4.4 Quadratic Forms

A homogeneous polynomial $P(\mathbf{x})$ of degree two of the form

$$P(\mathbf{x}) \equiv a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots + 2a_{n-1,n}x_{n-1}x_n, \tag{21}$$

real quadratic form

in which the coefficients a_{ij} and the variables in $\mathbf{x}(x_1, x_2, \ldots, x_n)$ are real numbers, is called a **real quadratic form** in the variables $x_1, x_2, \ldots, x_n$. The term *homogeneous* of degree two or, more precisely, *algebraically* homogeneous of degree two, means that each term in P is quadratic in the sense that it involves a product of precisely two of the variables $x_1, x_2, \ldots, x_n$. The terms involving the products $x_i x_j$ with $i \neq j$ are called the **mixed product** or **cross-product terms**.

Real quadratic forms

A real quadratic form $P(\mathbf{x})$ is a homogeneous polynomial in the real variables $x_1, x_2, \ldots, x_n$ of the form shown in (21). If $\mathbf{A}$ is a real symmetric $n \times n$ matrix and $\mathbf{x}$ is an n-element column vector defined as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}, \tag{22}$$

then $P(\mathbf{x})$ can be written in the matrix form

$$P(\mathbf{x}) \equiv \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}. \tag{23}$$

There is no loss of generality in requiring $\mathbf{A}$ to be a symmetric matrix, because if the coefficient of a cross-product term $x_i x_j$ equals b_{ij}, this can always be rewritten as $b_{ij} = 2a_{ij}$ allowing the terms a_{ij} to be positioned symmetrically about the leading diagonal, as shown in the matrix $\mathbf{A}$ in (22). Exercise 30 at the end of this section shows how the definition of a real quadratic form can be extended to any real $n \times n$ matrix.

EXAMPLE 4.18

Express the quadratic form

$$P(\mathbf{x}) \equiv 3x_1^2 - 2x_2^2 + 4x_3^2 + x_1x_2 + 3x_1x_3 - 2x_2x_3$$

as the matrix product $P(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$.

Solution By defining $\mathbf{x}$ and $\mathbf{A}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 3 & 1/2 & 3/2 \\ 1/2 & -2 & -1 \\ 3/2 & -1 & 4 \end{bmatrix},$$

we can write $P(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$. ■

Quadratic forms arise in various ways; for example, in mechanics a quadratic form can describe the ellipsoid of inertia of a solid body, the angular momentum of a solid body rotating about an axis, and the kinetic energy of a system of moving particles. Other areas in which quadratic forms occur include the geometry of conics in two space dimensions and of quadrics in three space dimensions, optimization problems, crystallography, and in the classification of partial differential equations (see Chapter 18).

We now give a general definition of a quadratic form that allows both the matrix **A** and the vector **x** to contain complex elements.

quadratic form and vectors with complex elements

General quadratic forms

Let the elements of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ and an n-element column vector $\mathbf{z}$ be complex numbers. Then a **quadratic form** $P(\mathbf{z})$ involving the variables $z_1, z_2, \dots, z_n$ of vector $\mathbf{z}$ is an expression of the form

$$P(\mathbf{z}) = \bar{\mathbf{z}}^{\mathrm{T}}\mathbf{A}\mathbf{z} = \sum_{i=1,\, j=1}^{n} a_{ij}\bar{z}_i z_j. \tag{24}$$

This definition is seen to include real quadratic forms, because when the elements of **A** and **z** are real, result (24) reduces to the real quadratic form defined in (23).

The structure of a quadratic form becomes clearer if a change of variables is made that removes the mixed product terms, leaving only the squared terms. This is called the **reduction** of the quadratic form to its **standard form**, also known as its **canonical form**. The next theorem shows how such a simplification can be achieved.

THEOREM 4.10

how to reduce a quadratic form to a sum of squares

Reduction of a quadratic form Let the $n \times n$ real symmetric matrix **A** have the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and let **Q** be an orthogonal matrix that diagonalizes **A**, so that $\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q} = \mathbf{D}$, where **D** is a diagonal matrix with the eigenvalues of **A** as the elements on its leading diagonal. Then the change of variable $\mathbf{x} = \mathbf{Q}\mathbf{y}$, involving the column vectors $\mathbf{x} = [x_1, x_2, \dots, x_n]^{\mathrm{T}}$ and $\mathbf{y} = [y_1, y_2, \dots, y_n]^{\mathrm{T}}$, transforms the real quadratic form $P(\mathbf{x}) \equiv \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ into the **standard form**

$$P(\mathbf{x}) \equiv \sum_{i=1,\, j=1}^{n} a_{ij}x_i x_j = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

Proof The proof uses the fact that because **Q** is an orthogonal matrix, $\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q} = \mathbf{D}$. Substituting $\mathbf{x} = \mathbf{Q}\mathbf{y}$ into the real quadratic form $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ gives

$$\begin{aligned} P(\mathbf{x}) \equiv \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} &= (\mathbf{Q}\mathbf{y})^{\mathrm{T}}\mathbf{A}\mathbf{Q}\mathbf{y} \\ &= \mathbf{y}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q}\mathbf{y} \\ &= \mathbf{y}^{\mathrm{T}}\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2. \end{aligned}$$ ■

It follows immediately from Theorem 4.10 that the standard form of $P(\mathbf{x})$ is determined once the eigenvalues of **A** are known and, when needed, the transformation of coordinates between **x** and **y** is given by $\mathbf{x} = \mathbf{Q}\mathbf{y}$ or, equivalently, by $\mathbf{y} = \mathbf{Q}^{\mathrm{T}}\mathbf{x}$.

