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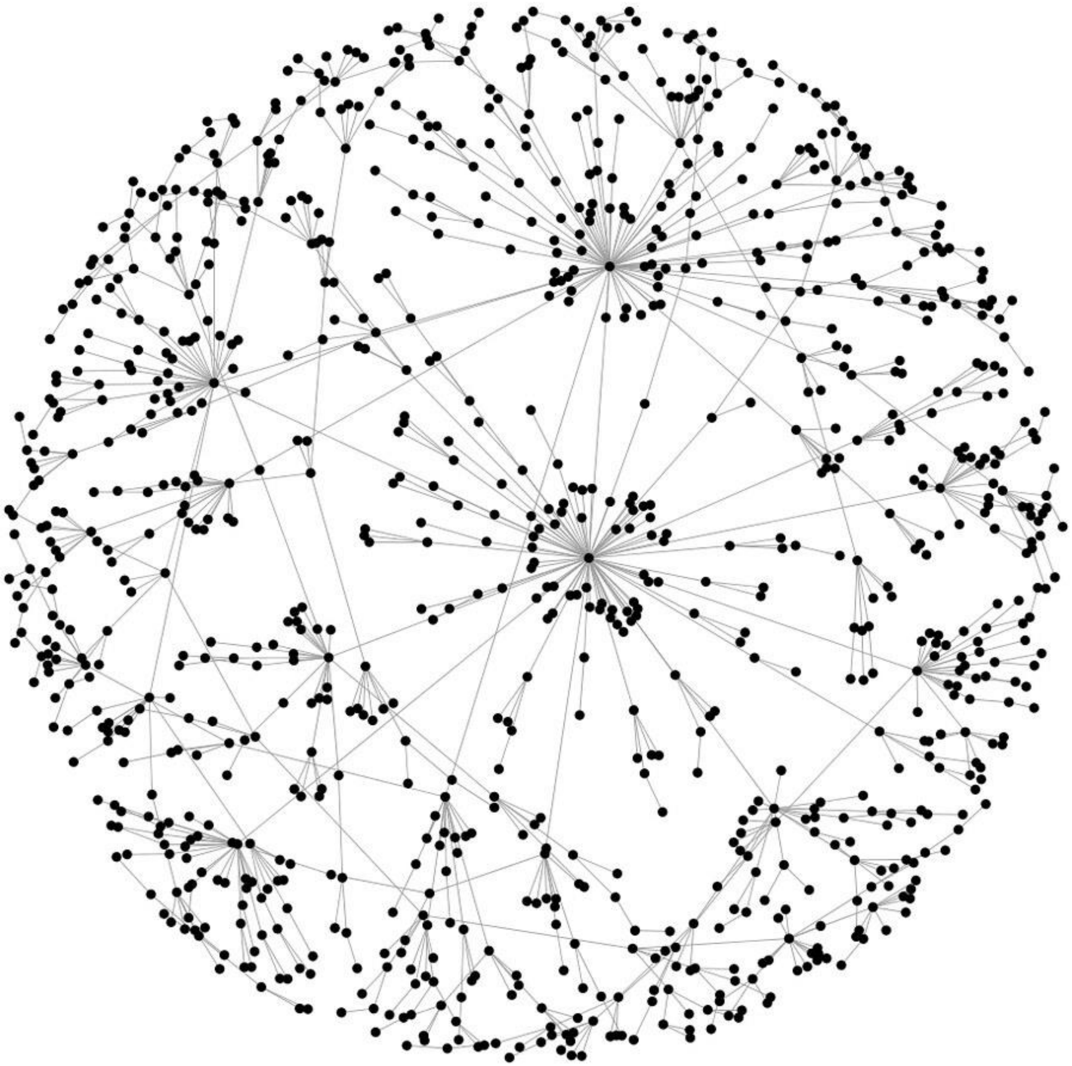
DEPARTAMENTO
DE ENGENHARIA INFORMÁTICA
TÉCNICO LISBOA

Network revolution(s)



Network Science, 2025/26

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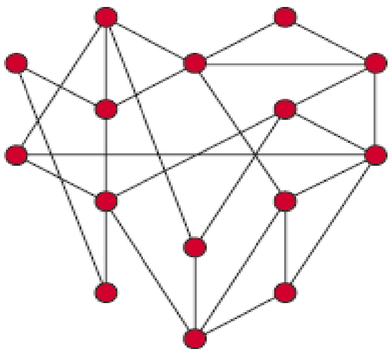
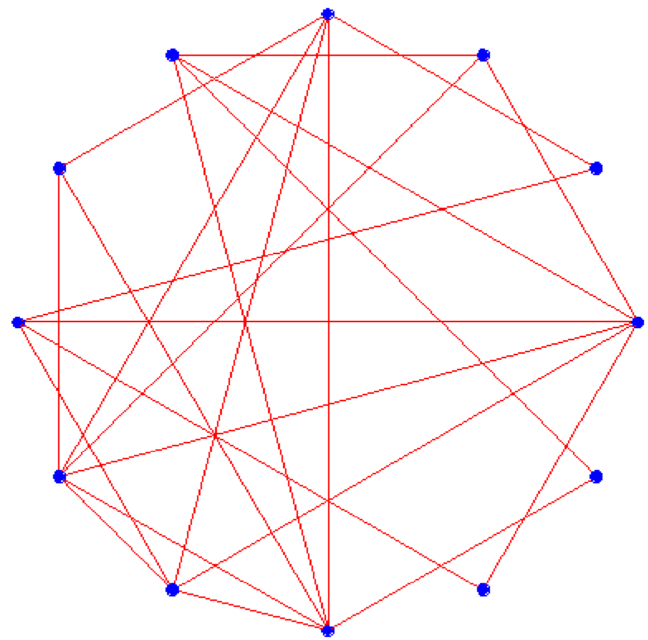
Last class:
Random graphs



Network Science, 2025/26

Random network model

- A random network consists of N nodes where each pair of nodes is connected with probability p.

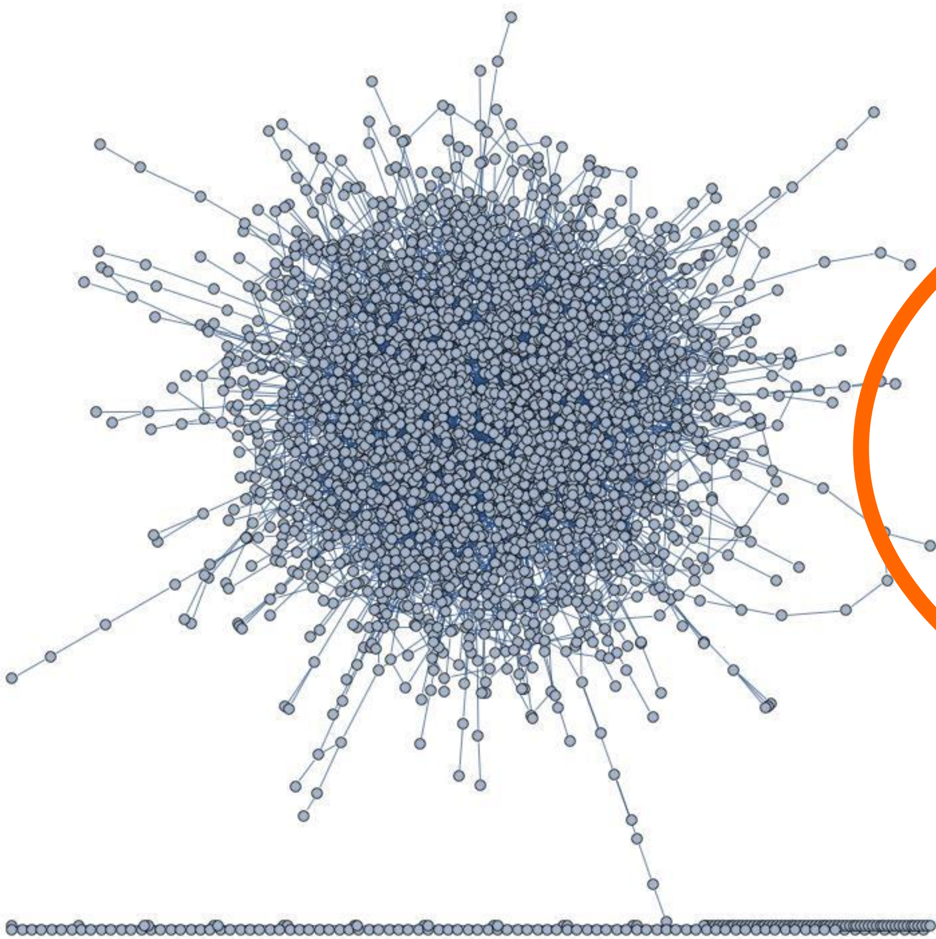


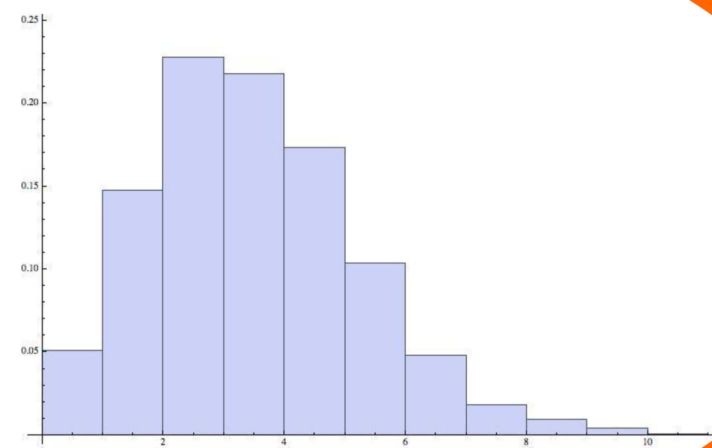
Random network model or the *Erdös and Renyi (ER) Model*

The null model of network science

Random network model

N = 3x10^3 nodes, p = 10^{-3}





Degree distribution of a random graph

- The probability that a node i has exactly k partners is given by ***a binomial distribution***

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

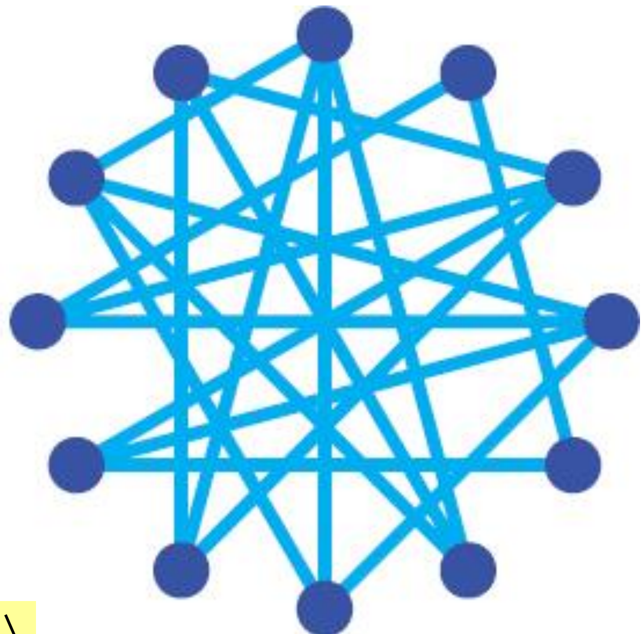
- Real networks are sparse *(i.e., $<k> << N$)*
- For large/sparse networks the binomial dist. is well approximated by ***a Poisson distribution***:

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

Random graphs portray a very low clustering coeff.

Average number of links e_i in the neighborhood of a node i with $\boldsymbol{k_i}$ partners?

$$p \times \frac{k_i(k_i-1)}{2}$$



Thus, the clustering coefficient reads

$$C_i = \frac{e_i}{k_i(k_i-1)/2} = p\frac{k_i(k_i-1)/2}{k_i(k_i-1)/2} = p = \frac{\langle k \rangle}{N}$$

which, for sparse & large graphs, C➔0 !!

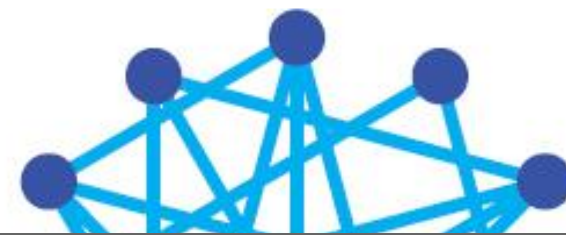
Random graphs portray a very low clustering coeff.

Real world networks often show large Clustering Coeff.

Film actors	➔	C=0.2
Company directors	➔	C=0.59
Math co-authorship	➔	C=0.15
Physics co-authorship	➔	C=0.45
Physics co-authorship	➔	C=0.45
WWW	➔	C=0.11
Neural networks	➔	C=0.18

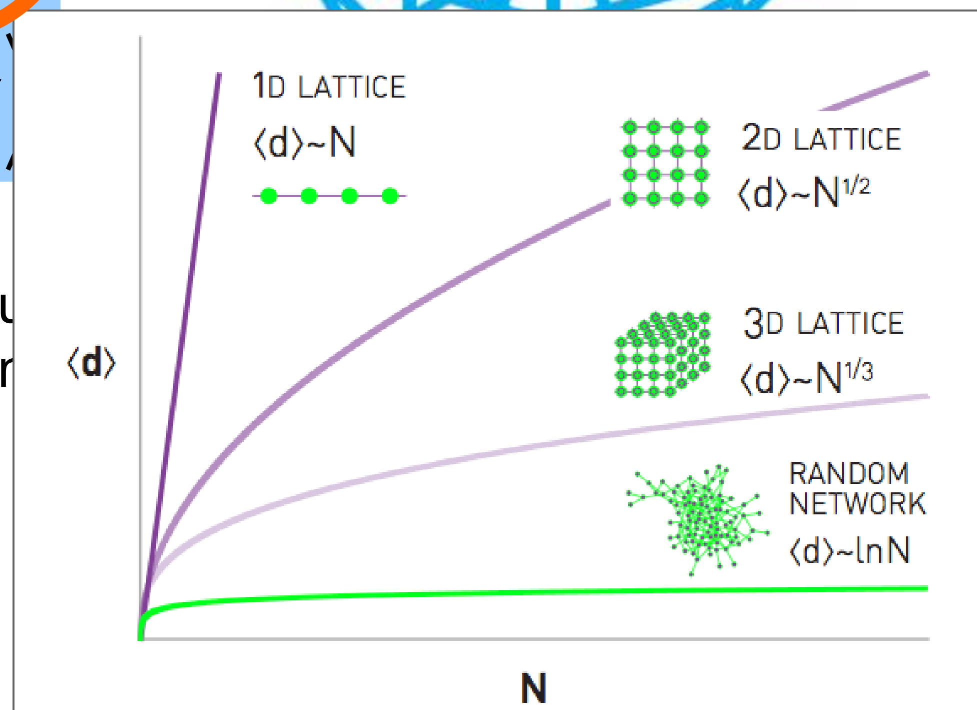
$$C = \frac{\langle k \rangle}{N} \quad ???$$

Average path length of a random graph



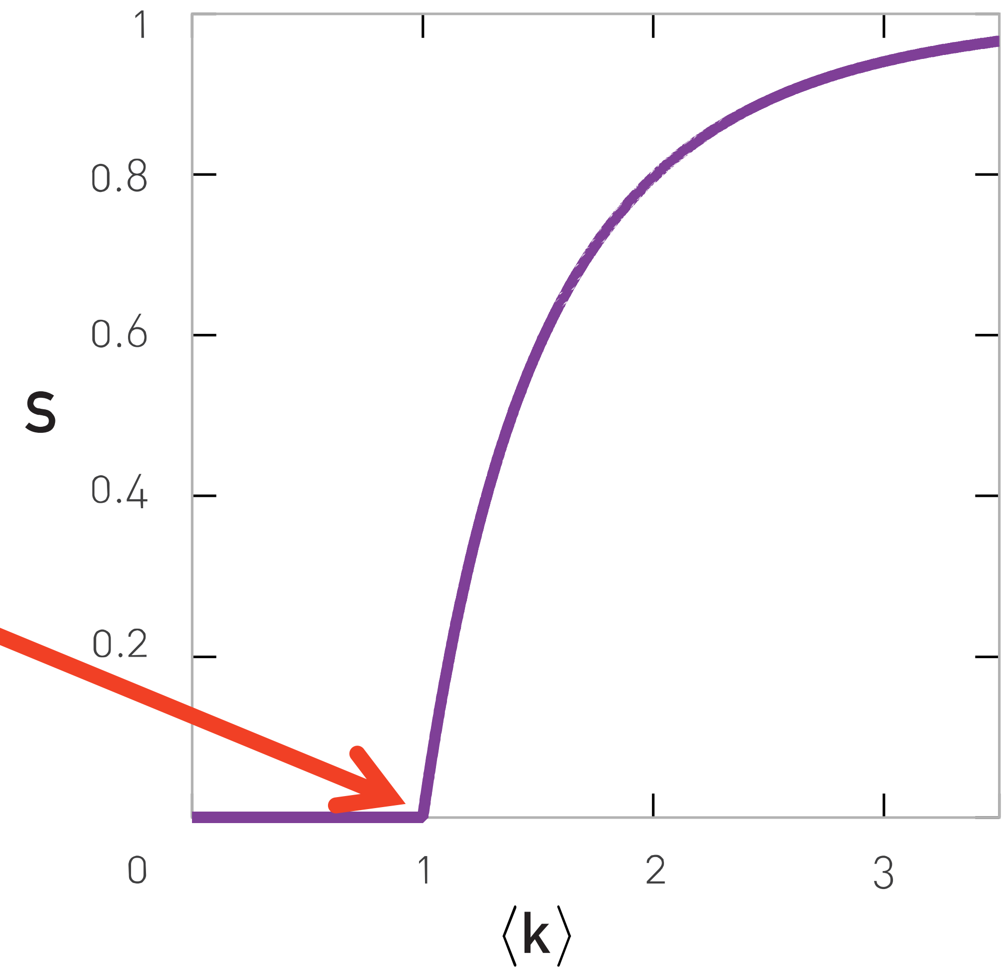
$$APL = \langle L \rangle \square \frac{\ln N}{\ln\langle k\rangle}$$

random network model can accou
emergence of small world phenor



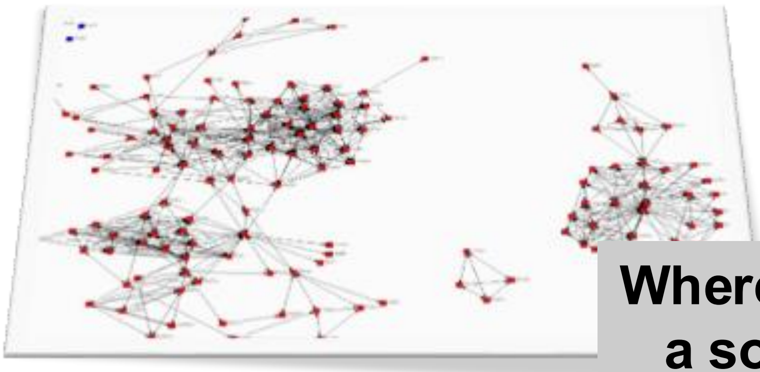
Critical transitions

What happens here?



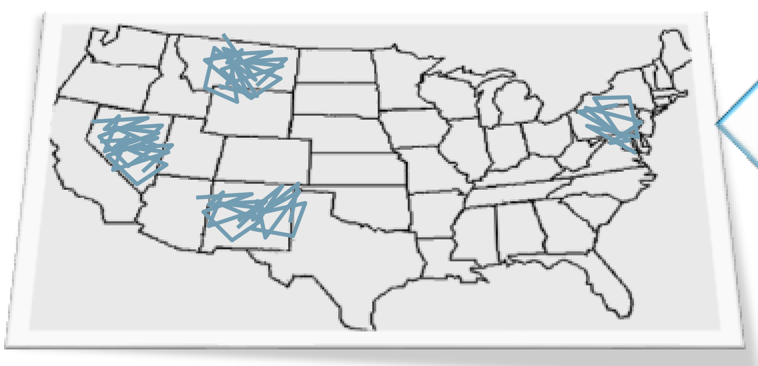
Clustering *vs* Randomness

Clustering implies locality



Randomness enables shortcuts

Where should we put a social network?



Locally Structured

Random

Could a network which is so strongly locally structured be at the same time a small world?

Merging structure and randomness

The Watts-Strogatz Small World model (Nature '98)

Watts & Strogatz invented a very simple model (***1 parameter!***) which interpolates between regular and random graphs.

Duncan J. Watts
Microsoft Research

Steven H. Strogatz
Cornell Univ.

Watts, Duncan J., and Steven H. Strogatz. "Collective dynamics of 'small-world' networks." Nature 393, no. 6684 (1998): 440.

Merging structure and randomness

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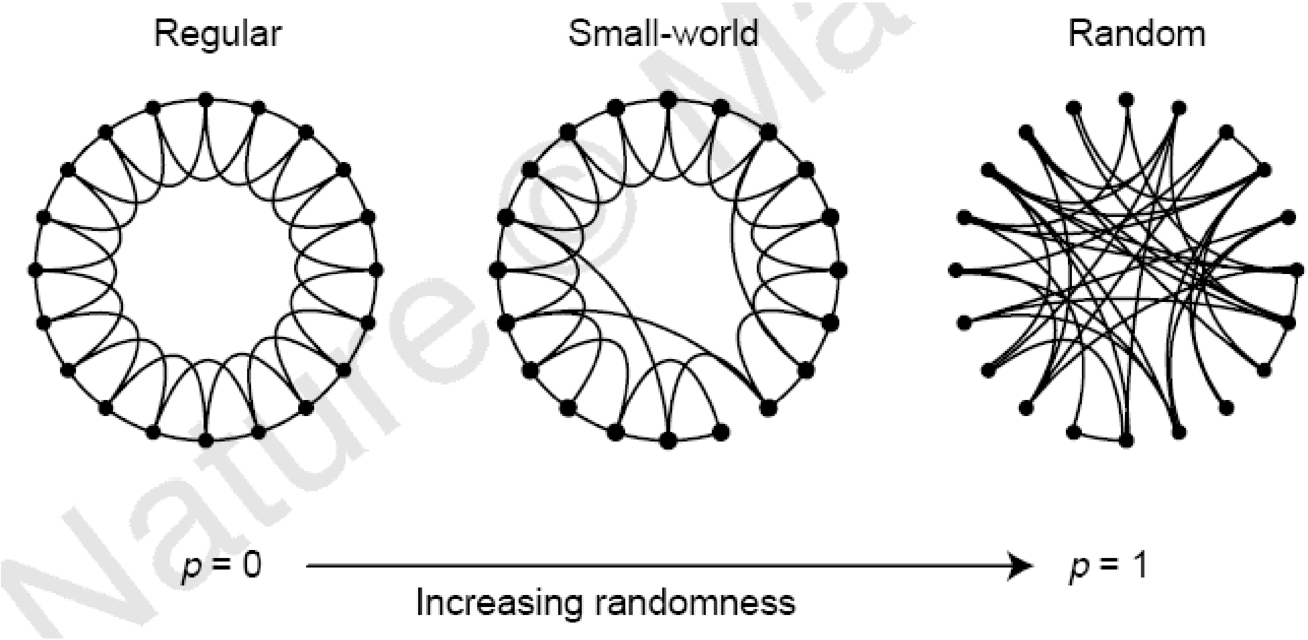
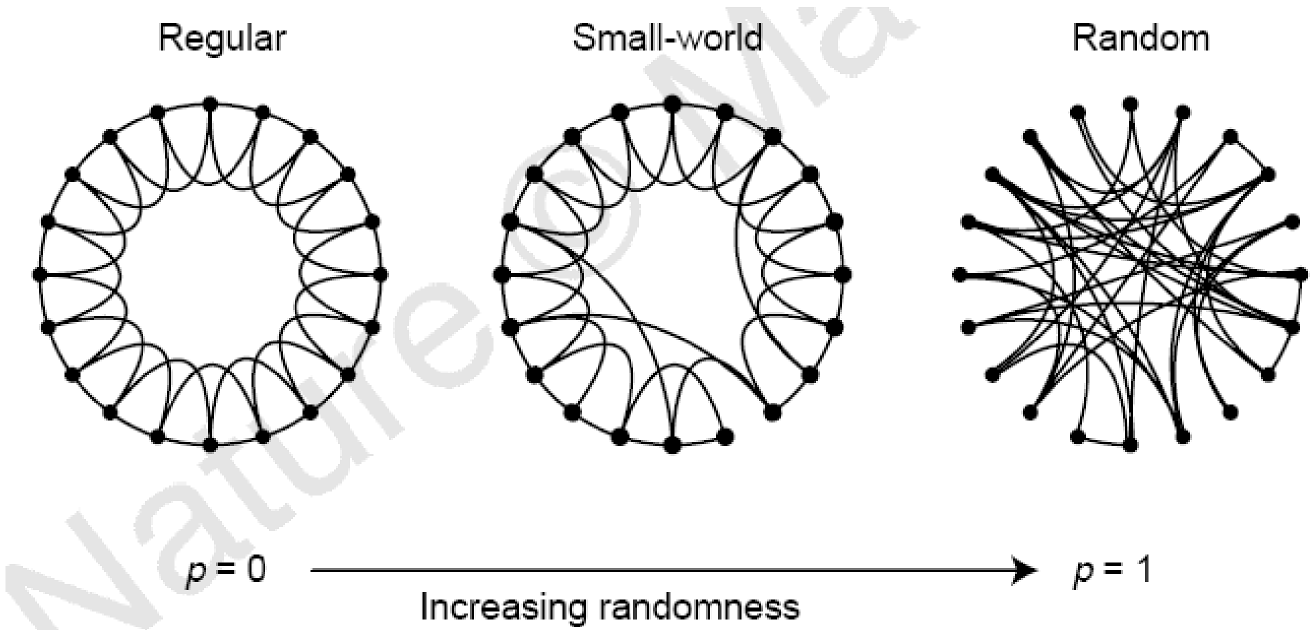


Merging structure and randomness

The Watts-Strogatz Small World model (Nature ‘98)

recipe : start from a regular graph (left); choose a circulating direction (say, clockwise); each edge one encounters is randomly re-directed with a probability ***p***; no repeated edges are allowed; stop when reaching the starting point;



Merging structure and randomness

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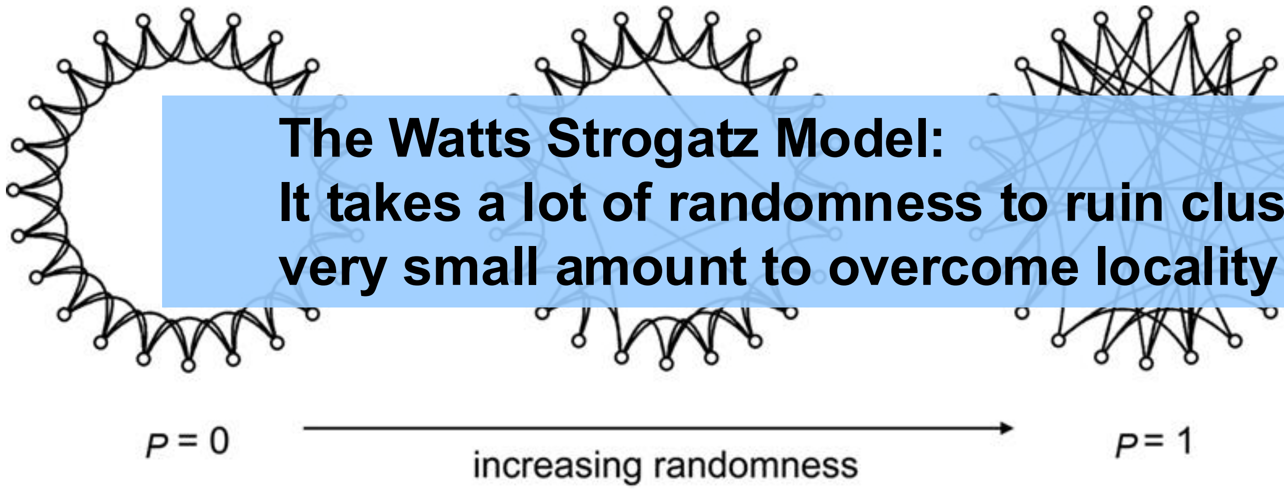
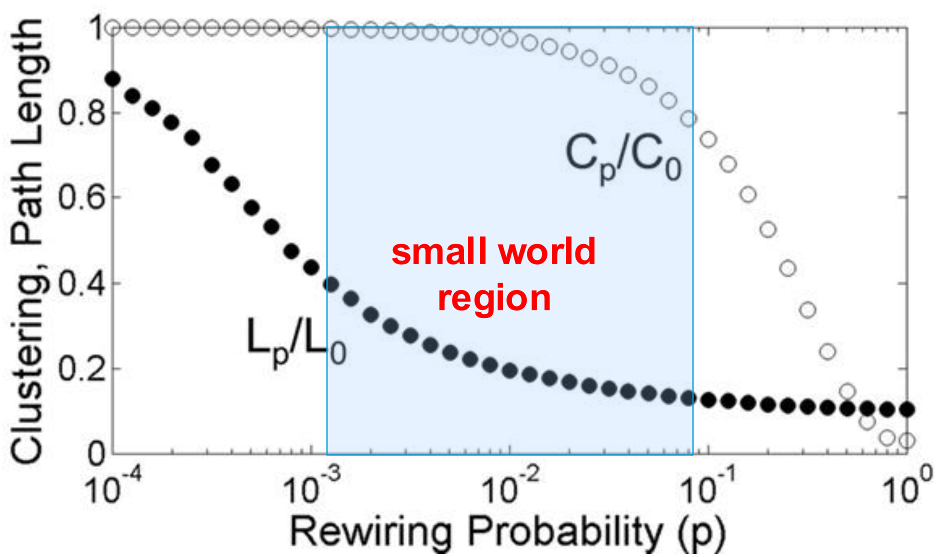


The Watts Strogatz Model:
It takes a lot of randomness to ruin clustering, but a very small amount to overcome locality



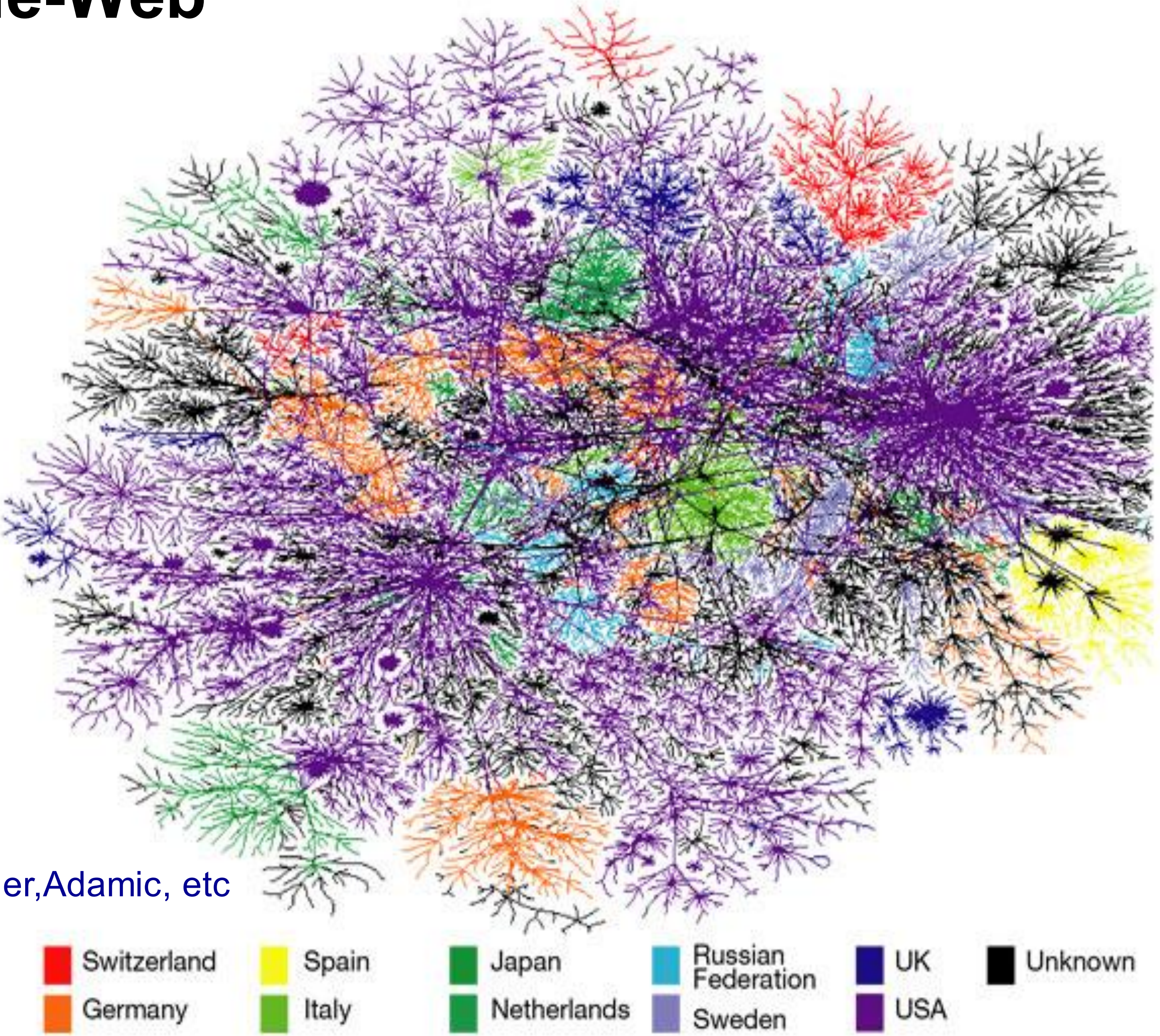
Challenge:
Can you reproduce this result?

2^{nd} network revolution

Barabási, Albert, Jeong, Dorogovtsev, Mendes, Havlin, Cohen…
(>1999)

Network Science, 2025/26

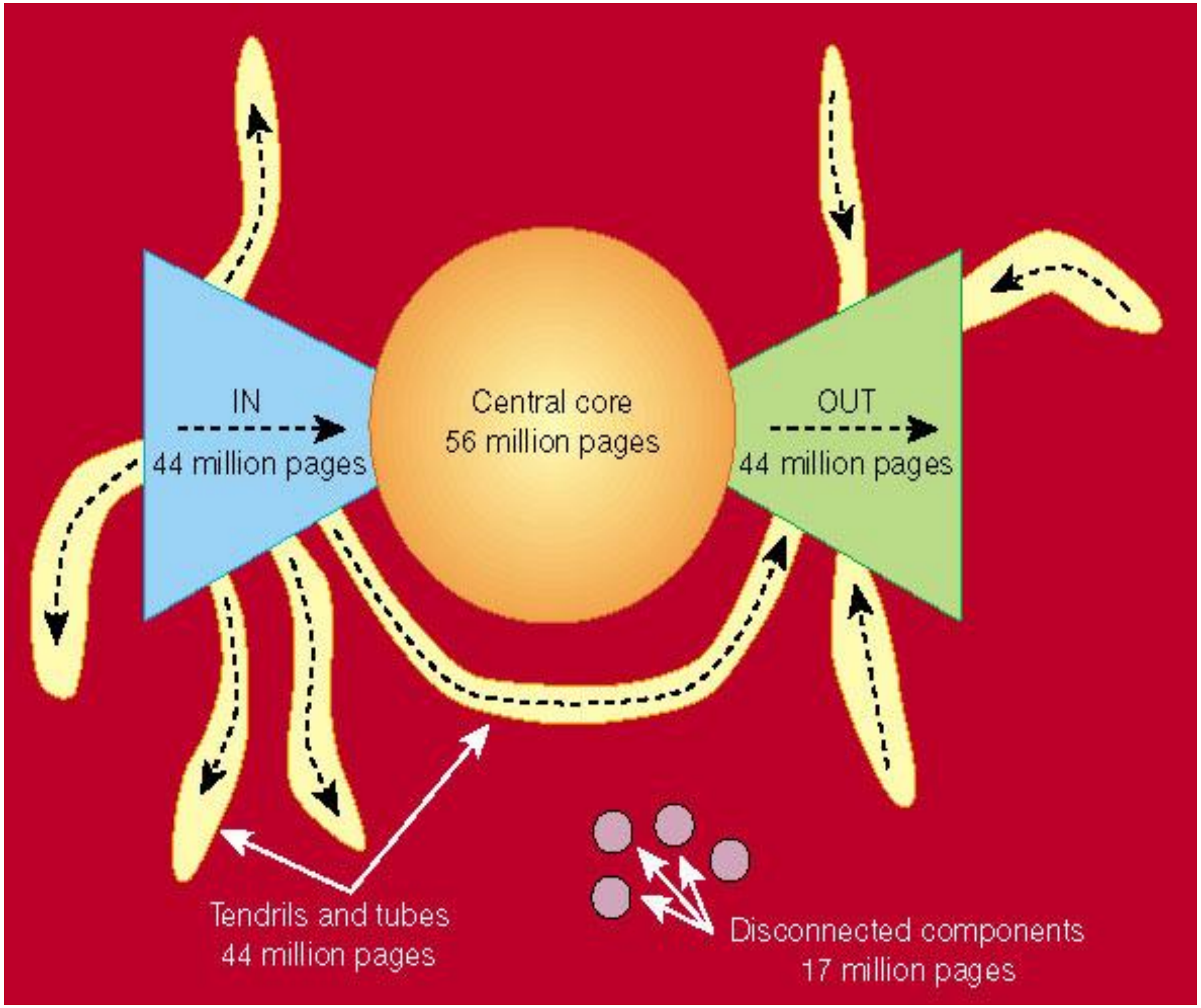
World-Wide-Web

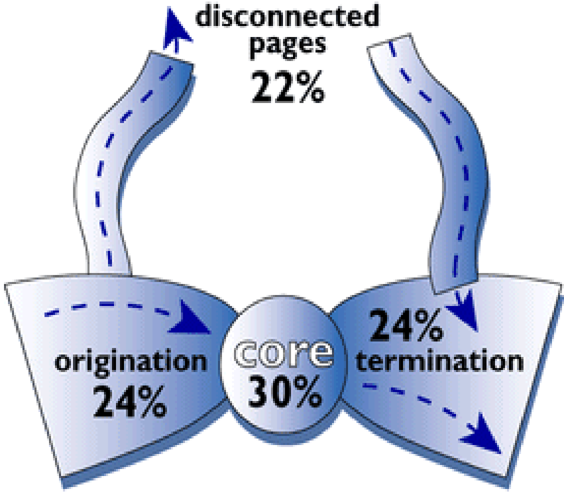


Kleinberg, Barabasi, Broder,Adamic, etc

World-Wide-Web is a “bow-tie”

200 million pages / 1500 million hyperlinks



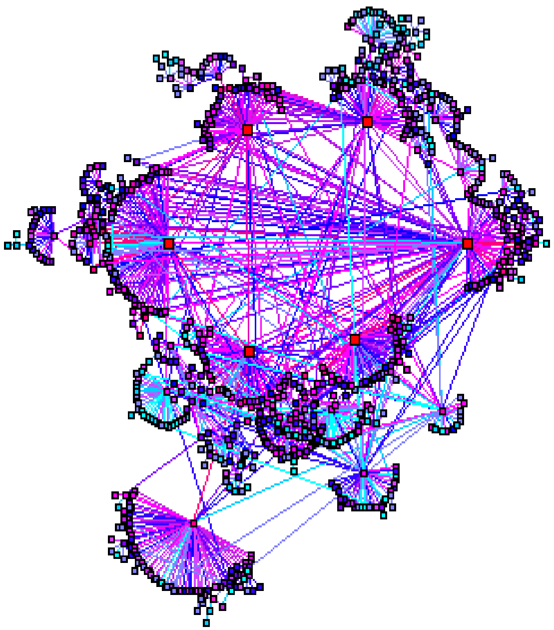


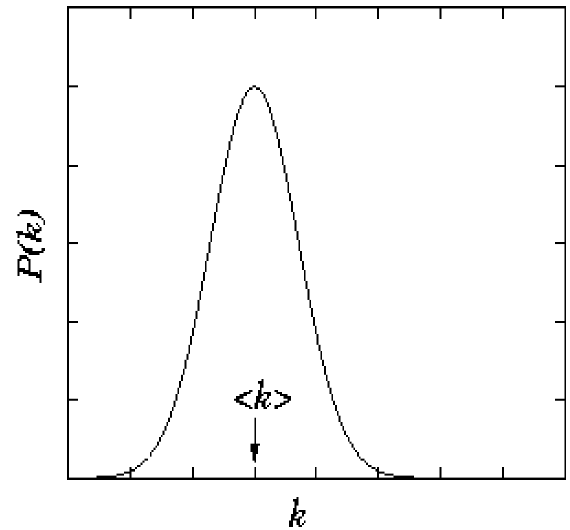
World-Wide-Web

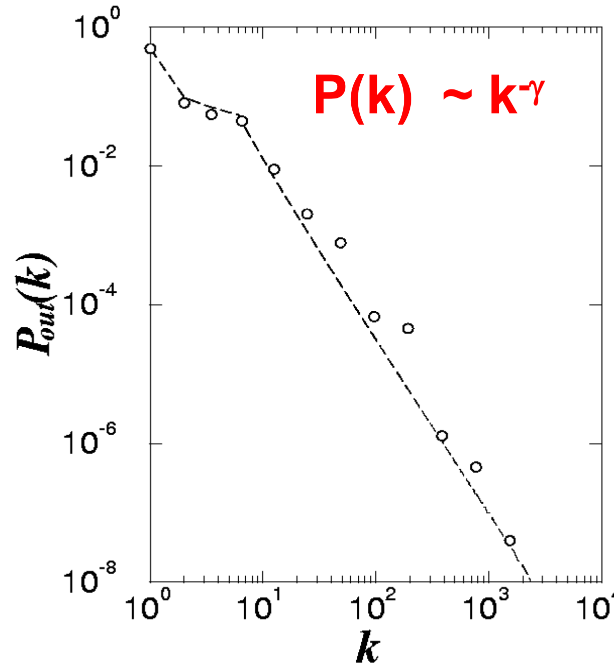
Nodes: **WWW documents**
Links: **URL links**

Over 3 billion documents

ROBOT: collects all URL's found in a document and follows them recursively



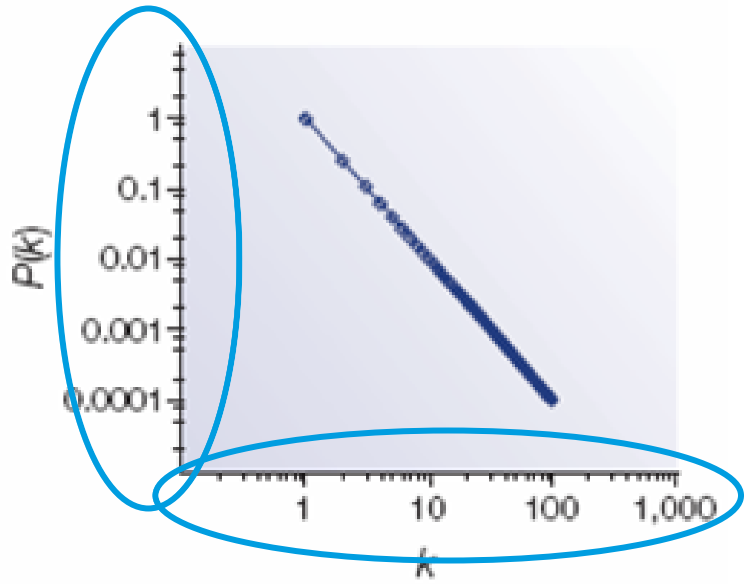




R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).

