Homework3

August 3, 2022

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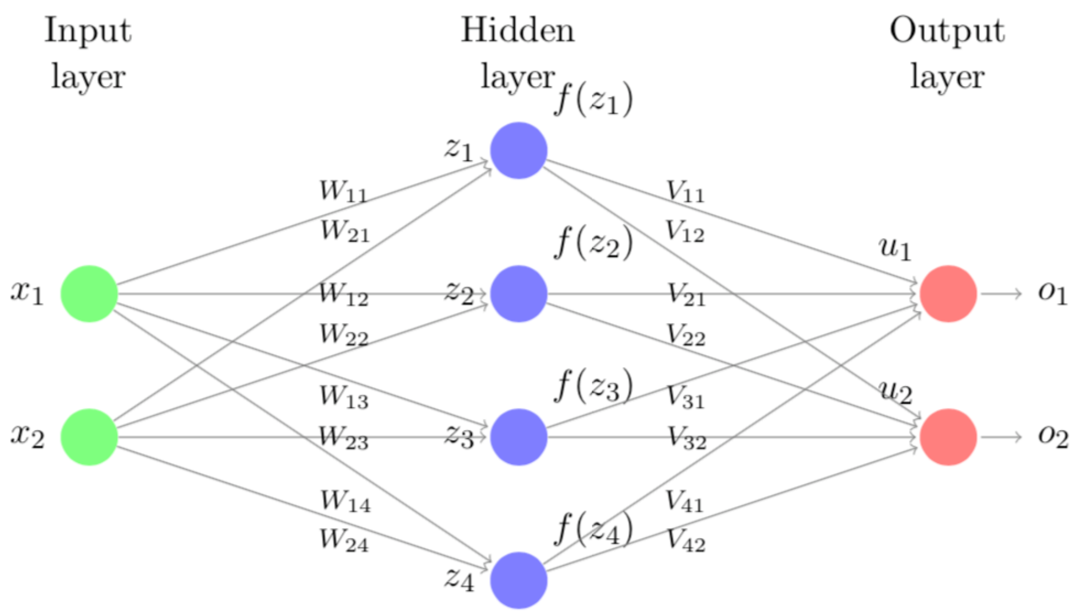
1 Neural Networks

In this problem we will analyze a simple neural network to understand its classification properties. Consider the neural network given in the figure below, with ReLU activation functions (denoted by f) on all neurons, and a softmax activation function in the output layer:



Given an input $x = [x_1, x_2]^T$, the hidden units in the network are activated in stages as described by the following equations:

$$\begin{aligned} z_1 &= x_1W_{11} + x_2W_{21} + W_{01} \qquad & f(z_1) &= \max\{z_1, 0\} \\ z_2 &= x_1W_{12} + x_2W_{22} + W_{02} \qquad & f(z_2) &= \max\{z_2, 0\} \\ z_3 &= x_1W_{13} + x_2W_{23} + W_{03} \qquad & f(z_3) &= \max\{z_3, 0\} \\ z_4 &= x_1W_{14} + x_2W_{24} + W_{04} \qquad & f(z_4) &= \max\{z_4, 0\} \end{aligned}$$

$$\begin{aligned} u_1 &= f(z_1)V_{11} + f(z_2)V_{21} + f(z_3)V_{31} + f(z_4)V_{41} + V_{01} \qquad & f(u_1) &= \max\{u_1, 0\} \\ u_2 &= f(z_1)V_{12} + f(z_2)V_{22} + f(z_3)V_{32} + f(z_4)V_{42} + V_{02} \qquad & f(u_2) &= \max\{u_2, 0\} \end{aligned}$$

The final output of the network is obtained by applying the softmax function to the last hidden layer,

$$\begin{aligned} o_1 &= \frac{e^{f(u_1)}}{e^{f(u_1)} + e^{f(u_2)}} \\ o_2 &= \frac{e^{f(u_2)}}{e^{f(u_1)} + e^{f(u_2)}} \end{aligned}$$

In this problem we will consider the following set of parameters:

$$\begin{bmatrix} W_{11} & W_{21} & W_{01} \\ W_{12} & W_{22} & W_{02} \\ W_{13} & W_{23} & W_{03} \\ W_{14} & W_{24} & W_{04} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} V_{11} & V_{21} & V_{31} & V_{41} & V_{01} \\ V_{12} & V_{22} & V_{32} & V_{42} & V_{02} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 2 \end{bmatrix}$$

1.1 Feed forward step

Consider the input $x_1 = 3$, $x_2 = 14$. What is the final output (o_1, o_2) of the network?