The next example provides a geometrical interpretation of Theorem 4.10 in the context of rigid body mechanics. In order to understand its implications it is necessary to know that if an origin O is taken at an arbitrary point inside a solid body, and an orthogonal set of axes $O\{x_1, x_2, x_3\}$ is located at O, nine moments and products of inertia of the body can be defined relative to these axes and displayed in the form of a 3×3 inertia matrix. The moment of inertia of the body about any line passing through the origin O is proportional to the length of the segment of the line that lies between O and the point where it intersects a three-dimensional surface defined by a quadratic form determined by the inertia matrix.

When the surface determined by the inertia matrix is scaled so the length of the line from O to its point of intersection with the surface equals the reciprocal of the moment of inertia about that line, the surface is called the **ellipsoid of inertia**. If the orientation of the $O\{x_1, x_2, x_3\}$ axes is chosen arbitrarily, the resulting quadratic form will be complicated by the presence of mixed product terms, but a suitable rotation of the axes can always remove these terms and lead to the most convenient orientation of the new system of axes $O\{y_1, y_2, y_3\}$. In the geometry of both conics and quadrics, and also in mechanics, new axes obtained in this way that lead to the elimination of mixed product terms are called the **principal axes**, and it is because of this that Theorem 4.10 is often known as the **principal axes theorem**.

quadratic forms and principal axes

EXAMPLE 4.19

The ellipsoid of inertia of a solid body is given by

$$P(\mathbf{x}) \equiv 4x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_2.$$

Find its standard form in terms of a new orthogonal set of axes $O\{y_1, y_2, y_3\}$, and find the linear transformation that connects the two sets of coordinates.

Solution The quadratic form $P(\mathbf{x})$ can be written as $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ by defining

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$, $\lambda_2 = 5$, and $\lambda_3 = 3$, so the standard form of $P(\mathbf{x})$ is

$$P(\mathbf{x}) \equiv y_1^2 + 5y_2^2 + 3y_3^2.$$

The eigenvalues and corresponding normalized eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = 1, \quad \hat{\mathbf{x}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \lambda_2 = 5, \quad \hat{\mathbf{x}}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \lambda_3 = 3, \quad \hat{\mathbf{x}}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix},$$

so the orthogonal diagonalizing matrix for $\mathbf{A}$ is

$$\mathbf{Q} = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix},$$

and the change of variables between $\mathbf{x}$ and $\mathbf{y}$ determined by $\mathbf{x} = \mathbf{Q}\mathbf{y}$ becomes

$$x_1 = (-y_2 + y_3)/\sqrt{2}, \quad x_2 = (y_2 + y_3)/\sqrt{2}, \quad x_3 = y_1.$$

The equation $P(\mathbf{x}) =$ constant is seen to be an *ellipsoid* for which $O\{y_1, y_2, y_3\}$ are the *principal axes*. ■

EXAMPLE 4.20

Reduce the quadratic part of the following expression to its standard form involving the principal axes $O\{y_1, y_2\}$, and hence find the form taken by the complete expression in terms of y_1 and y_2:

$$x_1^2 + 4x_1x_2 + 4x_2^2 + x_1 - 2x_2.$$

Solution The quadratic part of the expression is $x_1^2 + 4x_1x_2 + 4x_2^2$, and this can be expressed in the form $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ by setting

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

The eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = 5, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 0, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

so the orthogonal diagonalizing matrix is

$$\mathbf{Q} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}.$$

Making the variable change $\mathbf{x} = \mathbf{Q}\mathbf{y}$ shows the standard form of the quadratic terms to be $5y_1^2$. The variables x_1 and x_2 are related to y_1 and y_2 by the expressions $\mathbf{x}_1 = y_1/\sqrt{5} - 2y_2/\sqrt{5}$ and $x_2 = 2y_1/\sqrt{5} + y_2/\sqrt{5}$, so $x_1 - 2x_2 = -(3y_1 + 4y_2)/\sqrt{5}$. In terms of the principal axes involving the coordinates y_1 and y_2, the complete expression $x_1^2 + 4x_1x_2 + 4x_2^2 + x_1 - 2x_2$ reduces to

$$x_1^2 + 4x_1x_2 + 4x_2^2 + x_1 - 2x_2 = 5y_1^2 - (3y_1 + 4y_2)/\sqrt{5}.$$

Quadratic forms $P(\mathbf{x})$ are classified according to the behavior of the sign of $P(\mathbf{x})$ when $\mathbf{x}$ is allowed to take all possible values. In terms of vector spaces, this amounts to saying that if the vector $\mathbf{x}$ in $P(\mathbf{x})$ is an n vector, then $\mathbf{x} \in R^n$.