Power-law degree distributions

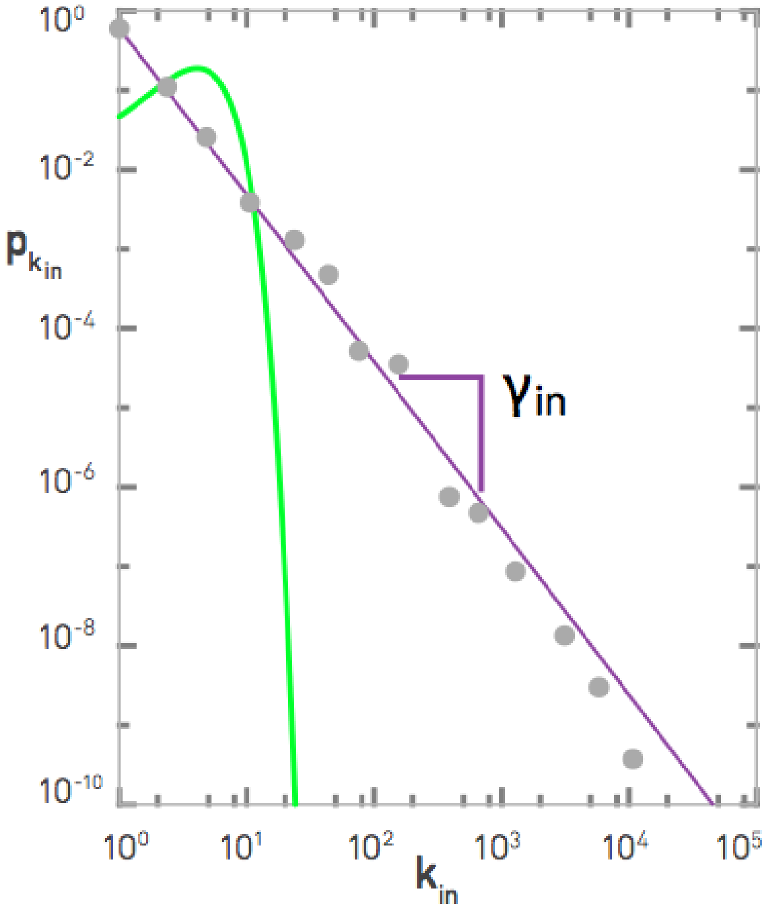
$$P_k \sim k^{-\gamma}$$

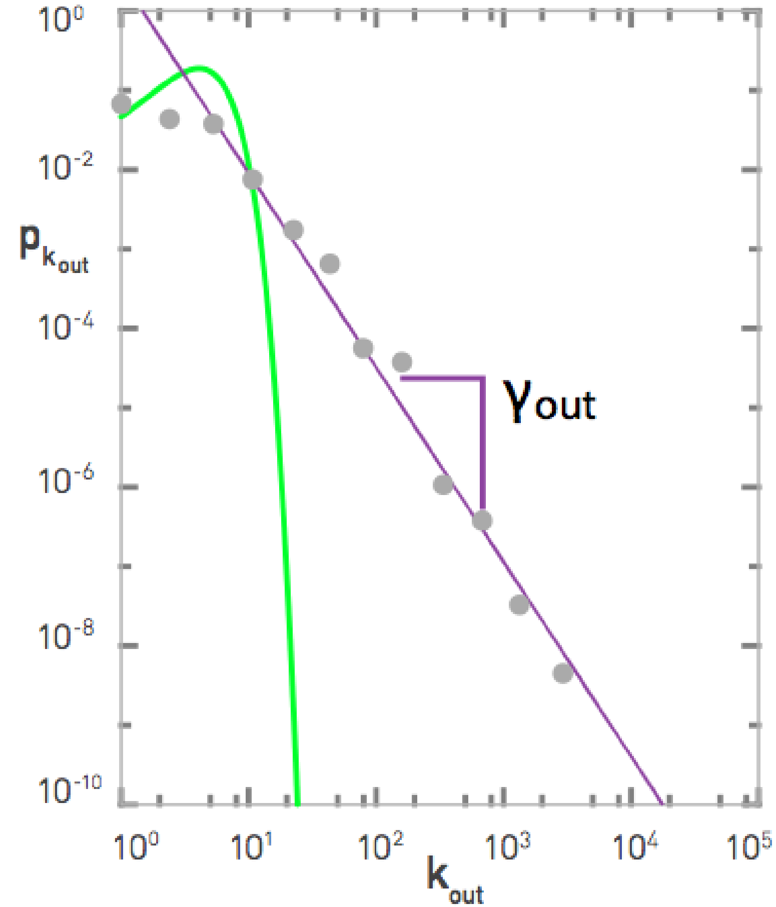


$$\ln P_k \sim \ln k^{-\gamma}$$

$$\ln P_k \sim -\gamma \ln k$$

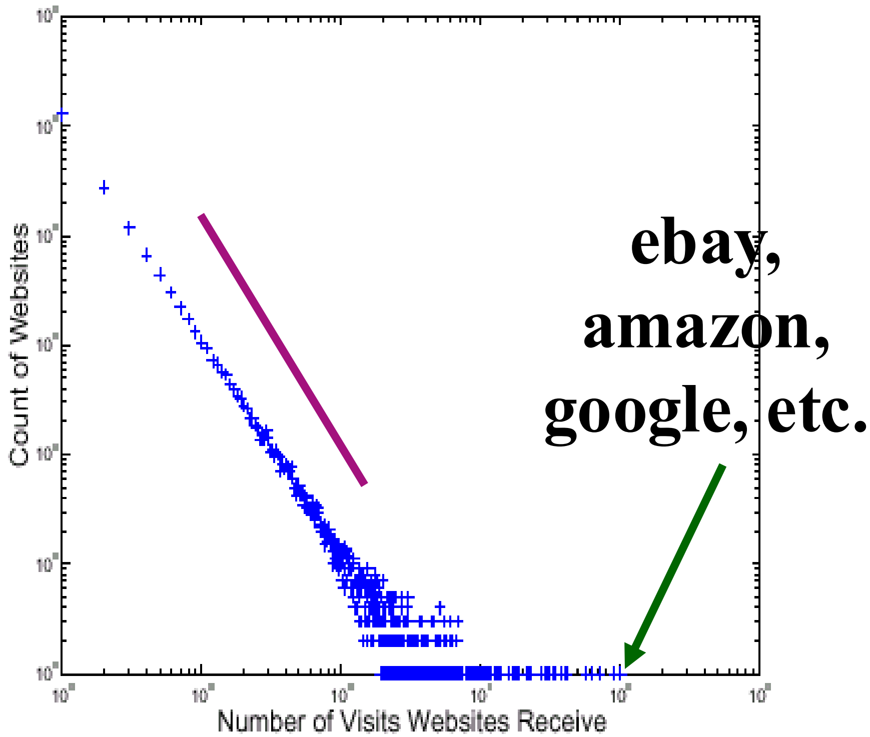
World-Wide-Web as a directed graph:
in & out degree distributions

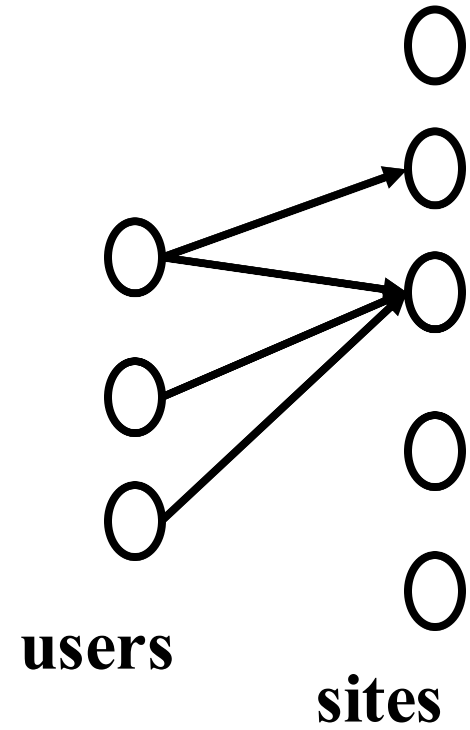




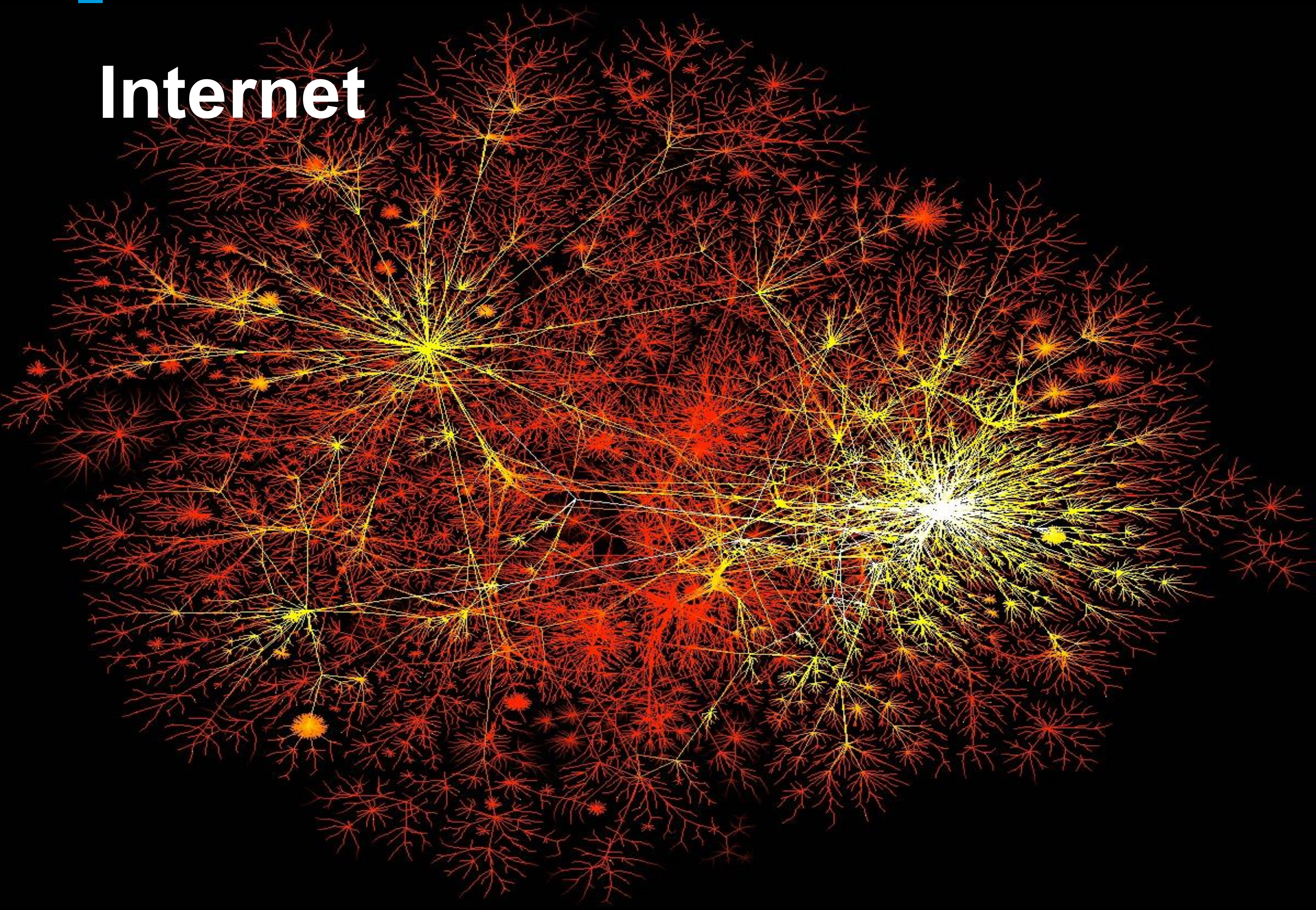
Power-laws are everywhere in the WWW

Web Site Traffic





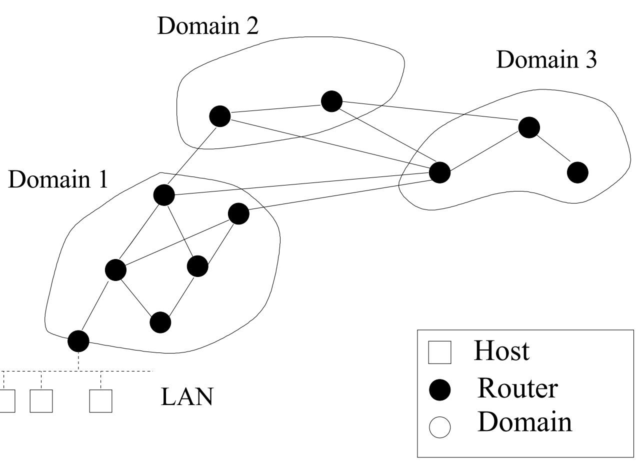
Internet

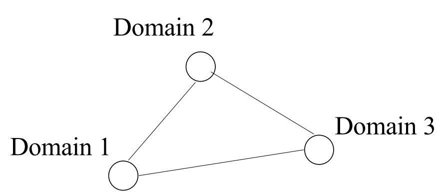


Internet

Internet (Faloutsos et al, 1999)
Nodes = computers, and edges physical connections among them … quite difficult

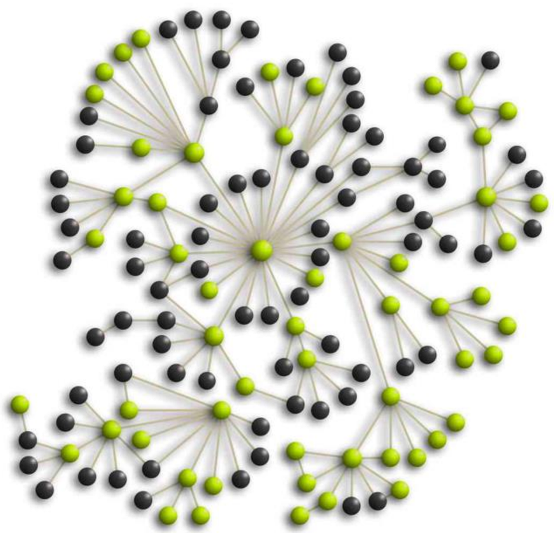
2 levels
Nodes = routers or
Nodes = domains





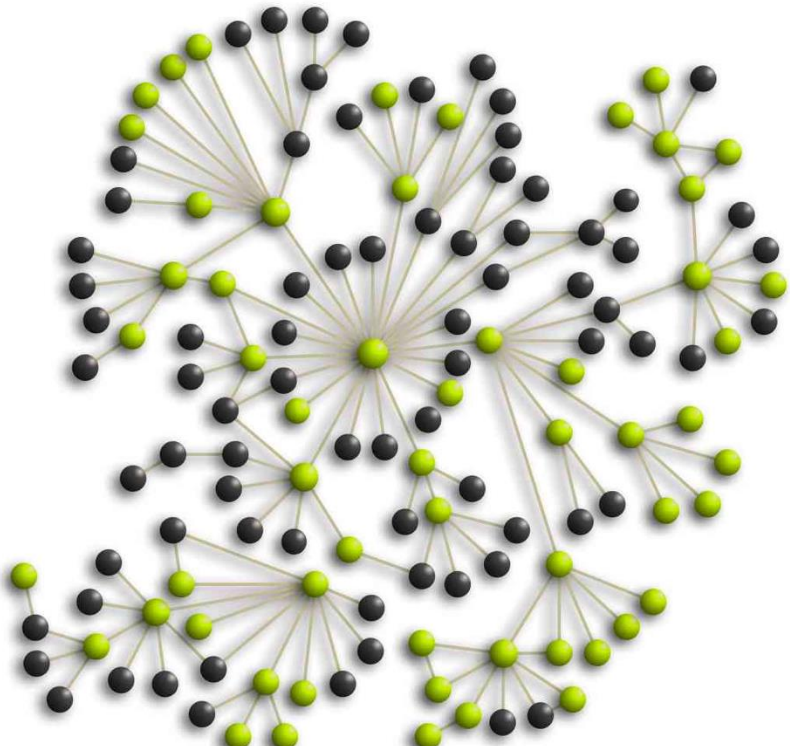
Both showed a scale-free degree dist.

Many real-world nets are scale-free

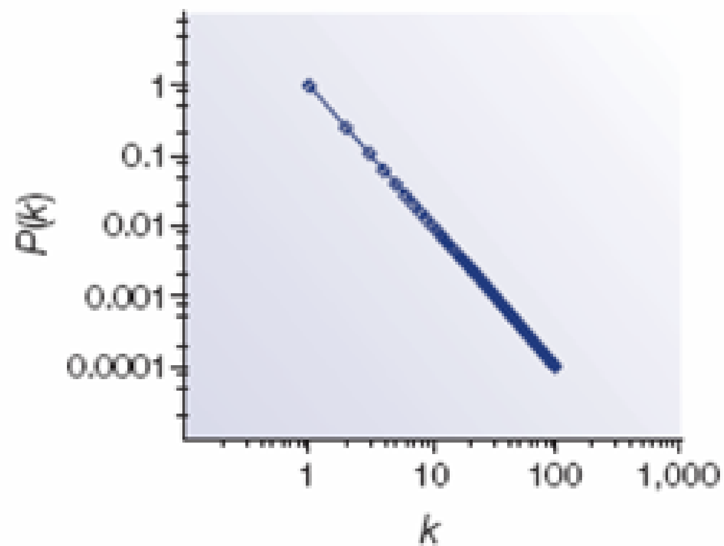


WWW, Internet (routers and domains), electronic circuits, computer software, movie actors, coauthorship networks, sexual web, instant messaging, email web, citations, phone calls, metabolic, protein interaction, protein domains, brain function web, linguistic networks, comic book characters, international trade, bank system, encryption trust net, energy landscapes, astrophysical network…

Universality? I will get back to this point...



$$P_k \sim k^{-\gamma}$$



WWW	actors	citations	sex	cellular	phones	linguistics
$\gamma = 2.1$	$\gamma = 2.3$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 2.1$	$\gamma = 2.1$	$\gamma = 2.8$

A closer look at power-law distributions

discrete representation: $P_k = Ck^{-\gamma}$ with C given by

$$\sum_{k_{\min}}^{\infty} P_k = 1 \Rightarrow \quad C = \frac{1}{\sum_{k_{\min}}^{\infty} k^{-\gamma}} = \frac{1}{\varsigma(\gamma)}$$

Riemann-zeta function

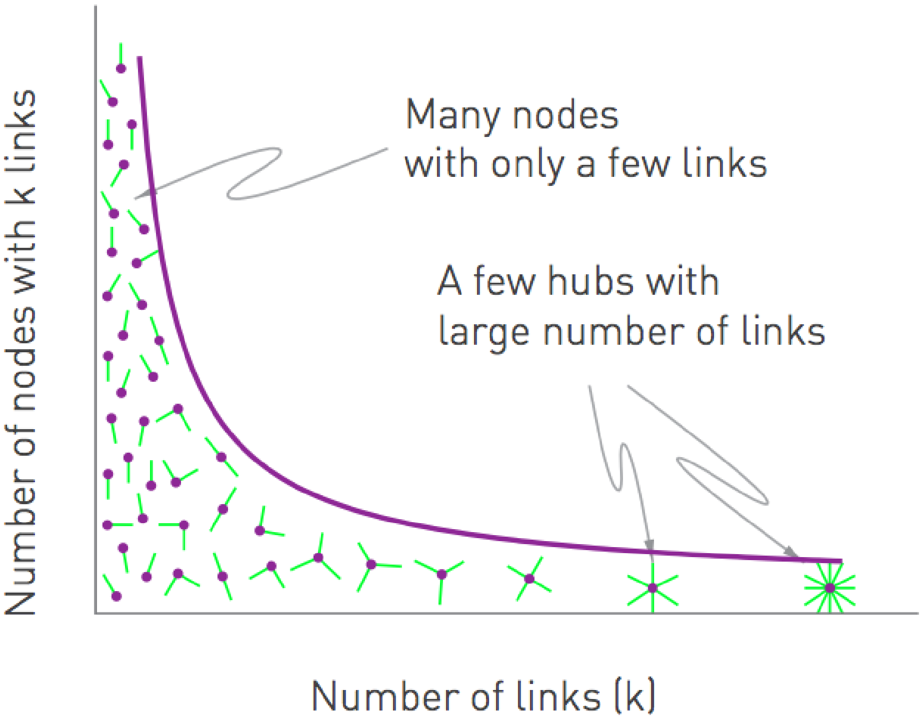
continuum description: $P_k = Ck^{-\gamma}$ with C given by

$$C = \frac{1}{\int_{K_{\min}}^{\infty} k^{-\gamma}\, dk} = (\gamma - 1) k_{\min}^{\gamma-1}$$

Largest hub

- Just for the fun of it: Imagine that we have a **scale-free network** with the size and average degree of the WWW ($N=10^{12}$,$\langle k \rangle=4.6$ and $\gamma=2.1$). What's the probability of having a node with a degree $k \geq 100$?

$$P_k = \left[(\gamma-1)k_{\min}^{\gamma-1}\right]k^{-\gamma}$$



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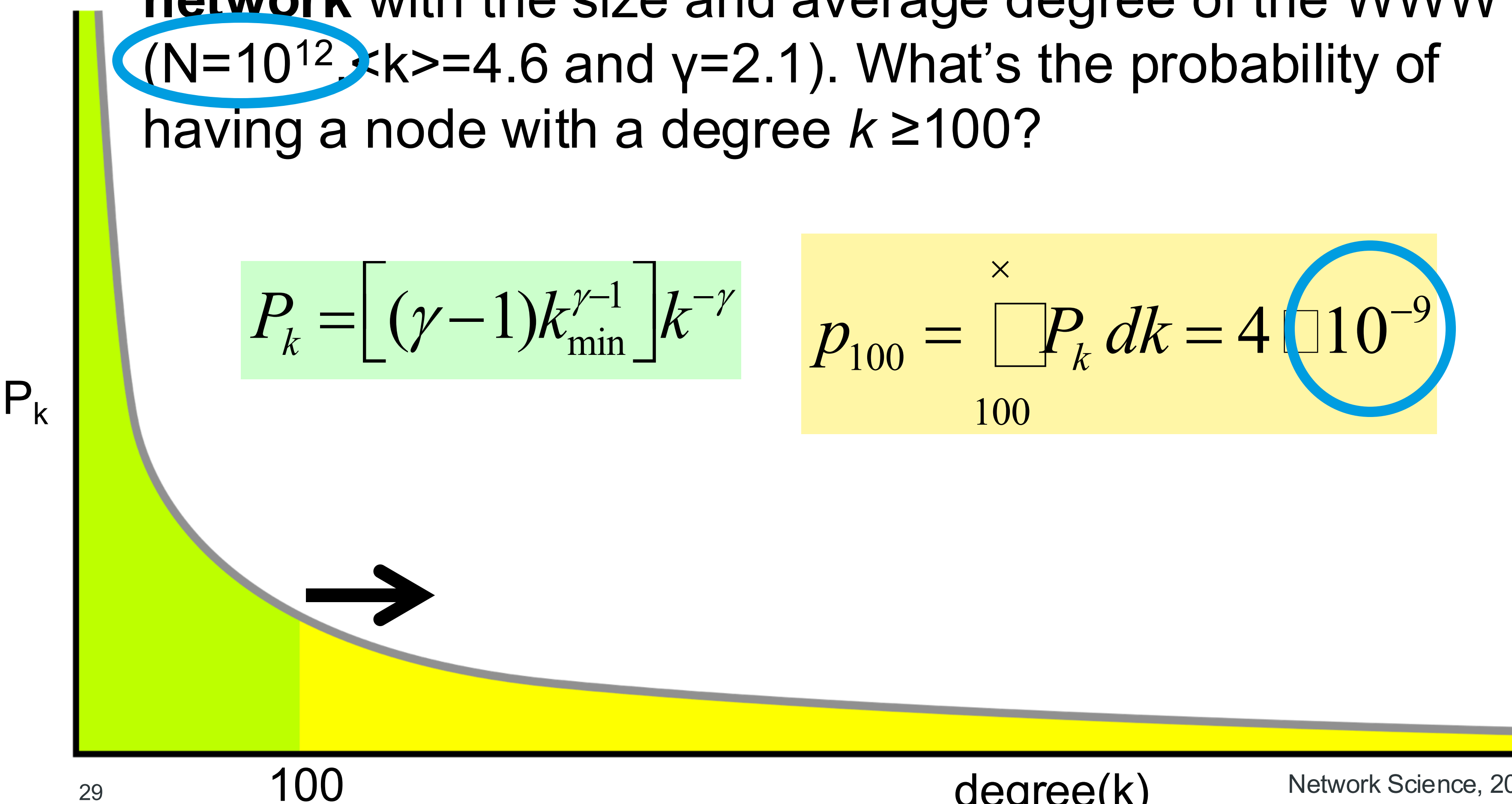
$$P_k = \left[(\gamma - 1)k_{\min}^{\gamma-1}\right]k^{-\gamma}$$

$$p_{100} = \int_{100}^{\infty} P_k \, dk = 4 \times 10^{-9}$$



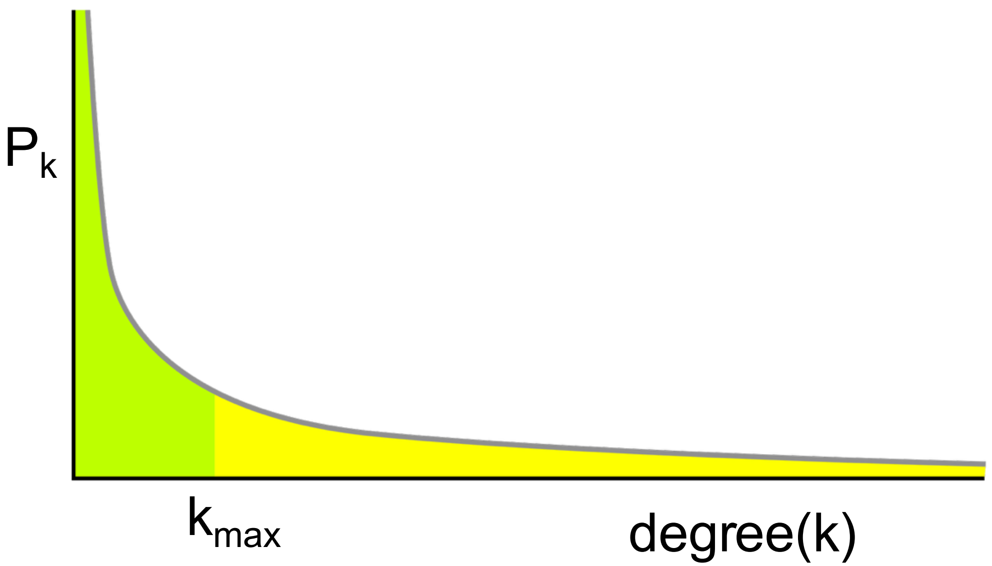
Can we estimate the largest hub?

- We have the degree dist:

$$P_k = \left[(\gamma - 1) k_{\min}^{\gamma-1}\right] k^{-\gamma}$$

- k_{max} will be given by

$$\int_{k_{\max}}^{\infty} P_k \, dk < \frac{1}{N}$$

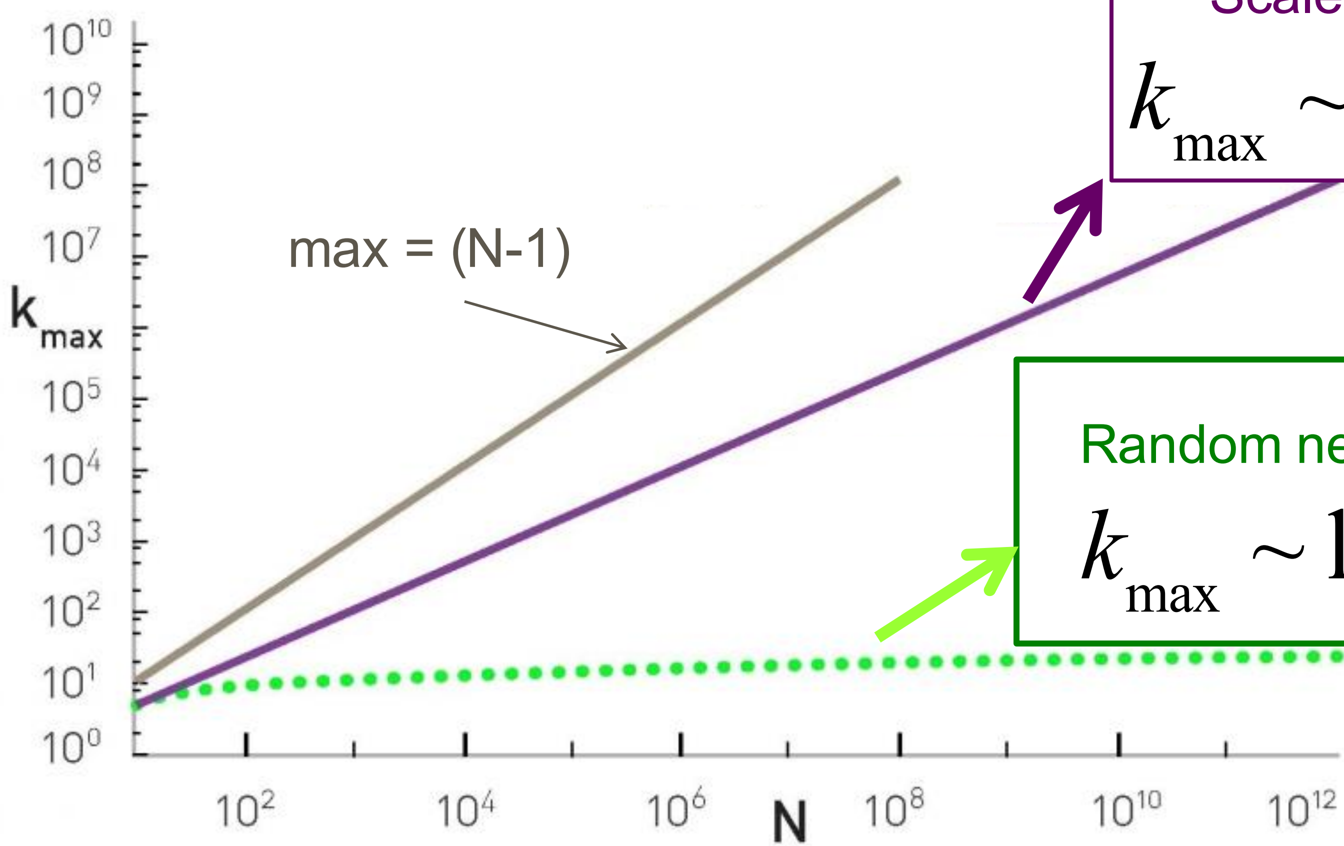


- Which yields

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

- Example: WWW gives k_{max}=95000

Can we estimate the largest hub?



Scale-free

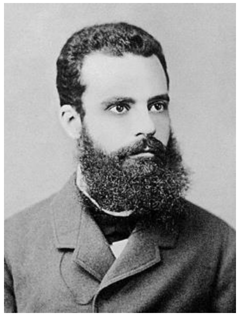
$$k_{\max} \sim N^{\frac{1}{\gamma - 1}}$$

Random network

$$k_{\max} \sim \ln N$$

Revisiting Pareto

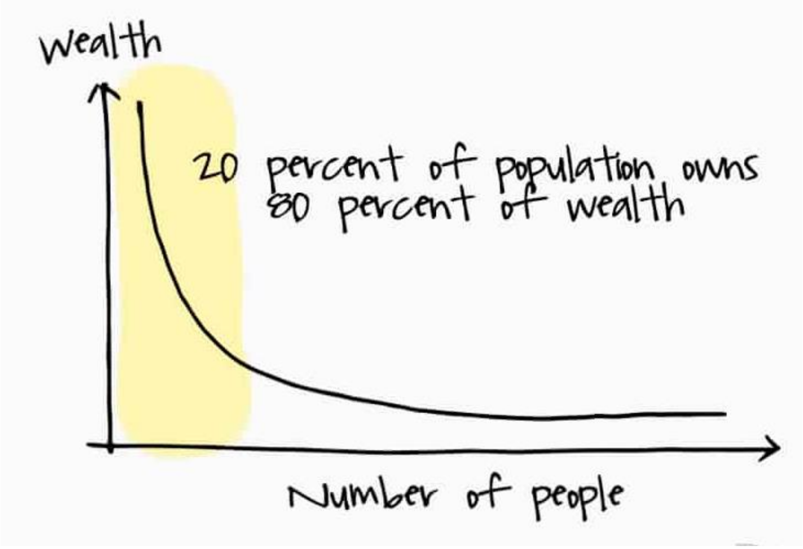
Vilfred Pareto
(1848-1923)



- Power-law distributions are also called Pareto distributions.
- Power-laws is the principle behind the 80/20 rule: *Roughly 80 percent of money is earned by only 20 percent of the population*.

Distribution of world GDP

Quintile of population	Income
Richest 20%	82.70%
Second 20%	11.75%
Third 20%	2.30%
Fourth 20%	1.85%
Poorest 20%	1.40%



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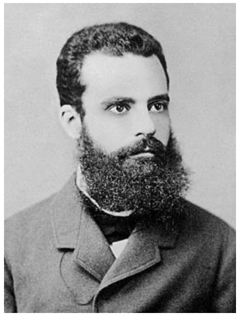
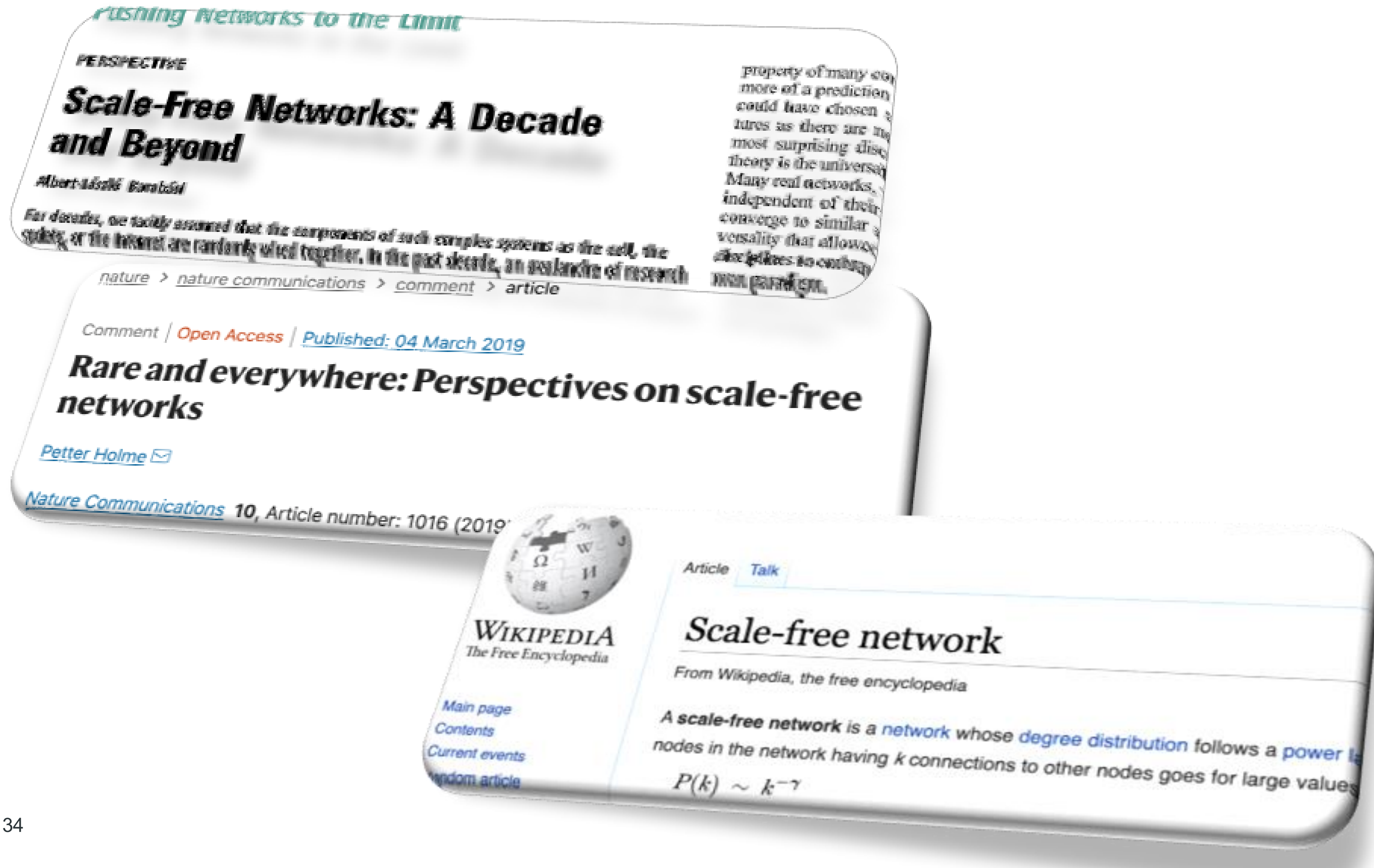


- Power-law distributions are also called Pareto distributions.
- Power-laws is the principle behind the 80/20 rule: *Roughly 80 percent of money is earned by only 20 percent of the population*.
  - WWW: 80% of the links point to 15% of the pages.
  - Citations: 80% of all citations point to 38% of scientists
  - Hollywood: 80% of all links connect 30% of actors.
  - USA: 1% of the population earns 15% of the total US income.

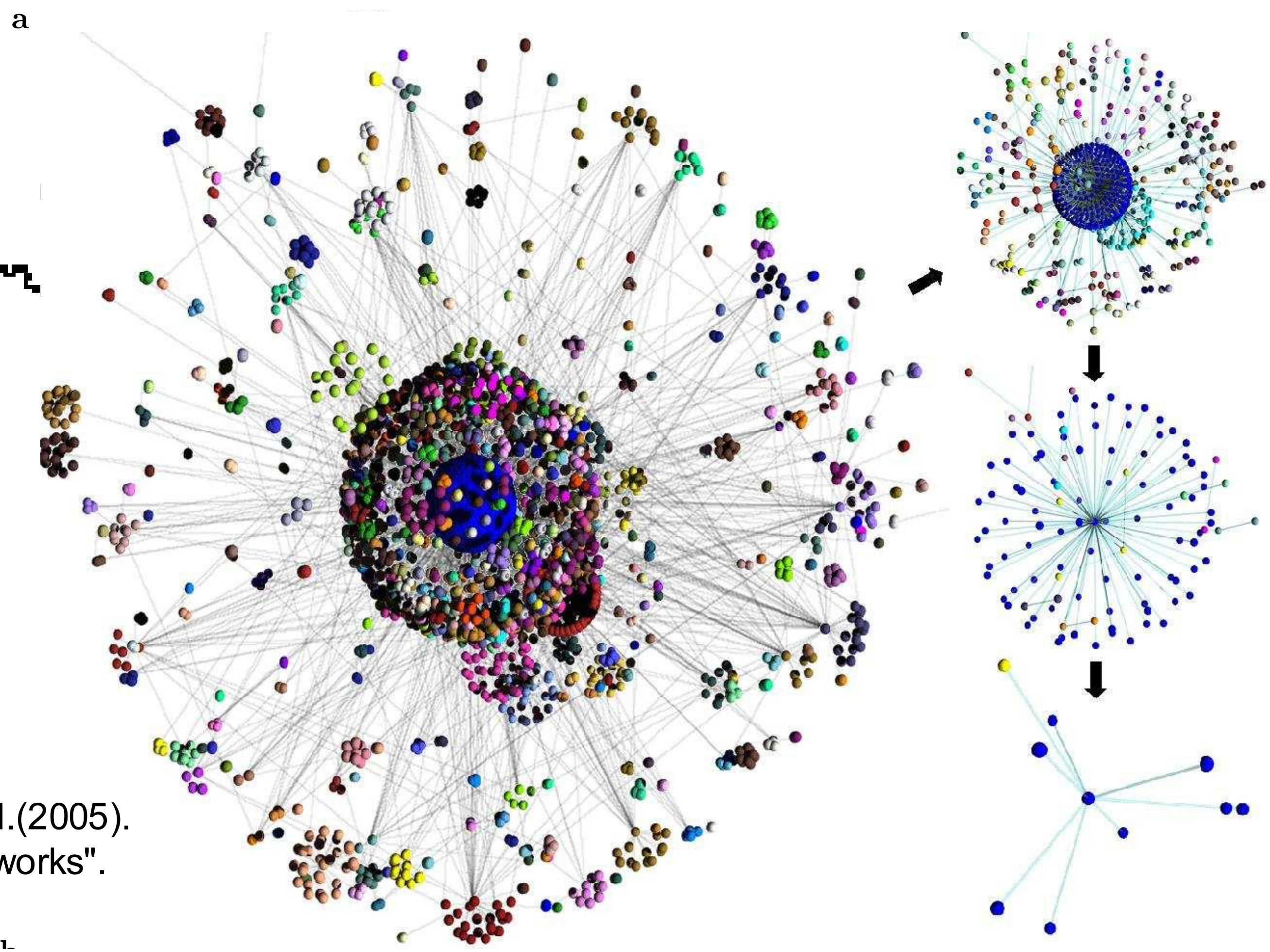
Scale-free networks? What does it mean?



Scale-free? What does it mean?

1st explanation: Scale invariance

Scale-invariance, self-similarity, etc.



Song, C.; Havlin, S.; Makse, H.(2005).
"Self-similarity of complex networks".
Nature 433 (7024): 392–395.

Scale-free? What does it mean?

1st explanation: Scale invariance

Scale-invariance?

$$P(k) = k^{-\gamma}$$

$$P(ck) = (ck)^{-\gamma} = Const.P(k) \propto P(k)$$

- Scaling the argument ***k*** by a constant factor ***c*** causes only a proportionate scaling of the function itself.
- In other words, all power laws (with a given exponent) are equivalent up to constant factors, since each is simply a scaled version of the others.

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

- ***n*th moments of a distribution**

$$\left\langle k^n \right\rangle = \sum_{k_{\min}}^{\infty} k^n P_k \approx \int_{k_{\min}}^{\infty} k^n P_k \, dk$$

- n=0 sums to one.
- n=1 gives the ***average*** degree
- n=2 helps us to calculate the ***variance***
- n=3 determines the ***skewness***…

$$\sigma^2 = \left\langle k^2 \right\rangle - \left\langle k \right\rangle^2$$

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

- For Scale-free networks we have

$$\langle k^n \rangle \sim \int_{k_{min}}^{k_{max}} k^n P_k \, dk = C \frac{k_{max}^{n-\gamma+1} - k_{min}^{n-\gamma+1}}{n-\gamma+1}$$

- n=0 sums to one.
- n=1 gives the ***average*** degree
- n=2 helps us to calculate the ***variance***
- n=3 determines the ***skewness***

$$k_{max} = k_{min} N^{\frac{1}{\gamma-1}}$$

Scales with N

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

- For Scale-free networks we have

$$\langle k^n \rangle \approx \int_{k_{\min}}^{k_{\max}} k^n P_k \, dk = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n-\gamma+1}$$

Do not scale with N

- n=0 sums to one.
- n=1 gives the **average** degree
- n=2 helps us to calculate the **variance**
- n=3 determines the **skewness**

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

Scales with N

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

- For Scale-free networks we have (for large N)

$$\langle k^n \rangle \xrightarrow{N\to\infty} N^{\frac{n-\gamma+1}{\gamma-1}}$$

For large N, it diverges for γ<2

$$\langle k \rangle \sim N^{\frac{2-\gamma}{\gamma-1}}$$

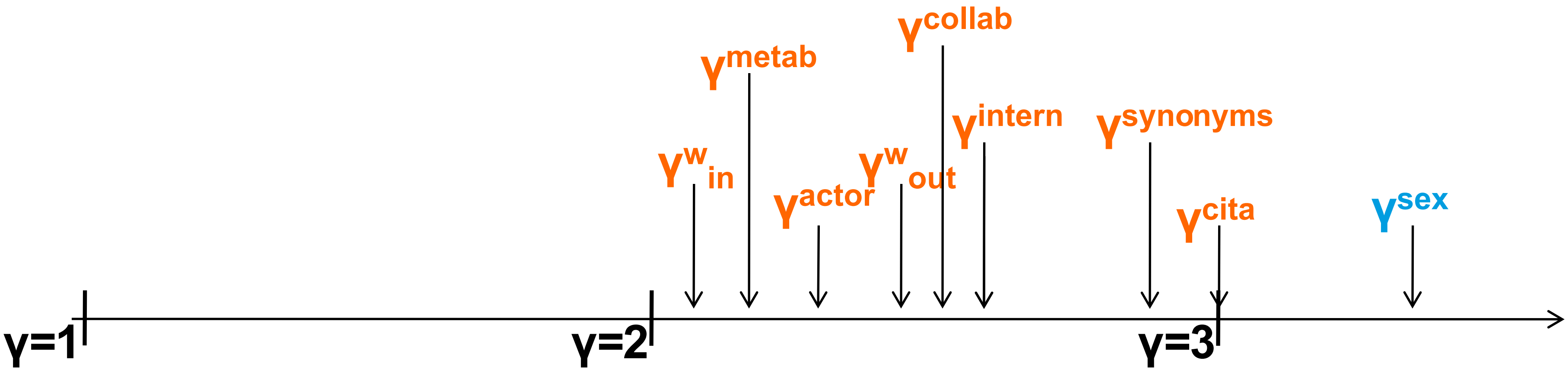
$$\langle k^2 \rangle \sim N^{\frac{3-\gamma}{\gamma-1}}$$

Diverges for γ<3

- n=0 sums to one.
- n=1 gives the ***average*** degree
- n=2 helps us to calculate the ***variance***
- n=3 determines the ***skewness***

$$k = \langle k \rangle \pm \times$$

Scale-free? What does it mean?