[121]:
```
import math
from math import exp
f = lambda x: max(0,x)
fexp = lambda input: map(lambda x: exp(f(x)), input)
softmax = lambda input: list(map(lambda x: x/sum(fexp(input)), fexp(input)))
inner_vv = lambda v1, v2: sum(map(lambda x, y: x*y, v1, v2))
inner_Mv = lambda M, v: [inner_vv(M_row, v) for M_row in M]
```

```
W = [[1, 0, -1], [0, 1, -1], [-1, 0, -1], [0, -1, -1]]
V = [[1,1,1,1,0], [-1,-1,-1,-1,2]]
x = [3, 14]

z = inner_Mv(W, [*x, 1])
fz = list(map(f, z))

u = inner_Mv(V, [*fz, 1])

print(softmax(u))

print("Alternatively:")
from functools import reduce
# print(f"o1 = e^{f(u[0])}/({reduce(lambda x,y: x+'+'+ y, ('e^' + str(f(x)) for␣
↪x in u))})")
# print(f"o2 = e^{f(u[1])}/({reduce(lambda x,y: x+'+'+ y, ('e^' + str(f(x)) for␣
↪x in u))})")

print(reduce(lambda x,y: x+'\n' + y, (f"o{the_index +1} = e^{f(the_u)}/
↪({reduce(lambda x,y: x+'+'+ y, ('e^' + str(f(x)) for x in u))})" for␣
↪the_index, the_u in enumerate(u))))
```

```
[0.9999996940977731, 3.059022269256247e-07]
Alternatively:
o1 = e^15/(e^15+e^0)
o2 = e^0/(e^15+e^0)
```

1.2 Decision Boundaries

In this problem we visualize the "decision boundaries" in x-space, corresponding to the four hidden units. These are the lines in x-space where the values of z_1, z_2, z_3, z_4 are exactly zero. Plot the decision boundaries of the four hidden units using the parameters of W provided above.

Enter below the area of the region of your plot that corresponds to a negative (< 0) value for all of the four hidden units.

Answer

Line1:

$$\begin{aligned} x_1 W_{11} + x_2 W_{21} + W_{01} &= 0 \\ x_1 - 1 &= 0 \\ x_1 &= 1 \end{aligned}$$

Line2:

$$
\begin{aligned}
x_1W_{12} + x_2W_{22} + W_{02} &= 0 \\
x_2 - 1 &= 0 \\
x_2 &= 1
\end{aligned}
$$

Line3:

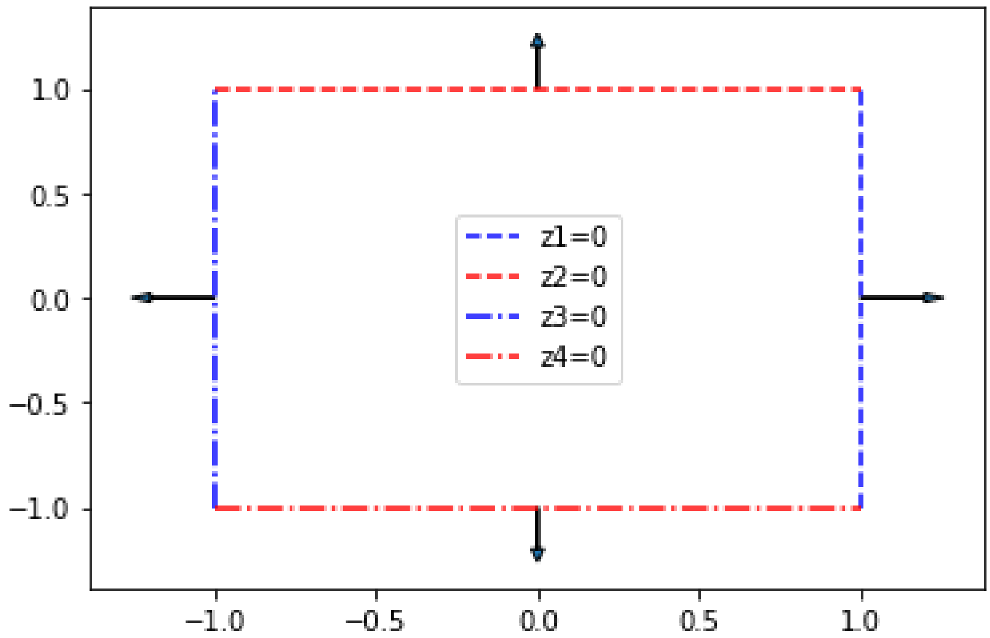
$$
\begin{aligned}
x_1W_{13} + x_2W_{23} + W_{03} &= 0 \\
-x_1 - 1 &= 0 \\
x_1 &= -1
\end{aligned}
$$

