The non-homogeneous equation of order n

Consider the n^{th} order LDE with constant coeff

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_{n-1} y' + a_n y = b(x), \longrightarrow (1)$$

where $a_1, a_2, \dots a_n$ are constants, and $b(x)$ is continuous on I.

If ψ_p is a particular soln of (1) and ψ is any other soln, then

$$L(\varphi - \varphi_p) = L(\varphi) - L(\varphi_p) = 0$$

$\Rightarrow$ $\varphi - \psi_p$ is a soln of $L(y) \equiv 0$,

$\Rightarrow$ Any soln of (1) can be written as

$$\psi = \psi_p + c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n.$$

where ψ_p is a p. soln of $L(y) = b(x)$, $c_1, c_2, \dots c_n$ are constants.

Method of variation of parameters

Theorem (To find a particular soln ψ_p).

Let $\varphi_1, \varphi_2, \dots \varphi_n$ are n l. independent solns of $L(y) \equiv 0$ on I. Then every soln $\psi(x)$ of $L(y) = b(x)$ can be written as

$$\psi = \psi_p + c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n.$$

where ψ_p is a p. soln of $L(y) = b(x)$.

A particular soln ~~of~~ ψ_b can be obtained by the following formula:

$$\psi_p(x) = \sum_{k=1}^{n} \varphi_k(x)\, u_k(x),$$

where

$$u_k(x) = \int_{x_0}^{x} \frac{W_k(t)\, b(t)}{W(\varphi_1, \varphi_2, \dots \varphi_n)(t)}\, dt, \qquad x_0 \in I.$$

with

$$W_k(t) = \begin{vmatrix} \varphi_1(t) & \varphi_2(t) & \dots & \varphi_{k-1}(t) & 0 & \varphi_{k+1}(t) & \dots & \varphi_n(t) \\ \varphi_1'(t) & \varphi_2'(t) & \dots & \varphi_{k-1}'(t) & 0 & \varphi_{k+1}'(t) & \dots & \varphi_n'(t) \\ & & & \vdots & & & & \\ \varphi_1^{(n-1)}(t) & \varphi_2^{(n-1)}(t) & \dots & \varphi_{k-1}^{(n-1)}(t) & 1 & \varphi_{k+1}^{(n-1)}(t) & \dots & \varphi_n^{(n-1)} \end{vmatrix}$$

$k = 1, 2 \dots n.$

Further, $\psi_p(x_0) = \psi_p'(x_0) = \dots = \psi_p^{(n-1)}(x_0) = 0.$

Example

$$y''' + y'' + y' + y = 1.$$

Soln The char poly is

$$p(r) = r^3 + r^2 + r + 1$$

The roots are $i, -i, \cancel{+}1.$

$\therefore \phi_1(x) = \cos x, \quad \phi_2(x) = \sin x, \quad \phi_3(x) = e^{-x}$

are l. independent solns of $L(y)=0$.

$$W(\phi_1,\phi_2,\phi_3)(x) = \begin{vmatrix} \cos x & \sin x & e^{-x} \\ -\sin x & \cos x & -e^{-x} \\ -\cos x & -\sin x & e^{-x} \end{vmatrix}$$

(W.K.T

$$W(\phi_1,\phi_2,\phi_3)(x) = e^{-x} W(\phi_1,\phi_2,\phi_3)(0) \quad (\because a_1 = 1))$$

$$= e^{-x} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = 2e^{-x}.$$

$\boxed{x_0=0}$

$$u_1(x) = \int_0^x \frac{\begin{vmatrix} 0 & \sin t & e^{-t} \\ 0 & \cos t & -e^{-t} \\ 1 & -\sin t & e^{-t} \end{vmatrix}}{2e^{-t}} \cdot 1 \cdot dt$$

$$= \int_0^x \frac{1}{2} e^{t} \left(-e^{-t}\sin t - \cos t\, e^{-t}\right) dt$$

$$= \int_0^x -\frac{1}{2}(\cos t + \sin t)\, dt = -\frac{1}{2}\left(\sin t - \cos t\right)\Big|_0^x$$

