Differential Equations.

A differential equation is an equation that relates one (or more) functions and their derivatives.

DE

ODE (eqn that relates a function $y(x)$ of one variable and its derivatives)

PDE (eqn that relates a function y depending on more than one variable and its partial derivatives)

A general ODE can be written as

$$F(x, y, y', \dots, y^{(n)}) = 0 \quad \leftarrow (1) \qquad \text{where } y = y(x), \quad y^{(k)}(x) = \frac{d^k y}{dx^k}.$$

- The order of the ODE (1) is defined as the order of the highest derivative of y that occurs in the equation.

For example,

$$x^2 + 2x y'' + 2x y' = 0$$

The order of the above eqn is 2.

- The degree of the ODE (1) is defined as greatest power of the highest order derivative occuring in the eqn after the equation has been made free of radicals and fractions in its derivatives.

For example

$$k\frac{d^2y}{dx^2} = f(x)\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}, k>0$$

The degree of the above eqn is 2.

Consider the n^{th} order ODE

$$F(x, y, y', \ldots y^{(n)}) \equiv 0 \longrightarrow ①$$

(or)

$$y^{(n)} = f(x, y, y', \ldots y^{(n-1)}) \longrightarrow ②$$

The ODE ① is said to be linear if it can be written as

$$③a \leftarrow a_0(x)\, y^{(n)}(x) + a_1(x)\, y^{(n-1)}(x) + \ldots + a_{n-1}(x)\, y'(x) + a_n(x)\, y(x) = f(x)$$

with $a_0(x) \neq 0$. We assume that the functions $a_0(x), \ldots a_n(x), f(x)$ are defined over some interval $a \le x \le b$. The functions $a_0(x), a_1(x), \ldots a_n(x)$ are called coefficients of the equation and $f(x)$ is called a forcing function.

The equation ③ is often written as

$$L(y) = f(x), \longrightarrow ③b$$

where $L \equiv a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{d}{dx} + a_n(x)$

If $f(x) \equiv 0$ on $a \le x \le b$, then ③ is called a homogeneous linear ODE.

- When one or more coefficients fns of ③ depends on x, then ③ is called variable coefficient equation.

- When all the coefficient functions are constants, then ③ is called constant coefficient equation.

Solutions of ODE.

- A function $\varphi(x)$ is said to be a solution of ① if it satisfies the equation ① for $y=\varphi(x)$. In this case, φ is called an integral curve.

- A solution of n^{th} order ODE ① that contains 'n' arbitrary constants is called the general solution of ①. If the arbitrary constant are assigned specific value, then it is called a particular solution of ①

- A solution that can not be obtained from the general solution by assigning suitable value for arbitrary constants is called a singular solution of ①.

Soln

General soln　　Singular soln

Theorem An n^{th} order linear ODE has no singular solutions

Lemma

If $y_1, y_2, \ldots y_n$ are solutions of the linear homogeneous DE

$$L(y) \equiv 0, \longrightarrow ①$$

then $c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is also soln of ①.

Proof.

$$L(c_1y_1 + \ldots + c_ny_n) = c_1 L(y_1) + c_2L(y_2) + \ldots + c_nL(y_n)$$

$$= 0$$

Consider the non-homogeneous linear DE

$$L(y) = f(x), \longrightarrow ③$$

The soln of ①, ie $\underbrace{c_1y_1 + \ldots + c_ny_n}_{y_c(x)}$, is called the complementary solution (function) of ③.

A solution $y_p(x)$ of ③ which does not contain any arbitrary constants is called a particular integral of ③.

The complete solution of ③ is given by

$$\boxed{y(x) = y_c(x) + y_p(x)}$$

First order ODE.

Consider the first order ODE

$$F(x, y, y') = 0 \longrightarrow (*)$$

or $\quad y' = f(x, y)$

Separable equations.

Suppose that $f(x,y) = g(x)h(y)$ in ⊛.

Then ① can be written as

$$y'(x) \equiv \frac{dy}{dx} = g(x)\,h(y)$$

$$\Rightarrow \quad \frac{y'(x)}{h(y(x))} = g(x)$$

Integrating both sides, we get

$$\int \frac{y'(x)}{h(y(x))}\,dx = \int g(x)\,dx + C$$

Let $u = y(x)$, $\frac{du}{dx} = y'(x)$ $\Big\} \Rightarrow \int \frac{1}{h(u)}\,du = \int g(x)\,dx + C$

Symbolically, we write

$$\frac{1}{h(u)}\,du = g(x)\,dx$$

Example $y' - 2y + a = 0$, a is constant.

Soln

$$y' = 2y - a$$

$$\frac{dy}{dx} = 2y - a$$

$$\frac{dy}{2y-a} = dx$$

Integrating both sides, we get

$$\frac{1}{2}\int \frac{2\,dy}{2y-a} = \int dx + C$$

$\Rightarrow \frac{1}{2}\ln|2y-a| = x + C$, if $y \neq a/2$.

$\ln|2y-a| = 2x + C$, C → generic const

$\Rightarrow |2y-a| = k e^{2x}$, $k > 0$.

$\Rightarrow 2y - a = \pm k e^{2x}$

$$\boxed{y = \pm c\, e^{2x} + a/2}$$