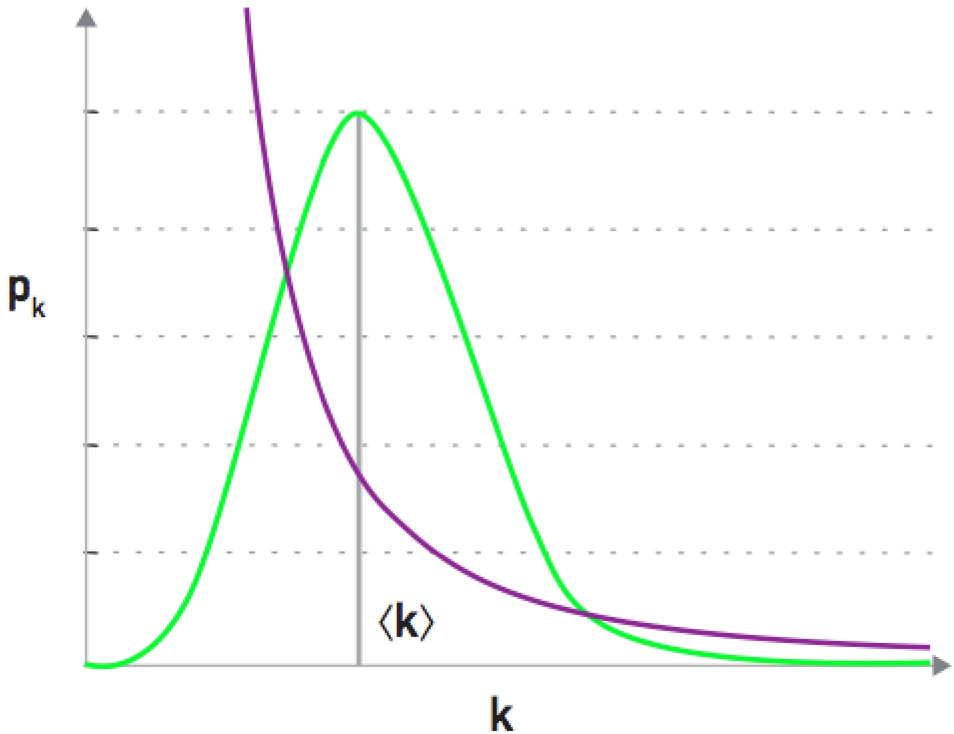
$\langle k^2 \rangle$ diverges	$\langle k^2 \rangle$ finite
$\langle k \rangle$ diverges	$\langle k \rangle$ finite

Most networks are in a regime in which the variance diverges for large N ☺

$$k = \langle k \rangle \pm \times$$

For large N, average values are not meaningful, as fluctuations are too large!

Scale-free? What does it mean?



Random Network

Randomly chosen node: $k = \langle k \rangle \pm \langle k \rangle^{1/2}$

Scale: $\langle k \rangle$

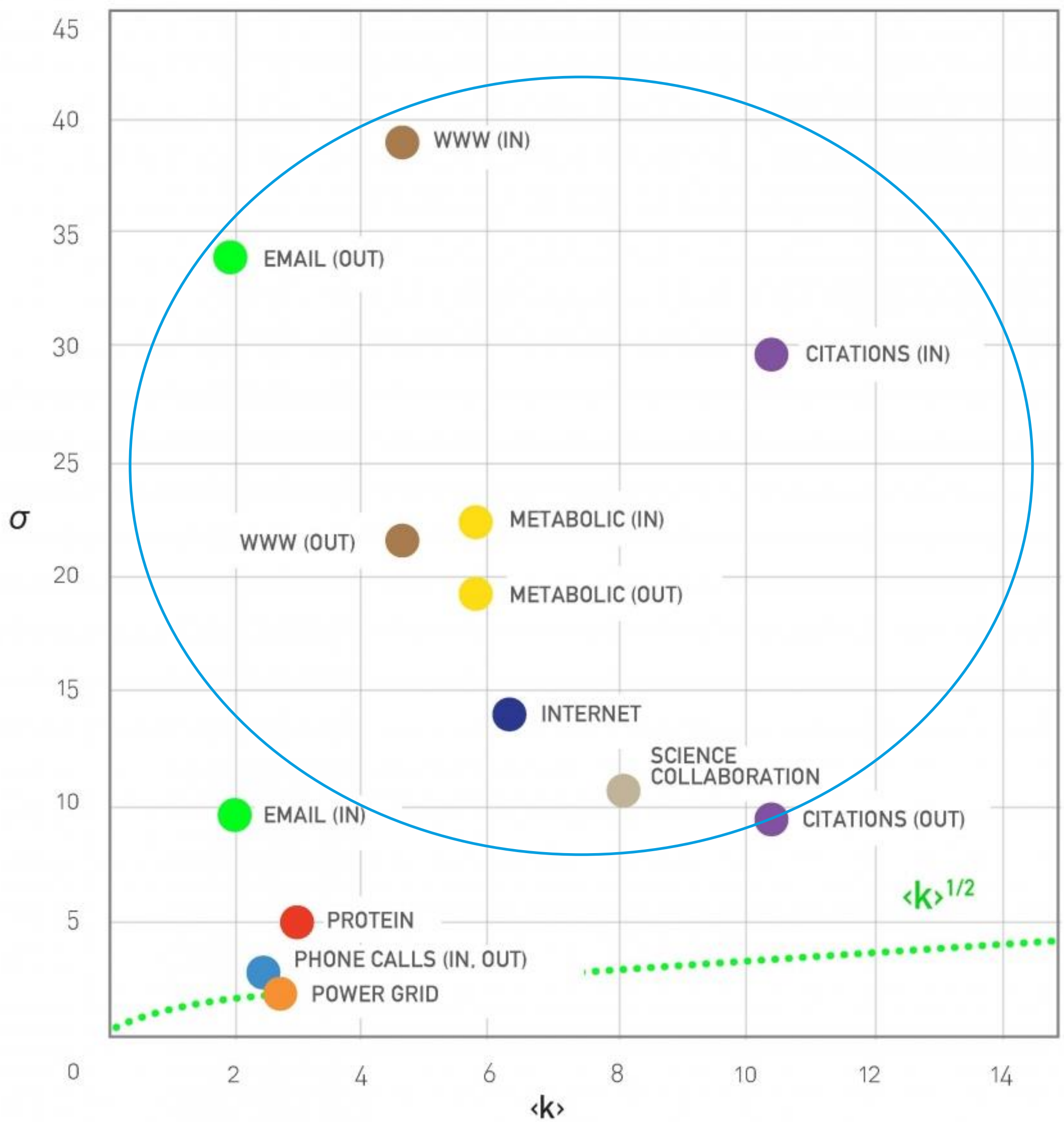
Scale-Free Network

Randomly chosen node: $k = \langle k \rangle \pm \infty$

Scale: none

Scale-free? Networks are finite, yet…

Standard deviation is very large in real networks



Do hubs affect the small world property?

Let’s compute the average path length (APL) for a scale-free network...

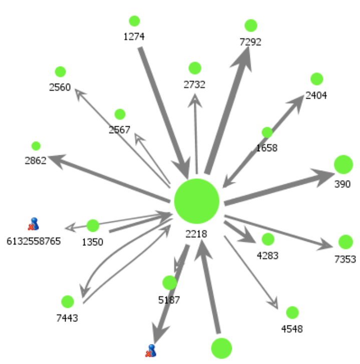
Do we live in a ultra-small-world?

$$APL = \langle L \rangle \sim \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

Anomalous regime (γ≤2)

$$k_{max} = k_{min} N^{\frac{1}{\gamma - 1}}$$

K_{max} grows faster than *N*



const.	$\gamma = 2$
$\ln \ln N$	$2 < \gamma < 3$
$\dfrac{\ln N}{\ln \ln N}$	$\gamma = 3$
$\ln N$	$\gamma > 3$

Ultra-small world regime

$$APL = \langle L \rangle \sim \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

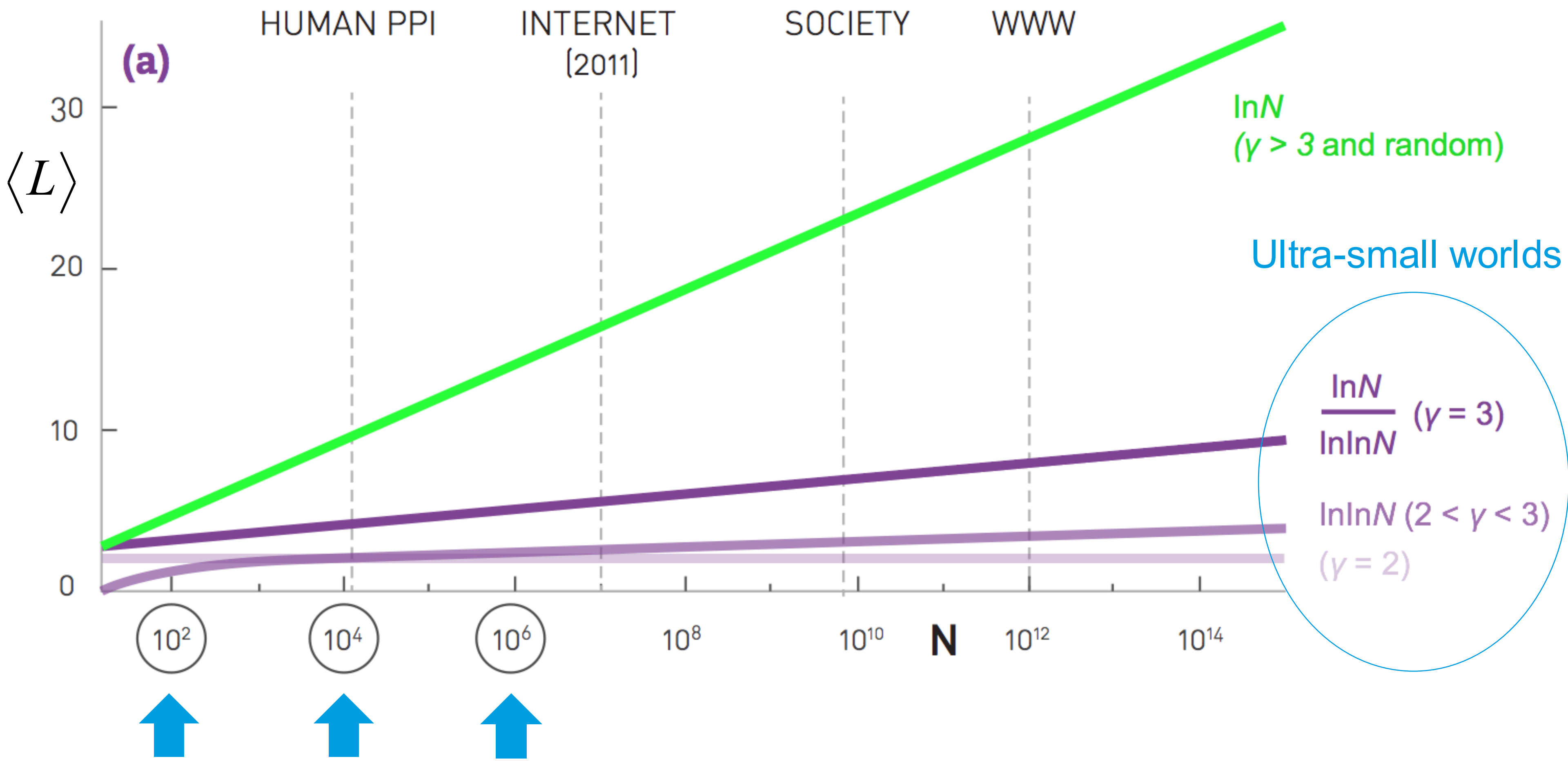
Critical regime

$$APL = \langle L \rangle \sim \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

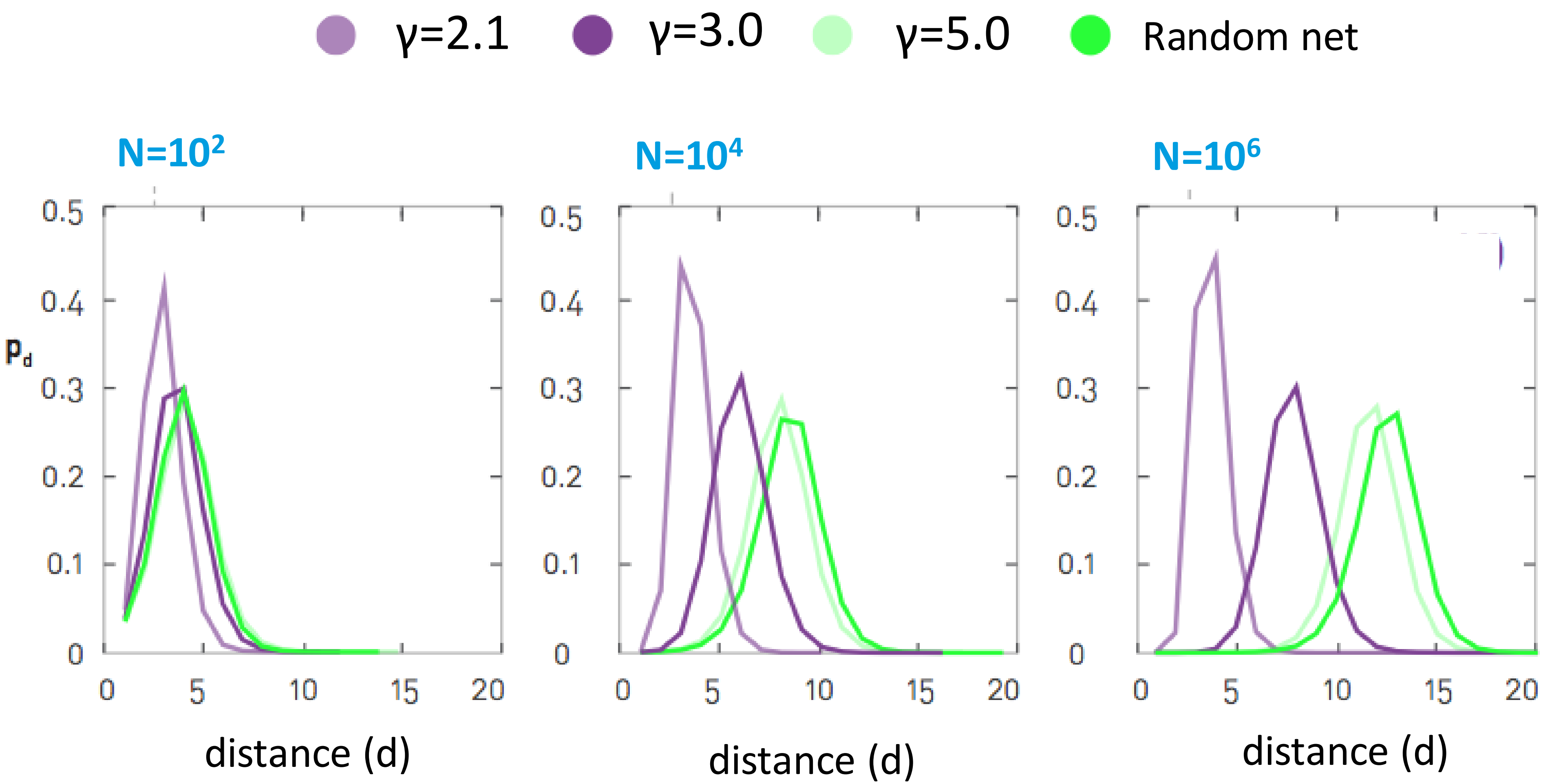
Small-world regime (same as random graphs)

$$APL = \langle L \rangle \square \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

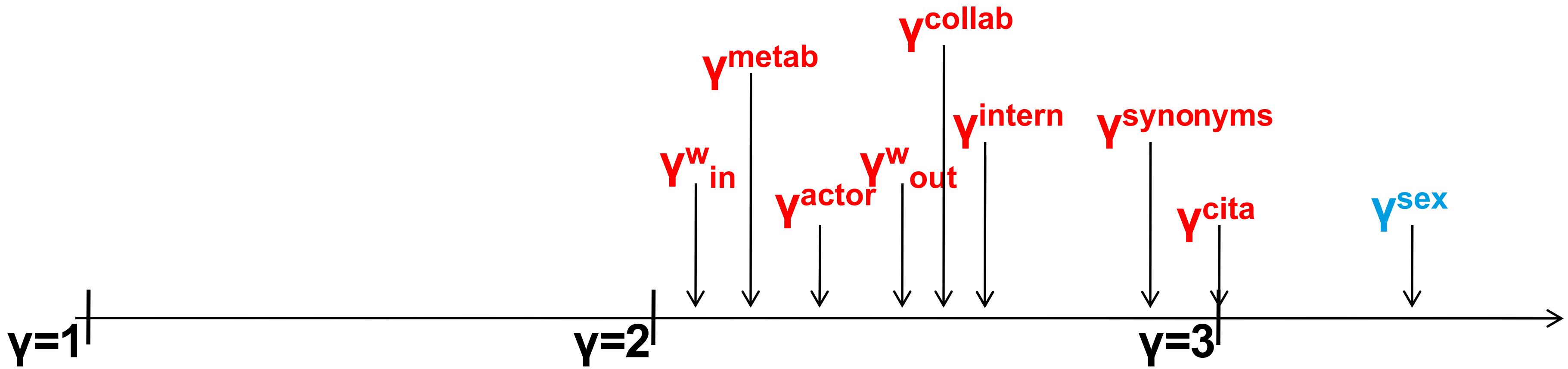
Do we live in a ultra-small-world?



Do we live in a ultra-small-world?



The universe of scale-free networks



$\langle k^2 \rangle$ diverges | $\langle k^2 \rangle$ finite

$\langle k \rangle$ diverges | $\langle k \rangle$ finite

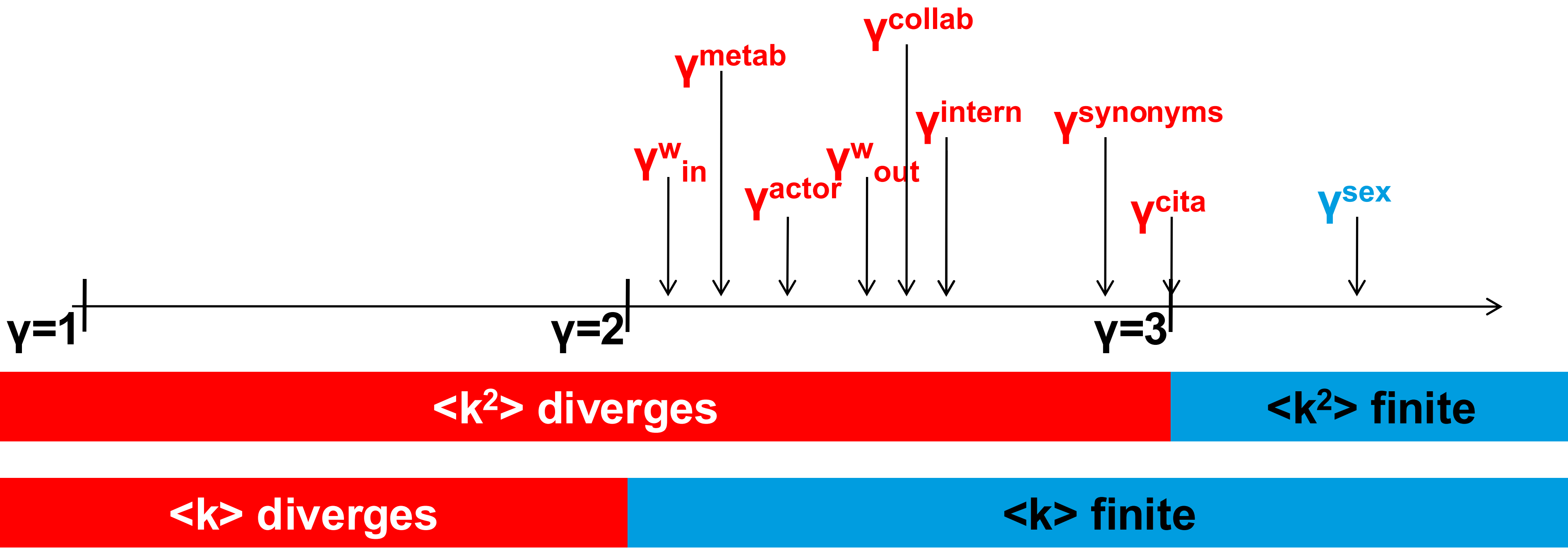
Ultra small world behavior | Small world

Regime full of anomalies… | The scale-free behavior is relevant | Behaves like a random network

Why are most exponents in this regime?

A world of magic exponents

Can you find a good argument which justifies this picture?

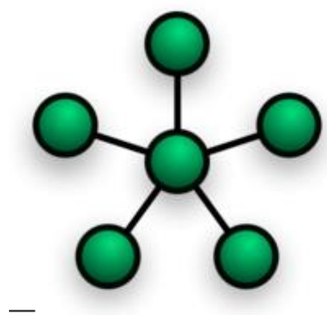


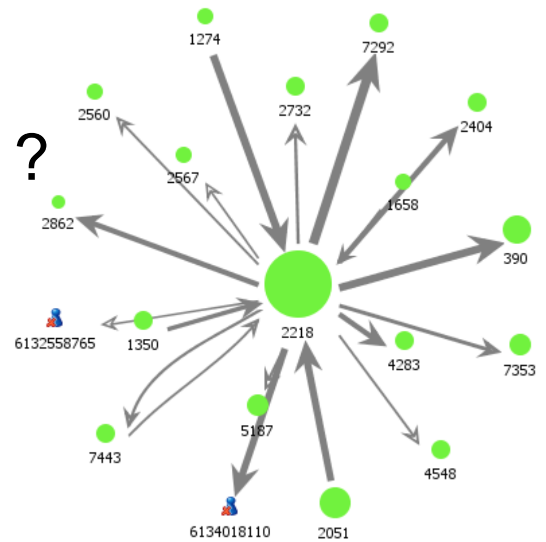
Magic exponents?

- Why is it hard to find networks with γ≤2 ?

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma - 1}}$$

K_{max} grows faster than *N*





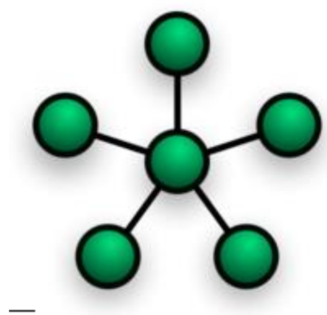
- Why is it hard to find networks with γ>>3 ?

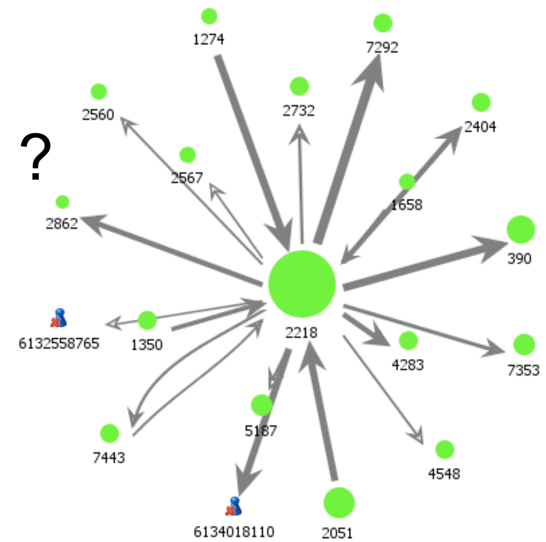
Magic exponents?

- Why is it hard to find networks with γ≤2 ?

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

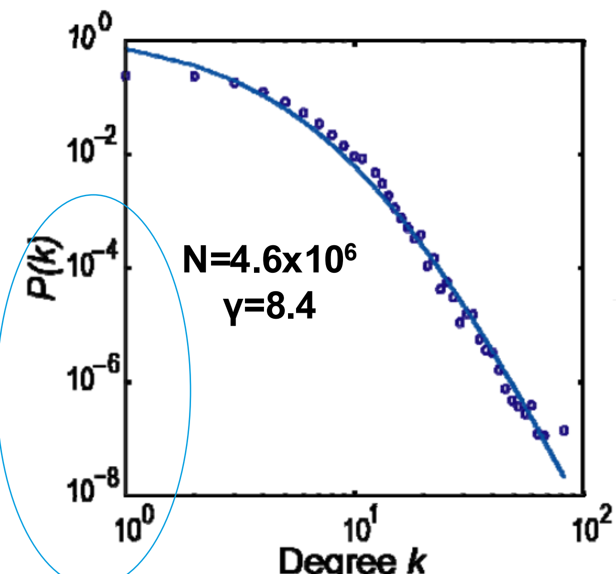
K_{max} grows faster than *N*





- **Why is it hard to find networks with γ>>3 ?**

Mobile Call Network

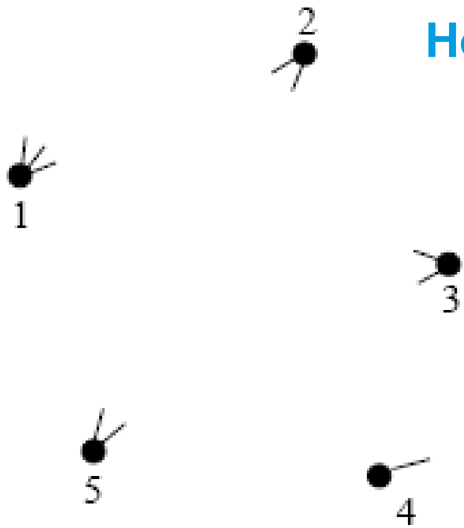


Onella et al. PNAS 2007

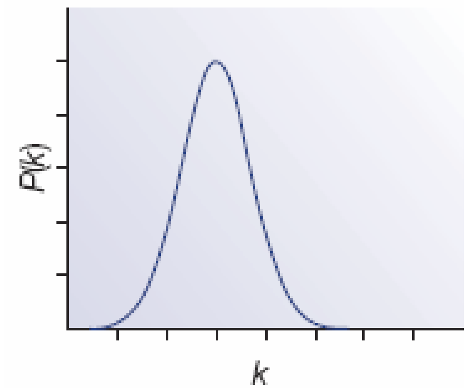
Configuration model

Challenge:
Can you imagine an algorithm capable of creating a random graph with an arbitrary degree distribution?

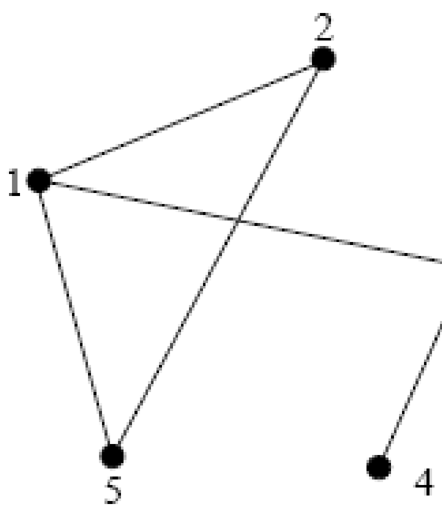
Configuration model



How to generate a graph compatible with a given $P(k)$



1. Create a histogram of $P(k)$ (discretization) the sum of which gives N<k>
2. add stubs to n's according to histogram
3. connect the stubs at random.
4. this leads to a random graph with a pre-defined $P(k)$.

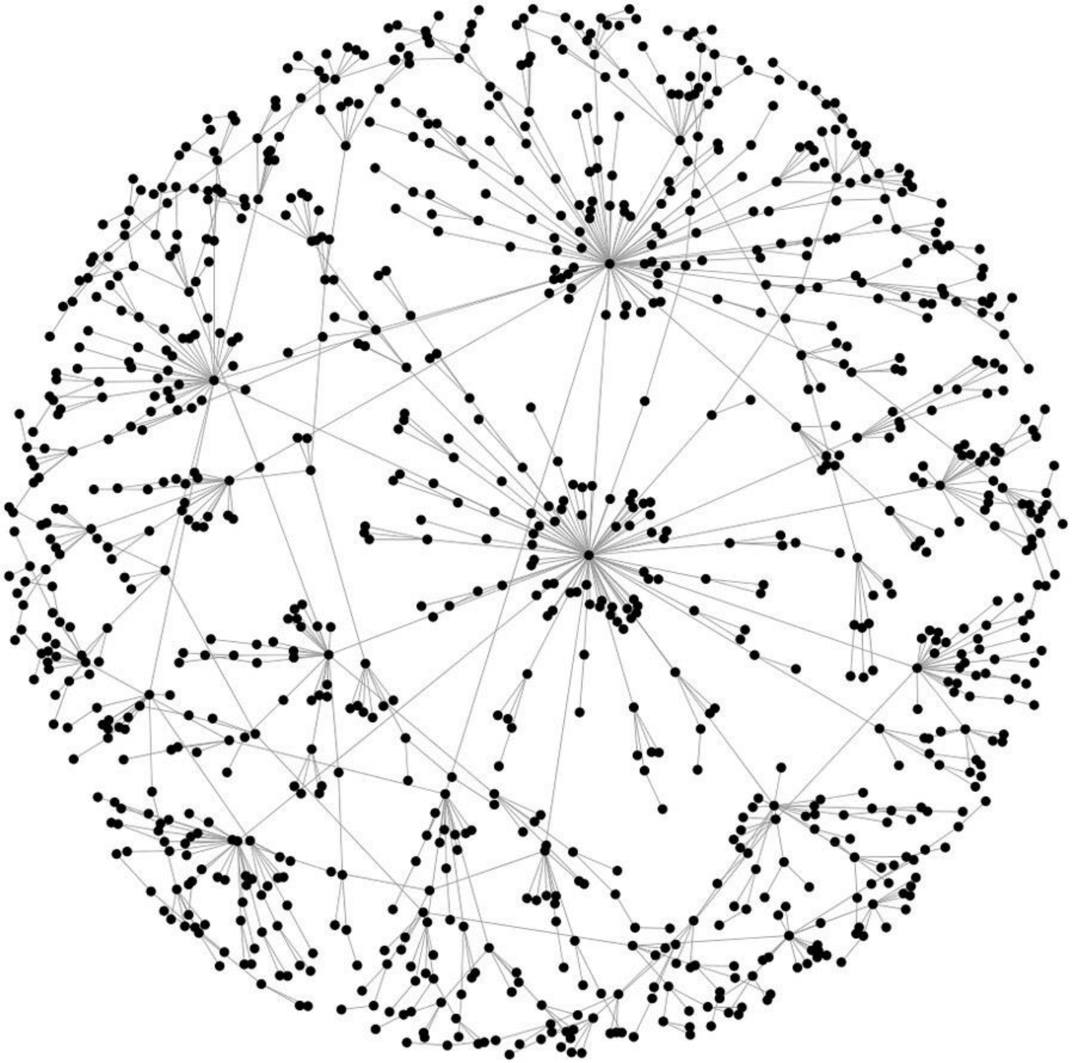


2nd challenge of the day

Challenge:
Can you create an algorithm capable of increasing its clustering coefficient of a network without changing its degree distribution?

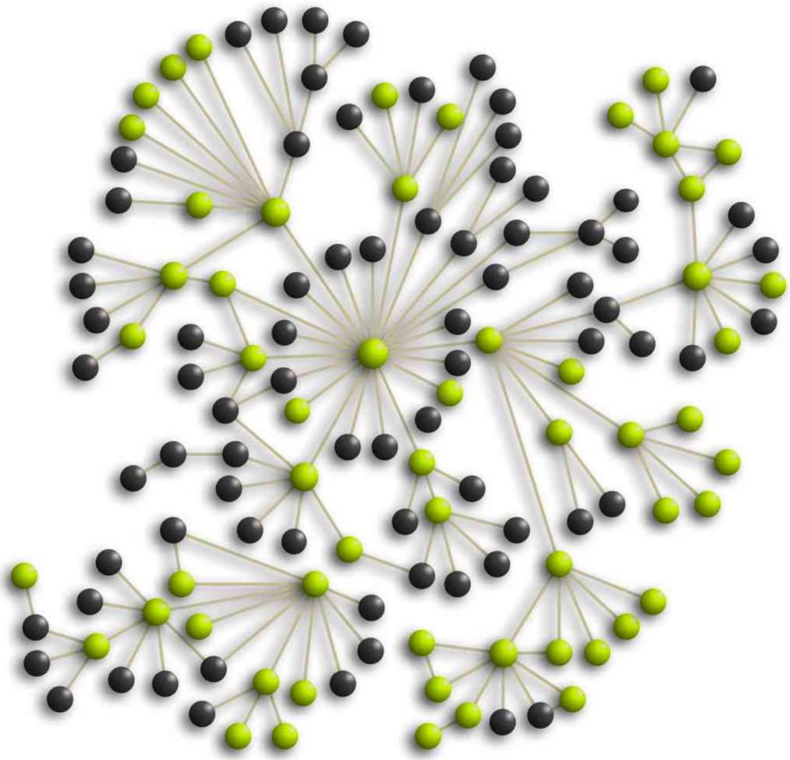
DEI
DEPARTAMENTO
DE ENGENHARIA INFORMÁTICA
TÉCNICO LISBOA

Models of Evolving Networks

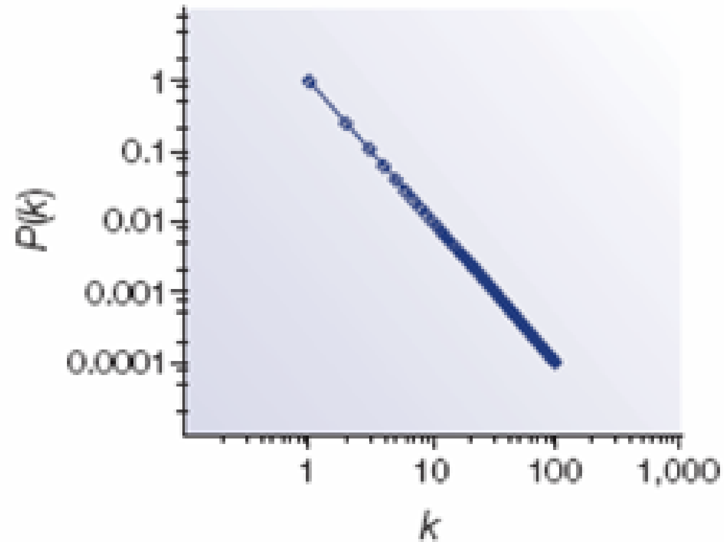


Network Science, 2025/26

Next challenge: Universality?



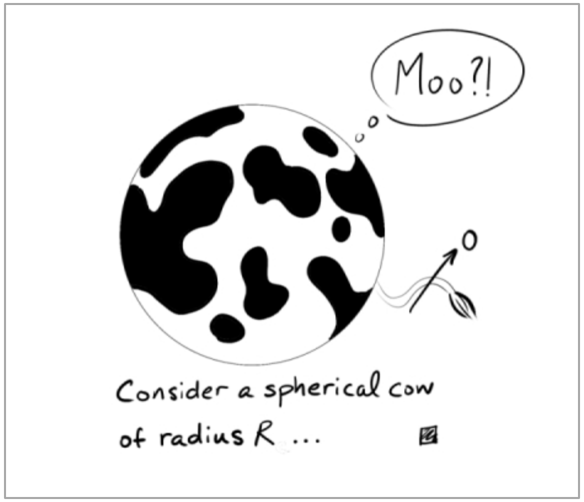
$$P_k \sim k^{-\gamma}$$



WWW	actors	citations	sex	cellular	phones	linguistics
$\gamma = 2.1$	$\gamma = 2.3$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 2.1$	$\gamma = 2.1$	$\gamma = 2.8$

Next challenge: Universality?

Challenge:
Can we identify the main principles leading to the emergence of scale-free networks?

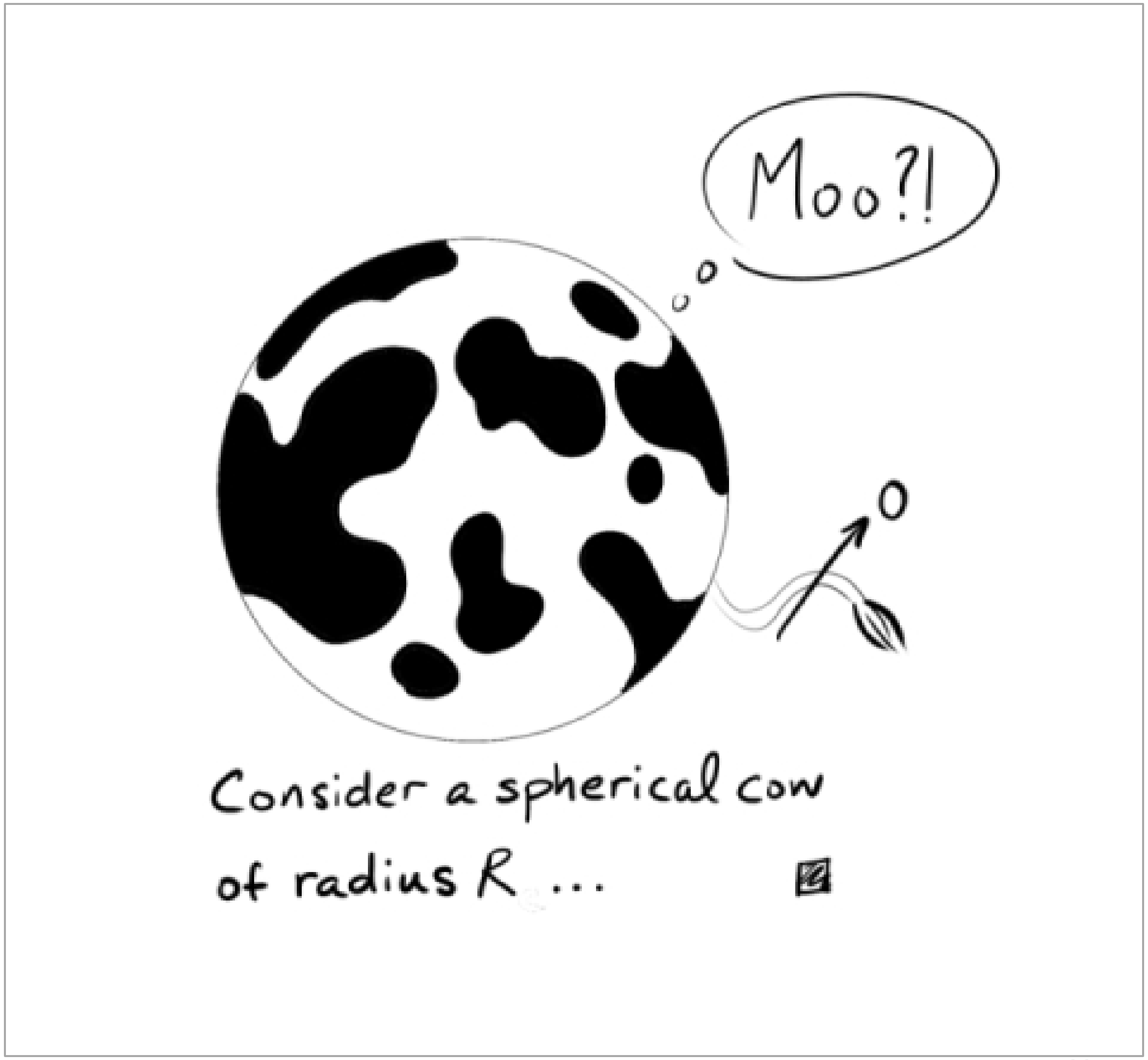


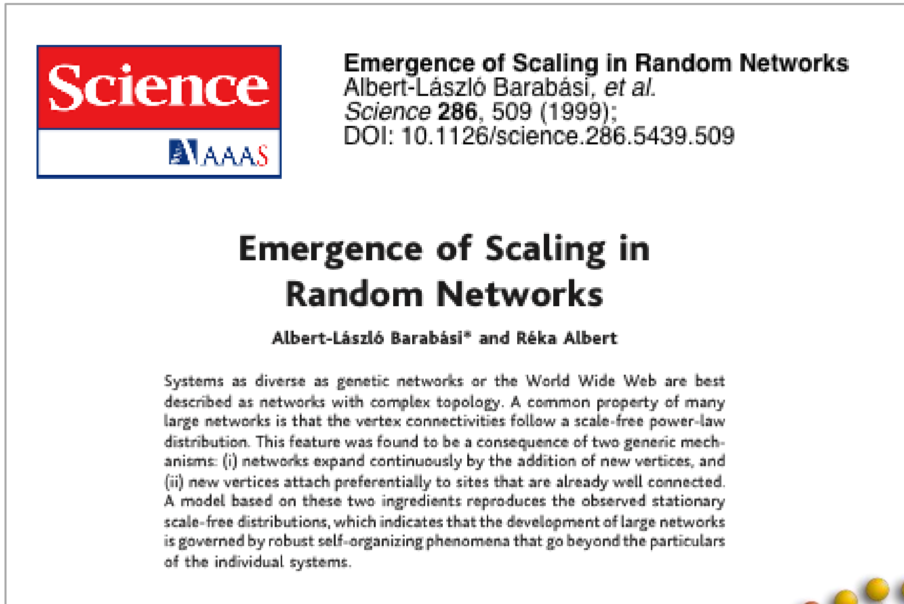
Moving forward: modeling complex systems

Self-organization: process where collective patterns arise out from local interactions between smaller component parts.

The whole is greater than the sum of its parts

The beauty of modeling core principles





Science

AAAS

Emergence of Scaling in Random Networks
Albert-László Barabási, *et al.*
Science **286**, 509 (1999);
DOI: 10.1126/science.286.5439.509

Emergence of Scaling in Random Networks

Albert-László Barabási* and Réka Albert

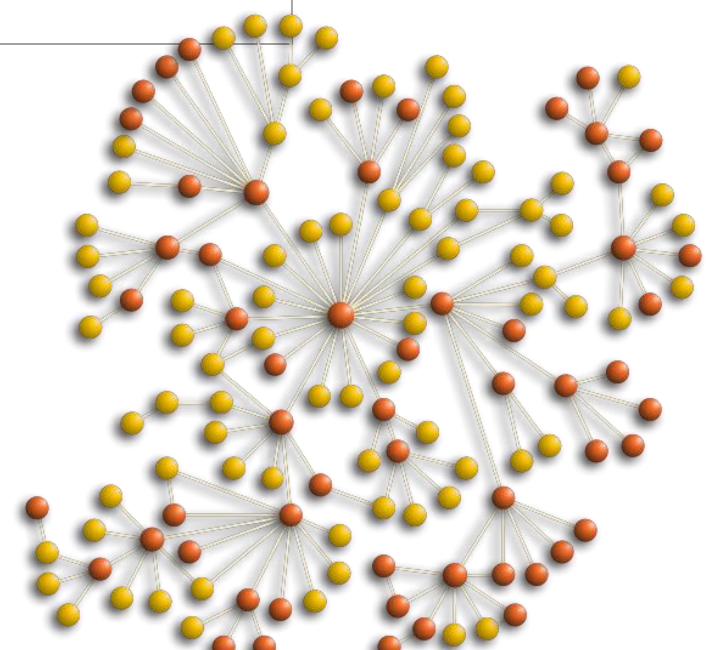
Systems as diverse as genetic networks or the World Wide Web are best described as networks with complex topology. A common property of many large networks is that the vertex connectivities follow a scale-free power-law distribution. This feature was found to be a consequence of two generic mechanisms: (i) networks expand continuously by the addition of new vertices, and (ii) new vertices attach preferentially to sites that are already well connected. A model based on these two ingredients reproduces the observed stationary scale-free distributions, which indicates that the development of large networks is governed by robust self-organizing phenomena that go beyond the particulars of the individual systems.