$$= -\frac{1}{2}\left[\sin x - \cos x + 1\right]$$

$$u_2(x) = \frac{1}{2}(\sin x + \cos x) - \frac{1}{2}$$

$$u_3(x) = \frac{1}{2}e^{x} - \frac{1}{2}.$$

$$\therefore \psi_p(x) = \varphi_1(x)u_1(x) + \varphi_2(x)u_2(x) + \varphi_3(x)u_3(x)$$

$$= \cos x\left(\cancel{-\frac{1}{2}\sin x} + \frac{1}{2}\cos x - \frac{1}{2}\right)$$

$$+ \sin x\left(\frac{1}{2}\sin x + \cancel{\frac{1}{2}\cos x} - \frac{1}{2}\right)$$

$$+ e^{-x}\left(\frac{1}{2}e^{x} - \frac{1}{2}\right)$$

$$= \frac{1}{2}(\cos^2 x) - \frac{1}{2}\cos x + \frac{1}{2}\sin^2 x - \frac{1}{2}\sin x$$

$$+ \frac{1}{2} - \frac{1}{2}e^{-x}$$

$$= \frac{1}{2} - \frac{1}{2}\cos x - \frac{1}{2}\sin x - \frac{1}{2}e^{-x} + \frac{1}{2}$$

$$= 1 - \frac{1}{2}(\cos x + \sin x + e^{-x}).$$

$\therefore$ The g. soln is

$$\varphi(x) = 1 - \frac{1}{2}(\cos x + \sin x + e^{-x}) + \tilde{c}_1 \cos x + \tilde{c}_2 \sin x + \tilde{c}_3 e^{-x}$$

$$= 1 + c_1 \cos x + c_2 \sin x + c_3 e^{-x}$$

$$u_2(x) = \int_0^x \frac{\begin{vmatrix} \cos x & 0 & e^{-x} \\ -\sin x & 0 & -e^{-x} \\ -\cos x & 1 & e^{-x} \end{vmatrix}}{2e^{-t}} \, dt$$

$$= -\int_0^x \left(-e^{-t}\cos t + e^{-t}\sin t\right) \frac{1}{2e^{-t}} \, dt$$

$$= -\int_0^x \frac{(\sin t - \cos t)}{2} \, dt$$

$$= \left. \frac{+\cos t + \sin t}{2} \right|_0^x$$

$$= \cancel{\cos t + \sin} \quad \frac{\cos x + \sin x}{2} - \frac{1}{2}.$$

Theorem. Let $\varphi_1, \varphi_2, \ldots \varphi_n$ be 'n' (~~linearly independent~~) solns of

$$L(y) \equiv y^{(n)} + a_1(x)\, y^{(n-1)} + \ldots + a_{n-1}(x)\, y' + a_n(x) = 0 \text{ on } I.$$

Let $x_0 \in I$. Then

$$W(\varphi_1, \varphi_2, \ldots \varphi_n)(x) = e^{-\int_{x_0}^{x} a_1(t)\,dt} \cdot W(\varphi_1, \varphi_2, \ldots \varphi_n)(x_0)$$

Corollary. Let $\varphi_1, \varphi_2, \ldots \varphi_n$ be 'n' solns of $L(y) = 0$ on I containing x_0. Then they are l. independent on I iff

$$W(\varphi_1, \varphi_2, \ldots \varphi_n)(x_0) \neq 0.$$

Method of undetermined coefficients. Consider the LODE with const $L(y) = b(x)$.

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$b(x) \to$ linear combination of elementary functions

It is not applicable when

Example. $b(x) = \ln x, \frac{1}{x}, \tan x, \sin^{-1} x$

~~Suppose~~ Assume that

$$b(x) = \begin{cases} P(x) \to \text{polynomial} \\ P(x)\, e^{\alpha x} \\ P(x)\, e^{\alpha x} \sin \beta x \\ P(x)\, e^{\alpha x} \cos \beta x \end{cases}$$

If y_p is a p. soln of (1), then

$$L(y_p) = y_p^{n} + a_1 y_p^{(n-1)} + \dots + a_{n-1} y_p' + a_n y_p$$

$= b(x)$

$\Rightarrow$ y_p has the same form as $b(x)$.

