Lemma.

Eigenvalues of Hermitian matrix are real

Proof.

Let λ be an eigenvalue of a Hermitian matrix A. ie, there exists a non-zero vector v such that

$$Av = \lambda v \longrightarrow (1)$$

Taking Hermitian conjugate in (1),

$$(Av)^* = (\lambda v)^*$$

$$\Rightarrow \quad v^* A^* = \bar{\lambda} v^* \quad \Rightarrow \quad v^* A = \bar{\lambda} v^* \quad (\because A = A^*)$$

Post-multiplying by v, we get

$$v^* A v = \bar{\lambda} v^* v$$

$$\Rightarrow v^* \lambda v = \bar{\lambda} v^* v$$

$$\Rightarrow \lambda v^* v - \bar{\lambda} v^* v = 0$$

$$\Rightarrow (\lambda - \bar{\lambda})(v^* v) = 0$$

$$\Rightarrow \lambda = \bar{\lambda} \quad (\because v \neq 0)$$

Lemma. Eigenvalues of skew-Hermitian matrices are purely imaginary. $(A = -A^*)$

Proof. Exercise. (Show $-\lambda = \bar{\lambda}$).

Lemma Eigenvalues of unitary matrix have unit modulus.

Proof. Exercise (See class Notes)

Diagonalization

Definition

(i) Let $T: V \to V$ be a linear operator on the finite dimensional vector space V over $\mathbb{K}$. Then we say that T is diagonalizable if there exists a basis $\mathcal{B}$ for V each vector of $\mathcal{B}$ is an eigenvector of T.

(ii) Let A be an $n \times n$ matrix over $\mathbb{K}$. Then A is said to be diagonalizable if there exists a basis $\mathcal{B}$ for $\mathbb{K}^{n \times 1}$ such that each vector in $\mathcal{B}$ is an eigenvector of A.

Theorem

An $n \times n$ matrix A is diagonalizable iff there exists an invertible matrix P such that

$$P^{-1}AP = D.$$

Theorem Let T be a linear operator on a finite dimensional vector space V. Then TFAE.

(i) T is diagonalizable

(ii) The characteristic polynomial of T is of the form

$$P(\lambda) = (\lambda-\lambda_1)^{d_1}(\lambda-\lambda_2)^{d_2} \cdots (\lambda-\lambda_k)^{d_k} \quad (\text{Here } \lambda_i \neq \lambda_j,\ i \neq j)$$

and $\dim N(T-\lambda_i I) = \text{G.M of } \lambda_i = \text{A.M of } \lambda_i = d_i$, $\forall\ i = \overline{1,k}$.

(iii) $\dim N(T-\lambda_1 I) + \cdots + \dim(N(T-\lambda_k I)) = \dim V$

Theorem: (Sufficient conditions)

(i) If all the eigenvalues of T (or A) are distinct, then then T (or A) is diagonalizable

(ii) Every real symmetric matrix is diagonalizable

Example

Consider the matrix $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$

Show that A is diagonalizable. Find P and D s.t $P^{-1}AP = D$.

Soln

The char. polynomial of A is

$$p(\lambda) = (\lambda-2)^2(\lambda-1)$$

The eigenvalues of A are 2 (twice), 1.

For $\lambda = 1$

$(A-I)X = 0$

$$\Rightarrow \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

rank $(A-I) = 2$

$\therefore$ nullity $(A-I)$ = G.M of 1 $= 1$ by rank-nullity thm)

$$\Rightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & \\ -4 & -6 & 4 & -6 \\ 3 & 2 & -1 & 3 \end{array} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$$

For $\lambda = 2$

$(A - 2I)X = 0$

$$\Rightarrow \begin{pmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\text{rank}(A - 2I) = 1$

$\therefore \text{Nullity}(A - 2I) = 2.$

$\Rightarrow \quad 3x_1 - 6x_2 - 6x_3 = 0$

Take $x_2 = 0$, $\cancel{x_3 = 0}$. Then $3x_1 = 6x_3$

$$\frac{x_1}{2} = \frac{x_3}{1}$$

$$\therefore x = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Take $x_3 = 0$, $\quad 3x_1 - 6x_2 = 0 \Rightarrow \dfrac{x_1}{2} = \dfrac{x_2}{1}$.

$$\therefore x = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore P = \begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} ; \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Verify that $P^{-1}AP = D$.

Solution: Let A = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic polynomial of A is $\lambda^2 - S_1\lambda + S_2$ where $S_1 = sum of the main diagonal elements$ = 3 + 2 = 5 and $S_2 = Determinant of A =$ |A| = 3(2) – 1(-1) = 7

Therefore, the characteristic polynomial is $\lambda^2 - 5\lambda + 7$

CAYLEY-HAMILTON THEOREM:

Statement: Every square matrix satisfies its own characteristic equation.

Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A
(2) To calculate the inverse of a square matrix A

Problems:

1. **Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation**

Solution:Let A = $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 = 2$

$S_2 = |A| = 1 - (-4) = 5$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$

$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Therefore, the given matrix satisfies its own characteristic equation

2. **If A = $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ $write\ A^2\ interms\ of\ A\ and\ I, using\ Cayley - Hamilton\ theorem$**

Solution:Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 6$

$S_2 = |A| = 5$