Line4:

$$
\begin{aligned}
x_1W_{14} + x_2W_{24} + W_{04} &= 0 \\
-x_2 - 1 &= 0 \\
x_2 &= -1
\end{aligned}
$$

[191]:

```
import matplotlib.pyplot as plt

plt.figure()
plt.plot([1, 1],[-1,1], 'b--', label='z1=0')
plt.arrow(1,0, 0.2* W[0][0], 0.2 * W[0][1], head_width=.04)
plt.plot([-1, 1],[1,1], 'r--', label='z2=0')
plt.arrow(0,1, 0.2* W[1][0], 0.2 * W[1][1], head_width=.04)
plt.plot([-1, -1],[-1,1], 'b-.', label='z3=0')
plt.arrow(-1,0, 0.2* W[2][0], 0.2 * W[2][1], head_width=.04)
plt.plot([-1, 1],[-1,-1], 'r-.', label='z4=0')
plt.arrow(0,-1, 0.2* W[3][0], 0.2 * W[3][1], head_width=.04)
plt.legend()
plt.show()
plt.close()

print('Area = 4 units')
```




```
Area = 4 units
```

1.3 Output of Neural Network

Using the same matrix V as above, what is the value of o_1 (accurate to at least three decimal places if responding numerically) in the following three cases?

Assuming that $f(z_1) + f(z_2) + f(z_3) + f(z_4) = 1$:

[202]:
```
sums = 1
u = [sums +0 , -sums +2]
print(reduce(lambda x,y: x+'\n' + y, (f"o{the_index +1} = e^{f(the_u)}/
↪({reduce(lambda x,y: x+'+'+ y, ('e^' + str(f(x)) for x in u))})" for␣
↪the_index, the_u in enumerate(u))))
print('Values:', softmax(u))
```

```
o1 = e^1/(e^1+e^1)
o2 = e^1/(e^1+e^1)
Values: [0.5, 0.5]
```

Assuming that $f(z_1) + f(z_2) + f(z_3) + f(z_4) = 0$:

[203]:
```
sums = 0
u = [sums +0 , -sums +2]
print(reduce(lambda x,y: x+'\n' + y, (f"o{the_index +1} = e^{f(the_u)}/
↪({reduce(lambda x,y: x+'+'+ y, ('e^' + str(f(x)) for x in u))})" for␣
↪the_index, the_u in enumerate(u))))
```

```
print('Values:', softmax(u))
```

```
o1 = e^0/(e^0+e^2)
o2 = e^2/(e^0+e^2)
Values: [0.11920292202211755, 0.8807970779778824]
```

Assuming that $f(z_1) + f(z_2) + f(z_3) + f(z_4) = 3$:

[204]:
```
sums = 3
u = [sums +0 , -sums +2]
print(reduce(lambda x,y: x+'\n' + y, (f"o{the_index +1} = e^{f(the_u)}/
 ↪({reduce(lambda x,y: x+'+'+ y, ('e^' + str(f(x)) for x in u))})" for␣
 ↪the_index, the_u in enumerate(u))))
print('Values:', softmax(u))
```