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6 \longrightarrow (*)$

Soln

The auxiliary eqn $p(r) = m^2 + 4m - 2$

$P(r) = 0 \Rightarrow -2-\sqrt{6}, -2+\sqrt{6}$

The C.F. is $e^{(-2+\sqrt{6})x}$, $e^{(-2-\sqrt{6})x}$ are fund. soln

$$y_h(x) = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6}x)}$$

Let $y_p(x) = Ax^2 + Bx + C \longrightarrow (1)$

be the p. soln of (*). Then

$y_p'(x) = 2Ax + B$

$y_p''(x) = 2A$

$\therefore\ y_p'' + 4y_p' - 2y_p = 2x^2 - 3x + 6$

$$\Rightarrow 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C$$

$$= 2x^2 - 3x + 6$$

$$\Rightarrow -2Ax^2 + (8A - 2B)x + (2A + 4B - 2C)$$

$$= 2x^2 - 3x + 6$$

$$\Rightarrow \quad -2A = 2 \Rightarrow \boxed{A = -1}$$

$$8A - 2B = -3 \Rightarrow -8 - 2B = -3$$

$$-2B = 5$$

$$\boxed{B = -\frac{5}{2}}$$

$$2A + 4B - 2C = 6$$

$$\Rightarrow -2 - 10 - 2C = 6$$

$$-2C = 18$$

$$\boxed{C = -9}$$

$$\therefore\ y_p(x) = -x^2 - \frac{5}{2}x - 9$$

$$y = y_c(x) + y_p(x)$$

Example 2

Find a p.s. $y'' - y' + y = 2\sin 3x$ $\longrightarrow (1)$

Soln

Let $y_p(x) = A\cos 3x + B\sin 3x$ be a p. soln of (1).

$y_p'(x) = -3A\sin 3x + 3B\cos 3x$

$y_p''(x) = -9A\cos 3x - 9B\sin 3x$

$\therefore \; y_p'' - y_p' + y' = 2\sin 3x$

$\Rightarrow -9A\cos 3x - 9B\sin 3x + 3A\sin 3x - 3B\cos 3x$
$+ A\cos 3x + B\sin 3x = 2\sin 3x$

$\Rightarrow (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$

(w.k.T $\cos 3x$, $\sin 3x$ are l.i.) $\therefore$

$-8A - 3B = 0$ and

$3A - 8B = 2$

$\Rightarrow A = 6/73, \; B = -16/73.$

$\therefore \; y_p(x) = \dfrac{6}{73}\cos 3x - \dfrac{16}{73}\sin 3x$

(13)

Ex.

$$y'' - 2y' - 3y = 4x - 5 + \boxed{6xe^{2x}}$$

Soln

$$p(r) = r^2 - 2r - 3$$

$$\Rightarrow \quad y_c(x) = c_1 e^{-x} + c_2 e^{3x}$$

$$y_p(x) = Ax + B + Cxe^{2x} + Ee^{2x}$$

$$\Rightarrow -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x}$$

$$\Rightarrow \quad -3A = 4$$

$$-2A - 3B = -5$$

$$-3C = 6$$

$$2C - 3E = 0$$

$$\Rightarrow \quad A = -4/3, \quad B = \frac{23}{9}, \quad C = -2, \quad E = -4/3.$$

Trial p. soln

$b(x)$	Form of y_p
1	A
$5x+7$	$Ax+B$
$3x^2-7$	Ax^2+Bx+C
$x^3 - x + 1$	Ax^3+Bx^2+Cx+E
$8\sin 4x$	$A\cos 4x + B\sin 4x$
$8\cos x$	"
e^{5x}	Ae^{5x}
$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$

Ex. A glitch in the method

$$y'' - 5y' + 4y = 8e^{x}$$

$y_c(x) = c_1 e^{x} + c_2 e^{4x}$

(132)