Reka Albert

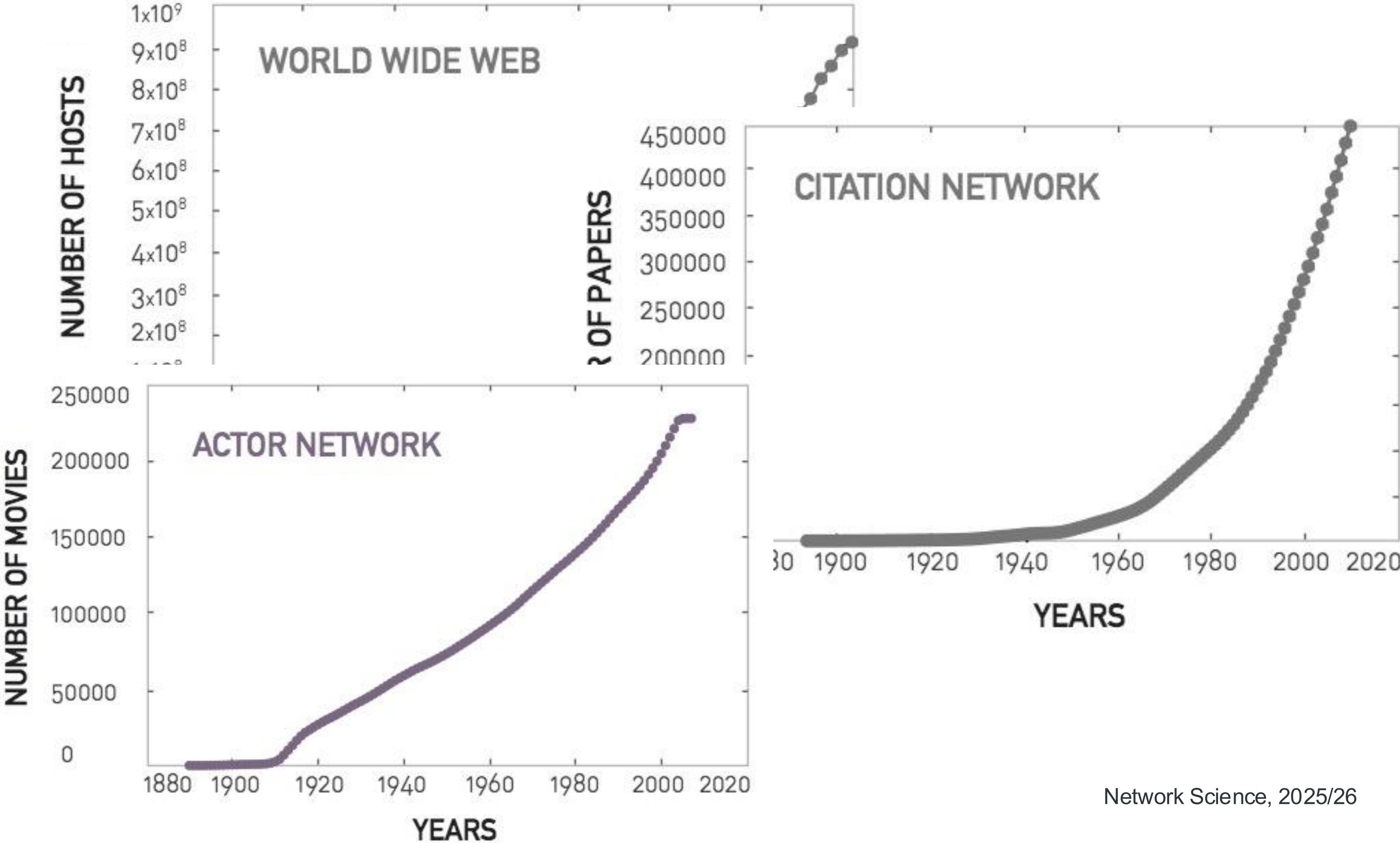
Penn State University

Albert-László Barabási

Northeastern University



First idea: Networks grow!



Second idea: popularity is attractive

- **WWW**: the nodes we know are not entirely random. Our knowledge is biased towards the most popular web documents. Thus, we are more likely to link (e.g., our webpage) to high degree nodes.

- **Citation network**: Our time is limited. We cannot read all papers published in the world! The more cited a paper is, the more likely that we hear about it, and read it. As we cite what we read, we tend to cite paper with high number of citations.

Second idea: popularity is attractive

- **Actor network**: the more movies an actor has played, the higher the chances that she/he will be considered for a new role…

- **This idea is often referred in the literature as Preferential Attachment.**

Related concepts

- ***Herbert Simon*** (Turing award ‘75) showed how ***preferential attachment*** can give rise to fat-tailed distributions describing ***city sizes and wealth distributions***.

- ***Proportional growth*** in business. Larger firms grow faster than smaller firms (R. Gibrat).

- ***Rich get richer effect*** in wealth distributions (Zipf law).

- Derek Prize’s “***Cumulative advantage***” principle, applied to citation statistics.

Related concepts

- ***Matthew effect*** (R. Merton) in sociology:

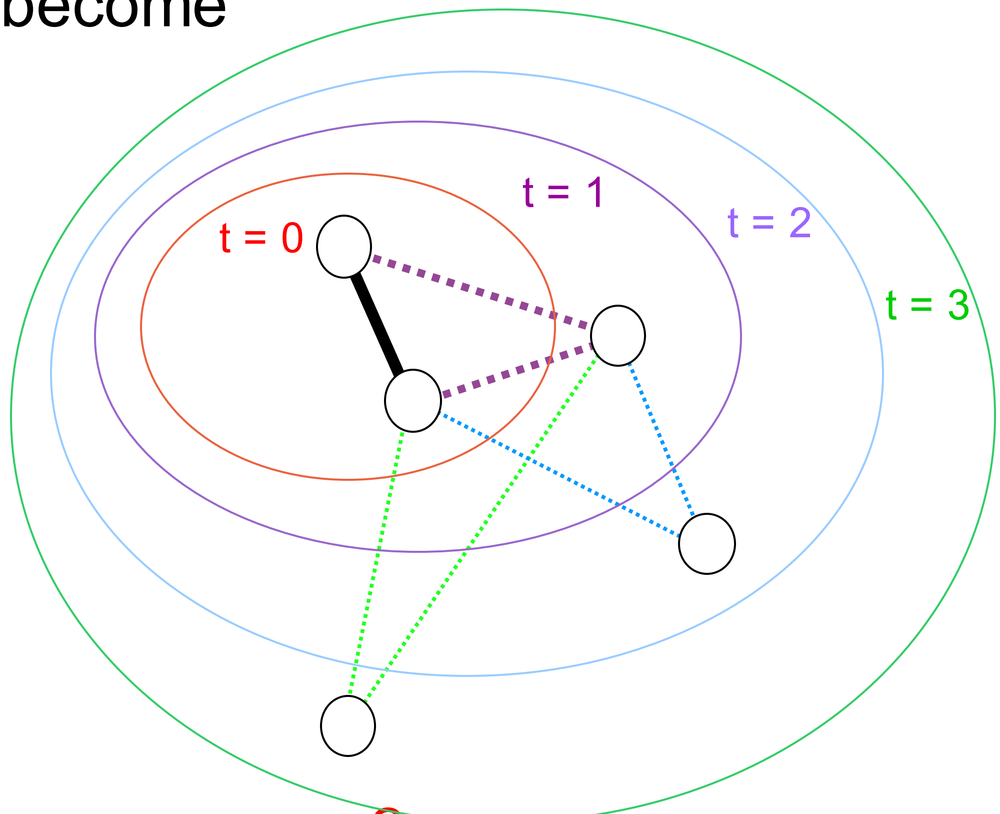
"For everyone who has will be given more, and he will have an abundance."

Matthew 25:29, King James Version.

Pois a quem tem mais se lhe dará, e terá em abundância; mas ao que não tem, até aquilo que tem ser-lhe-á tirado.
Mateus 25:29, Novo testamento

The Barabási-Albert model

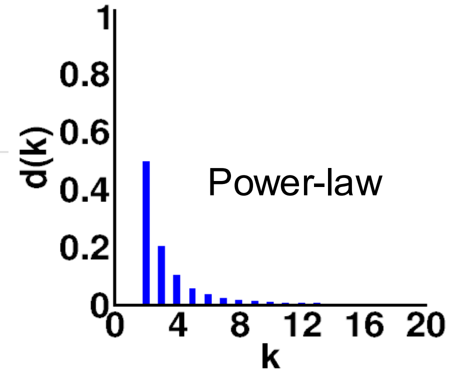
- ***Growth***: add nodes sequentially.
- ***Preferential attachment***: the more popular you are, the more popular you become

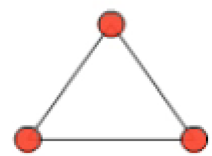


n	k
1	2
2	4
3	4
4	2
5	2

$m = m_0 = 2$

Growth + Preferential Attachment = Scale-free networks!





The Barabási-Albert model

- ***Growth***: add nodes sequentially.

At t=0 consider (e.g.) a ring with m_0 nodes; for each time-step, add 1 node and connect it via m new edges with existing nodes;

The Barabási-Albert model

- ***Growth***: add nodes sequentially.

At t=0 consider (e.g.) a ring with m_0 nodes; for each time-step, add 1 node and connect it via m new edges with existing nodes;

- ***Preferential attachment***:

when establishing the connections, each new node is connected to older nodes with a probability proportional to the degree of the older nodes.

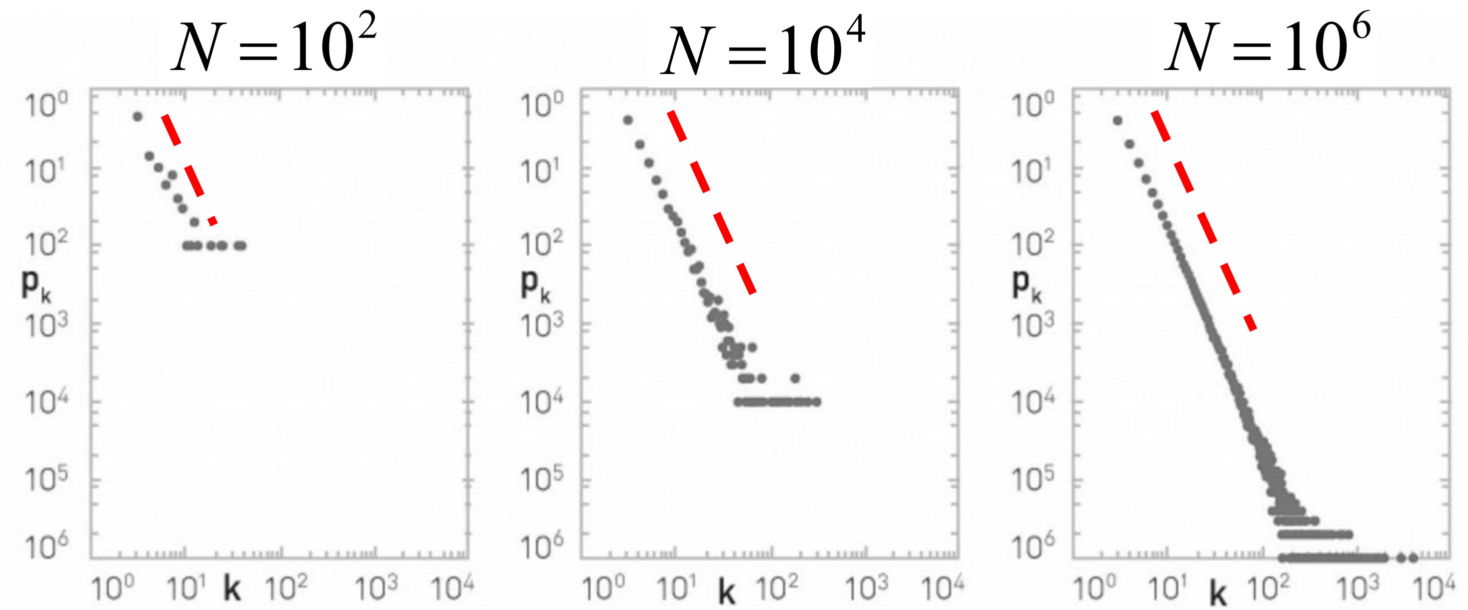
$$\Pi_i = \frac{k_i}{\sum_j k_j}$$

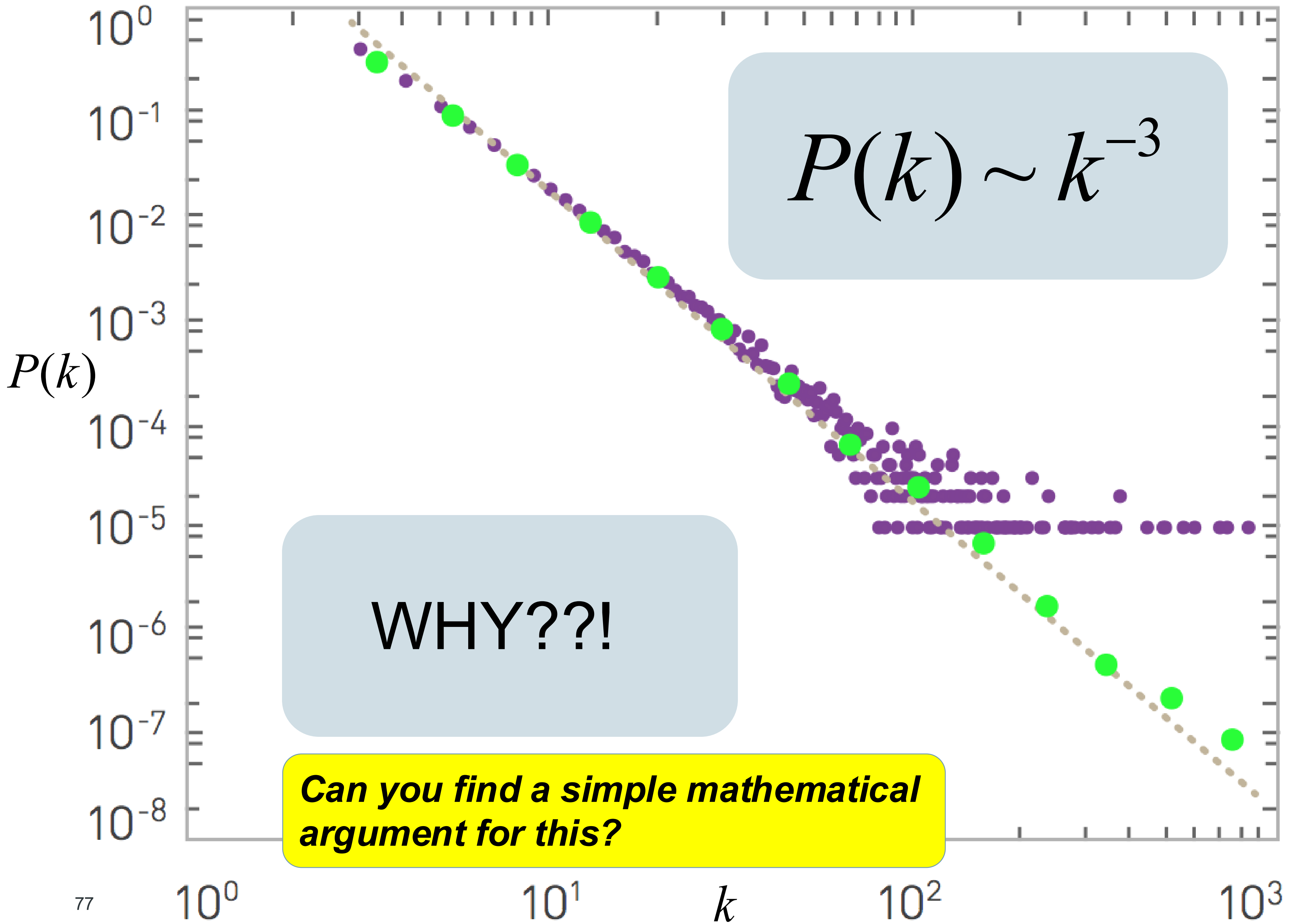
the more popular you are, the more popular you become

The Barabási-Albert model – Average degree?

- After t time-steps (t>>1) the graph has m_0+t nodes and *mt* edges; hence z =<k>≈ ??

<k>~2m





Can you find a simple mathematical argument for this?

BA-model: Degree distribution

- For an exact analysis of the structure of growing networks please check the works by

Sergey Dorogovtsev
Dep. Physics, Univ. Aveiro

José Fernando Mendes
Dep. Physics, Univ. Aveiro

Ex: Dorogovtsev et al. (2000). Structure of growing networks with preferential linking. Phys. Rev. Lett. , 85(21), 4633. (*note: this would be a neat analytical project*)

BA-model: Time evolution of degrees

Back-of-the-envelope argument (Barabasi & Albert, Science 1999)

- The rate at which an existing node *i* acquires links as a result of new nodes (with *m* links) connecting to it is

$$\frac{dk_i}{dt} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

k_i = degree of node i

BA-model: Time evolution of degrees

Back-of-the-envelope argument (Barabasi & Albert, Science 1999)

- The rate at which an existing node *i* acquires links as a result of new nodes (with *m* links) connecting to it is

$$\frac{dk_i}{dt} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

<k> × N = 2m × *t*

m = number of links of each new node.

In other words, each existing node *i* has *m* chances to be chosen

BA-model: Time evolution of degrees

- Since $$\sum_{j=1}^{N-1} k_j = 2mt$$

- We have $$\frac{dk_i}{dt} = m\frac{k_i}{2mt} = \frac{k_i}{2t}$$

- We get $$\boxed{k_i(t) = Const.\left(t\right)^{1/2}}$$

i joins the network at time t_i

$$\left(k_i(t_i) = m\right)$$

$$\boxed{Const = \frac{m}{t_i^{1/2}}}$$

BA-model: **Time evolution of degrees**

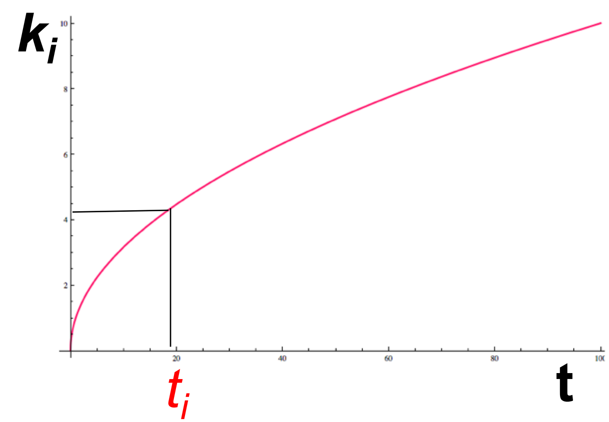
- Since $$\sum_{j=1}^{N-1} k_j \approx 2mt$$

i joins the network at time t_i

$$\left(k_i(t_i) = m\right)$$

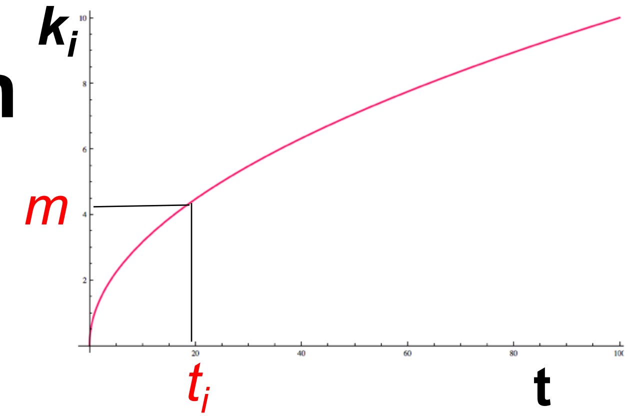
- We have $$\frac{dk_i}{dt} = m\frac{k_i}{2mt} = \frac{k_i}{2t}$$

- We get $$\boxed{k_i(t) = m\left(\frac{t}{t_i}\right)^{1/2}}$$



BA-model: Degree distribution

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{1/2}$$



I can also invert this eq. to say that, <u>if the degree of *i* is smaller than *k*, then *i* appeared at the network after time t_i</u>

$$k_i < k \Rightarrow t_i > t\left(\frac{m}{k}\right)^2$$

Since we add a node at each time-step ($N = t$),
the ***number of nodes appearing after t_i*** is given by $N - t_i$

$\Updownarrow$

number of nodes with degree < k = $N - t\left(\frac{m}{k}\right)^2$

BA-model: Degree distribution

- (normalizing by N) the ***probability*** *$P(k_i<k)$* of having a node with degree lower than k is

$$P(k_i < k) = \frac{N - t\left(\frac{m}{k}\right)^2}{N} \approx 1 - \left(\frac{m}{k}\right)^2$$

For large N

$$\left(N = m_0 + t \square t\right)$$

i.e., we get the ***C***umulative ***D***istribution ***F***unction (CDF):

$$CDF(k) = P(k_i < k) = \int_0^k P(k')\,dk' = 1 - \left(\frac{m}{k}\right)^2$$

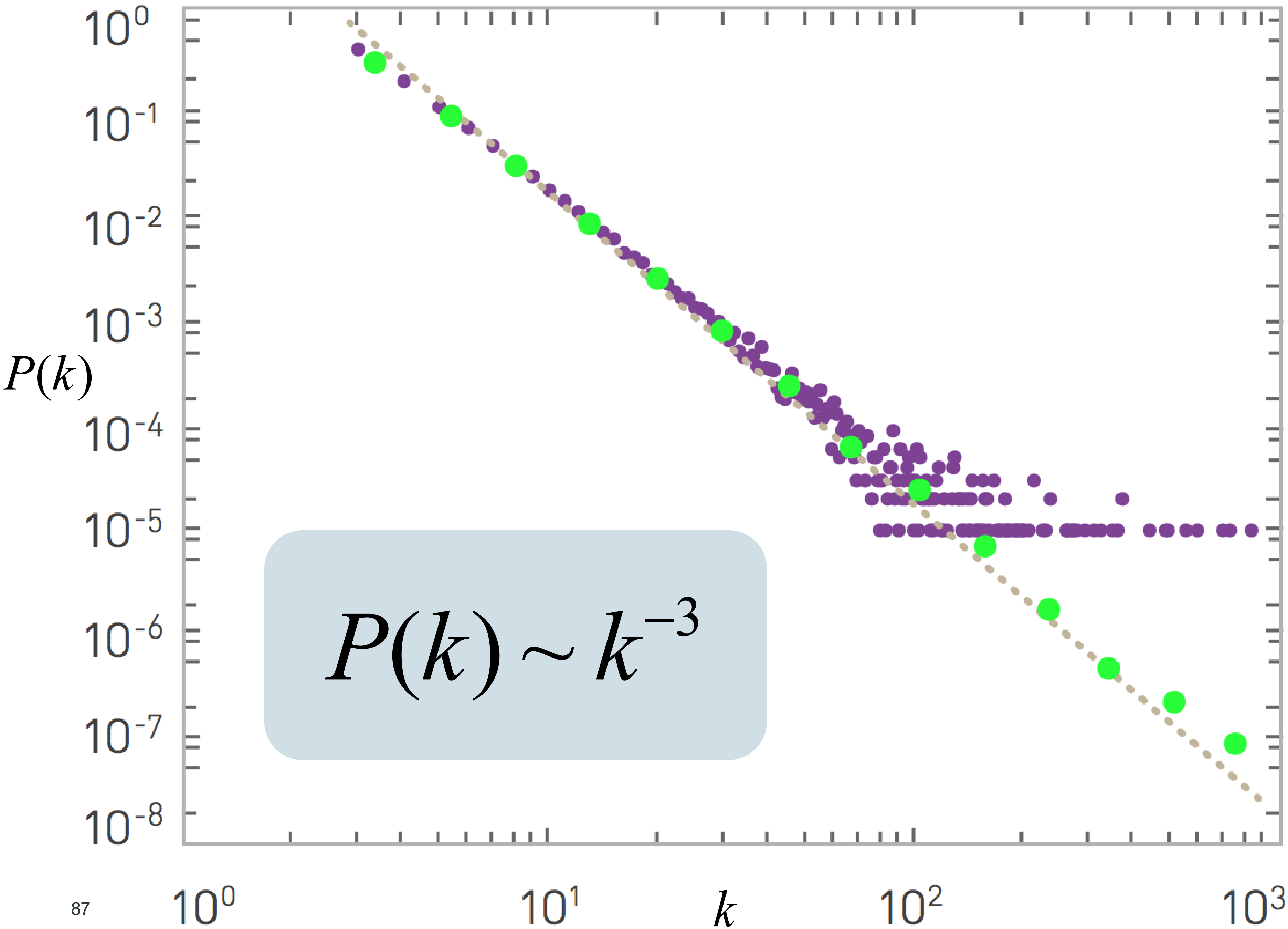
BA-model: Degree distribution

- …if we take its derivative, we get the degree distribution

$$P(k) = \frac{\partial CDF(k)}{\partial k} = 2m^2 \left(\frac{1}{k} \right)^3$$

Et voilà ☺!

$$P(k) \sim k^{-3}$$

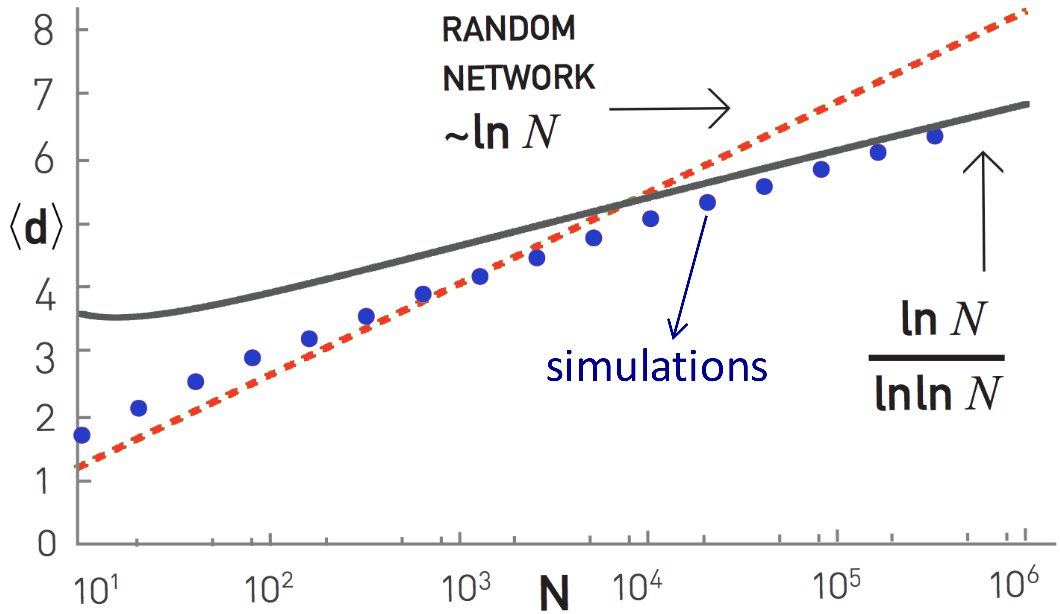


10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
$P(k)$
$P(k) \sim k^{-3}$
10^0
10^1
k
10^2
10^3

APL of the Barabási-Albert model?

- Average-Path-Length (γ=3):

$$\langle L \rangle \sim \frac{\ln N}{\ln\ln N}$$



Clustering of the Barabási-Albert model

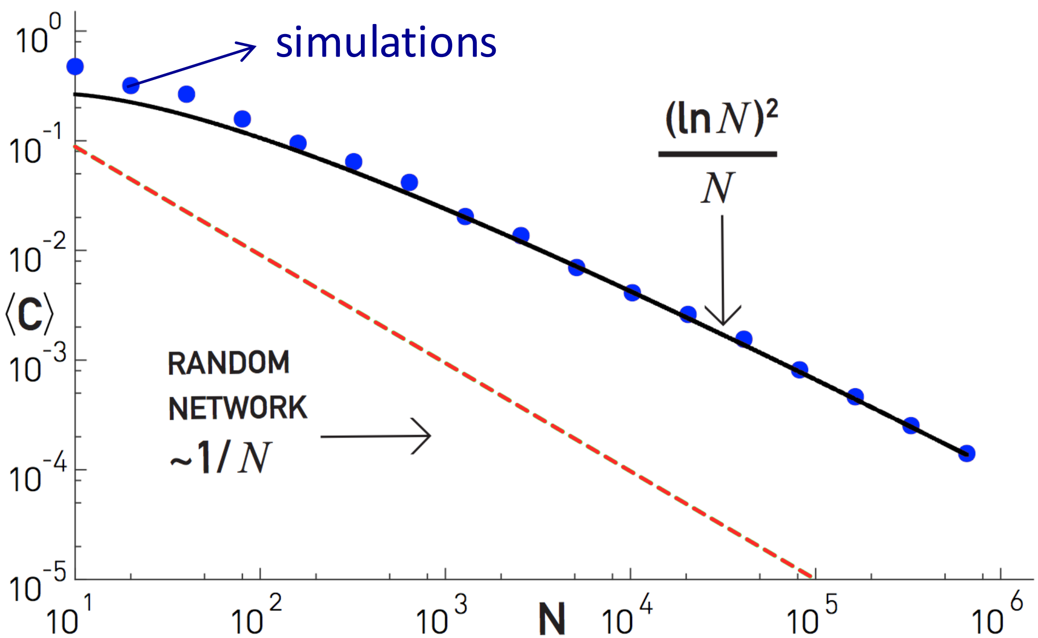
$$C_i \sim \frac{(\ln N)^2}{N}$$

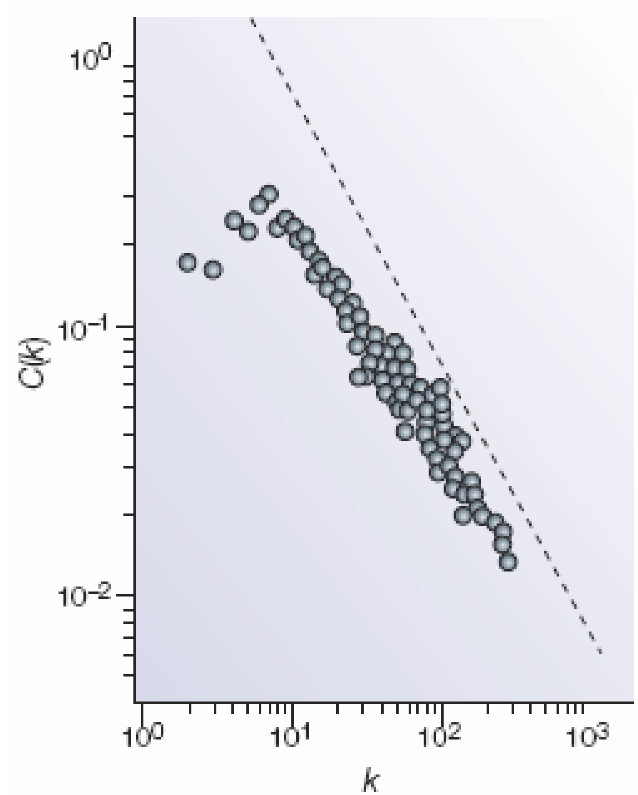
1. For a fixed <k>, larger the network, smaller the clustering coefficient.
2. Clustering coefficient is independent of the node's degree.

Real nets

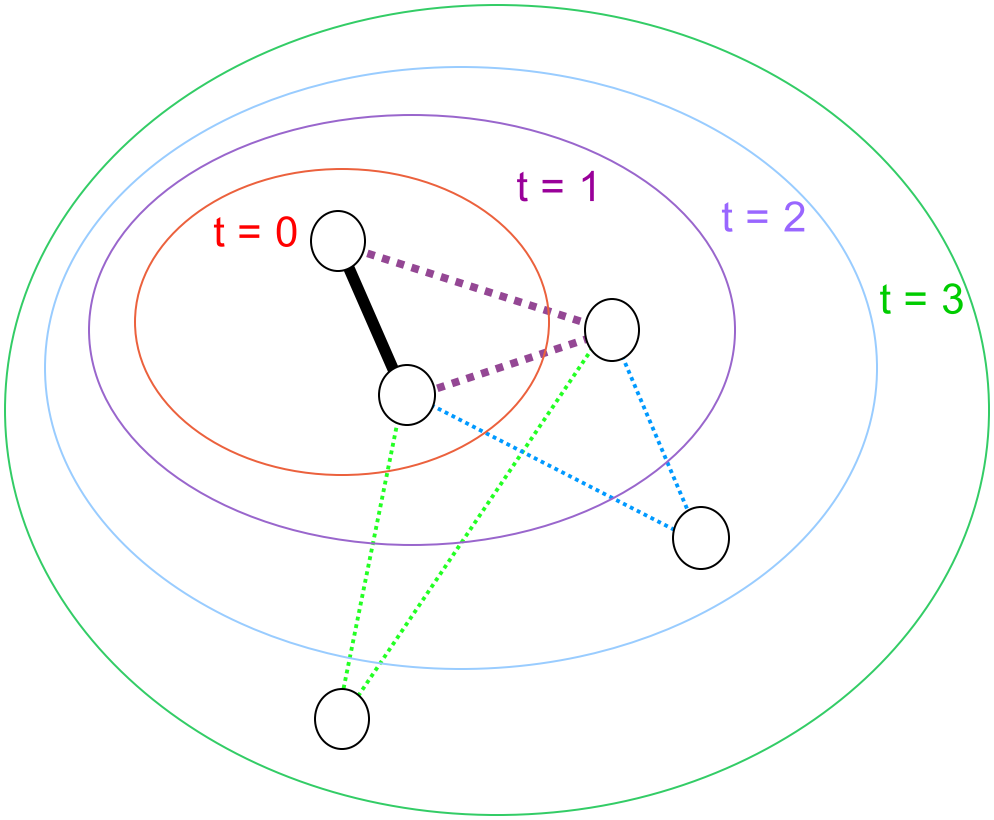
X

X



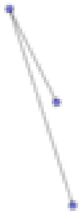


Other features: Age correlations



We are creating a league of gentleman !

Try it!!! How would you implement a preferential attachment model?



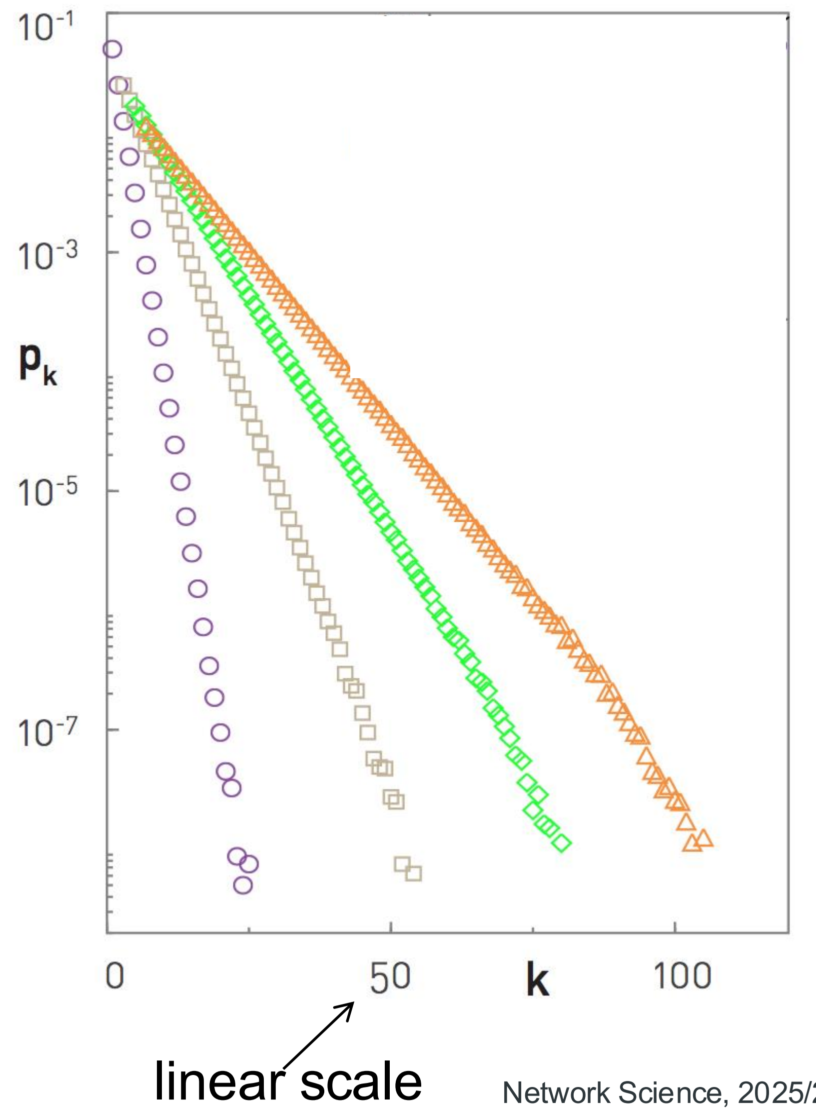
Do we need both growth and Preferential attachment?

- Model A (growth only)

$$P(k) \sim e^{-k/m}$$

Can you explain why ? ☺

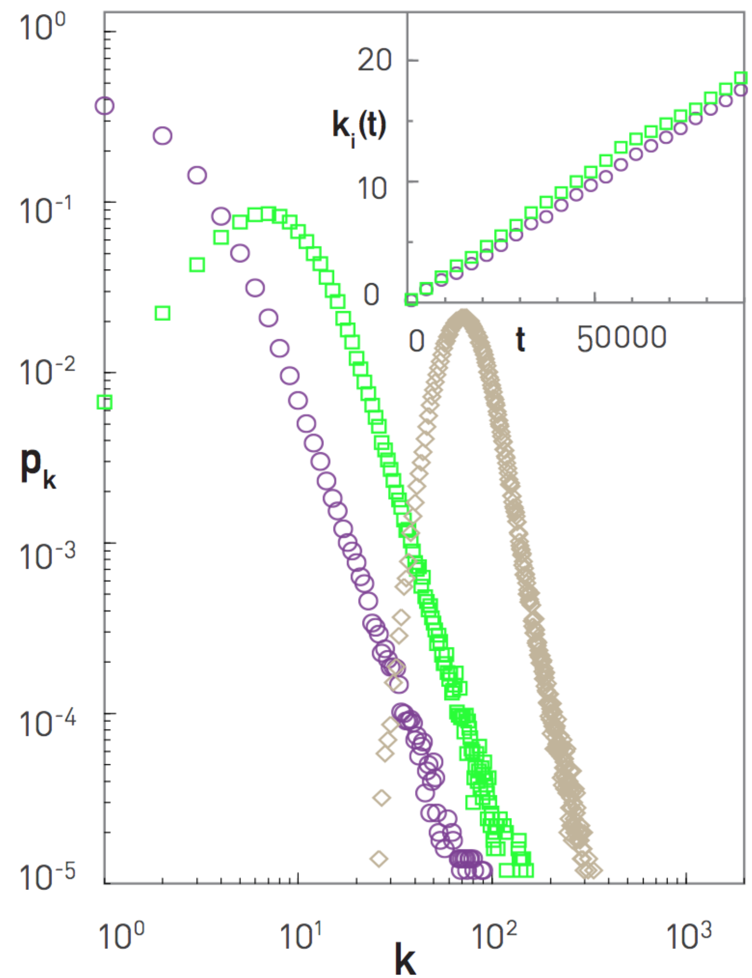
$$\frac{dk_i}{dt} = m\Pi(k_i) = m\frac{1}{N(t)-1} \sim m\frac{1}{t}$$



Do we need both growth and Preferential attachment?

- Model B
  (only preferential attachment)

- Begin with a fixed number of disconnected nodes and add links, preferentially choosing high degree nodes as link destinations.
- Though the degree distribution early in the simulation looks scale-free, the distribution is not stable, and it eventually becomes nearly Gaussian as the network nears saturation.
- Thus, preferential attachment alone is not sufficient to produce a scale-free structure.



Can we be sure of having a preferential attachment mechanism?