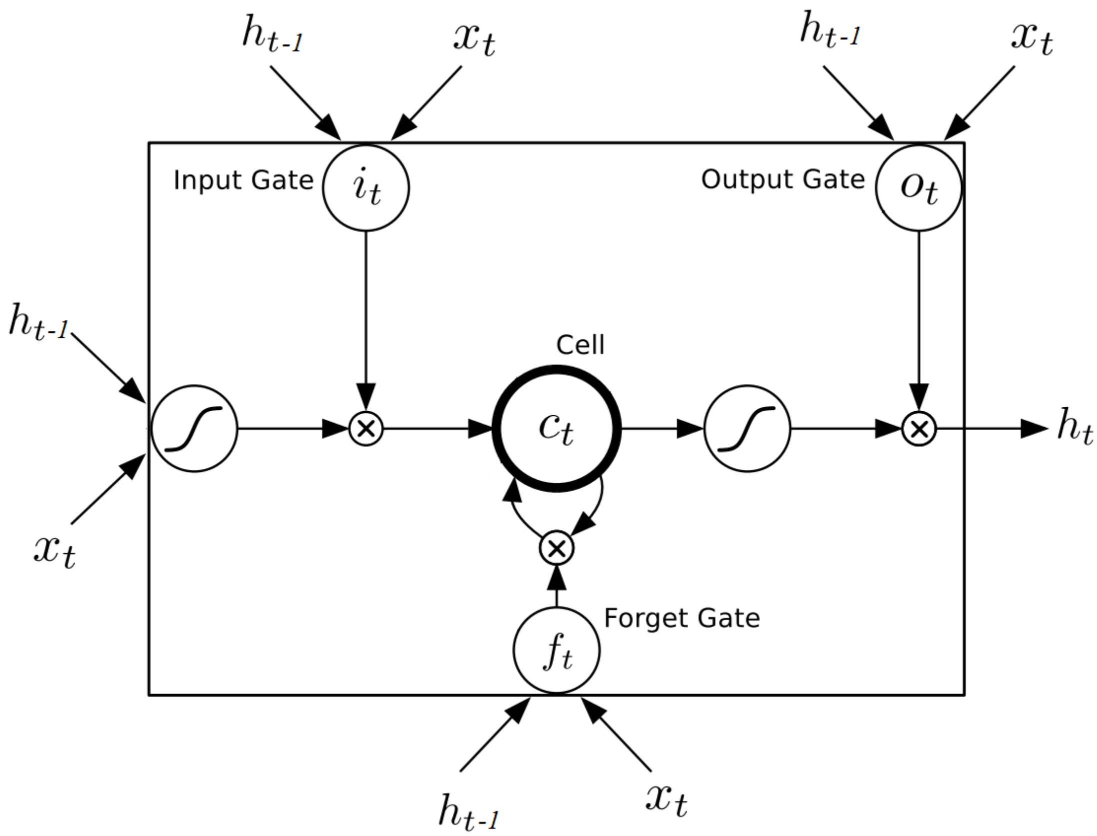
```
o1 = e^3/(e^3+e^0)
o2 = e^0/(e^3+e^0)
Values: [0.9525741268224333, 0.04742587317756678]
```

1.4 Inverse temperature

Just math.. fuck it

2 LSTM

The diagram below shows a single LSTM unit that consists of Input, Output, and Forget gates.



The behavior of such a unit as a recurrent neural network is specified by a set of update equations. These equations define how the gates, "memory cell" c_t and the "visible state" h_t are updated in response to input x_t and previous states c_{t-1}, h_{t-1}. For the LSTM unit,

$$
\begin{aligned}
f_t &= \text{sigmoid}(W^{f,h}h_{t-1} \ + \ W^{f,x}x_t \ + \ b_f) \\
i_t &= \text{sigmoid}(W^{i,h}h_{t-1} \ + \ W^{i,x}x_t \ + \ b_i) \\
o_t &= \text{sigmoid}(W^{o,h}h_{t-1} \ + \ W^{o,x}x_t \ + \ b_o) \\
c_t &= f_t \odot c_{t-1} + i_t \odot \tanh(W^{c,h}h_{t-1} + W^{c,x}x_t + b_c) \\
h_t &= o_t \odot \tanh(c_t)
\end{aligned}
$$

