + Tv_2$$

$$= cw_1+w_2$$

$$\Rightarrow cv_1+v_2 \in R(T).$$

$\therefore$ $R(T)$ is a subspace of W

- The space $N(T)$ is called the null space of T.
- The space $R(T)$ is called the range space of T.
- The dimension of $N(T)$ is denoted by nullity(T)
- The dimension of $R(T)$ is denoted by rank(T).