If $y_p(x) = Ae^{x}$ is a p. soln, then

$0 = 8e^{x} \longrightarrow \Leftarrow$

wrong guess

$$y_p(x) = Axe^{x}$$

$$\Longrightarrow \quad y_p(x) = -\frac{8}{3}xe^{x}.$$

$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
$5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x + (Ex^2+Fx+G)\sin 4x$
$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x + (Cx+E)e^{3x}\sin 4x$

$$y'' + 5y' + 6y = 4e^{-x} + 5\sin x$$

$$y_c(x) = c_1 e^{-2x} + c_2 e^{-3x}$$

$$y_p(x) = Ae^{-x} + B\sin x + C\cos x$$

$$\Rightarrow A = 2,\ B = \tfrac{1}{2},\ C = -\tfrac{1}{2}.$$

Inverse operator Technique — Grewal

Consider

$$f(D)\,y = b(x) \longrightarrow (1)$$

If $y = \frac{1}{f(D)} b(x)$ then it satisfies (1).

$\Rightarrow f(D), \frac{1}{f(D)}$ are inverse operators.

$\boxed{X = b(x)}$

(1). $f(D) = D$. Then

$$\frac{1}{D} f(x) = \int b(x)\,dx$$

Proof

$D\left(\frac{1}{D} b(x)\right) = Dy \Rightarrow D\left(\int b(x)\,dx\right) = \frac{dy}{dx}$

$b(x) = \frac{dy}{dx}$

$\Rightarrow b(x) = \frac{dy}{dx}$

$\Rightarrow \boxed{y = \int b(x)\,dx}$

$\therefore \boxed{\frac{1}{D} b(x) = \int b(x)\,dx}$

$$\boxed{\begin{aligned} &D\left(\frac{1}{D} b(x)\right) \\ &= D\left(\int b(x)\,dx\right) \\ &= b(x) \end{aligned}}$$

(2). $\frac{1}{(D-a)} b(x) = e^{ax} \int b(x)\, e^{-ax}\,dx$ $\quad f(D) = D - a$

$(D-a)\left(e^{ax}\int b(x) e^{-ax}\,dx\right)$

$= \cancel{e^{ax}} \cdot b(x) \cancel{e^{-ax}} + \int b(x) e^{-ax}\,dx \cdot a e^{ax}$

$= b(x) \quad - a e^{ax} \int b(x) e^{-ax}\,dx$

Rules for find P.I.

Consider the D.E

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = b(x) \longrightarrow (1)$$

(or) $f(D)y = b(x)$,

where $f(D) \equiv D^n + a_1 D^{n-1} + \dots + a_n I$

Case(i) : $b(x) = e^{ax}$

$$De^{ax} = ae^{ax}$$

$$\vdots$$

$$D^n e^{ax} = a^n e^{ax}$$

$$\therefore f(D)(e^{ax}) = (a^n + a_1 a^{n-1} + \dots + a_n) e^{ax}$$

$$\Rightarrow f(D) e^{ax} = f(a) e^{ax}$$

$$\Rightarrow \frac{1}{f(D)}\left(f(D) e^{ax}\right) = \frac{1}{f(D)}\left(f(a) e^{ax}\right)$$

$$\Rightarrow e^{ax} = f(a) \frac{1}{f(D)}(e^{ax})$$

$$\Rightarrow \boxed{\frac{1}{f(D)}(e^{ax}) = \frac{1}{f(a)} e^{ax} \text{ if } f(a) \neq 0}$$

If $f(a)=0$,

$$f(D)=(D-a)\phi(D), \quad \phi(a)\neq 0.$$

$$\Rightarrow \frac{1}{f(D)}e^{ax} = \frac{1}{D-a}\left[\frac{1}{\phi(D)}e^{ax}\right]$$

$$= \frac{1}{D-a}\left(\frac{1}{\phi(a)}e^{ax}\right)$$

$$= \frac{1}{\phi(a)}\left(\frac{1}{D-a}e^{ax}\right)$$

$$= \frac{1}{\phi(a)}e^{ax}\int e^{ax}e^{-ax}\,dx$$

$$= \frac{1}{\phi(a)}\cdot x e^{ax}$$

(Since $f'(D)=(D-a)\phi'(D)+1.\phi(D)$

$f'(a)=0+1.\phi(a)$)

$$\boxed{\frac{1}{f(D)}e^{ax} = \frac{1}{f'(a)}xe^{ax}} \quad \text{if } f(a)=0,\ f'(a)\neq 0$$