where symbol $\odot$ stands for element-wise multiplication. The adjustable parameters in this unit are matrices $W^{f,h}$, $W^{f,x}$, $W^{i,h}$, $W^{i,x}$, $W^{o,h}$, $W^{o,x}$, $W^{c,h}$, $W^{c,x}$ as well as the offset parameter vectors b_f, b_i, b_o, and b_c. By changing these parameters, we change how the unit evolves as a function of inputs x_t.

To keep things simple, in this problem we assume that x_t, c_t, and h_t are all scalars. Concretely, suppose that the parameters are given by:

$$\begin{array}{llll} W^{f,h} = 0 & W^{f,x} = 0 & b_f = -100 & W^{c,h} = -100 \\ W^{i,h} = 0 & W^{i,x} = 100 & b_i = 100 & W^{c,x} = 50 \\ W^{o,h} = 0 & W^{o,x} = 100 & b_o = 0 & b_c = 0 \end{array}$$

We run this unit with initial conditions $h_{-1} = 0$ and $c_{-1} = 0$, and in response to the following input sequence: [0, 0, 1, 1, 1, 0] (For example, $x_0 = 0$, $x_1 = 0$, $x_2 = 1$, and so on).

2.1 LSTM states

Calculate the values h_t at each time-step and enter them below as an array $[h_0, h_1, h_2, h_3, h_4, h_5]$.

(Please round h_t to the closest integer in every time-step. If $h_t = \pm 0.5$, then round it to 0. For ease of calculation, assume that $\text{sigmoid}(x) \approx 1$ and $\tanh(x) \approx 1$ for $x \geq 1$, and $\text{sigmoid}(x) \approx 0$ and $\tanh(x) \approx -1$ for $x \leq -1$.)

[233]:
```
Wfh = 0
Wih = 0
Woh = 0
Wfx = 0
Wix = 100
Wox = 100
b_f = -100
b_i = 100
b_o = 0
Wch = -100
Wcx = 50
b_c = 0

h_1 = 0
c_1 = 0
x = [0, 0, 1, 1, 1, 0]

sigmoid = lambda x: 1 if x>=1 else 0 if x<=-1 else 0.5*x + 0.5
tanh    = lambda x: 1 if x>=1 else -1 if x<=-1 else x

f = lambda ht_1, x_t, Wfx=Wfx, Wfh=Wfh, b_f=b_f:    sigmoid(Wfh * ht_1 + Wfx *
 ↪x_t + b_f)
i = lambda ht_1, x_t, Wix=Wix, Wih=Wih, b_i=b_i:    sigmoid(Wih * ht_1 + Wix *
 ↪x_t + b_i)
o = lambda ht_1, x_t, Wox=Wox, Woh=Woh, b_o=b_o:    sigmoid(Woh * ht_1 + Wox *
 ↪x_t + b_o)

c = lambda ft, it, ct_1, ht_1, x_t, Wch=Wch, Wcx=Wcx, b_c=b_c:        ft * ct_1
 ↪+ it * tanh(Wch * ht_1 + Wcx * x_t + b_c)
h = lambda o_t, c_t:                                o_t * tanh(c_t)
```

```
ht_1 = [h_1]
ct_1 = [c_1]
for j in range(6):
    ft = f(ht_1[j], x[j])
    it = i(ht_1[j], x[j])
    ot = o(ht_1[j], x[j])
    ct_1.append( c(ft, it, ct_1[j], ht_1[j], x[j]) )
    ht_1.append( round(h(ot, ct_1[j+1])) )



print(ht_1[1:])
```

```
[0, 0, 1, -1, 1, 0]
```

2.2 LSTM states 2

Now, we run the same model again with the same parameters and same initial conditions as in the previous question. The only difference is that our input sequence in now: [1, 1, 0, 1, 1].

