Existence and Uniqueness of of first order IVP.

Consider the first order IVP

$$① \longleftarrow \quad y' = f(x,y) \; ; \; y(x_0) = y_0 .$$

Definition

Let $f(x,y)$ be a function defined in a set $S \subseteq \mathbb{R}^2$.
We say that f satisfies Lipschitz condition w.r.to y on S if there exists a +ve constant K such that

$$|f(x_0, y_1) - f(x, y_2)| \leq K\,|y_1 - y_2|, \quad \forall \; (x,y_1),(x,y_2) \in S.$$

The constant K is called a Lipschitz constant.

Theorem

Suppose S is either a rectangle

$$|x - x_0| \leq a, \quad |y - y_0| \leq b, \quad (a, b > 0)$$

or a strip

$$|x - x_0| \leq a, \; |y| < \infty \quad (a > 0).$$

If $\cancel{f \in C(S)}$ f is real valued function on S such that

(i) $\dfrac{\partial f}{\partial y}$ exists

(ii) $\dfrac{\partial f}{\partial y}$ is continuous on S

(iii) $\left| \dfrac{\partial f(x,y)}{\partial y} \right| \leq K, \; \forall \; (x,y) \in S$ for some $K > 0$.

Then f satisfies Lipschitz condition w.r.to y on S with Lipschitz constant K.

Example 1

S : $|x| \leq 1$ & $|y| \leq 1$

$f(x,y) = xy^2$.

$\frac{\partial f}{\partial y} = 2xy$

$\left|\frac{\partial f}{\partial y}\right| = |2xy| = 2|x||y|$

≤ 2 in S

$\therefore$ f satisfies the Lipschitz condition w.r.to y on S.

Example 2

S : $|x| \leq 1$, $|y| < \infty$.

$f(x,y) = xy^2$.

We show that f does not satisfies the Lipschitz condition w.r.to y on S.

If possible, f satisfies the Lipschitz condition w.r.to y on S. Then $\exists$ a $k > 0$ s.t

$$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|, \ \forall \ (x,y_1),(x,y_2) \in S.$$

In particular,

$$|f(x,y_1) - f(x,0)| \leq K|y_1|$$

$$\Rightarrow \frac{|f(x,y_1) - f(x,0)|}{|y_1|} \leq K$$

$$\Rightarrow \frac{|x y_1^2|}{|y_1|} \le K$$

$$\Rightarrow |x||y_1| \le K \quad , \forall \ (x, y_1) \in S$$

Choose $x_0 \ne 0$, Then

$$|y_1| \le \frac{K}{|x_0|}, \ \forall \ y_1 \in \mathbb{R}$$

When $|y_1| \to \infty$,

$$\infty \le \frac{K}{|x_0|} \longrightarrow \leftarrow.$$

<u>Example</u> (Continuous but <u>not</u> Lipschitz)

$$f(x,y) = y^{2/3} \quad \text{on} \quad R: |x| \le 1, |y| \le 1.$$

It is clear that f is continuous on R.

If possible f is Lipschitz w.r.to y on R. Then $\exists$ a constant $K>0$ such that

$$|f(x,y_1) - f(x,y_2)| \le K|y_1 - y_2| \ , \forall \ (x,y_1),(x,y_2) \in R.$$

In particular,

$$|f(x,y_1) - f(x,0)| \le K|y_1|$$

$$\Rightarrow |y_1^{2/3}| \le K|y_1|$$

$$\Rightarrow |y_1|^{-1/3} \le K$$

$$\Rightarrow \frac{1}{|y_1|^{1/3}} \le K \quad , \forall \ |y_1| \le 1$$

As $|y_1| \to 0$

$$\Rightarrow \infty \le K \to \leftarrow.$$

Theorem

Consider the rectangle $R : |x - x_0| \leq a,\ |y - y_0| \leq b.\ (a, b > 0)$

Let $f \in C(R)$ such that f satisfies the Lipschitz condition w.r.to y on R. Then the IVP

$$y' = f(x, y)\ ,\ y(x_0) = y_0 \longrightarrow ①$$

has a unique solution in the interval

$$I : |x - x_0| \leq \alpha, \qquad |f(x,y)| \leq M,\ \forall (x,y) \in R$$

where $\alpha = \min\{a, b/M\}$ with $\cancel{M = \max\{|f(x,y)| : (x,y) \in R\}}$

The solution of ① is given by the method of successive approximations:

$$(*) \leftarrow \begin{cases} \text{Set} \quad \phi_0(x) = y_0 \\ \phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t))\, dt, \quad n = 0, 1, 2, 3 \ldots \end{cases}$$

Then $\phi_n(x)$ converges to the soln ϕ of the IVP ①.

Theorem

Consider the strip $S : |x - x_0| \leq a,\ |y| < \infty\ (a > 0)$

Let $f \in C(S)$ such that f satisfies the Lipschitz condition w.r.to y on S. Then IVP ① has a unique soln in the entire interval $|x - x_0| \leq a$. Furthermore, the solution of ① can be computed by $(*)$.

Theorem

Suppose $f \in C(\mathbb{R})$. If f satisfies a Lipschitz condition on each strip

$$S_a : |x| \le a, \ |y| < \infty \quad (a > 0),$$

then the IVP

$$y' = f(x,y); \ y(x_0) = y_0 \longrightarrow (1)$$

has a unique solution for all real x. The soln of (1) is given by the method of successive approximations.

Example

Find three successive approximations to the IVP

$$\frac{dy}{dx} = x^2 y - x; \ y(0) = 0. \longrightarrow (1)$$

Soln

$x_0 = 0, \ y_0 = 0$

$f(x,y) = x^2 y - x$

$$\frac{\partial f}{\partial y} = x^2$$

$$\left|\frac{\partial f}{\partial y}\right| = |x^2| \le a^2 \quad |x| \le a.$$

$\therefore$ f satisfies the Lipschitz condition on each strip S_a and hence (1) has unique soln for all real x.

$$\phi_0(x) = y_0 = 0$$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\,dt$$

$$= \int_0^x -t\,dt = -t^2/2 \Big|_0^x = -\frac{x^2}{2}.$$

$$\phi_2(x) = y_0 + \int_0^{x} f(t, \phi_1(t))\,dt$$

$$= \int_0^x f(t, -t^2/2)\,dt$$

$$= \int_0^x \left(t^2 \cdot \left(-\frac{t^2}{2}\right) - t\right) dt$$

$$= \int_0^x \left(-\frac{t^4}{2} - t\right) dt$$

$$= -\frac{t^5}{5 \cdot 2} - \frac{t^2}{2} \Big|_0^x$$

$$= -\frac{x^5}{2 \cdot 5} - \frac{x^2}{2}.$$

$$\phi_3(x) = \int_0^x f(t, \phi_2(t))\,dt$$

$$= \int_0^x f\left(t, -\frac{t^5}{10} - \frac{t^2}{2}\right) dt$$

$$= \int_0^x \left(t^2 \left(-\frac{t^5}{10} - \frac{t^2}{2}\right) - t\right) dt$$

$$= -\frac{x^8}{2 \cdot 5 \cdot 8} - \frac{x^5}{2 \cdot 5} - \frac{x^2}{2}.$$