- Changes in degree should follow

$$\Delta k_i = k_i(t+\Delta t) - k_i(t)$$

i.e.,

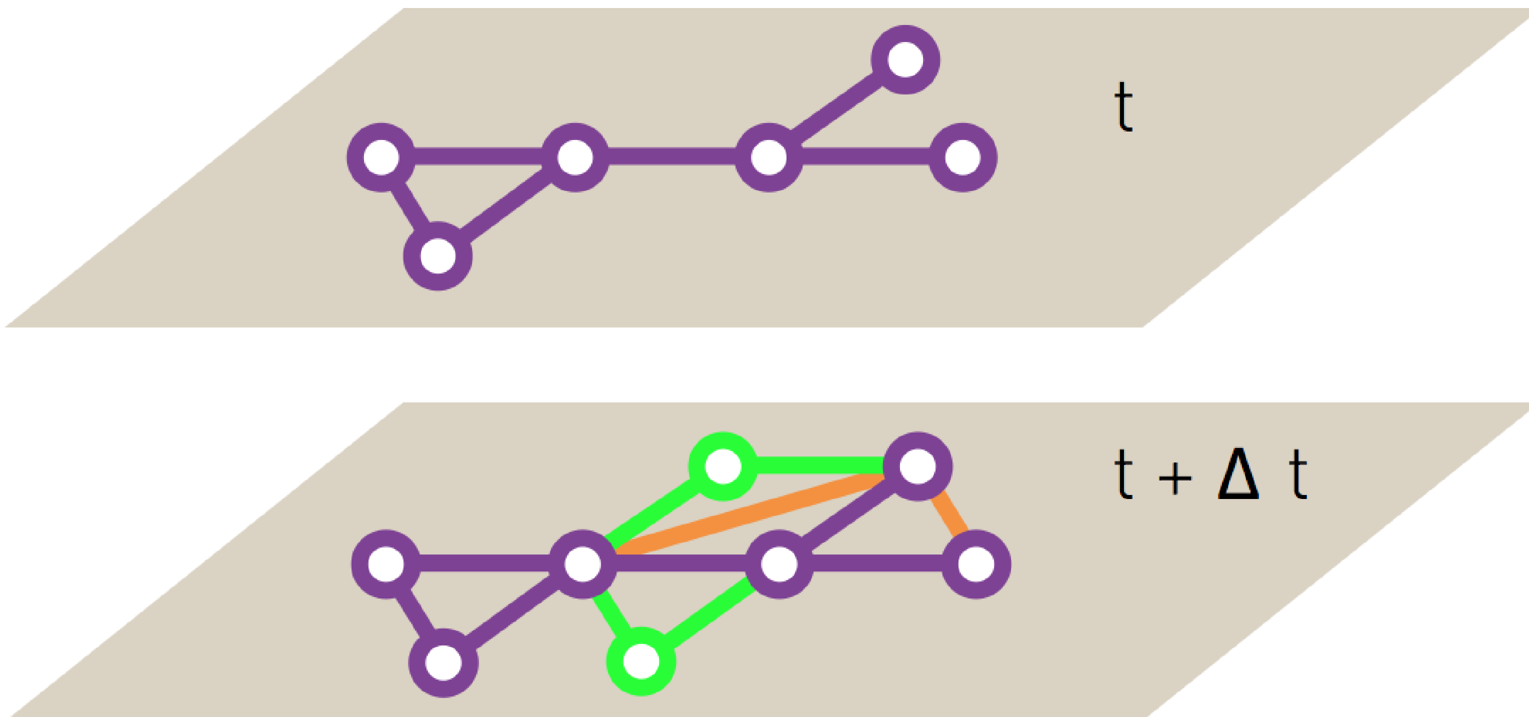
$$\frac{\Delta k_i}{\Delta t} \sim \frac{k_i}{\sum_j k_j}$$

→ **The BA model assumes a linear form of preferential attachment**

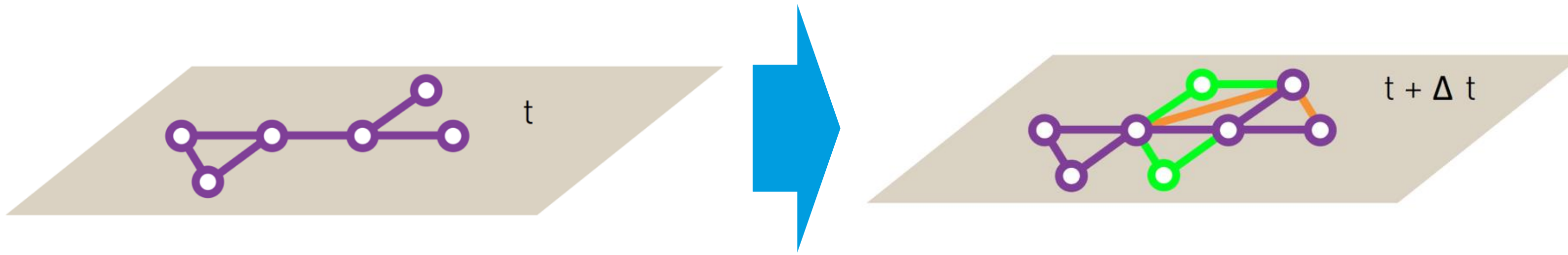
We can average this variation for empirical data and compare it with different scenarios

$$\frac{\Delta k_i}{\Delta t} \sim \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

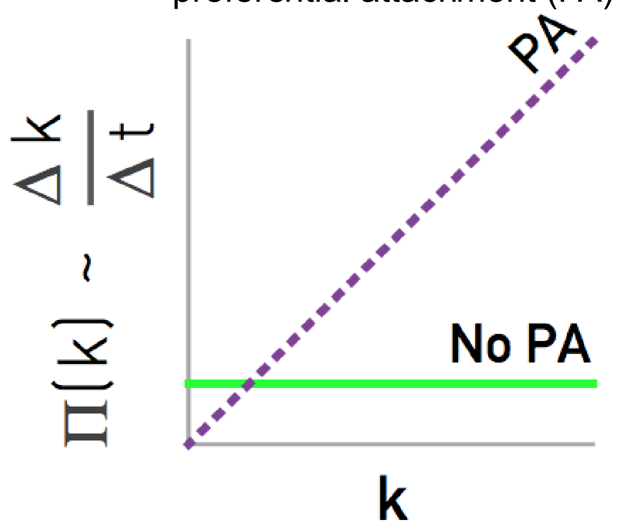
Can we be sure of having a preferential attachment mechanism?



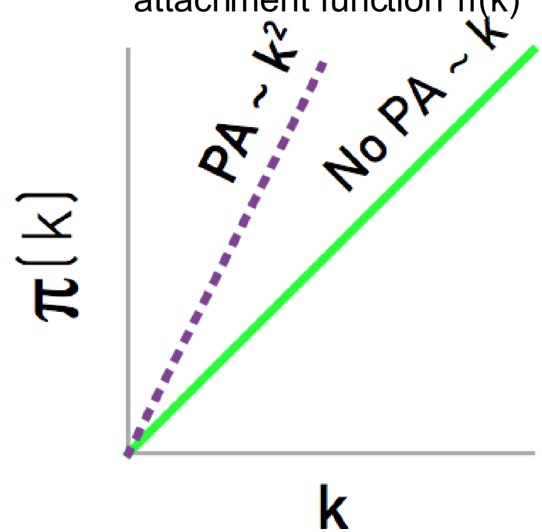
Can we be sure of having a preferential attachment mechanism?



preferential attachment (PA)



cumulative preferential attachment function π(k)



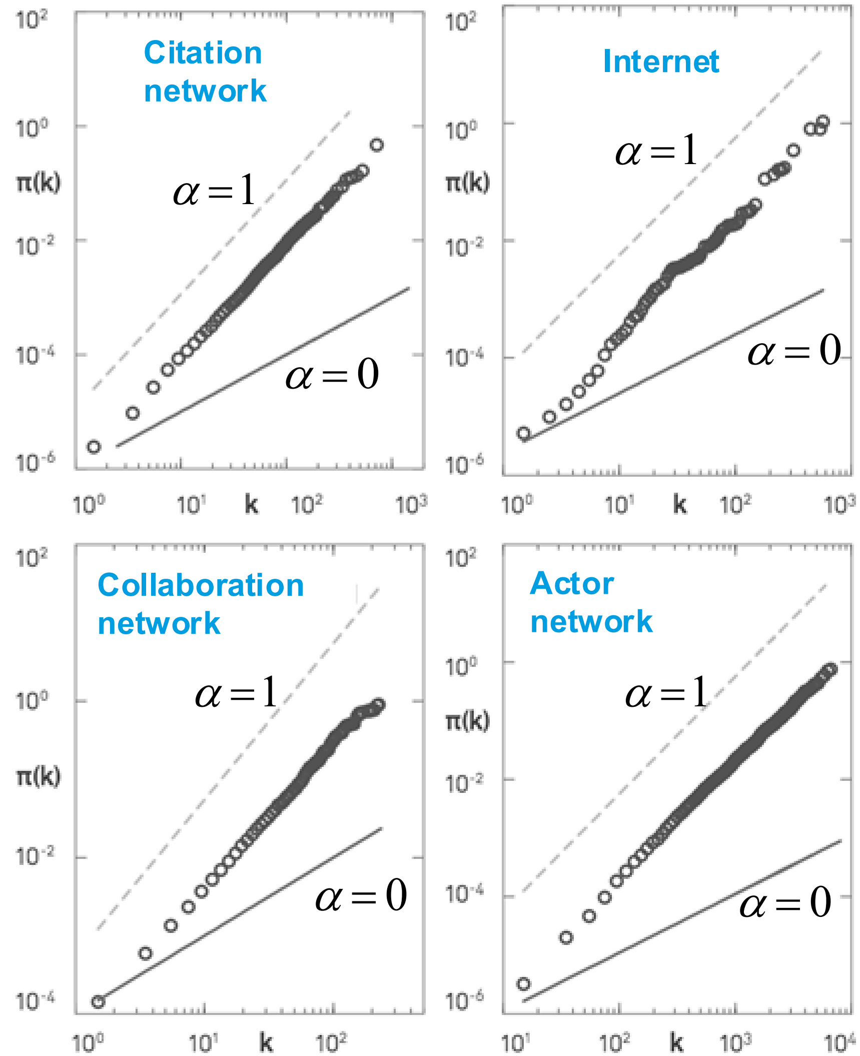
Log-log scales

Can we be sure of having a preferential attachment mechanism?

- Preferential attachment is present
- Yet, we may have a non-linear preferential attachment!!

$$\frac{\Delta k_i}{\Delta t} \sim \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

cumulative preferential attachment function π(k)



Non-linear preferential attachment:

Does it change anything?

- ***Growth***: add nodes sequentially.

At t=0 consider (e.g.) a ring with m_0 nodes; for each time-step, add 1 node and connect it via m new edges with existing nodes;

- ***Preferential attachment***:

when establishing the connections, each new node is connected to older nodes with a probability proportional to ***degree***$^{\alpha}$ of the older nodes.

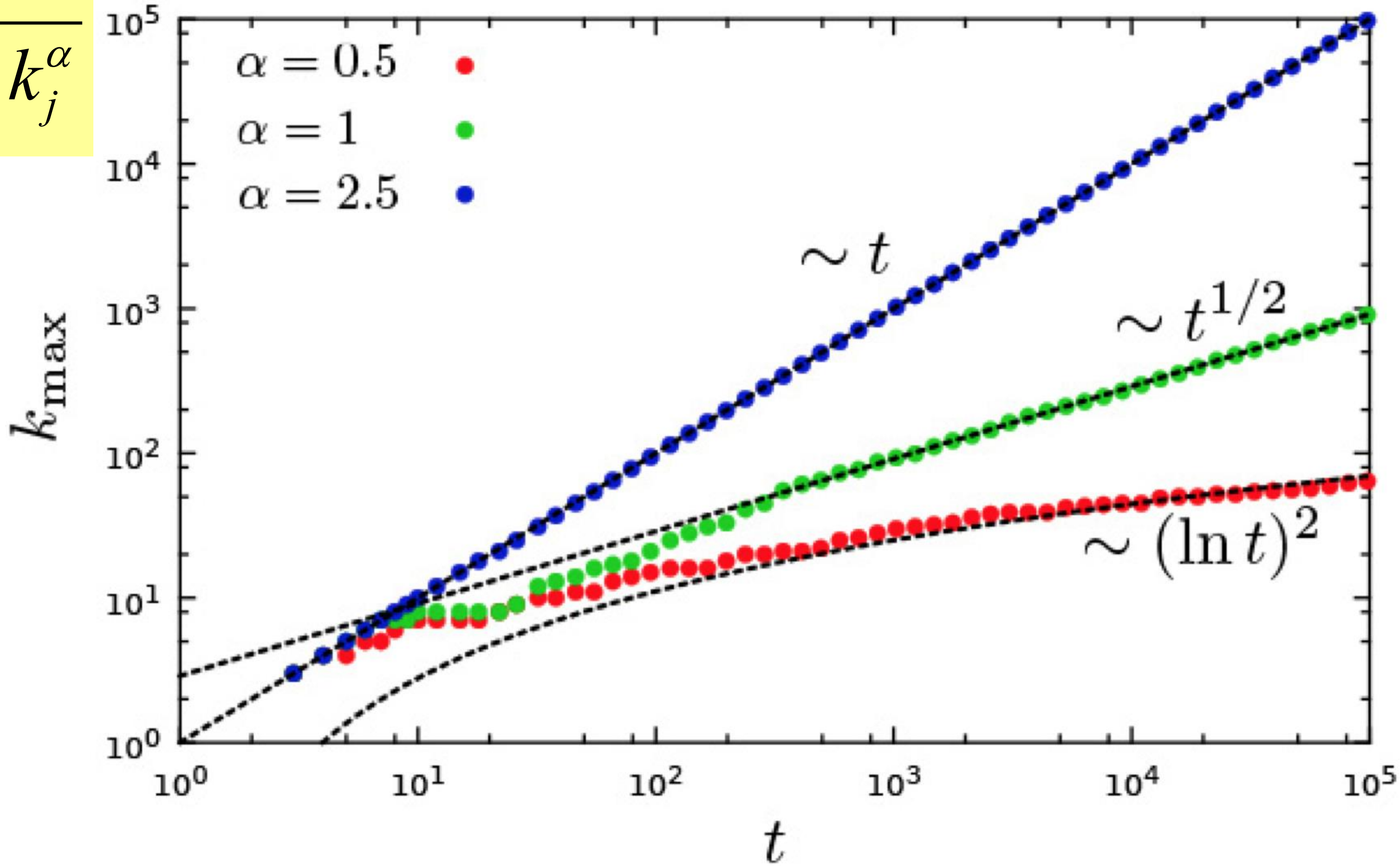
$$\Pi_i = \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

the more popular you are, the more popular you become, yet with different strengths

Hub's degree dynamics

At each time-step we add a node. What's the degree of the largest hub after t steps?

$$\Pi_i = \frac{k_i^\alpha}{\sum_j k_j^\alpha}$$



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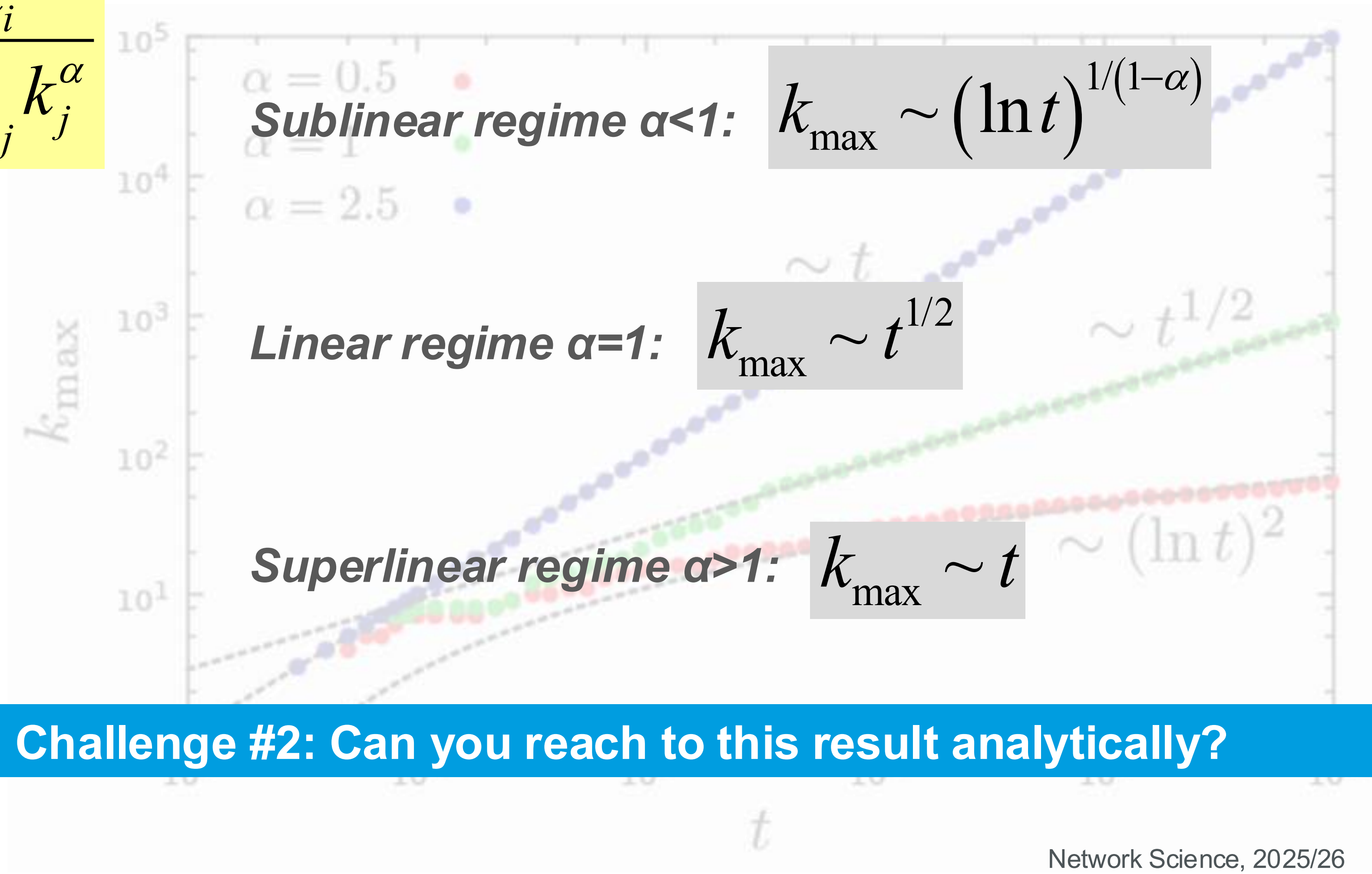


Sublinear regime α<1: $k_{max} \sim (\ln t)^{1/(1-\alpha)}$

Linear regime α=1: $k_{max} \sim t^{1/2}$

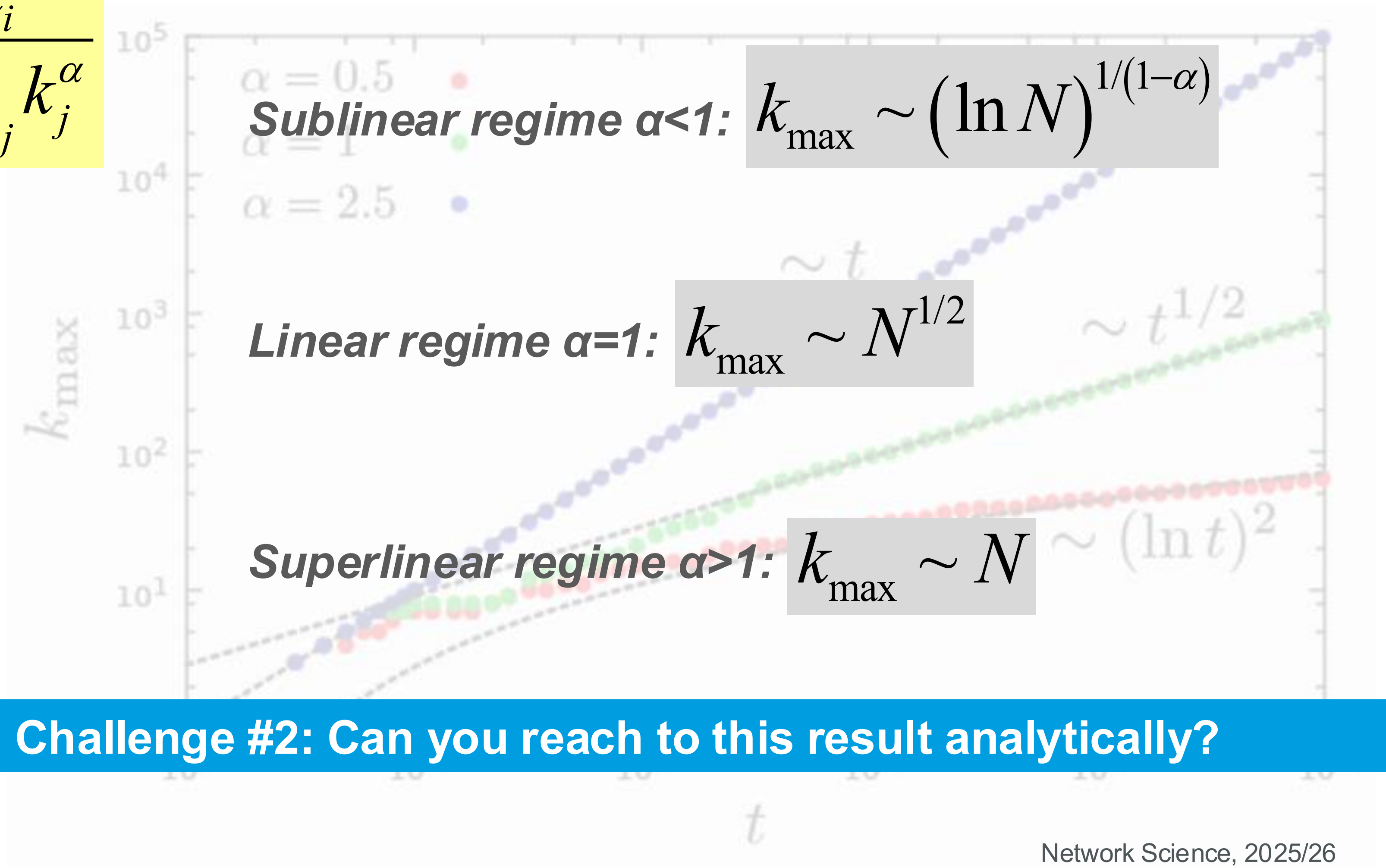
Superlinear regime α>1: $k_{max} \sim t$

Challenge #2: Can you reach to this result analytically?

Hub's degree dynamics

$$\Pi_i = \frac{k_i^\alpha}{\sum_j k_j^\alpha}$$

***At** each time-step we add a node. What's the degree of the largest hub after t steps?*



Sublinear regime α<1: $k_{\max} \sim (\ln N)^{1/(1-\alpha)}$

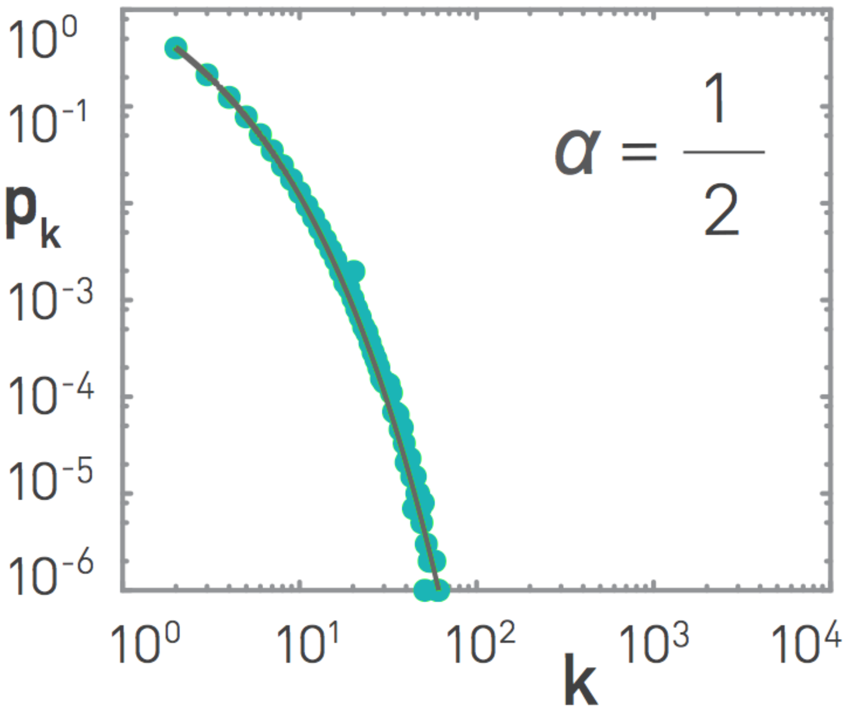
Linear regime α=1: $k_{\max} \sim N^{1/2}$

Superlinear regime α>1: $k_{\max} \sim N$

Challenge #2: Can you reach to this result analytically?

Degree distributions (sublinear regime)

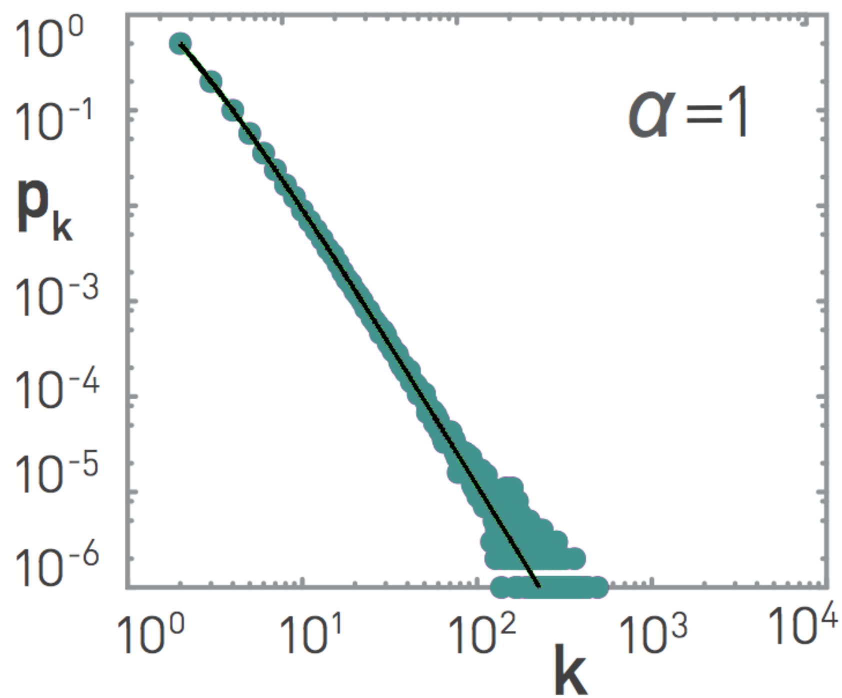
Sublinear



Stretched Exponential

$$k_{\max} \sim (\ln t)^{1/(1-\alpha)}$$

Linear

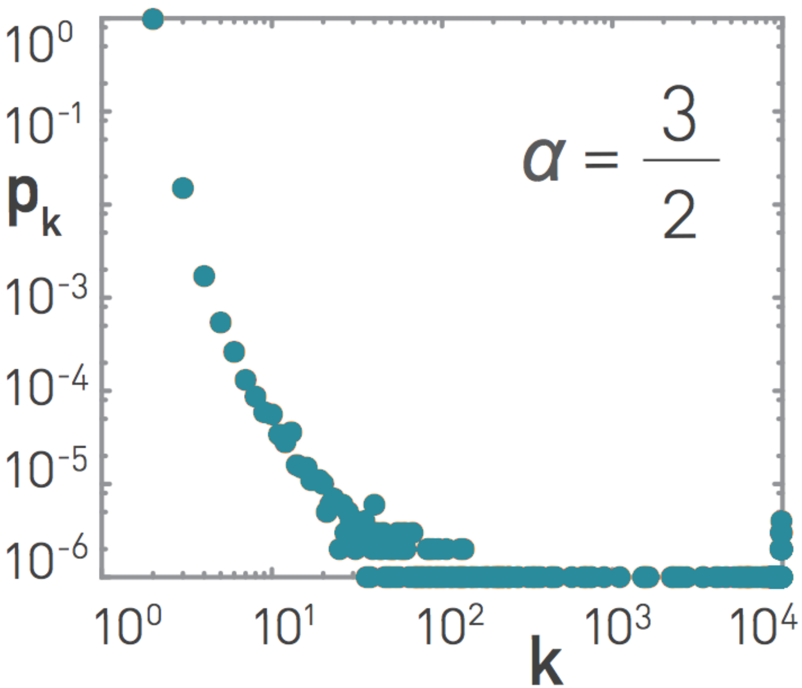


Power law

$$k_{\max} \sim t^{1/2}$$

Degree distributions (superlinear regime)

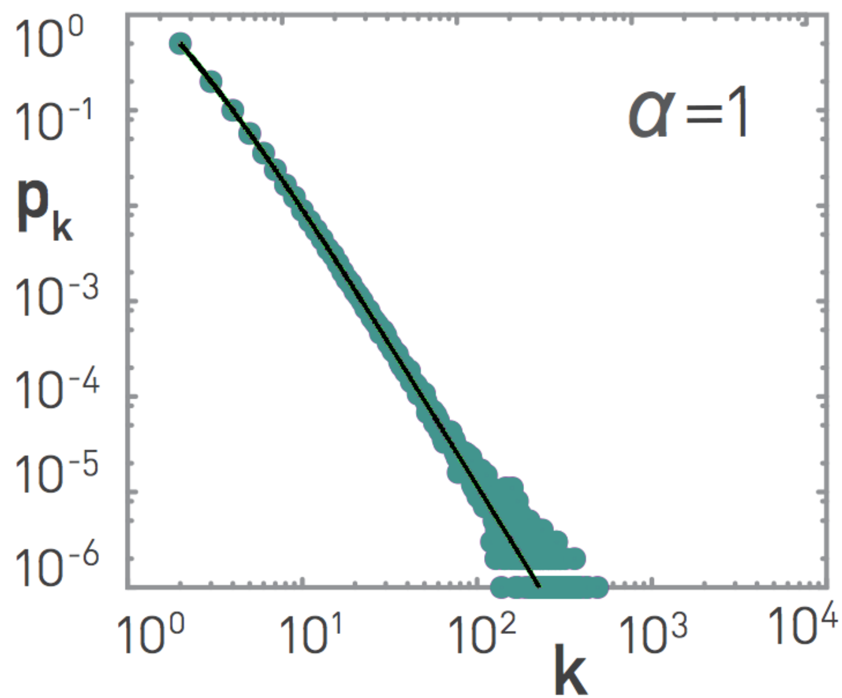
Superlinear



Winners take it all!

$k_{\max} \sim t$

Linear



Power law

$k_{\max} \sim t^{1/2}$

Non-linear preferential attachment

Conclusion:

Only linear preferential attachment produces scale-free growing networks. On the other hand, non-linear preferential attachment is more than sufficient to provide a wide spectrum of fat-tailed degree distributions.

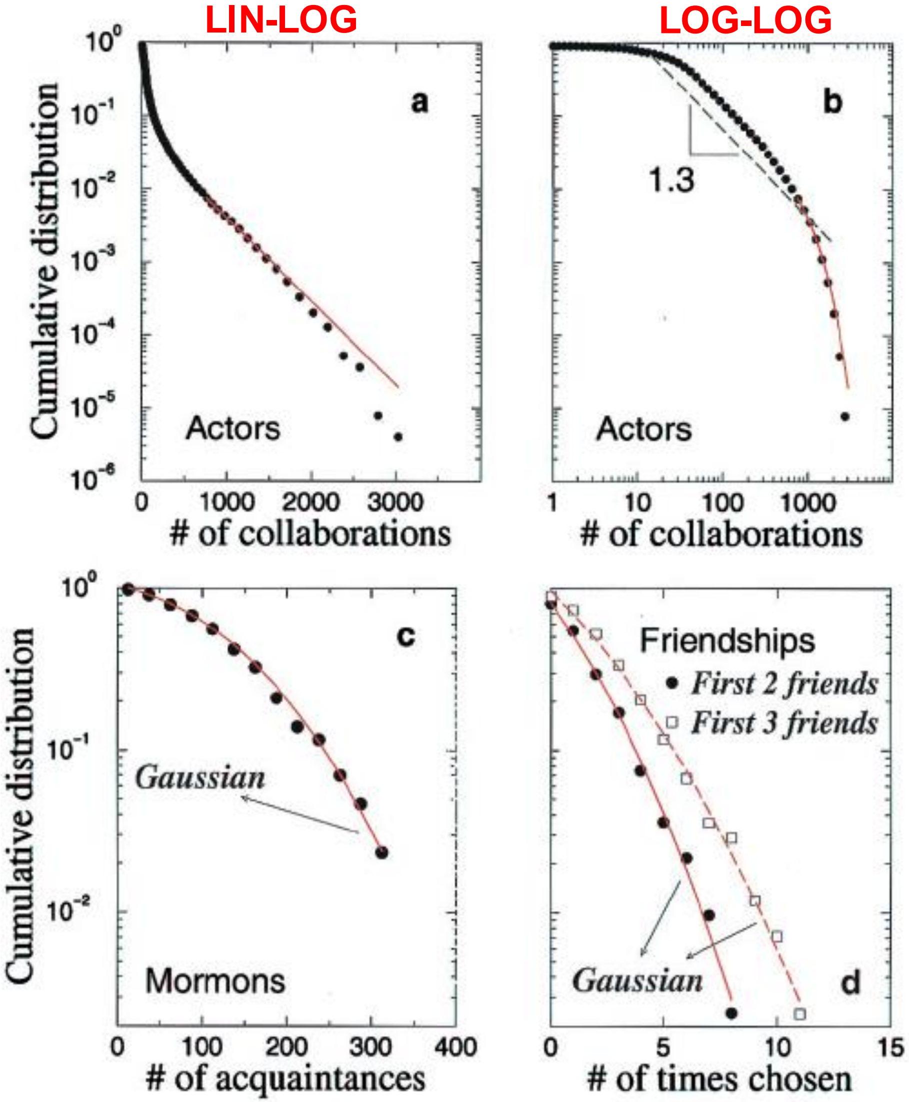
The idea of “Scale-free networks” is a subtle and fragile concept

Scale-free nets is a useful concept…

but be careful with fits

Luís Nunes Amaral
Northwestern University

Social networks . . .



Amaral, Scala, Barthélémy, and Stanley, Classes of small-world networks, PNAS 97 (21) 11149 (2000).

Scale-free nets is a useful concept…
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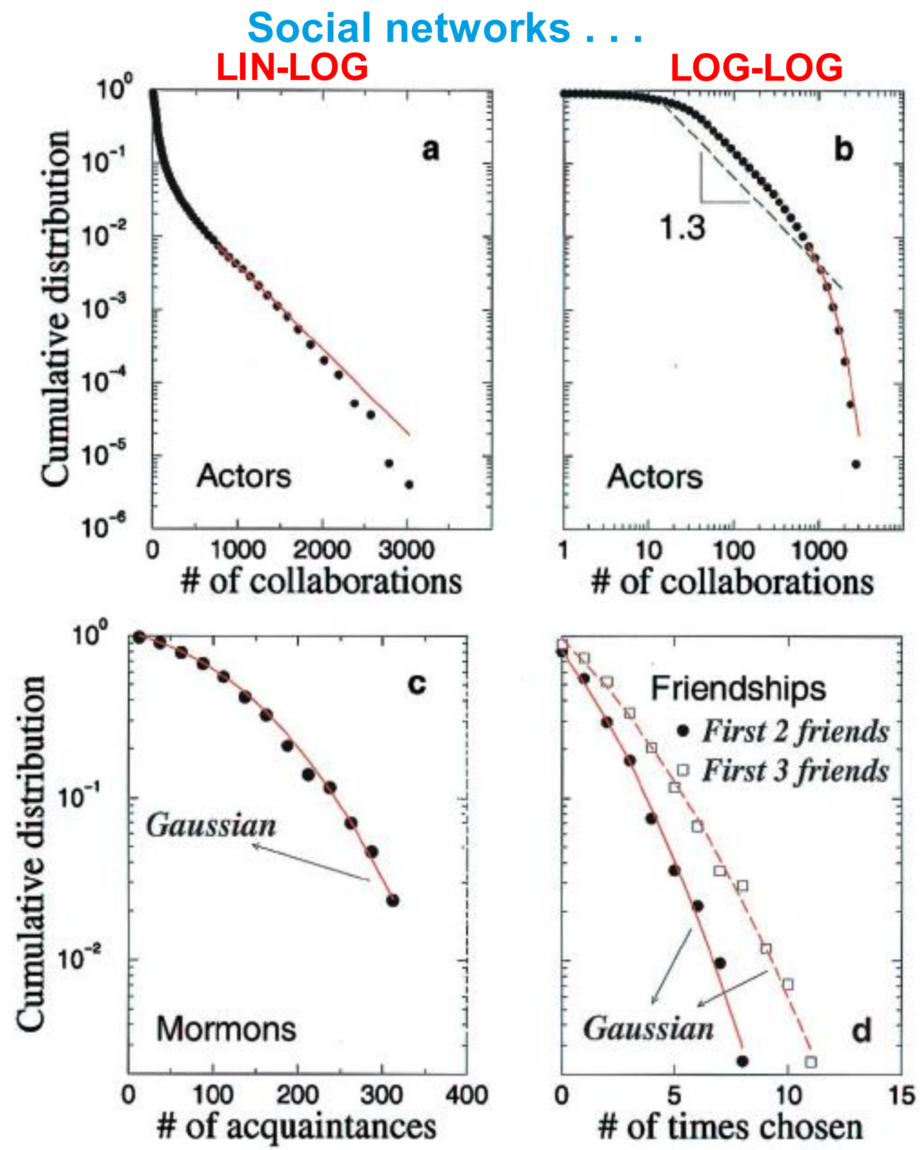

…exponential decays
are also common

Networks are finite!!

Amaral, Scala, Barthélémy, and Stanley, Classes of small-world networks, PNAS 97 (21) 11149 (2000).

Scale-free nets is a useful concept…
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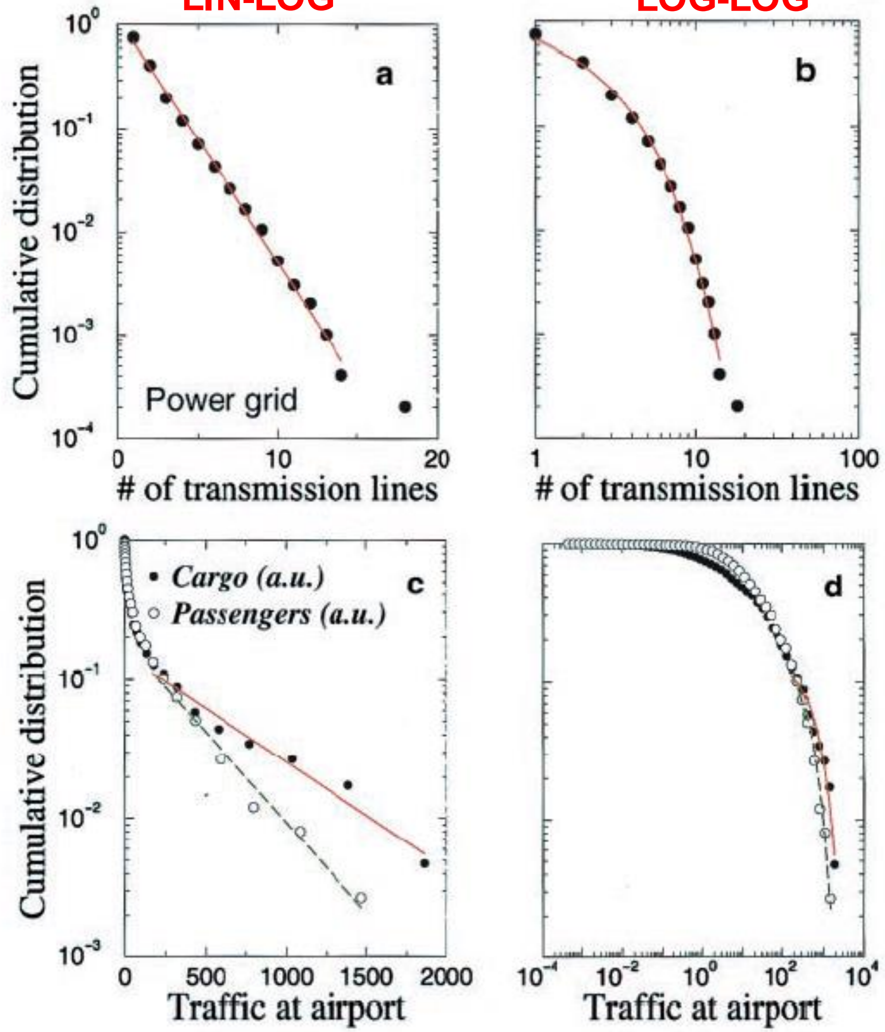
Technological networks . . .

LIN-LOG LOG-LOG



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biological networks . . .

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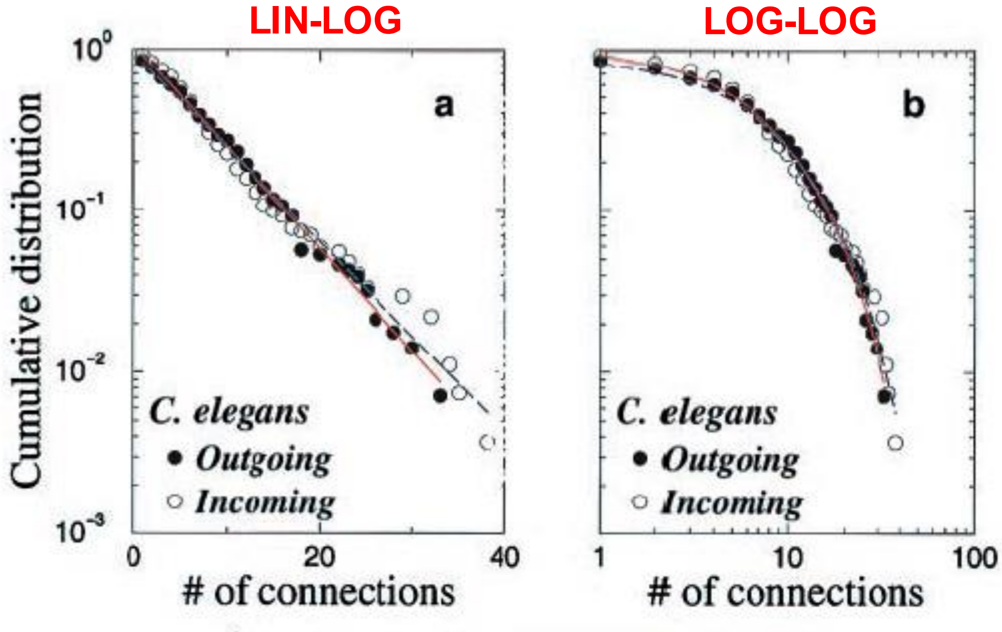

Amaral, Scala, Barthélémy, and Stanley,
Classes of small-world networks,
PNAS 97 (21) 11149 (2000).

Classes of small-world networks

We can state (based on existing empirical analysis of real nets) that there are 3 main classes of graphs :

single-scale : individual degrees do not deviate appreciably from the average degree of the graph (type most compatible with WS-model);

broad-scale : degrees span a wider interval, with degree distribution falling off exponentially for large k ;

scale-free : those graphs in which the degree distributions decays with a power law, exhibiting the same behavior at all scales.

Amaral, Scala, Barthélémy, and Stanley,
Classes of small-world networks,
PNAS 97 (21) 11149 (2000).

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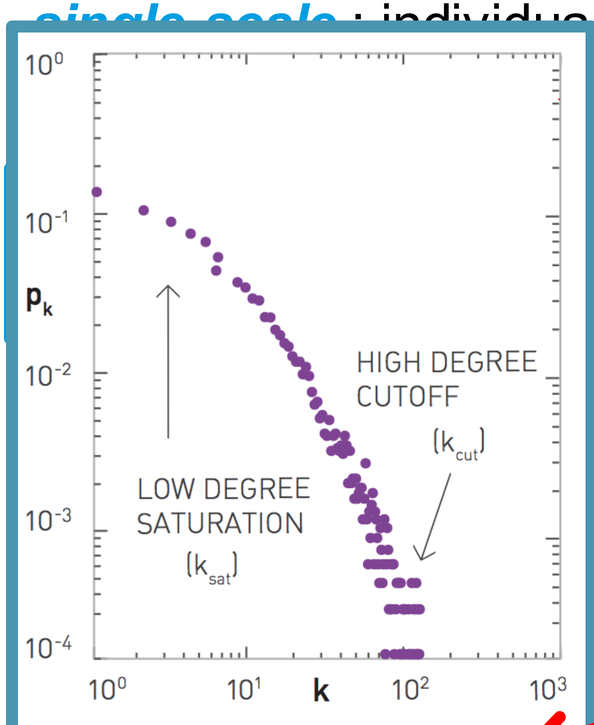


Dorogovtsev, Mendes, & Samukhin (2001). Size-dependent degree distribution of a scale-free growing network. Phys Rev E, 63(6), 062101.

Exponential cutoffs & finite size effects:

- A net is SF only in the limit of very large N (try it!).
- There's a maximum degree a node can have. This introduces natural (exponential) cutoffs in the degree dist.