Calculate the values h_t at each time-step and enter them below as an array $[h_0, h_1, h_2, h_3, h_4, h_5]$.

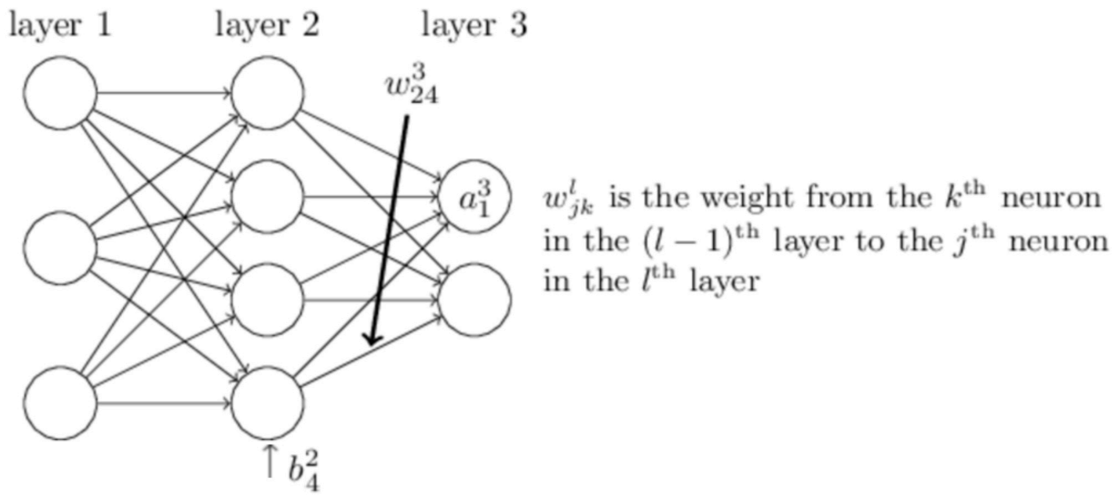
[236]:
```
x =  [1,1,0,1,1]

ht_1 = [h_1]
ct_1 = [c_1]
for j in range(5):
    ft = f(ht_1[j], x[j])
    it = i(ht_1[j], x[j])
    ot = o(ht_1[j], x[j])
    ct_1.append( c(ft, it, ct_1[j], ht_1[j], x[j]) )
    ht_1.append( round(h(ot, ct_1[j+1])) )

print(ht_1[1:])
```

```
[1, -1, 0, 1, -1]
```

3 Backpropagation

One of the key steps for training multi-layer neural networks is stochastic gradient descent. We will use the back-propagation algorithm to compute the gradient of the loss function with respect to the model parameters.

Consider the L-layer neural network below:



In the following problems, we will the following notation: b^l_j is the bias of the j^{th} neuron in the l^{th} layer, a^l_j is the activation of j^{th} neuron in the l^{th} layer, and w^l_{jk} is the weight for the connection from the k^{th} neuron in the $(l-1)^{th}$ layer to the j^{th} neuron in the l^{th} layer.

If the activation function is f and the loss function we are minimizing is C, then the equations describing the network are:

$$a^l_j = f\left(\sum_k w^l_{jk} a^{l-1}_k + b^l_j\right)$$
$$\text{Loss} = C(a^L)$$

Note that notations without subscript denote the corresponding vector or matrix, so that a^l is activation vector of the l^{th} layer, and w^l is the weights matrix in l^{th} layer.

For $l = 1, \dots, L$.

3.1 Computing the Error

Let the weighted inputs to the d neurons in layer l be defined as $z^l = w^l a^{l-1} + b^l$, where $z^l \in \mathbb{R}^d$. As a result, we can also write the activation of layer l as $a^l \equiv f(z^l)$, and the "error" of neuron j in layer l as $\delta^l_j \equiv \dfrac{\partial C}{\partial z^l_j}$. Let $\delta^l \in \mathbb{R}^d$ denote the full vector of errors associated with layer l.

Back-propagation will give us a way of computing δ^l for every layer.

Assume there are d outputs from the last layer (i.e. $a^L \in \mathbb{R}^d$). What is δ^L_j for the last layer?

Loads of arithmetic later....