In general,

$$\frac{1}{f(D)}e^{ax} = \frac{x^n}{f^{(n)}(a)}e^{ax} \quad \text{if } f(a)=0,\ f'(a)=0,\ \vdots,\ f^{(n)}(a)\neq 0$$

Example

Find P.I $(D^2+5D+6)y = e^{x}$

Soln

$$P.I = y_p(x) = \frac{1}{(D^2+5D+6)} e^{1x}$$

$$= \frac{1}{1^2+5+6} e^{x}$$

$$= \frac{e^{x}}{12},$$

Example

$$(D+2)(D-1)^2 y = e^{-2x} + \cancel{2\sinh x}$$

Soln

$$P.I = y_p(x) = \frac{1}{(D+2)(D-1)^2}\left(e^{-2x} + \cancel{e^{x} - e^{-x}}\right)$$

$$= \frac{1}{(D+2)(D-1)^2}\left(e^{-2x}\right) + \cancel{\frac{1}{(D+2)(D-1)^2} e^{x} - \frac{1}{(D+2)(D-1)^2} e^{-x}}$$

$$= \frac{1}{(D+2)}\left[\frac{1}{(-2-1)^2} e^{-2x}\right]$$

$$= \frac{1}{(D+2)}\left(\frac{1}{9} e^{-2x}\right) = \frac{1}{9}\frac{1}{D+2} e^{-2x}$$

$f(x) = x+2$
$f'(x) = 1$

$$= \frac{1}{9} \cdot x e^{-2x}$$

Case(ii) $b(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$\frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b) \quad \text{provided } f(-a^2) \neq 0$$

$$= \frac{x}{f'(-a^2)} \sin(ax+b) \quad \text{provided } f(-a^2)=0 \text{ but } f'(-a^2) \neq 0.$$

Example

Find the PI of

$$(D^3+1)y = \cos(2x-1)$$

Soln

$$y_p(x) = \frac{1}{(D^3+1)} \cos(2x-1)$$

$$= \frac{1}{(D \cdot D^2+1)} \cos(2x-1)$$

$$= \frac{1}{(-4D+1)} \cos(2x-1) \quad (\text{replace } D^2 = -2^2 = -4)$$

$$= \frac{(1+4D)}{(1-4D)(1+4D)} \cos(2x-1)$$

$$= \frac{(1+4D)}{(1-16D^2)} \cos(2x-1)$$

$$= \frac{(1+4D)}{65} \cos(2x-1) = \frac{1}{65}\left(\cos(2x-1) - 8\sin(2x-1)\right)$$

~~Example~~

Case iii) $b(x) = x^m$.

~~P.I~~

$$P.I = \frac{1}{f(D)} x^m = (f(D))^{-1} x^m.$$

Example

Find PI of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

Soln

$$P.I = \frac{1}{(D^2 + D)} (x^2 + 2x + 4)$$

$$= \frac{1}{D(D+1)} (x^2 + 2x + 4)$$

$$= \frac{1}{D} (1+D)^{-1} (x^2 + 2x + 4)$$

$$= \frac{1}{D} \left\{ (1 - D + D^2 - D^3 \ldots) (x^2 + 2x + 4) \right\}$$

$$= \frac{1}{D} \left(x^2 + 2x + 4 - (2x + 2) + 2 \right)$$

$$= \int (x^2 + 4) \, dx$$

$$= \frac{x^3}{3} + 4x$$

Case (iv) $b(x) = e^{ax} v(x)$, v being a fn of x.

$$\frac{1}{f(D)}(e^{ax} v) = e^{ax} \frac{1}{f(D+a)} v(x).$$

Ex. $(D^2 - 2D + 4)y = e^x \cos x$.

Sol

$$P.I = \frac{1}{(D^2 - 2D + 4)}(e^x \cos x)$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4}(\cos x)$$

$$= e^x \frac{1}{(D^2+3)} \cos x$$

$$= e^x \frac{1}{-1+3} \cos x$$

$$= \frac{e^x \cos x}{2}.$$