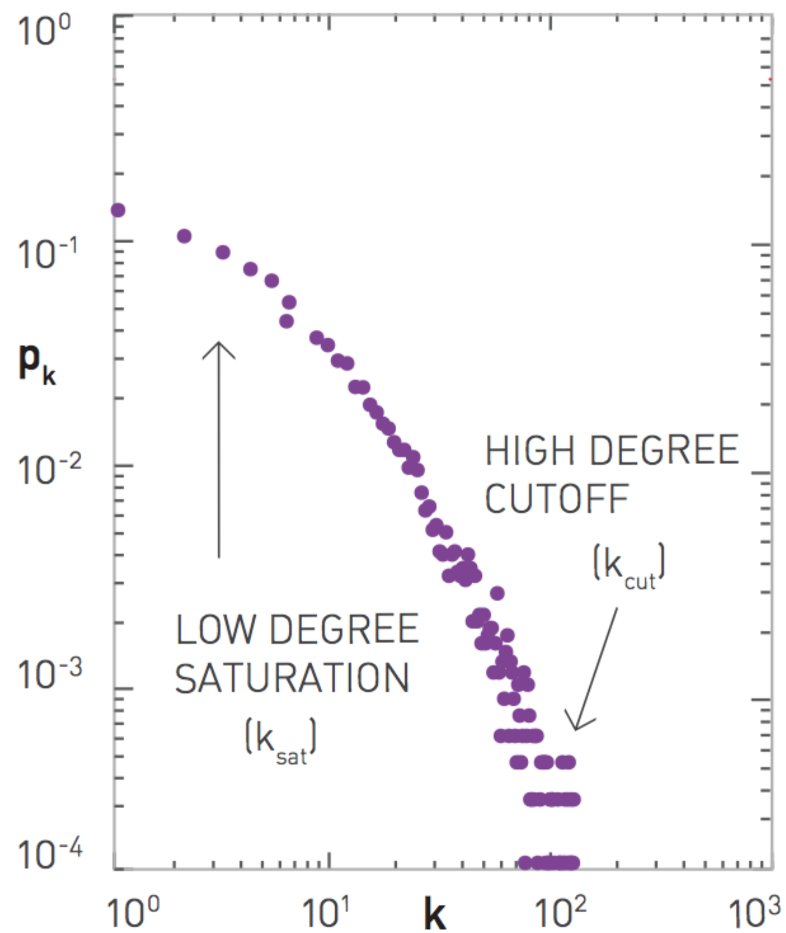
NAME	$p_x/p(x)$	$\langle x \rangle$	$\langle x^2 \rangle$
Poisson (discrete)	$e^{-\mu}\mu^x/x!$	μ	$\mu(1+\mu)$
Exponential (discrete)	$(1-e^{-\lambda})e^{-\lambda x}$	$1/(e^{\lambda}-1)$	$(e^{\lambda}+1)/(e^{\lambda}-1)^2$
Exponential (continuous)	$\lambda e^{-\lambda x}$	$1/\lambda$	$2/\lambda^2$
Power law (discrete)	$x^{-\alpha}/\zeta(\alpha)$	$\begin{cases}\zeta(\alpha-2)/\zeta(\alpha), & \text{if } \alpha > 2\\ \infty, & \text{if } \alpha \le 1\end{cases}$	$\begin{cases}\zeta(\alpha-1)/\zeta(\alpha), & \text{if } \alpha > 1\\ \infty, & \text{if } \alpha \le 2\end{cases}$
Power law (continuous)	$\alpha x^{-\alpha}$	$\begin{cases}\alpha/(\alpha-1), & \text{if } \alpha > 2\\ \infty, & \text{if } \alpha \le 1\end{cases}$	$\begin{cases}\alpha/(\alpha-2), & \text{if } \alpha > 1\\ \infty, & \text{if } \alpha \le 2\end{cases}$
Power law with cutoff (continuous)	$\frac{\lambda^{1-\alpha}}{\Gamma(1-\alpha)}x^{-\alpha}e^{-\lambda x}$	$\lambda^{-1}\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}$	$\lambda^{-2}\frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)}$
Stretched exponential (continuous)	$\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}$	$\lambda^{-1}\Gamma(1+\beta^{-1})$	$\lambda^{-2}\Gamma(1+2\beta^{-1})$
Log-normal (continuous)	$\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\ln x-\mu)^2/(2\sigma^2)}$	$e^{\mu+\sigma^2/2}$	$e^{2(\mu+\sigma^2)}$

take-home message

In real systems we rarely find perfect power-laws

Low-degree saturation: is a common deviation from the power-law behavior. Its signature is a flattened P(k) for $k < k_{sat}$. This indicates that we have fewer small degree nodes than expected for a pure power law.

High-degree saturation: High-degree cutoff appears as a rapid drop in P(k) for $k > k_{cut}$. These cutoffs emerge from i) having a finite network, ii) limitations in the number of links a node can have, or iii) high costs of maintaining a large number of links.



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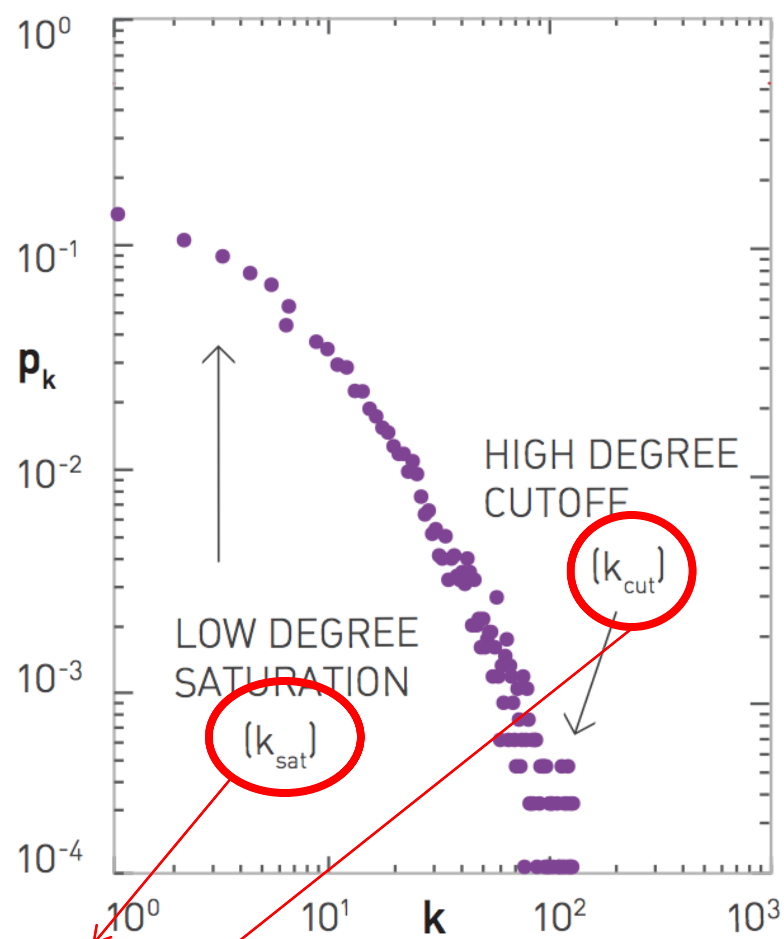
High-degree saturation: High-degree cutoff appears as a rapid drop in P(k) for $k > k_{cut}$. These cutoffs can emerge from different sources, some of which will be analyzed next.



Amaral, Scala, Barthélémy, and Stanley, Classes of small-world networks, PNAS 97 (21) 11149 (2000).

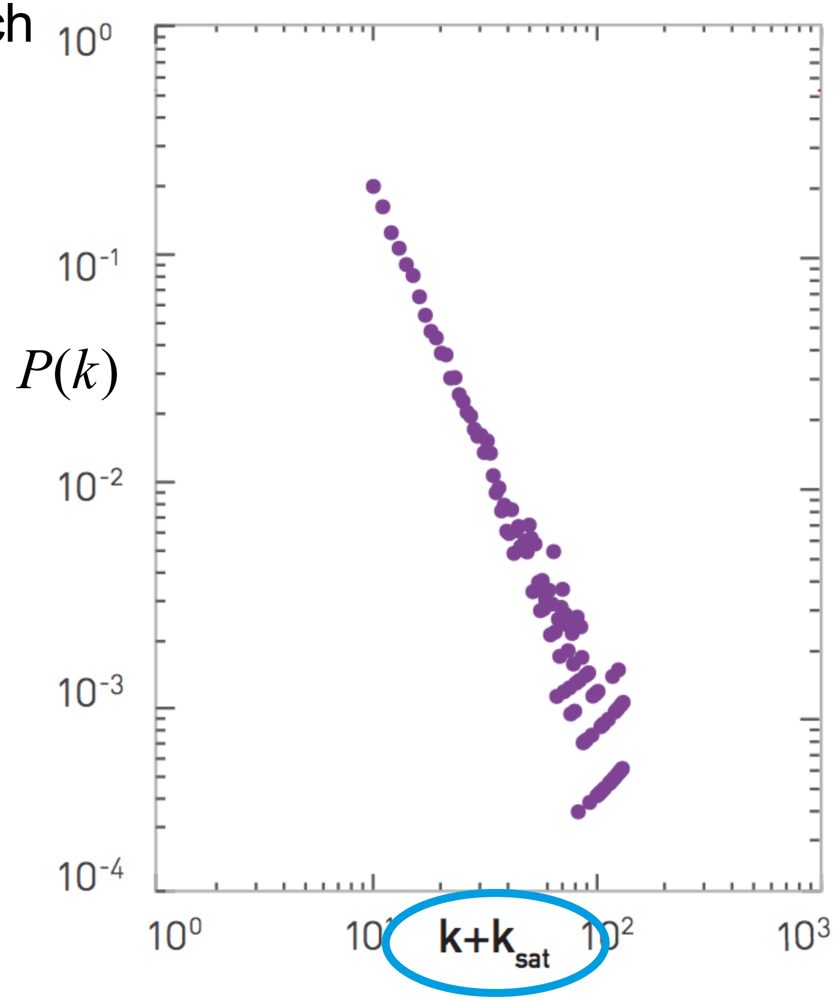
Note however that these cutoffs are hard to determine…

Estimating degree exponents

Given the widespread presence of such cutoffs the degree distribution is occasionally fitted to

$$P(k) = a(k + k_{sat})^{-\gamma} \exp\left(-\frac{k}{k_{cut}}\right)$$

Note however that these cutoffs are hard to determine…



Clauset et al. method: General idea

For details (and code) on this procedure, see: http://tuvalu.santafe.edu/~aaronc/powerlaws/

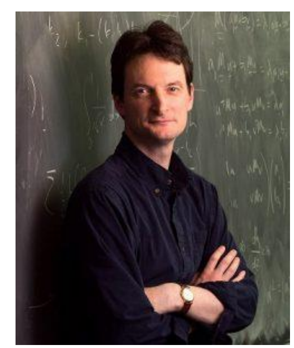
Goal: estimate γ from a discrete set of data points

1. Choose a k_{sat} and a k_{cut} between an interval of possible k_{min} and k_{max}. Optimize the best value of the degree exponent corresponding to this pair, using a statistical test to evaluate the quality of the fit.

1. Repeat 1, scanning the entire interval of possible values of k_{sat}, k_{cut} , keeping the best combination $\{\gamma, k_{sat,} k_{cut}\}$ in what concerns the "goodness" of the fit.

Aaron Clauset
Santa Fe Inst. & Univ. of Colorado

Mark E. J. Newman
Michigan Univ.



A. Clauset, et al. *Power-law distributions in empirical data*. SIAM Review S1: 661-703, 2009.

Clauset et al. method: General idea

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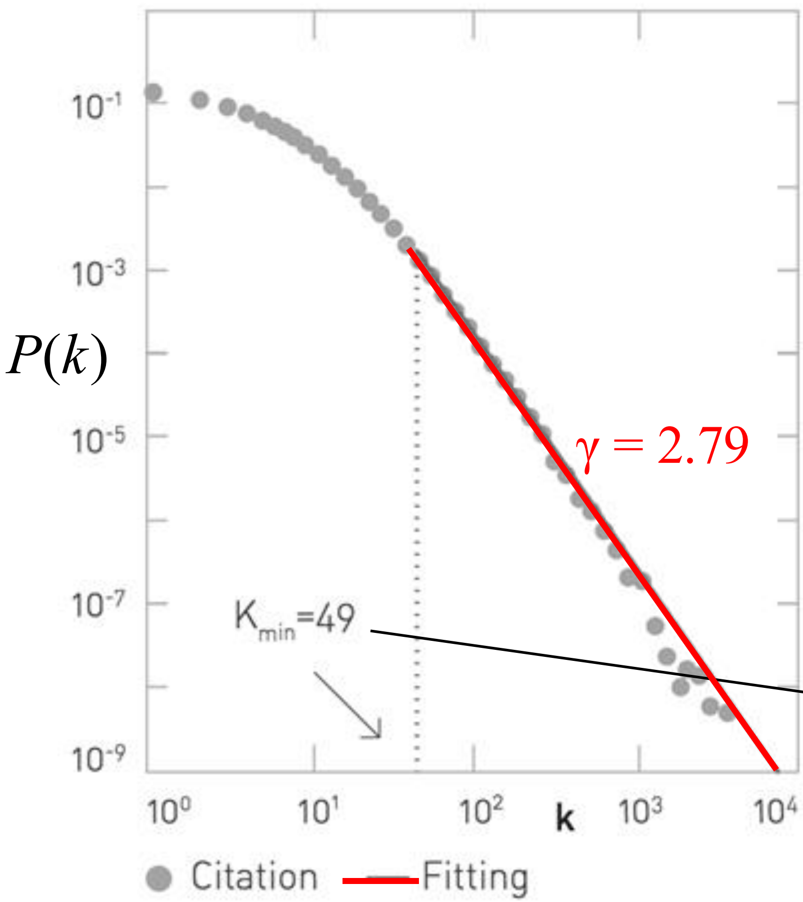
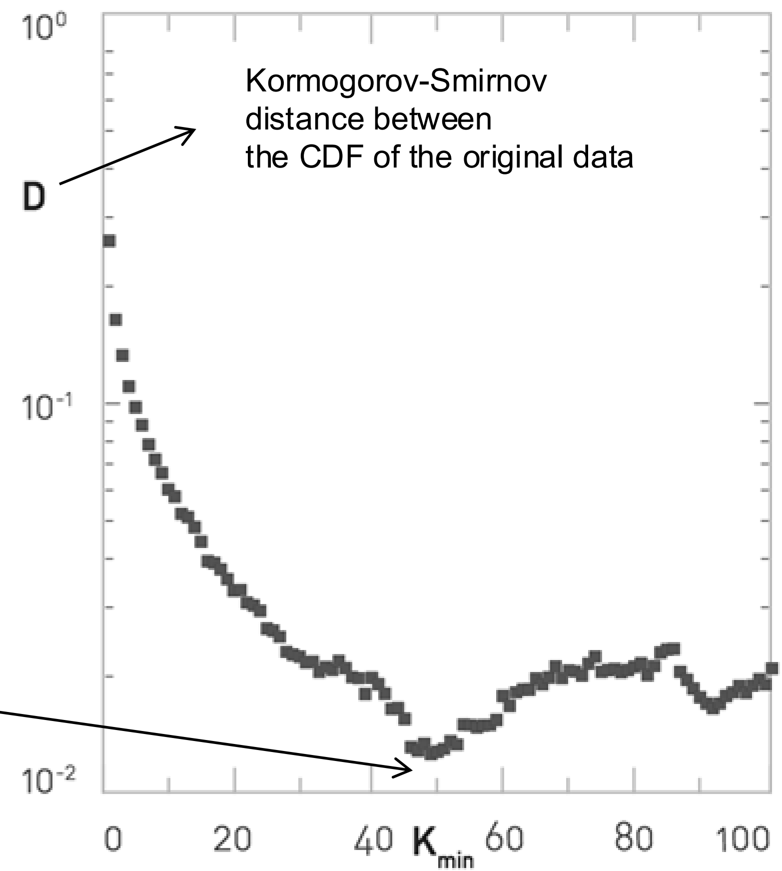
Example: citation networks





Clauset et al. method: General idea

For details (and code) on this procedure, see: http://tuvalu.santafe.edu/~aaronc/powerlaws/

Example: email networks

Let's resort to the input file enron.outdegrees
(available here: https://dl.dropboxusercontent.com/s/e1kr9ri2btfwg9v/enron.outdegrees?dl=0) and enron.outdegree
(available here: https://dl.dropboxusercontent.com/s/79n6w3p3cyat5lx/enron.outdegree?dl=0)
created with webgraph in exercise 6 of our problem set 1.

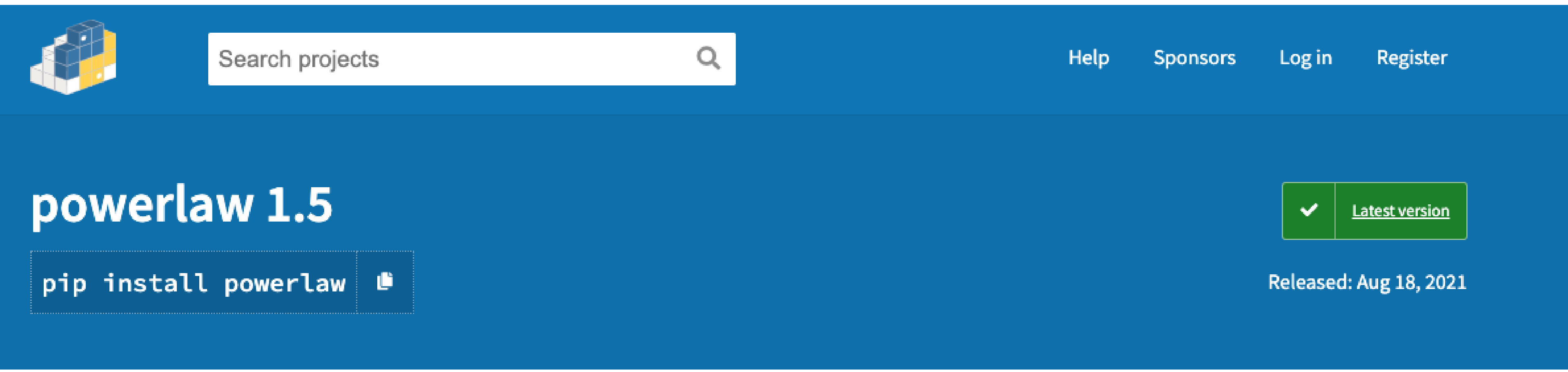
You may try the implementation in R of Clauset's algorithm (last problem in our 2nd set of exercises

```
[madonna@tecnico lab02]$ R
> install.packages("igraph")
> library('igraph')
> degs <- read.table("enron.outdegrees")
> deg_pmf <- read.table("enron.outdegree")
> degs_pl_fit <- power.law.fit(degs$V1)
> degs_pl_fit > plot(deg_pmf$V1, log="xy", xlab="degree", ylab="#vertices")
> plot(rev(cumsum(rev(deg_pmf$V1))), log="xy", xlab="degree", ylab="#vertices")
> plot(sort(degs$V1, decreasing = TRUE), 1:length(degs$V1), log="xy", xlab="degree",
ylab="rank")
```

Clauset et al. method: General idea

For details (and code) on this procedure, see: http://tuvalu.santafe.edu/~aaronc/powerlaws/

Same idea, but in Python:
https://pypi.org/project/powerlaw/



Conclusion:
Be careful with bold statements...

- The scale-free property is a rather fragile concept for small N.
- Moreover, it is unlikely to have a perfect ***linear*** preferential attachment, we may have constraints on the number of links and vertices, etc.
- **Take-home message:**

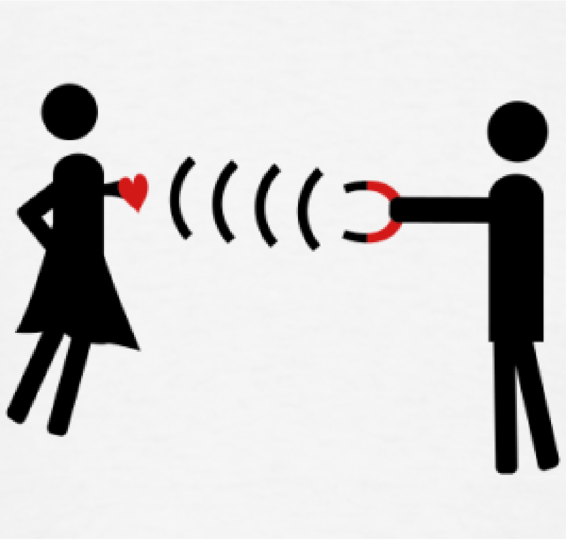
Perfect power-laws are useful concepts to have in mind as a reference point. Reality is often more complex, portraying different classes of complex networks.

Amaral, Scala, Barthélémy, and Stanley, PNAS 97 (21) 11149 (2000).

Broido, A. D., & Clauset, A.. Nature communications, 10(1), 1-10. (2019)

What are the mechanisms behind these variations?

Example:
Is the degree the only thing that matters?



Other variant: BA model with initial attractiveness

- ***Growth***: add nodes sequentially.

At t=0 consider (e.g.) a ring with m_0 nodes; for each time-step, add 1 node and connect it via m new edges with existing nodes;

- ***Preferential attachment***:

when establishing the connections, each new node is connected to m older nodes with a probability proportional to a constant plus the degree of the older nodes.

$$\Pi_i = \frac{A_i + k_i}{\sum_j (A_j + k_j)}$$

Irrespectively of your degree, you have always some attractiveness!

Dorogovtsev, et al. (2000), PRL 85: 4633.

Other variant: BA model with initial attractiveness

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when establishing the connections, each new node is connected to m older nodes with a probability proportional to a constant plus the degree of the older nodes.

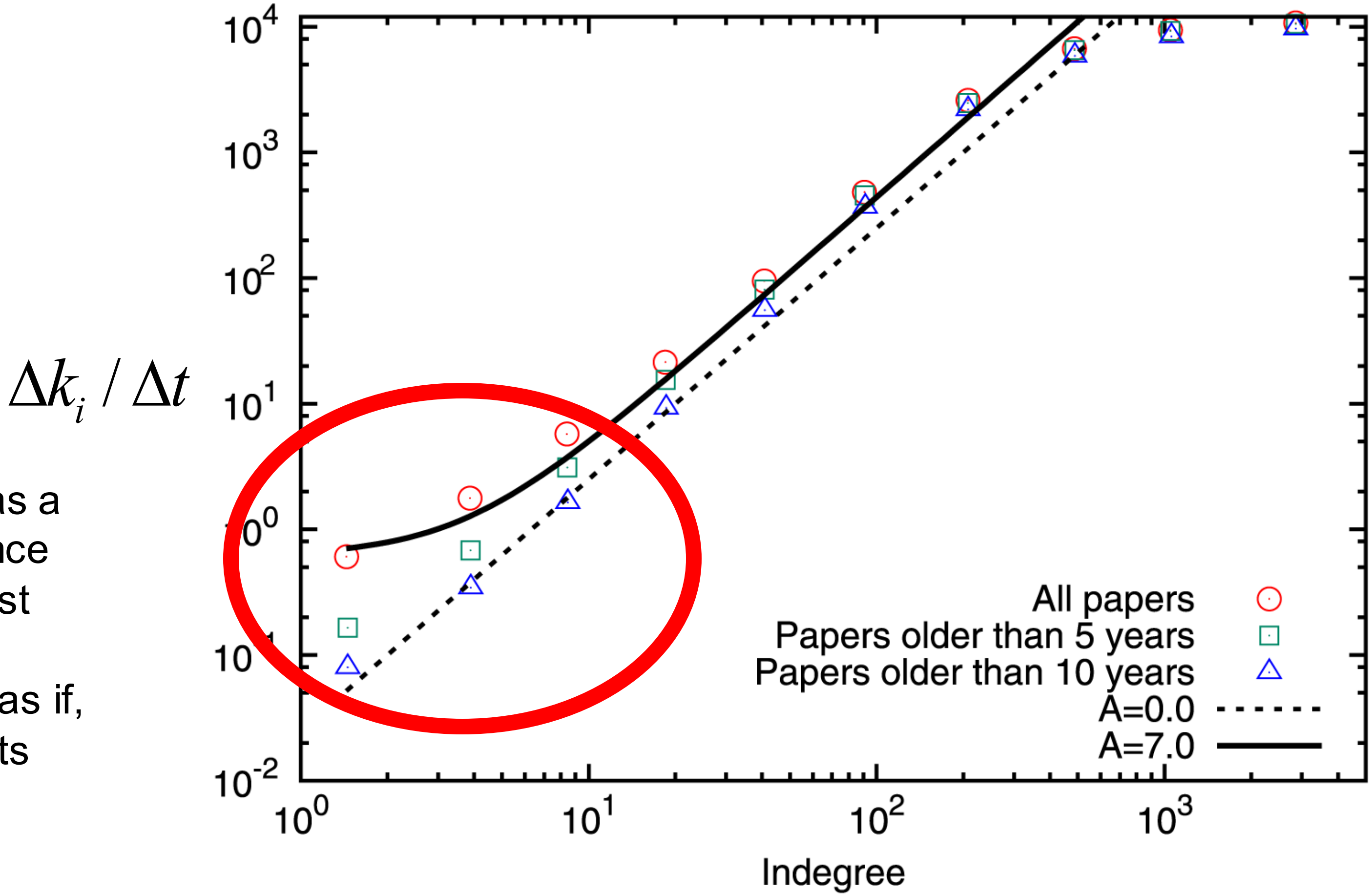
$$\Pi_i = \frac{A_i + k_i}{\sum_j \left(A_j + k_j\right)}$$

$$\gamma = 3 + \frac{A}{m} > 3$$

Dorogovtsev, et al. (2000), PRL 85: 4633.

Initial attractiveness & citation networks

Attractiveness can also be measured



Attractiveness has a significant influence only within the first few years after publication. It is, as if, a new paper starts with 7 citations…

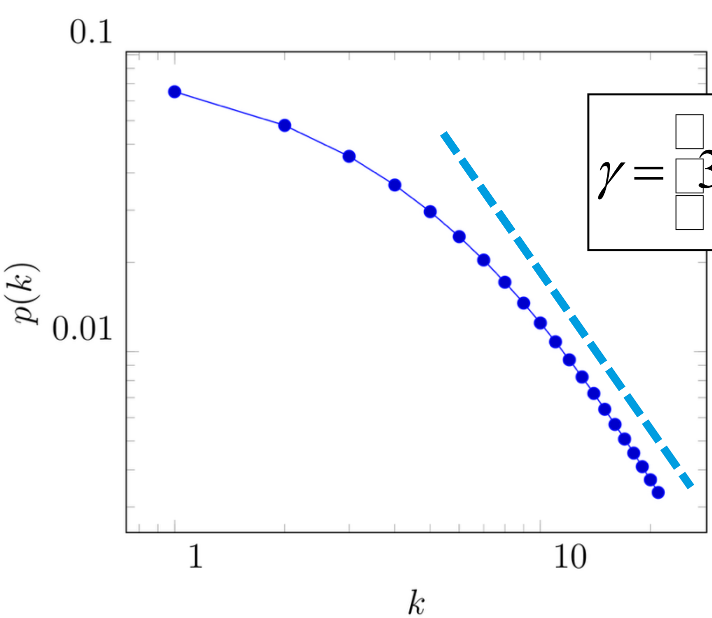
Y.-H. Eom and S. Fortunato. *Characterizing and Modeling Citation Dynamics*. PLoS ONE, 6: e24926, 2011.

BA model with initial attractiveness

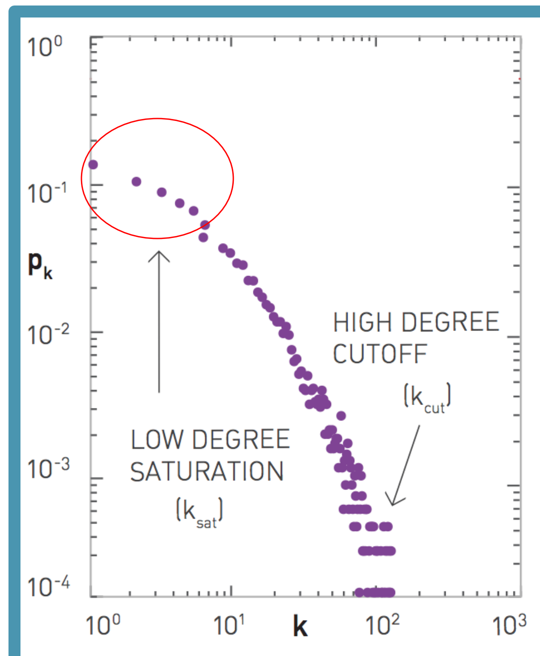
Dorogovtsev, et al. (2000), Phys Rev Lett 85: 4633.

- It also generates a small degree saturation and increases the value of γ:

$$\Pi_i = \frac{A + k_i}{\sum_j (A + k_j)} \Rightarrow P(k) \sim (A + k)^{-\gamma}$$



$$\gamma = \left(3 + \frac{A}{m}\right)$$



Other variant: BA model with initial attractiveness

Challenge #3: Can you analyze either through computer simulations or analytically the impact of initial attractiveness?

What would be the result if attractiveness depends on the age of the node?

Dorogovtsev, et al. (2000), PRL 85: 4633.

Fitness models

Ginestra Bianconi et al. Bose-Einstein Condensation in Complex Networks, Phys. Rev. Lett., 86: 5632–5635, 2001.

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- ***Preferential attachment***:

when establishing the connections, each new node is connected to older nodes with a probability proportional to the degree of the older nodes and to nodes intrinsic fitness η.

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

"fittest-get-richer" process

Fitness models

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$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

"fittest-get-richer" process

The intrinsic fitness is taken from a given distribution $\rho(\eta)$

New nodes may now become central, even if added at a later stage!

Result: The shape of the degree distribution depends on $\rho(\eta)$

Fitness models

"fittest-get-richer" process

Example: Let's consider a uniform distribution $\rho(\eta)$ taken from the interval [0,1].

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

$$P(k) \sim \frac{k^{-\gamma}}{\ln k}$$

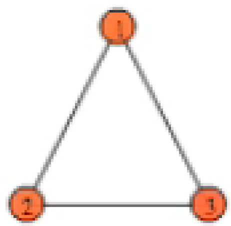
$$\gamma = 2.255 < 3 \text{ (BA-model)}$$

Fitness models

"fittest-get-richer" process

Challenge: Confirm this result simulating the growth of a network biased by a uniform fitness dist.

What if we do not have a uniform distribution of fitness values?



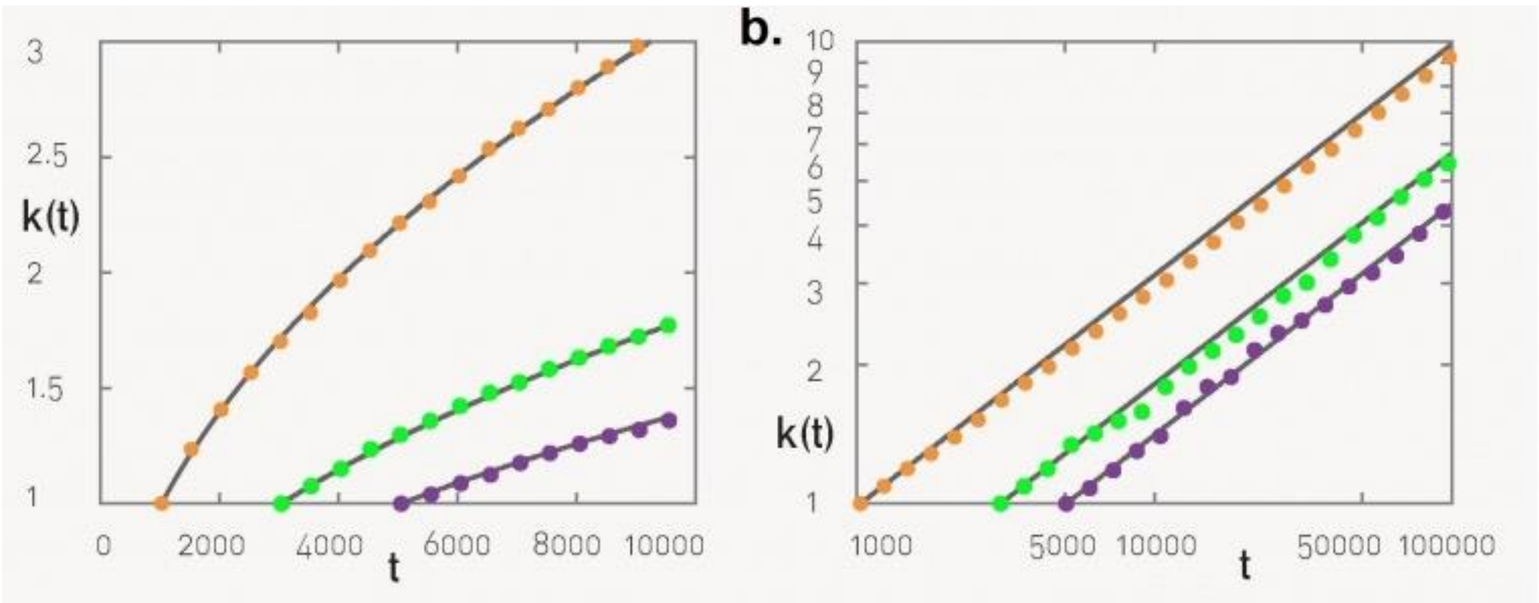
Growing network in which each new node acquires a randomly chosen fitness parameter at birth

Fitness models

"fittest-get-richer" process

Linear plot

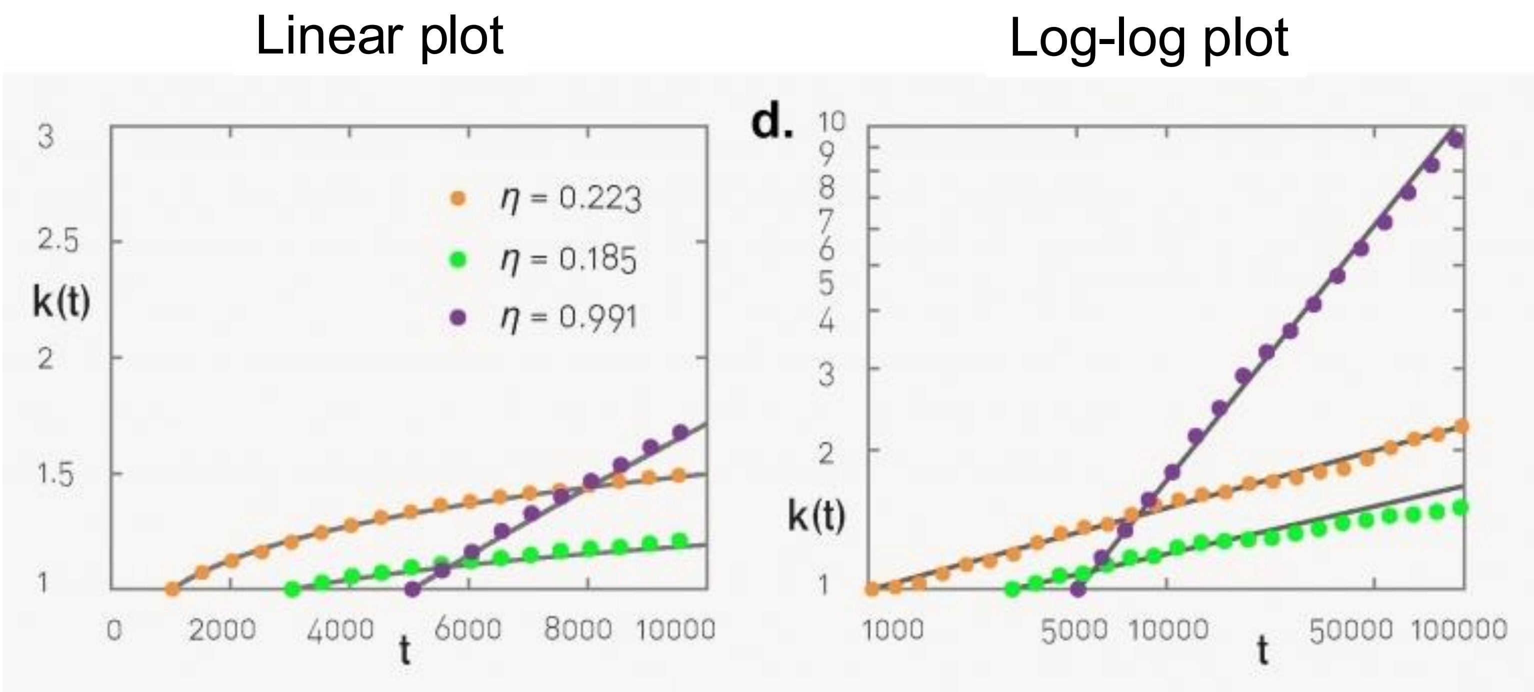
Log-log plot



In the Barabasi-Albert model, nodes that get into the network later are unable to pass the earlier nodes

Fitness models

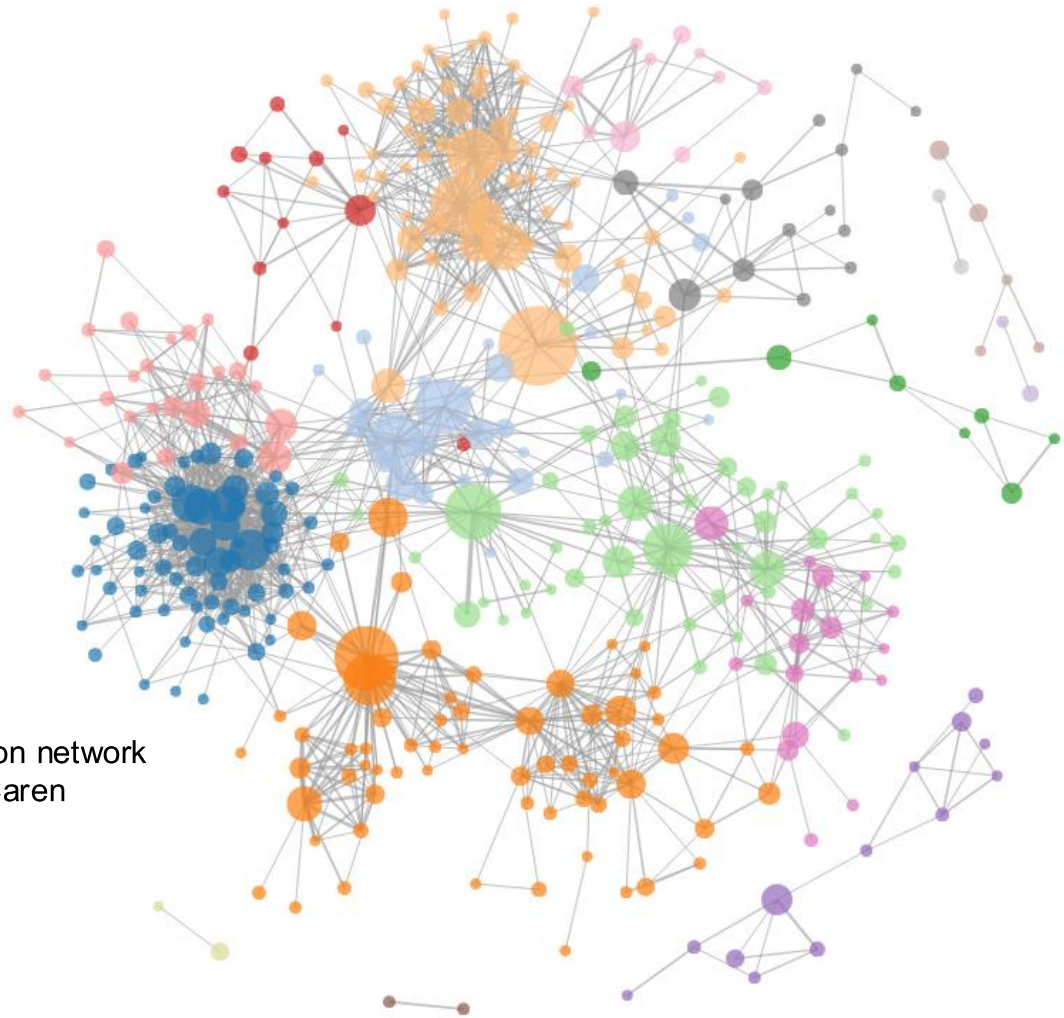
"fittest-get-richer" process



If we add fitness, a latecomer node with a higher fitness (purple symbols) can overcome the earlier nodes.