how to classify quadratic forms

Classification of quadratic forms

Let $P(\mathbf{x})$ be a quadratic form. Then:

1. $P(\mathbf{x})$ is said to be **positive definite** if $P(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in R^n, with $P(\mathbf{x}) = 0$ if, and only if, $\mathbf{x} = \mathbf{0}$. $P(\mathbf{x})$ is said to be **negative definite** if in this definition the inequality sign $>$ is replaced by $<$.
2. $P(\mathbf{x})$ is said to be **positive semidefinite** if $P(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in R^n, and to be **negative semidefinite** if in this definition the inequality sign $\geq$ is replaced by $\leq$.
3. $P(\mathbf{x})$ is said to be **indefinite** if it satisfies none of the above conditions.

It is an immediate consequence of Theorem 4.10 that if $P(\mathbf{x})$ is associated with a real symmetric matrix $\mathbf{A}$, then:

(a) $P(\mathbf{x})$ is positive definite if all the eigenvalues of $\mathbf{A}$ are positive, and it is negative definite if all the eigenvalues of $\mathbf{A}$ are negative.

(b) $P(\mathbf{x})$ is positive semidefinite if all the eigenvalues of $\mathbf{A}$ are nonnegative, and it is negative semidefinite if all the eigenvalues of $\mathbf{A}$ are nonpositive. So, in each semidefinite case, one or more of the eigenvalues may be zero.

- Example: The quadratic form $Q(x, y, z) = x^2 - y^2 - z^2$ on $\mathbb{R}^3$ has index 1 and signature -1.
- Example: The quadratic form $Q(x, y, z) = x^2 - z^2$ on $\mathbb{R}^3$ has index 1 and signature 0.
- Example: The quadratic form $Q(x, y, z) = 5x^2 + 4xy + 6y^2 + 4yz + 7z^2$ on $\mathbb{R}^3$ has index 3 and signature 3, since we computed its diagonalization to have diagonal entries $3, 6, 9$.

- Example: Find the index and signature of the quadratic form $Q(x, y, z) = -x^2 - 8xy + 4xz - y^2 + 4yz + 2z^2$.
  - The matrix associated to the corresponding bilinear form is $A = \begin{bmatrix} -1 & -4 & 2 \\ -4 & -1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.
  - The characteristic polynomial is $p(t) = \det(tI_3 - A) = t^3 - 27t + 54 = (t-3)^2(t+6)$.
  - Thus, since the eigenvalues are $\lambda = 3, 3, -6$, we see that the diagonalization will have two positive diagonal entries and one negative diagonal entry.
  - This means that the index is $\boxed{2}$ and the signature is $\boxed{1}$.

- As a corollary of Sylvester's law of inertia, we can read off the shape of a conic section or quadric surface (in all nondegenerate cases, and also in many degenerate cases) simply by examining the signs of the eigenvalues of the underlying quadratic form.

- Example: Determine the shape of the quadric surface $13x^2 - 4xy + 10y^2 - 8xz + 4yz + 13z^2 = 1$.
  - If $Q(x, y, z)$ is the quadratic form above, the bilinear form has associated matrix $A = \begin{bmatrix} 13 & -2 & -4 \\ -2 & 10 & 2 \\ -4 & 2 & 13 \end{bmatrix}$.
  - The characteristic polynomial is $p(t) = \det(tI_3 - A) = t^3 - 144t^2 + 6480t - 93312 = (t-36)^2(t-72)$.
  - This means, upon diagonalizing $Q(x, y, z)$, we will obtain the equation $36(x')^2 + 36(y')^2 + 72(z')^2 = 1$. This is the equation of an $\boxed{\text{ellipsoid}}$.
  - Note that the only information we needed here was the fact that all three eigenvalues were positive to make this observation: the quadric surfaces $Q(x, y, z) = 1$ that are ellipsoids are precisely those for which $Q(x, y, z)$ is a positive-definite quadratic form.

- We will close our discussion by observing that the study of quadratic forms touches on nearly every branch of mathematics: we have already examined some of its ties to linear algebra (in the guise of bilinear forms and diagonalization), analysis (in the classification of critical points), and geometry (in the analysis of quadratic varieties and the action of matrices on quadratic forms).
  - We will not discuss it much here, since the requisite tools do not really belong to linear algebra, but the study of quadratic forms over $\mathbb{Q}$ turns out to be intimately tied with many topics in number theory.
  - A very classical problem in elementary number theory is to characterize, in as much detail as possible, the integers represented by a particular quadratic form. For example: which integers are represented by the quadratic form $Q(x, y) = x^2 + y^2$ (i.e., which integers can be written as the sum of two squares)?
  - This family of problems, while seemingly quite simple, is actually intimately related to a number of very deep results in modern number theory, and (historically speaking) was a major motivating force in the development of a branch of algebraic number theory known as class field theory.

Well, you're at the end of my handout. Hope it was helpful.