Fitness models are problem dependent

- Example: Modeling citation networks



Sociology citation network
by Neal Caren

Fitness models are problem dependent

- Example: Modeling citation networks

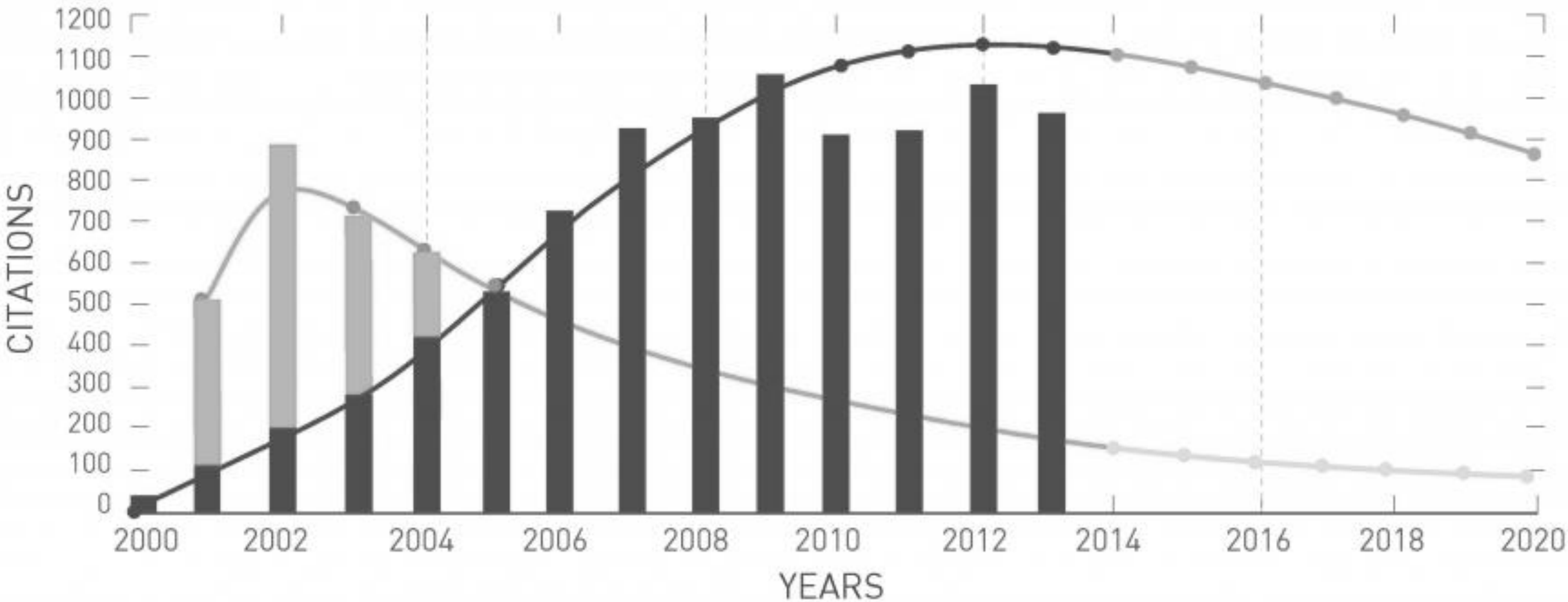
$$\Pi_i \sim \eta_i k_i P_i(t)$$

Fitness of a paper
(novelty, importance, etc)

Number of
citations at time t

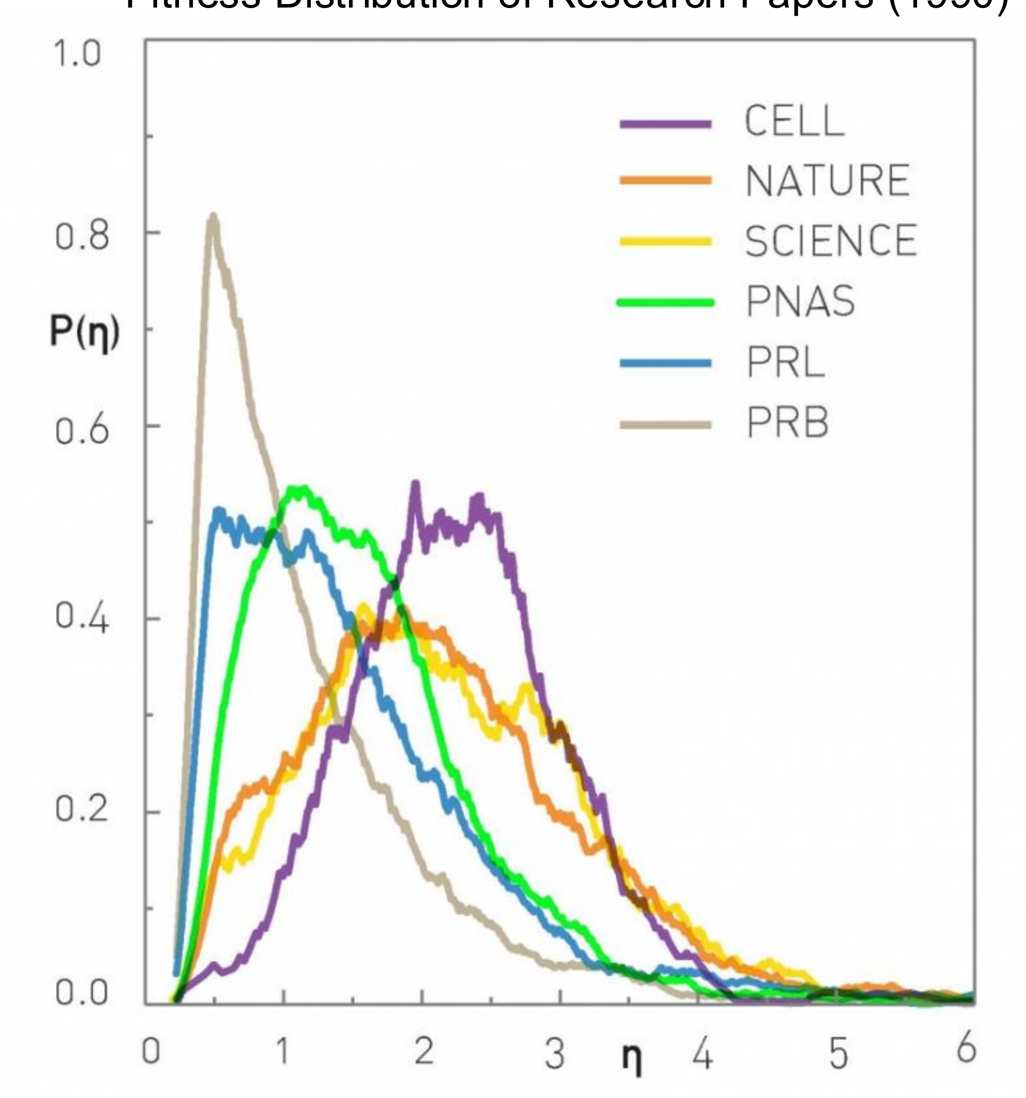
new ideas are
integrated into
subsequent work,
hence the novelty
fades with time

Fitness models are problem dependent

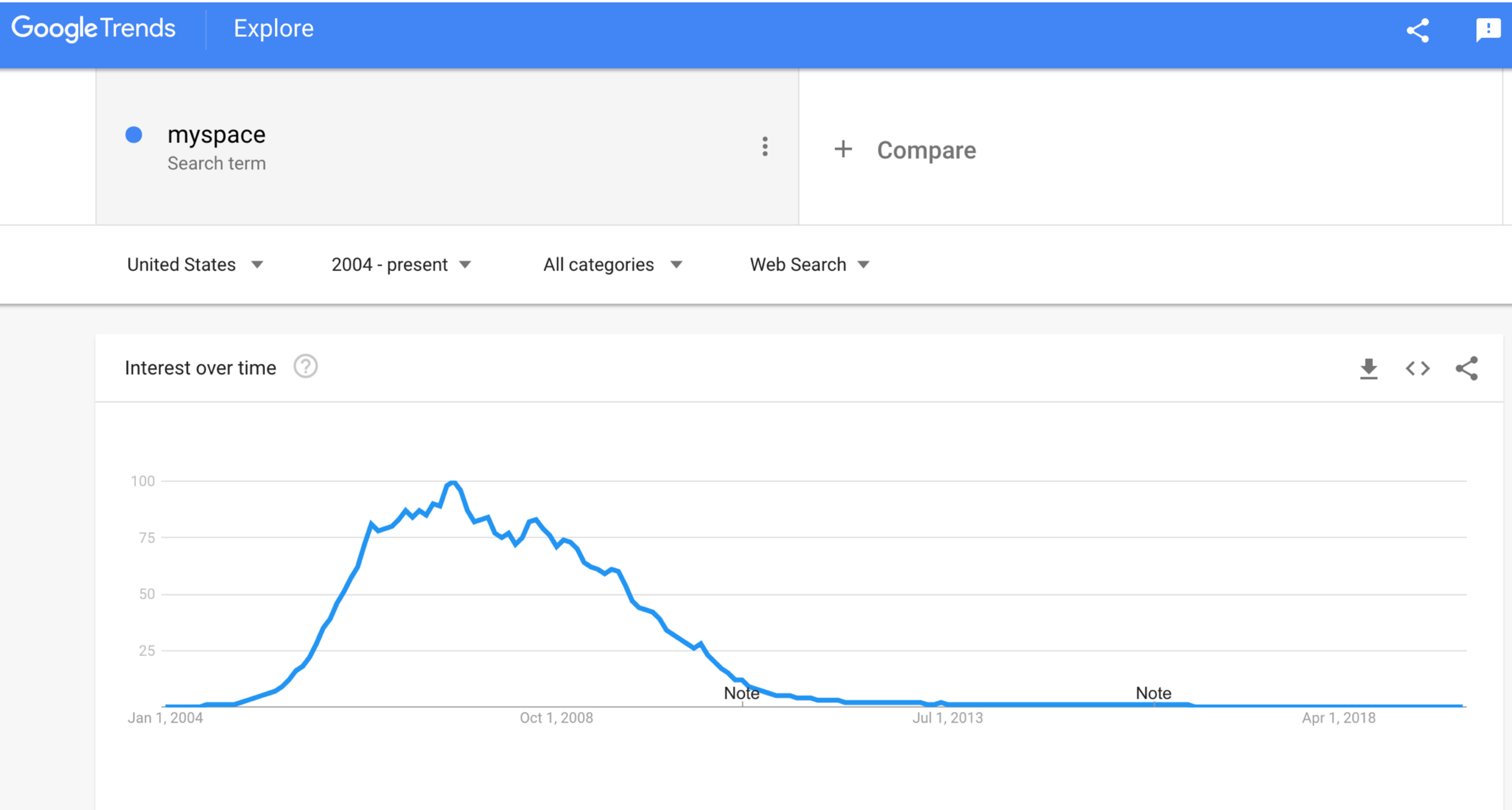


Fitness models are problem dependent

Fitness Distribution of Research Papers (1990)



Another example: WWW and MySpace



Fitness of the WWW

- Example: Modeling the WWW

$$\Pi_i \sim \eta_i k_i$$

Fitness of a document
(which can be measured)

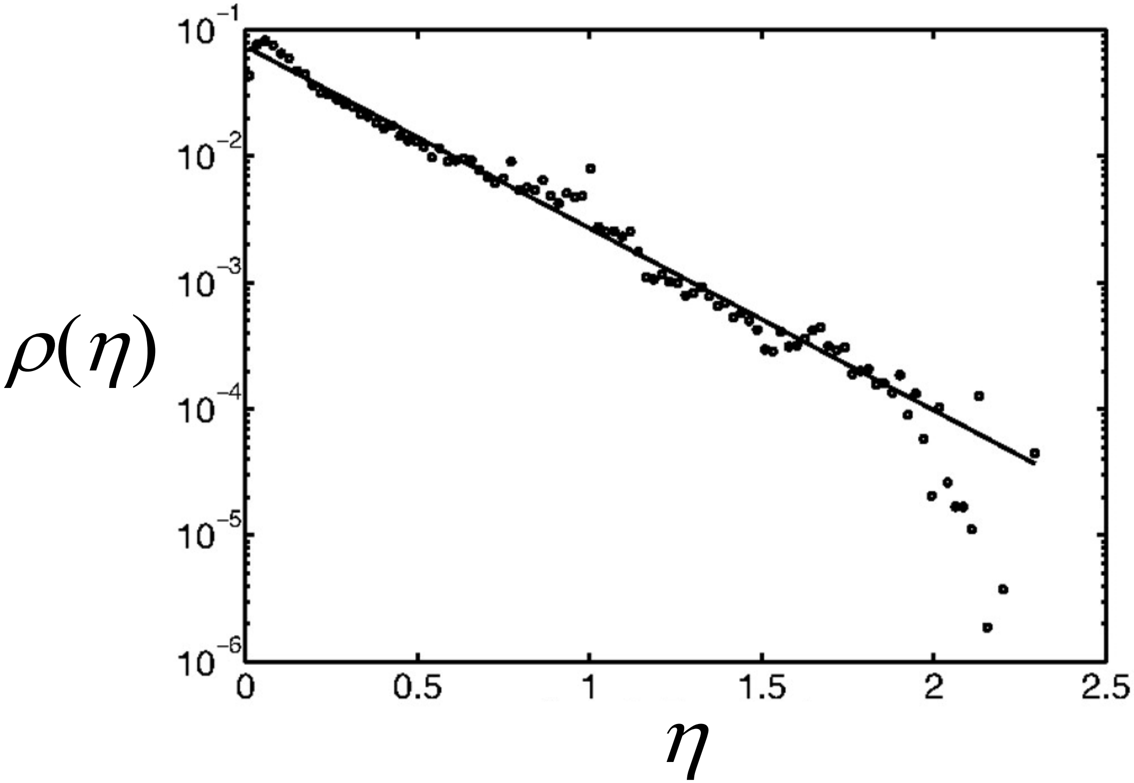
Number of in-links

It’s also neat to show that, small differences in fitness lead, far into the future, to large differences in degrees.

J. S. Kong, N. Sarshar, and V. P. Roychowdhury.
Experience versus talent shapes the structure of the Web.
PNAS, 105:13724-9, 2008.

Fitness of the WWW

- Distribution of fitness values:



- Fitness distribution is exponential
- Fitness of Web documents varies in a relatively narrow range.
- Growth+preferential attachment amplifies the small fitness differences, turning nodes with slightly higher fitness into much bigger nodes.

J. S. Kong, N. Sarshar, and V. P. Roychowdhury. Experience versus talent shapes the structure of the Web. PNAS, 105:13724-9, 2008.

Do only new nodes create links?

Other variant: BA model with internal links

- new links do not only arrive with new nodes but are added between pre-existing nodes (e.g., WWW).
- Consider an extension of the BA-model, where in each time step we ***add a new node with m links***, followed by ***n internal links***, each selected with probability

$$\Pi(k_i, k_j) \sim k_i \cdot k_j$$

(double preferential attachment)

$$\gamma = 2 + \frac{m}{m+2n} < 3$$

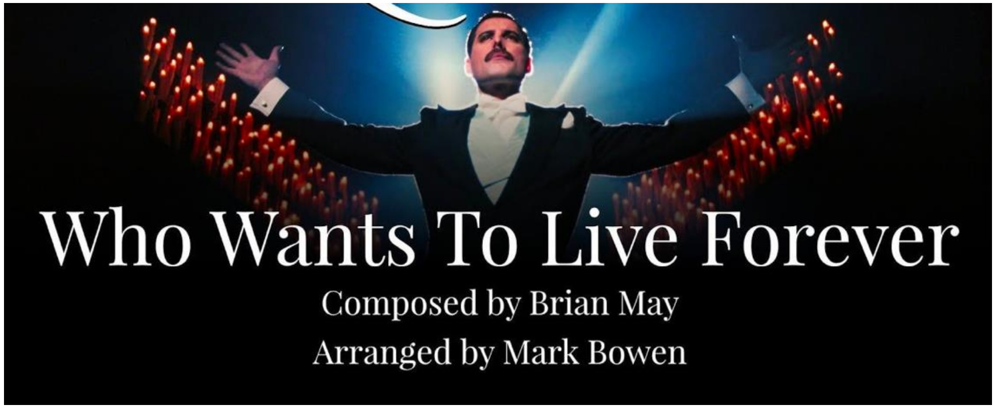
lowers the degree exponent from 3 to 2, hence increasing the network's heterogeneity

Other variant: BA model with internal links

- new links do not only arrive with new nodes but are added between pre-existing nodes (e.g., WWW).
- Consider an extension of the BA-model, where in each time step we add a new node with *m* links, followed by ***n* internal links**, each selected with ***random probability***

$$\gamma = 3 + \frac{2n}{m} > 3$$

the resulting network will be more homogenous than the network without internal links

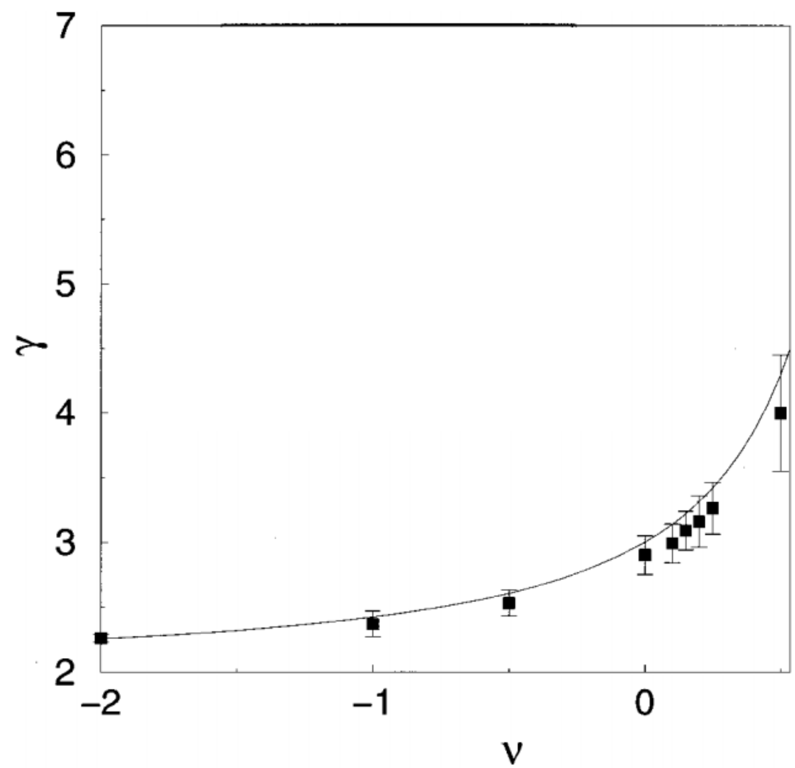

Who Wants To Live Forever
Composed by Brian May
Arranged by Mark Bowen

Other variant: BA model *with ageing* (model 1)

Amaral et al. (2000)
Dorogovtsev, et al. (2000)

Preferential attachment with preference for old/ young nodes

$$\Pi(k,t,t_i) \sim k(t-t_i)^{-\nu}$$



v>0 : young nodes get more attractive

v<0 : old nodes get (even) more attractive

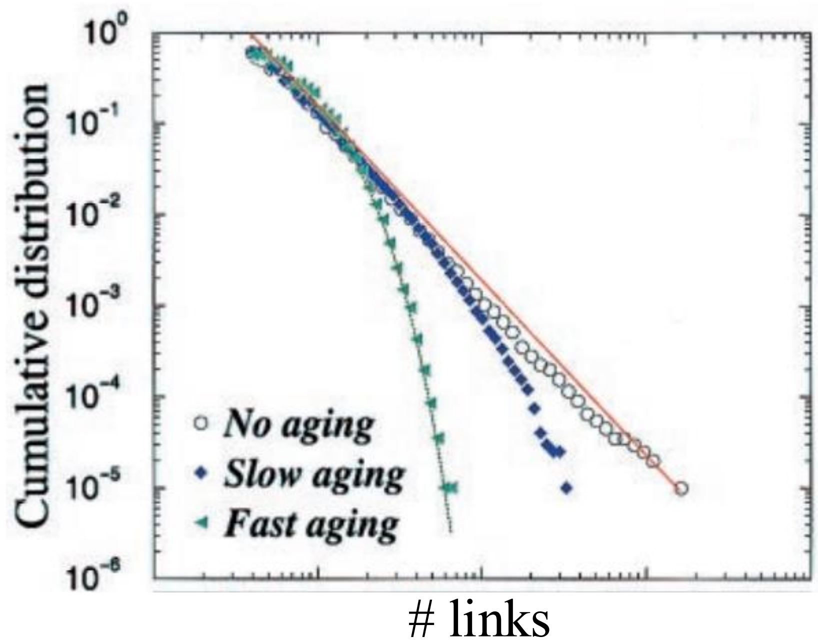
Other variant: BA model *with ageing* (model 2)

Amaral, Barthélémy, Stanley,
Classes of small-world networks, PNAS 2000

Same as the linear preferential attachment. Yet, with a probability that scales with

$$(t - t_i)^{-\nu}$$

an old node becomes inactive and cannot receive links from new nodes.



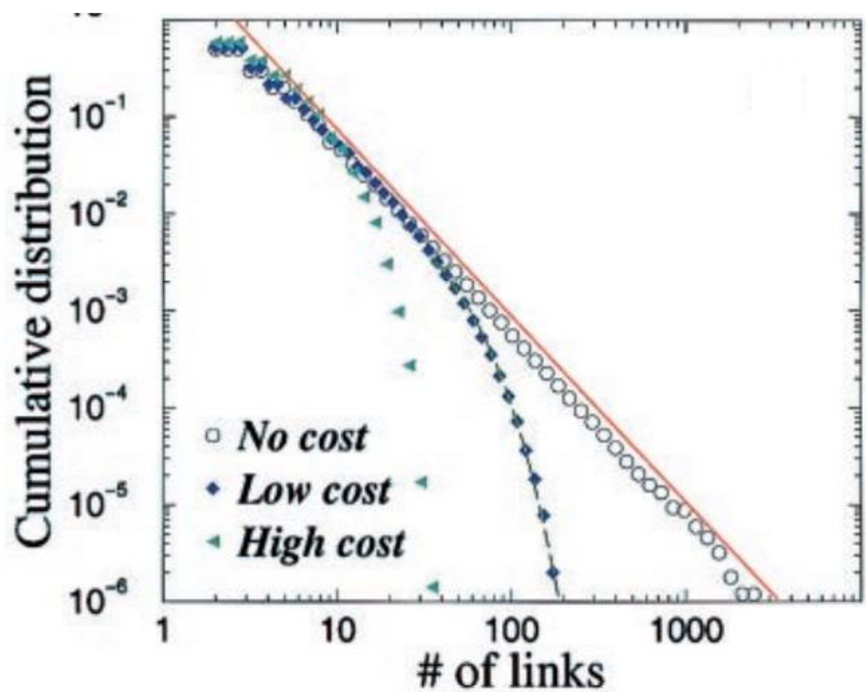
Other variant: BA model *with costs*

Amaral, Barthélémy, Stanley,
Classes of small-world networks, PNAS 2000

physical costs of adding links limits the number of possible links attaching to a given node.

a vertex becomes inactive when it reaches a maximum number of links k_{max}.

This creates natural exponential cut-offs in degree distributions.

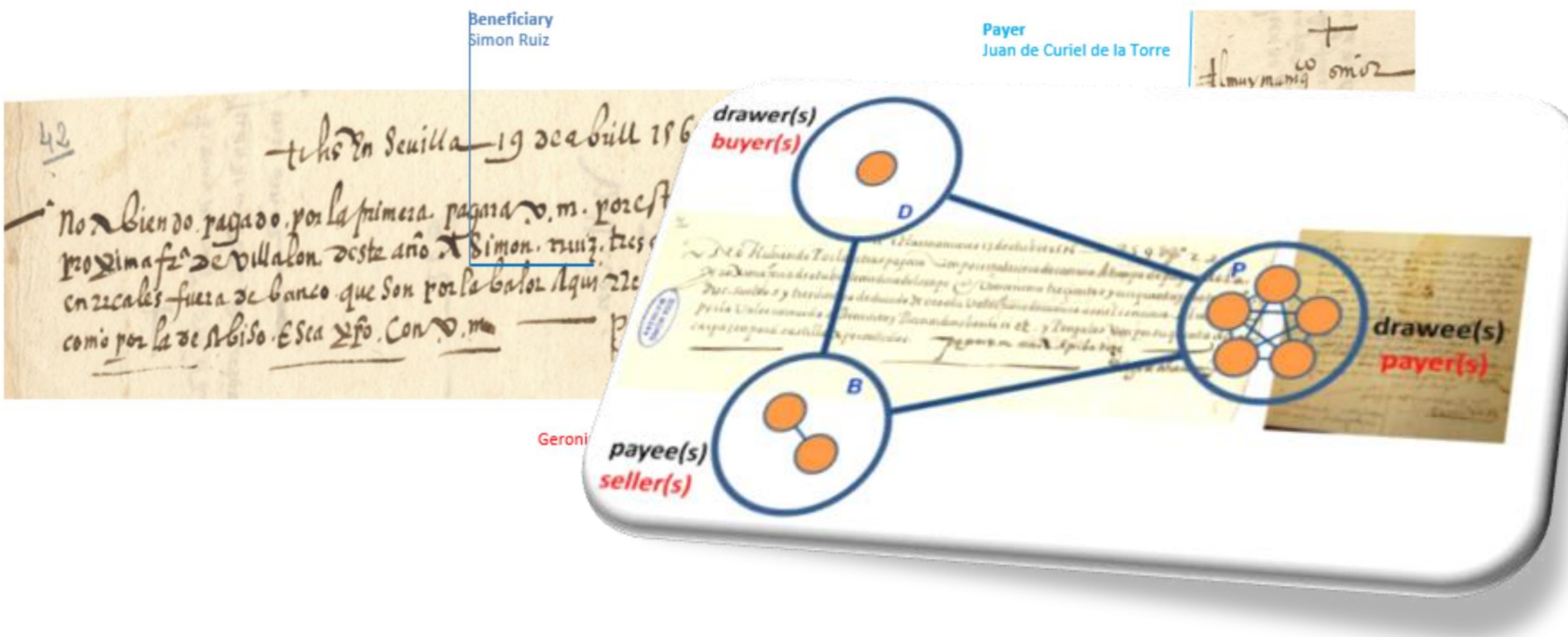


One more challenge

- **Networks grow, but also shrink…**
- Suggest a model in which the impact of this effect is tested?

Example #1

First global trading market using information contained in ~9000 Bills of Exchange



Ribeiro et al. 2018 Structural and temporal patterns of the first global trading market. R. Soc. open sci. 5: 180577.

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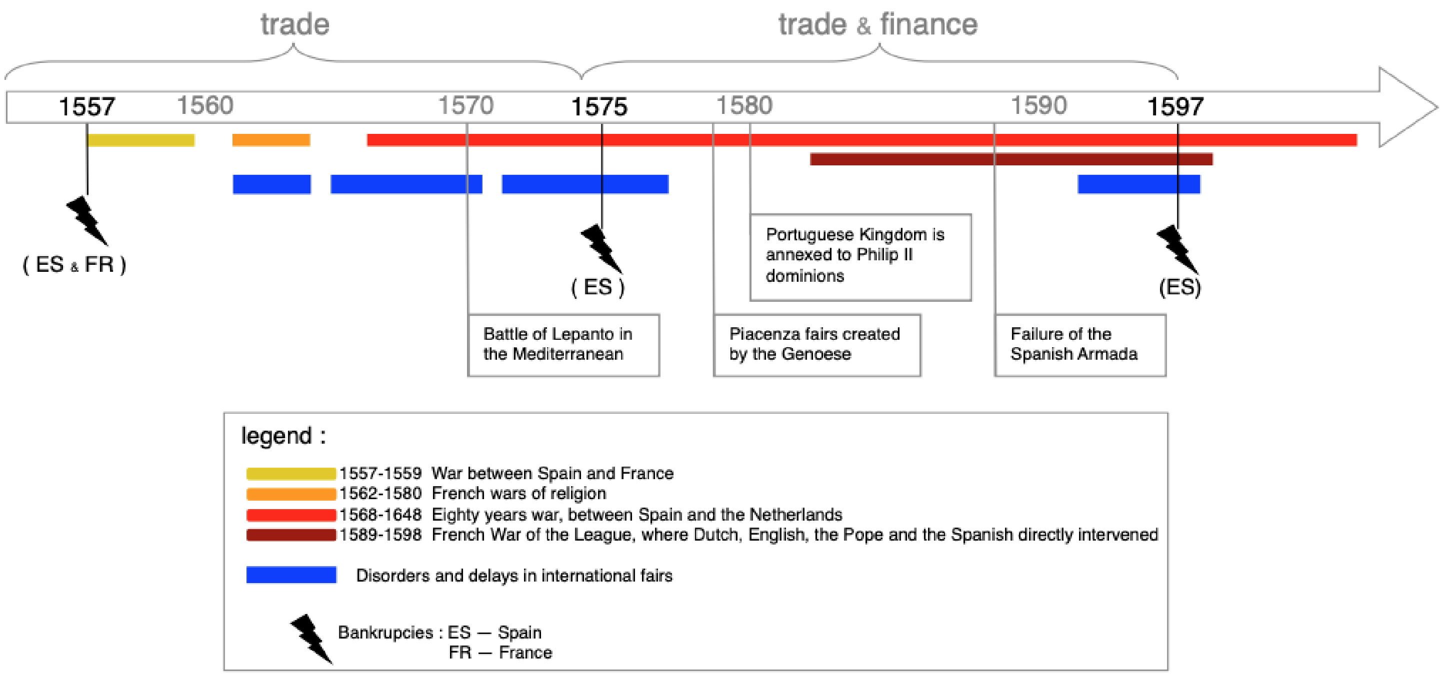

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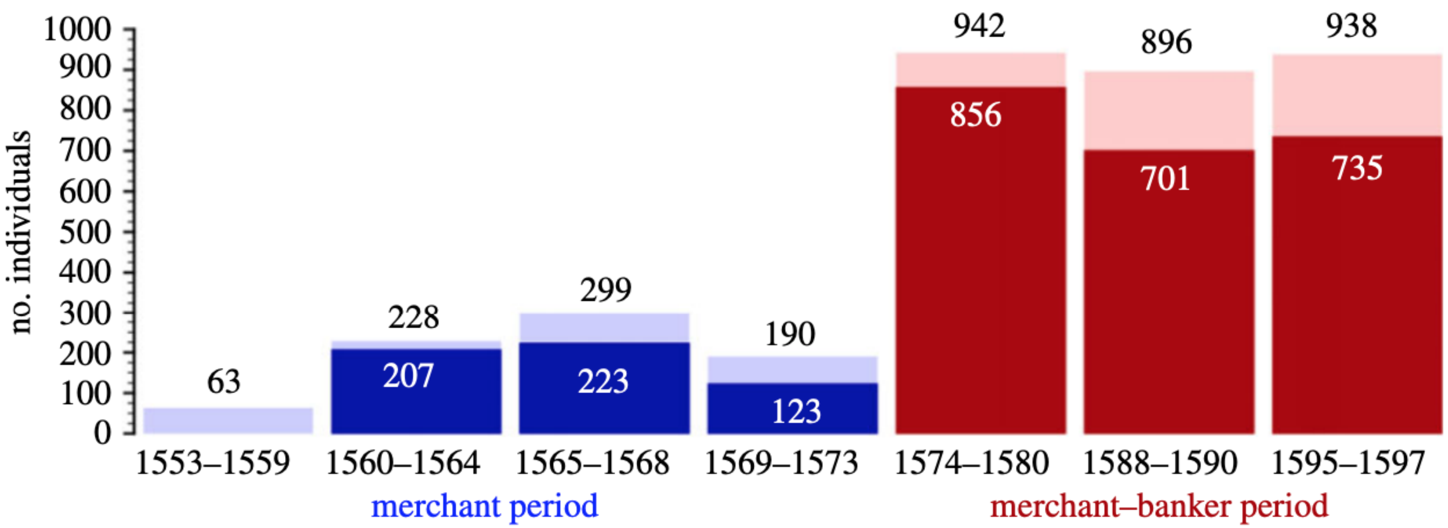

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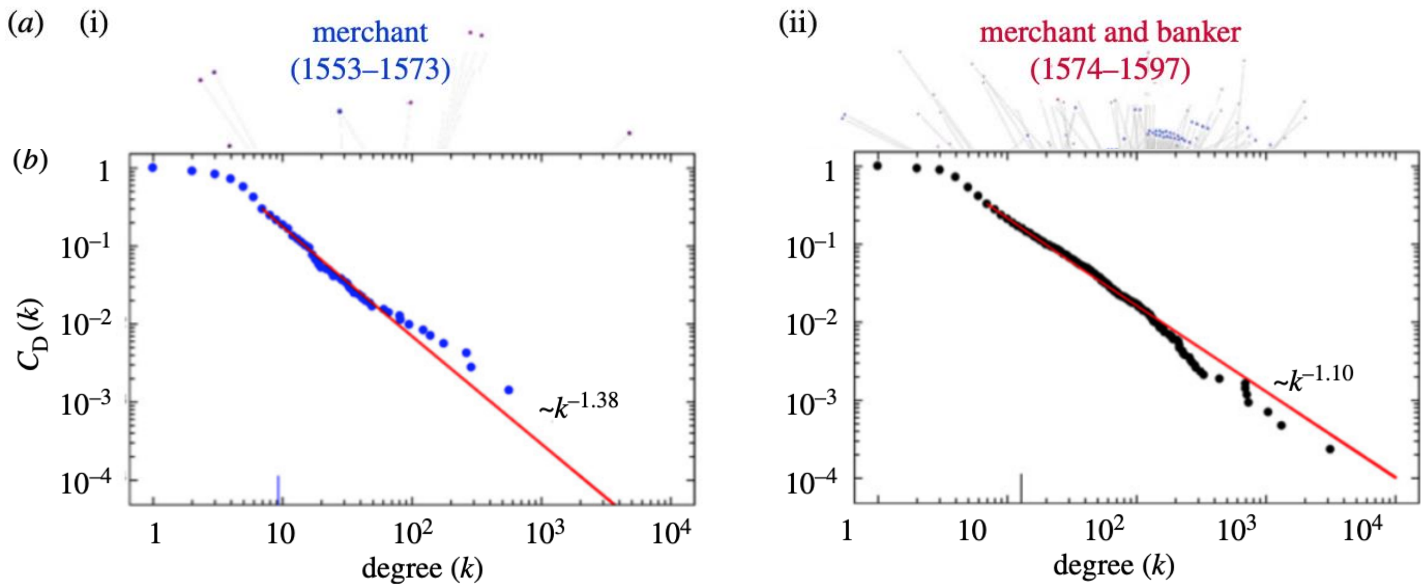

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Example #2

- Networks grow, but also shrink… Nodes and links can disappear.

Another example: NY fashion industry.

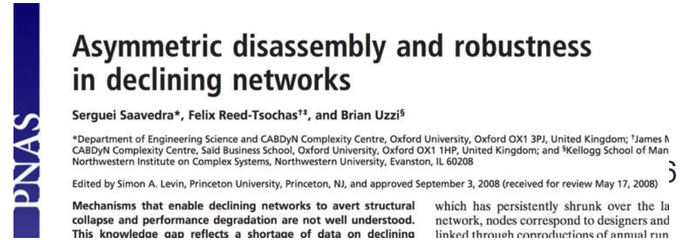
Nodes = designers and contractors

Links = annual co-production of lines of clothing

The industry has decayed persistently during the 90s:

$$N_{1985} = 3249 \text{ nodes}$$

$$N_{2003} = 190 \text{ nodes}$$



PNAS

Asymmetric disassembly and robustness in declining networks

Serguei Saavedra*, Felix Reed-Tsochas†‡, and Brian Uzzi§

*Department of Engineering Science and CABDyN Complexity Centre, Oxford University, Oxford OX1 3PJ, United Kingdom; †James M
CABDyN Complexity Centre, Saïd Business School, Oxford University, Oxford OX1 1HP, United Kingdom; and §Kellogg School of Man
Northwestern Institute on Complex Systems, Northwestern University, Evanston, IL 60208

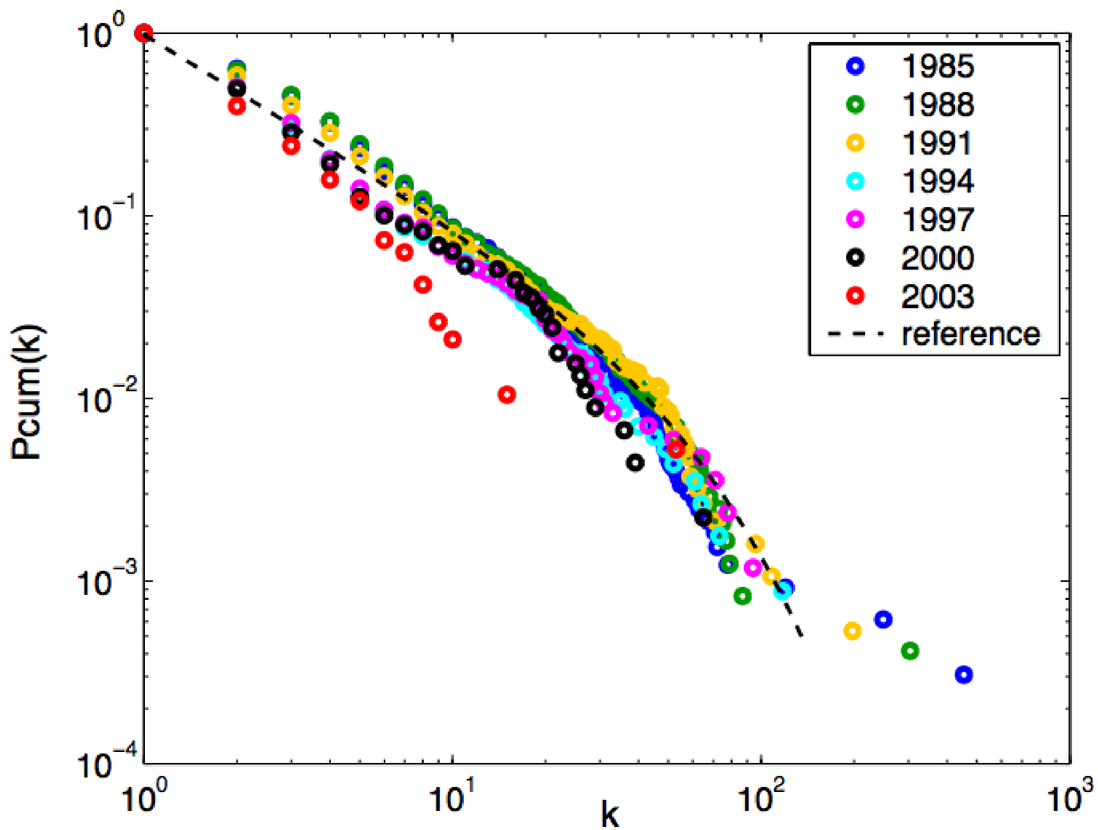
Edited by Simon A. Levin, Princeton University, Princeton, NJ, and approved September 3, 2008 (received for review May 17, 2008)

Mechanisms that enable declining networks to avert structural collapse and performance degradation are not well understood. This knowledge gap reflects a shortage of data on declining

which has persistently shrunk over the la
network, nodes correspond to designers and
linked through coproductions of annual run

Declining networks

Interestingly the network's degree dist. remained unchanged



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Let's create a model with death of nodes…

Example:

- With rate r_{death} we remove a node.
- With rate r_{birth} we add a node as in the BA model

For simplicity let's reduce the number of parameters such that

$$r = \frac{r_{death}}{r_{birth} = 1}$$

What would be the outcome of this model?

Let's create a model…

Results are not surprising:

r<1

the number of removed nodes is smaller than the number of new nodes, hence the network continues to grow. In this case the network is scale-free with degree exponent

$$\gamma = 3 + \frac{2r}{r-1}$$

r=1

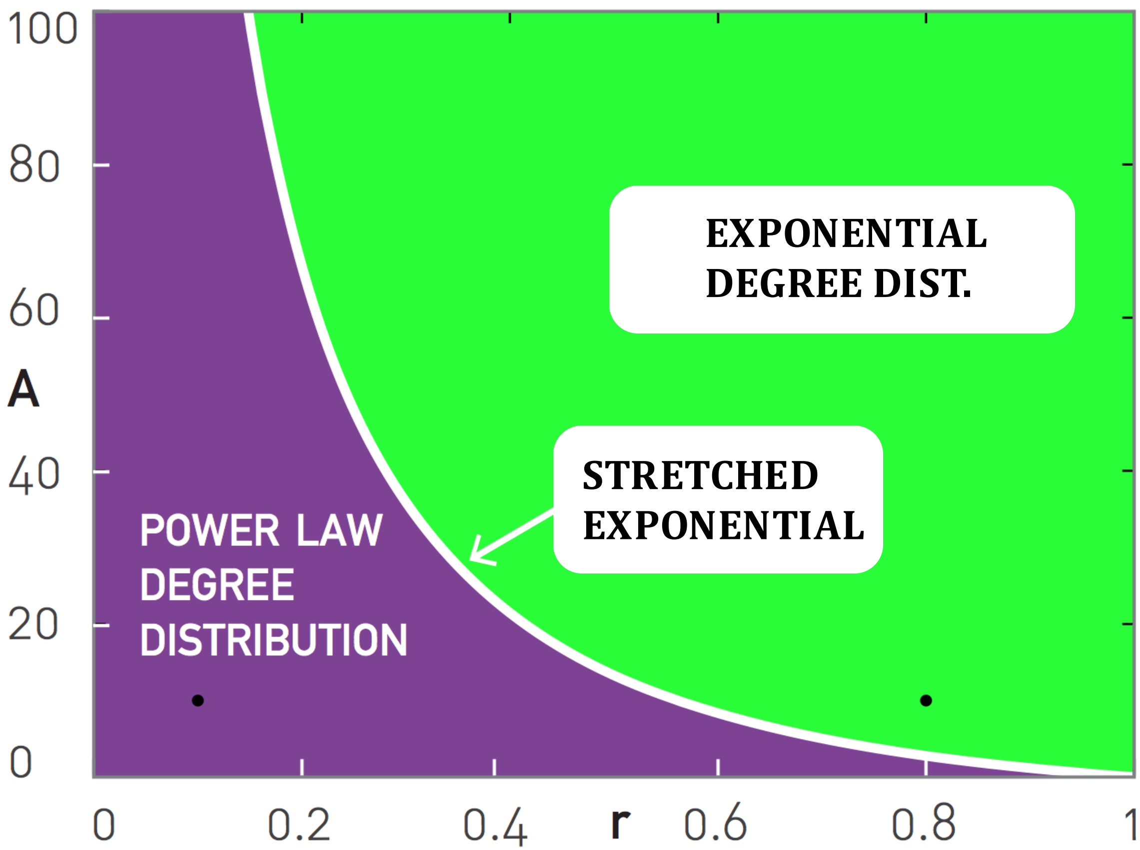
nodes arrive and are removed at the same rate, hence the network has a fixed size and would lose its scale-free nature.

r>1

number of removed nodes exceeds the number of new nodes, hence the network declines.

Let’s create a model…

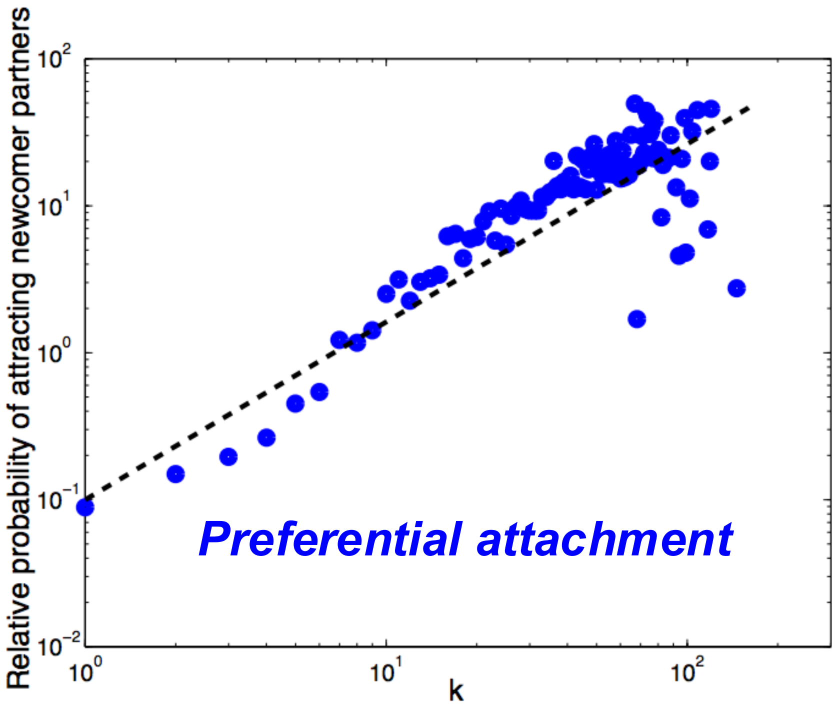
All gets counter-intuitive when combined with other real-world constraints (example: initial attractiveness, A):

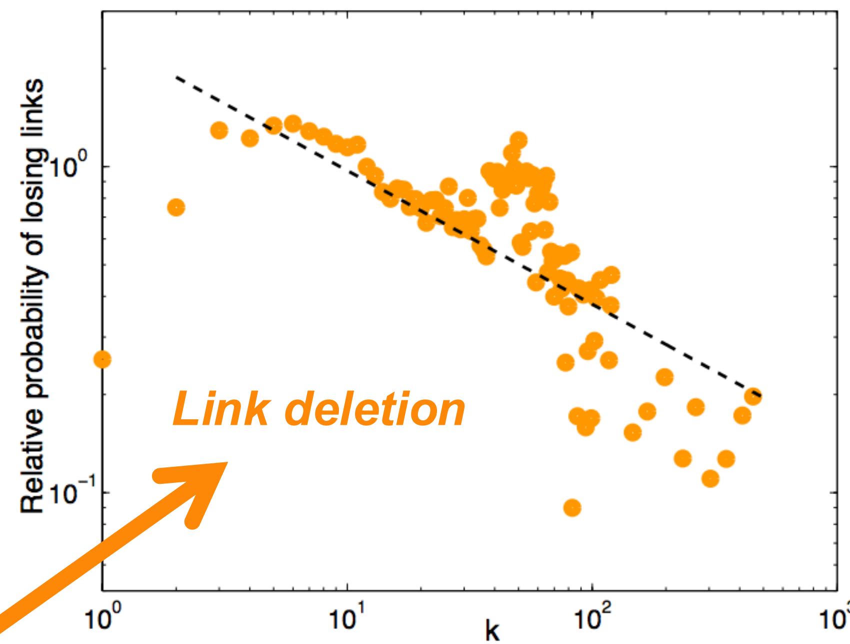


H. Bauke, C. Moore, J. Rouquier, and D. Sherrington. Topological phase transition in a network model with preferential attachment and node removal. The European Physical Journal B, 83: 519-524, 2011.

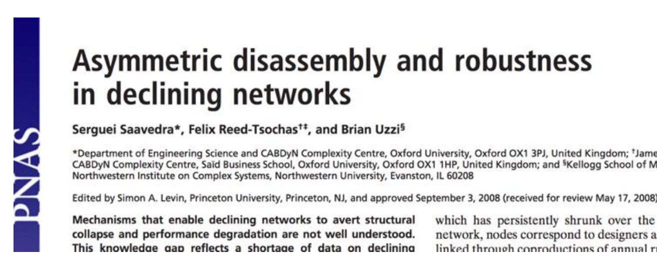
Open challenge

Now, let's return back to the fashion industry… Empirical results show





What would be the result of removing links (instead of nodes), taking into account the degrees of the nodes involved? Would you get the observed degree dists?


PNAS

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which has persistently shrunk over the la… network, nodes correspond to designers and… linked through coproductions of annual run…

Conclusions so far…

Models with Pref. Attachment

Barabási-Albert Model
Π~k ➔ γ=3

Non-linear Pref. Attachment

$$\Pi_i = \frac{k_i^\alpha}{\sum_j k_j^\alpha}$$

α<1 ➔ exponential dist
α=1 ➔ power-law
α>1 ➔ winners-take all

Growth with removal of nodes and/or links

Initial attractiveness
Π~A+k ➔ γ=3+A/m

n internal links

i) double pref. attachment

$$\gamma = 2 + \frac{m}{m+2n} < 3$$

ii) random attachment

$$\gamma = 3 + \frac{2n}{2m} > 3$$

Growth w/costs

$$k > k_{\max}$$

Fitness models

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

Ageing models

$$(t_i - t)^{-\nu}$$

Do we need degree-based preferential attachment to get to scale-free (or strongly heterogeneous) networks?

Can we get to power-laws following different principles?

Scale-free growth by ranking

Fortunato et al. PRL (2006)

- Growth + preferential attachment model in which attachment probability of a new node to an old vertex *s* given by the form

$$\Pi_s = \frac{R_s^{-\alpha}}{\sum_j R_j^{-\alpha}}$$

where R_s denotes the rank of the node *s* for some specific attribute and where α is a positive parameter.

Scale-free growth by ranking

Fortunato et al. PRL (2006)

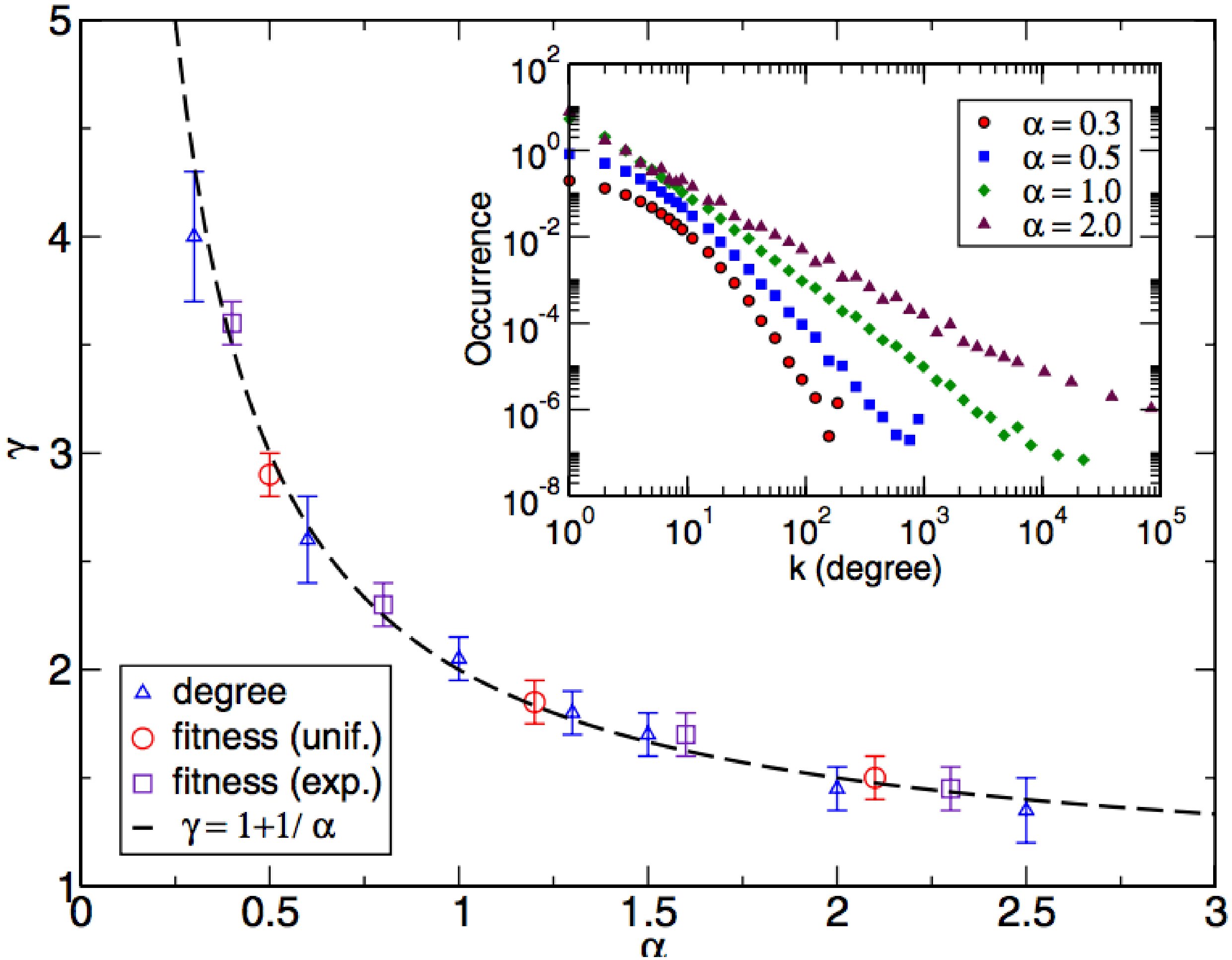
- *Example:* R_s denotes denotes the age-ranking of the nodes of a growing network.

$$P(k) \sim k^{-\left(1+\frac{1}{\alpha}\right)}$$

- Similar behavior will occur if, for instance, we consider the ranking in terms of the in-degree of a node (think for instance on the WWW).
- We get the same type of scaling if we assign a fitness value taken to each node from a given distribution. If we rank the nodes based on this, we get the same behavior.

Scale-free growth by ranking (simulations)

Fortunato et al. PRL (2006)



$$P(k) \sim k^{-\left(1+\frac{1}{\alpha}\right)}$$

Scale-free growth by ranking

Fortunato et al. PRL (2006)

- *Example: R_s* denotes denotes the age-ranking of the nodes of a growing network.

Challenge : Imagine a growth model Page Rank. Would you get to a power-law degree distribution?

- Similar behavior will occur if, for instance, we consider the ranking in terms of the in-degree of a node (think for instance on the WWW).
- A similar pattern will occur if we assign a fitness value taken to each node from a given distribution. If we rank the nodes based on this, we get the same behavior.

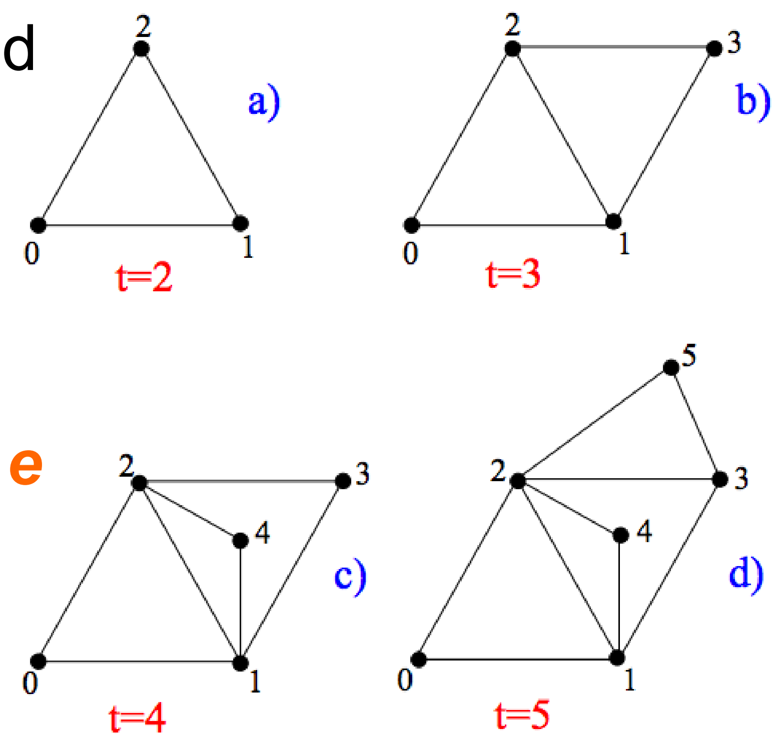
DMS minimal model or link-selection model

***D**orogovtsev, **M**endes, **S**amukhin Phys. Rev. E 63, 062101 (2001)*

- ***Growth***: At each time step we add a new node to the network.

- ***Link selection***: each new node selects a link ***e*** at random, and connects itself to the two ends of ***e***

Note: you may also assume that it connects to only one of the two ends of ***e***.

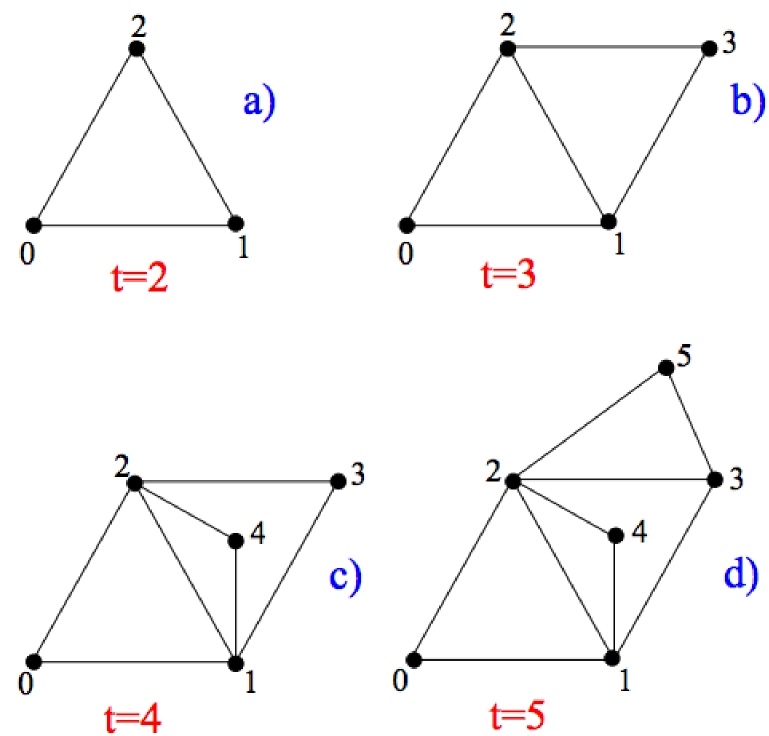


DMS minimal model: degree distribution

Dorogovtsev, *Mendes*, *Samukhin* Phys. Rev. E 63, 062101 (2001)

What's the probability that a node i, of degree k_i, gets a link from the new node?

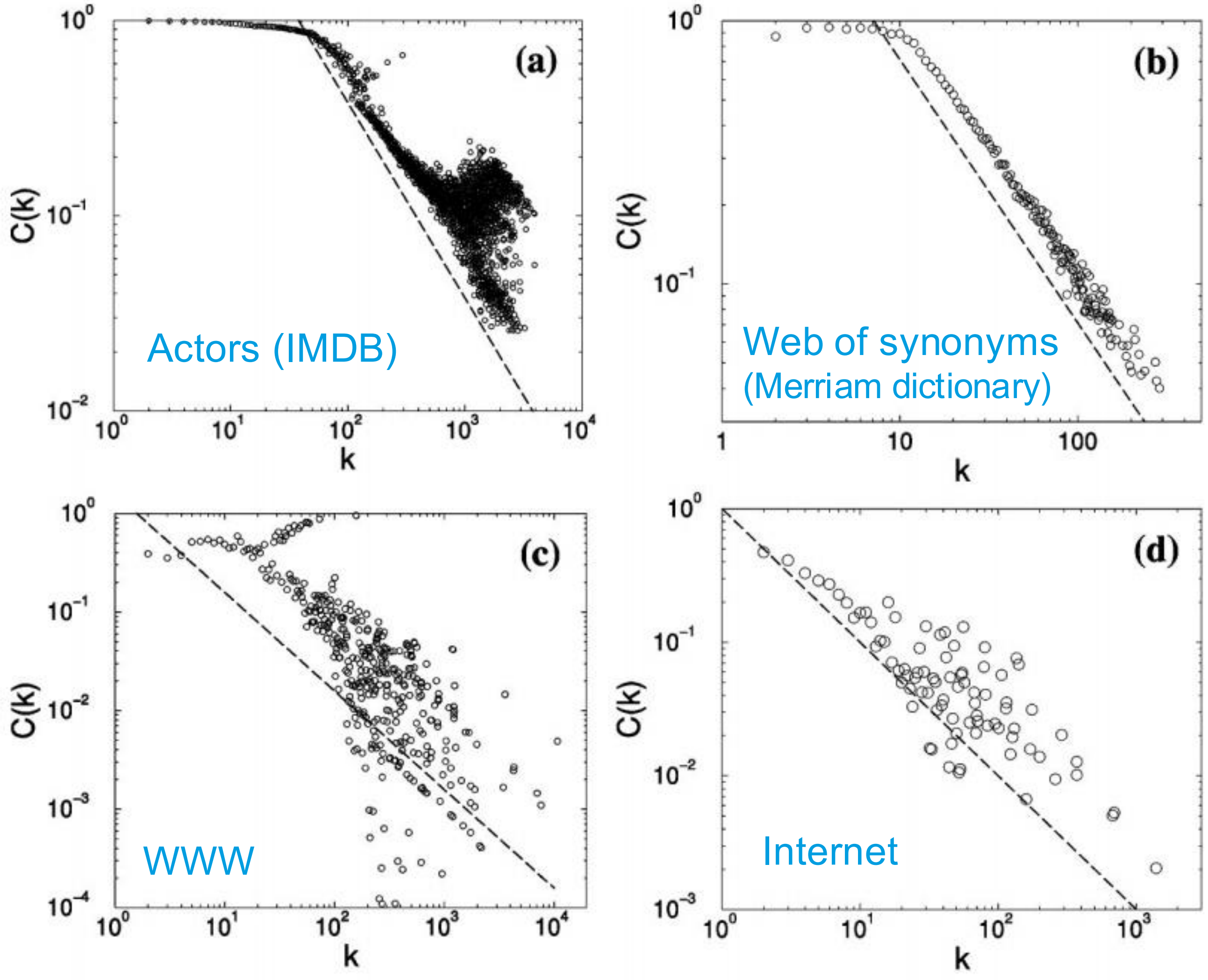
$$\Pi_i = \frac{k_i}{E} = \frac{k_i}{2t-1} \sim \frac{k_i}{\sum_j k_j}$$



i.e., linear preferential attachment, as the Barabási-Albert model ☺

$$P(k) \sim k^{-3}$$

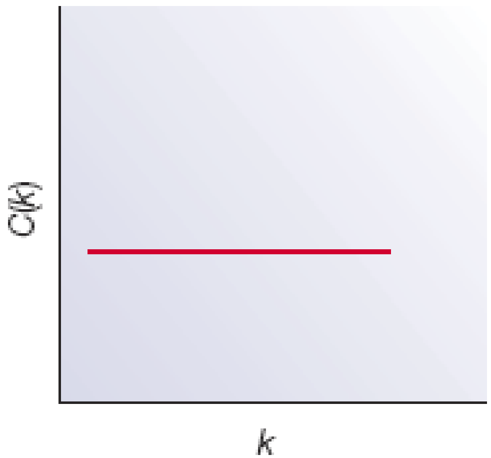
Scaling of clustering coefficient



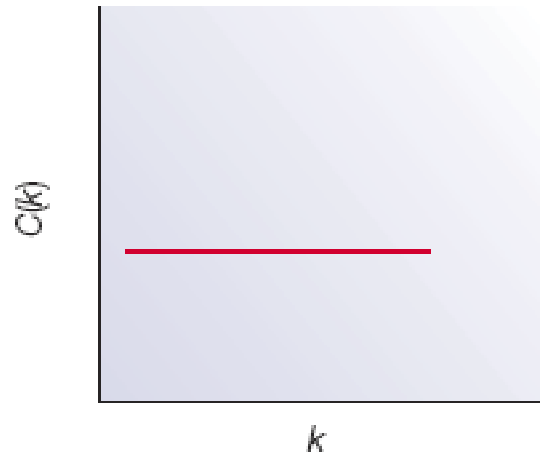
$$C(k) \sim k^{-\beta}$$

DMS Minimal model: Clustering coeff. distribution

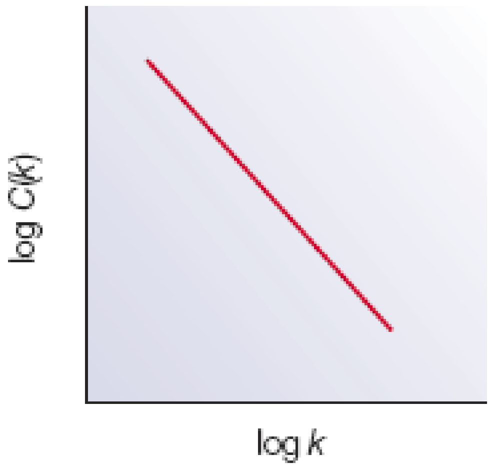
Random Networks

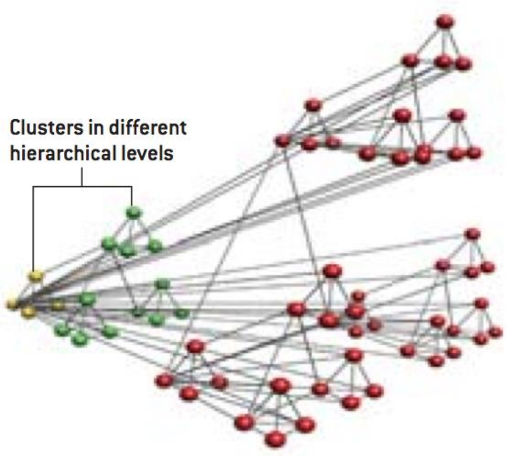


Barabási-Albert model

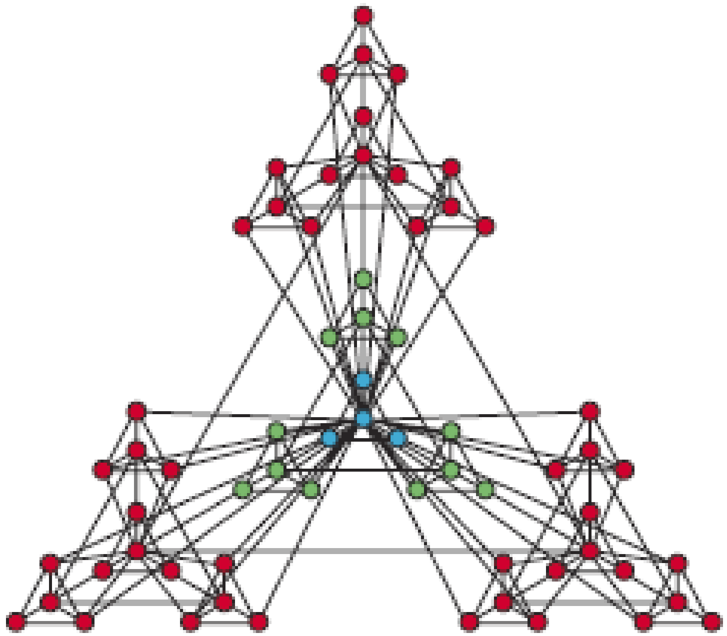


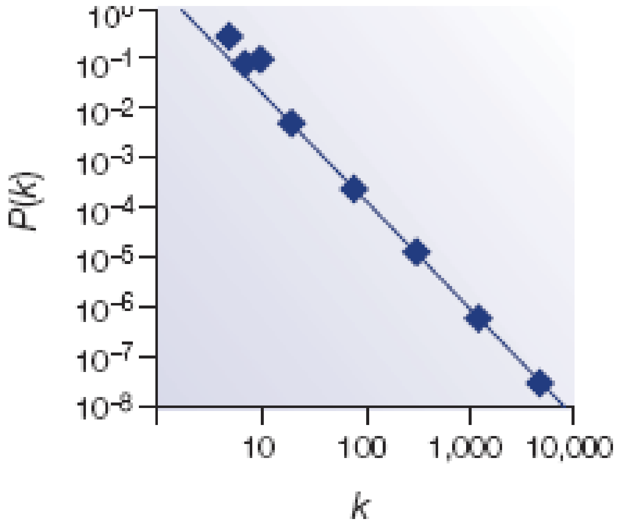
DMS Minimal model

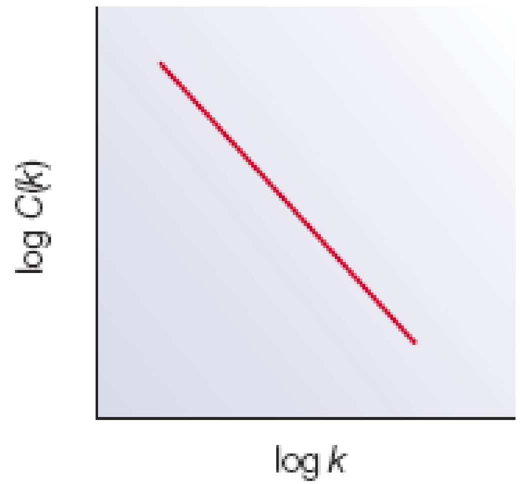




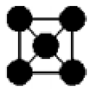
Hierarchical networks



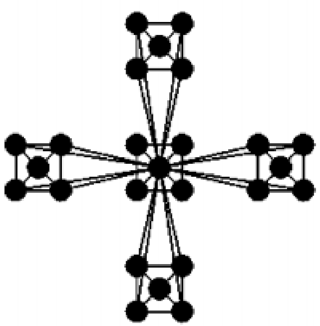




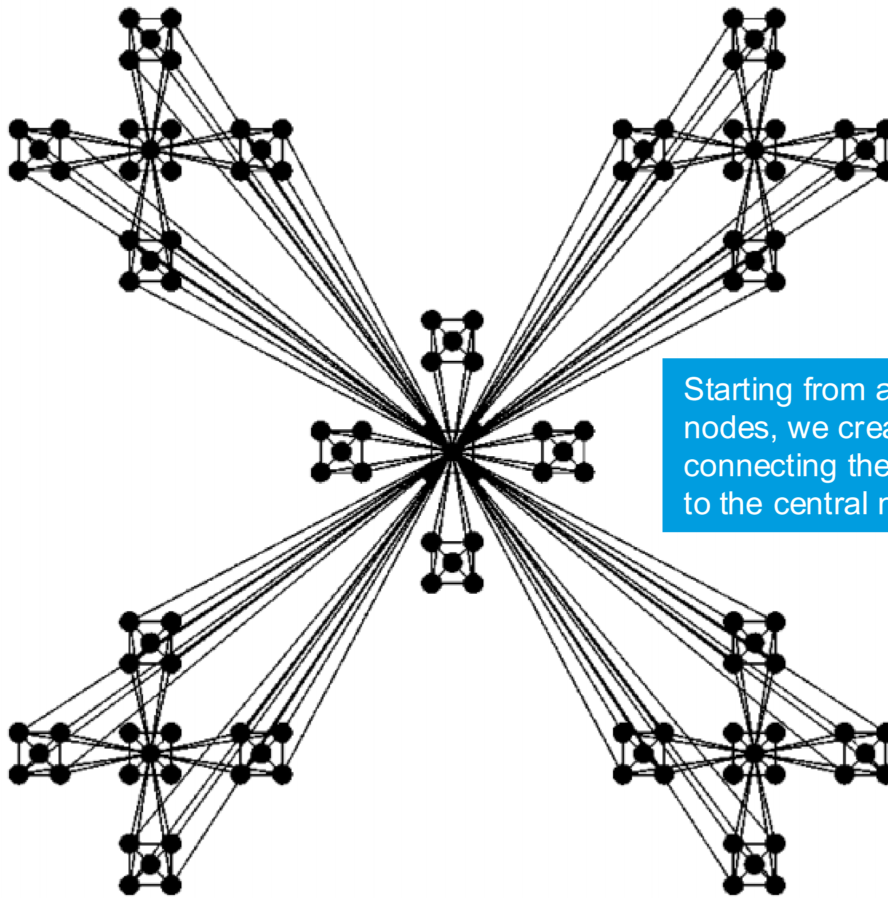
Deterministic Models of Hierarchical Networks



(a) n=0, N=5



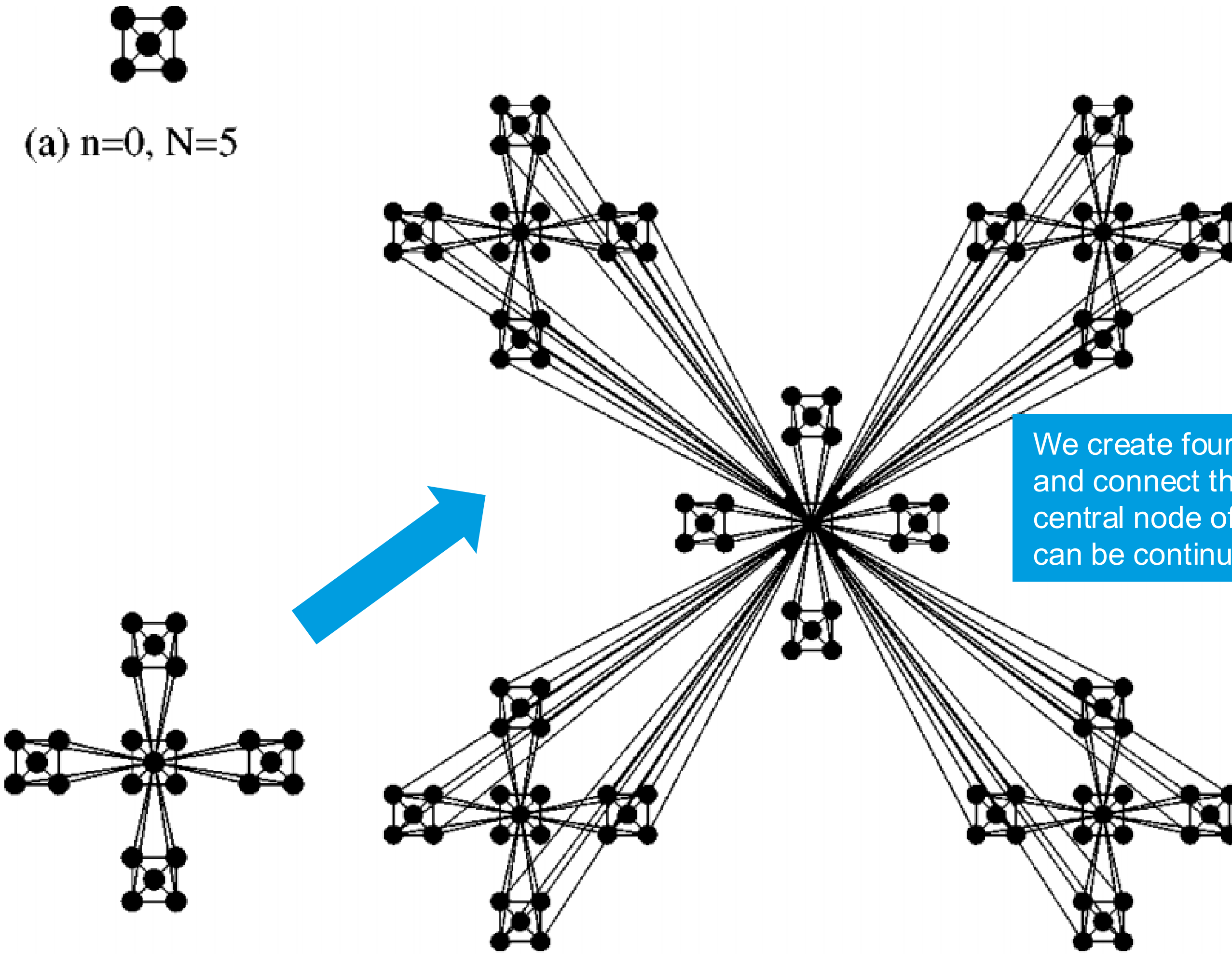
(b) n=1,
N=25



(c) n=2, N=125

Starting from a fully connected cluster of five nodes, we create four identical replicas, connecting the peripheral nodes of each cluster to the central node of the original cluster.

Deterministic Models of Hierarchical Networks



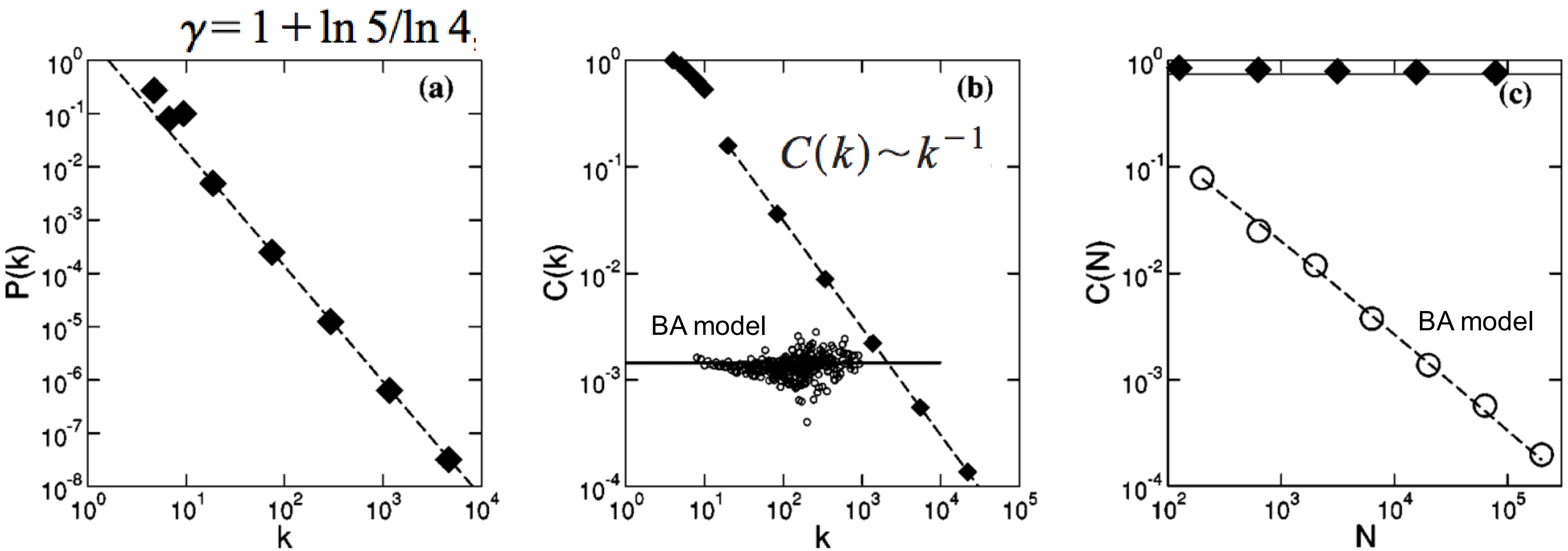
(a) n=0, N=5

(b) n=1, N=25

(c) n=2, N=125

We create four replicas of the obtained cluster, and connect the peripheral nodes again, to the central node of the original module. This process can be continued indefinitely

Hierarchical networks



Ravasz et al.,
Phys Rev E 67,
026112 (2003)

Duplication/copying models

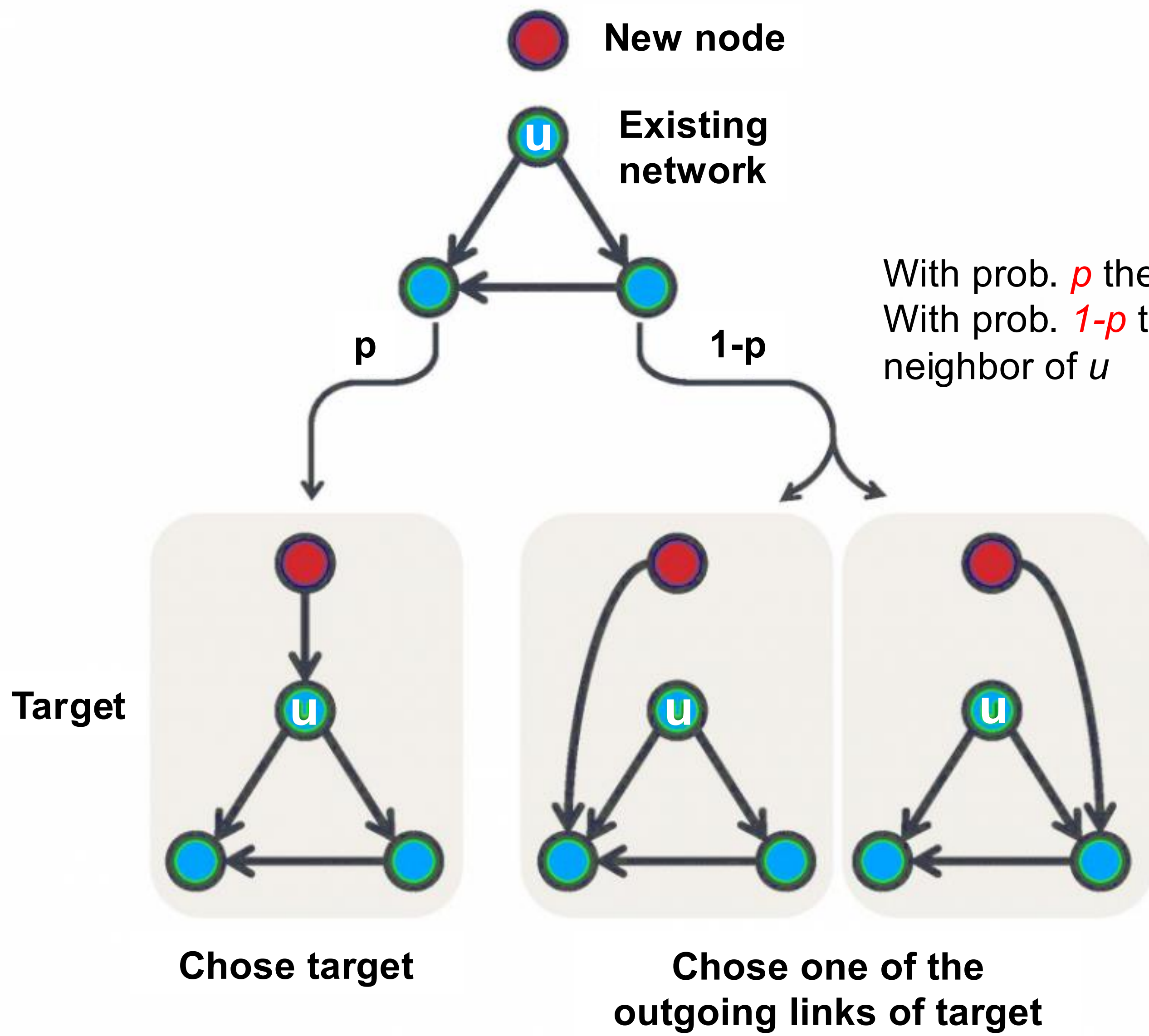
Wagner, Kleinberg, & others

Analytic results for copying models have been given by Chung et al.

Examples

- The authors of a new webpage tend to borrow links from other webpages on related topics.
- Similar arguments may be used for social networks.
- Genes that code for proteins duplicate. Since the proteins coded for by each copy are the same, their interactions are also the same, i.e., the new gene copies its edges in the interaction network from the old.
- Similar arguments have been used for Metabolic networks and other.
- Etc.

Copying model (simplest version)

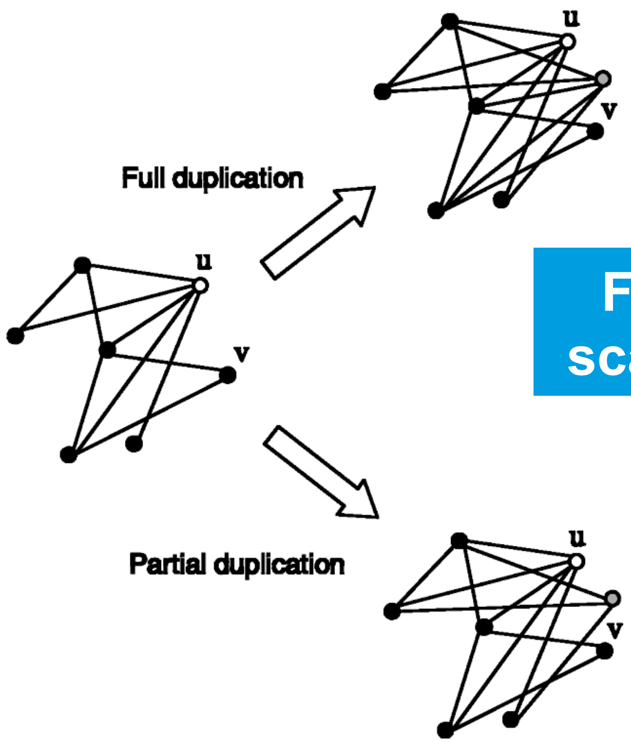


With prob. *p* the new node links to *u*
With prob. *1-p* the new node links to a neighbor of *u*

Other version: Partial duplication model

(Vazquez, Flammini, Maritan and Vespignani, 2003)

- At every time-step a randomly chosen vertex is duplicated at random creating a new node *s*.
- Each of *s* links is either kept with probability ***1−α*** or it is rewired (or removed) with probability ***α*** (equivalent to a mutation).



For low “mutation” probabilities (α), scale-free behavior ($<k^2> \rightarrow$) emerges.

Conclusion: Understanding topological variety

Models with Pref. Attachment

Non-linear Pref. Attachment

$$\Pi_i = \frac{k_i^\alpha}{\sum_j k_j^\alpha}$$

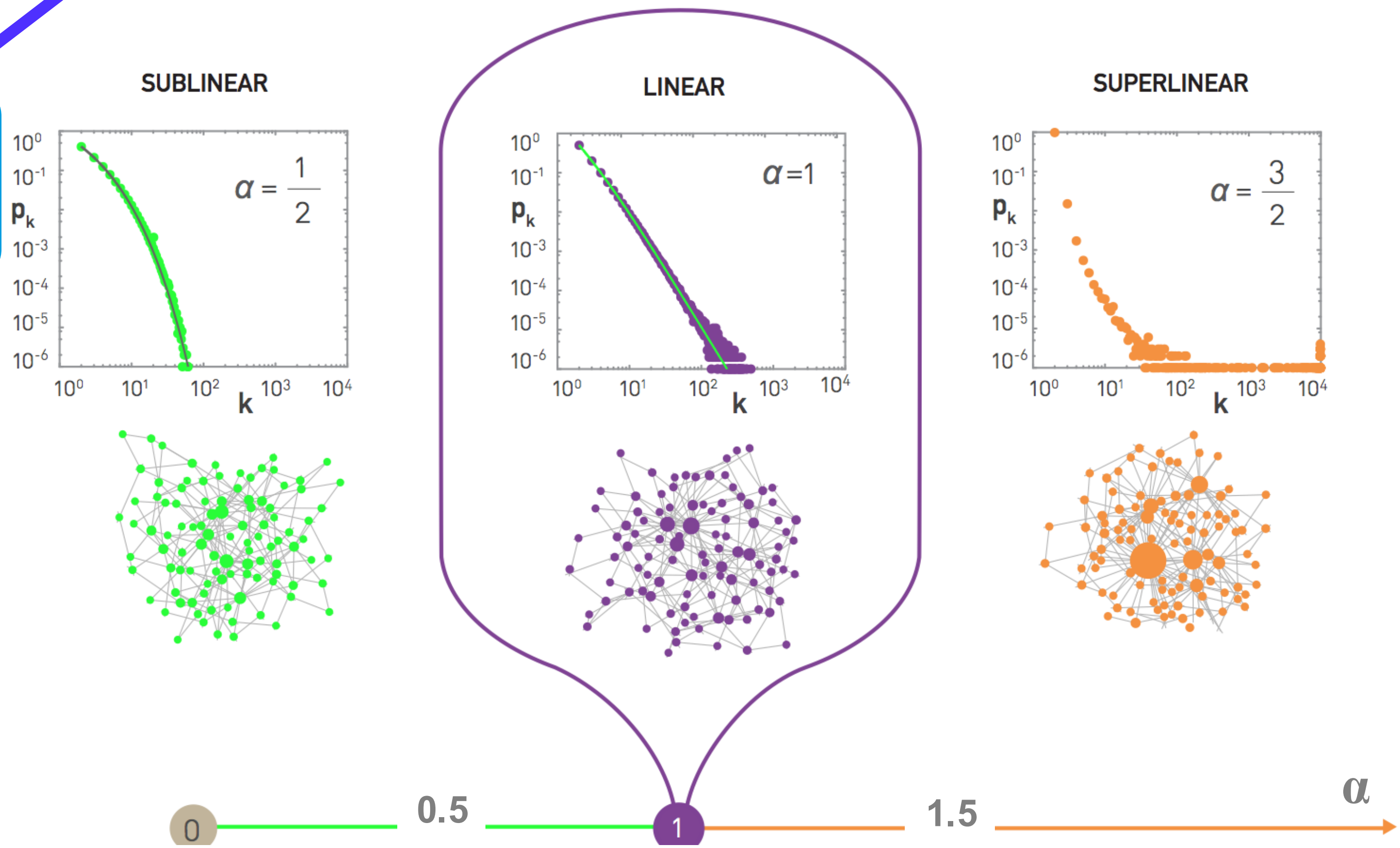
α<1 ➔ exponential dist

α=1 ➔ power-law

α>1 ➔ winners-take all

Barabási-Albert Model

Π~k ➔ γ=3



Conclusion: Understanding topological variety

Models with Pref. Attachment

Barabási-Albert Model

$\Pi \sim k \rightarrow \gamma = 3$

Non-linear Pref. Attachment

$$\Pi_i = \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

$\alpha < 1 \rightarrow$ exponential dist

$\alpha = 1 \rightarrow$ power-law

$\alpha > 1 \rightarrow$ winners-take all

Initial attractiveness

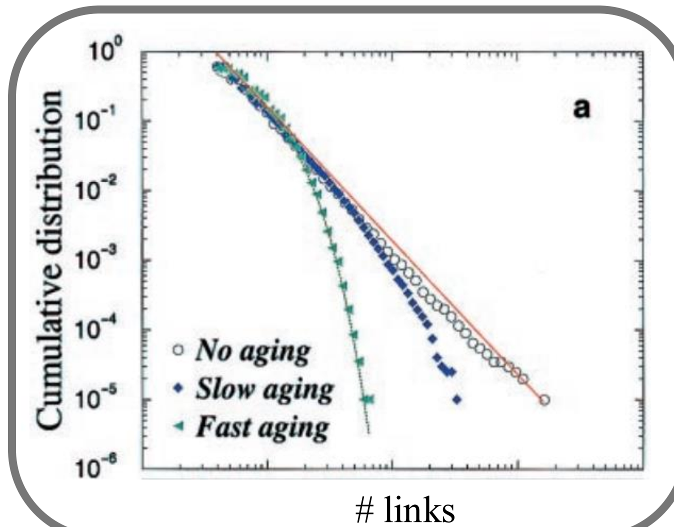
$\Pi \sim A + k \rightarrow \gamma = 3 + A/m$

Fitness models

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

Ageing models

$$(t_i - t)^{-\nu}$$



Conclusion: Understanding topological variety

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Barabási-Albert Model
$\Pi \sim k \Rightarrow \gamma=3$

Non-linear Pref. Attachment

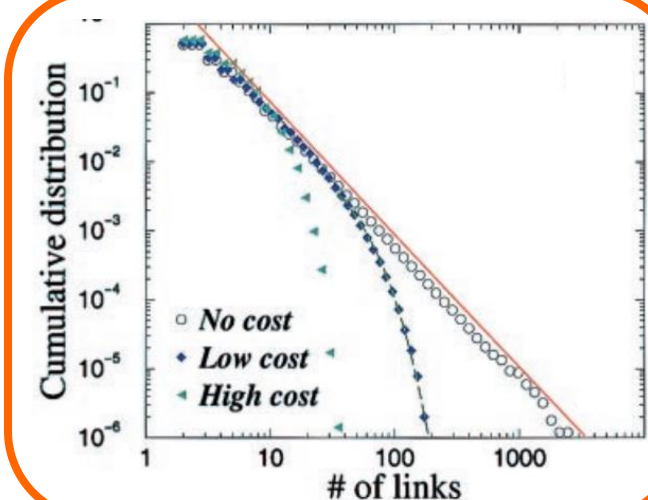
$$\Pi_i = \frac{k_i^\alpha}{\sum_j k_j^\alpha}$$

$\alpha<1 \Rightarrow$ exponential dist
$\alpha=1 \Rightarrow$ power-law
$\alpha>1 \Rightarrow$ winners-take all

Initial attractiveness
$\Pi \sim A+k \Rightarrow \gamma=3+A/m$

Growth w/costs

$$k > k_{\max}$$



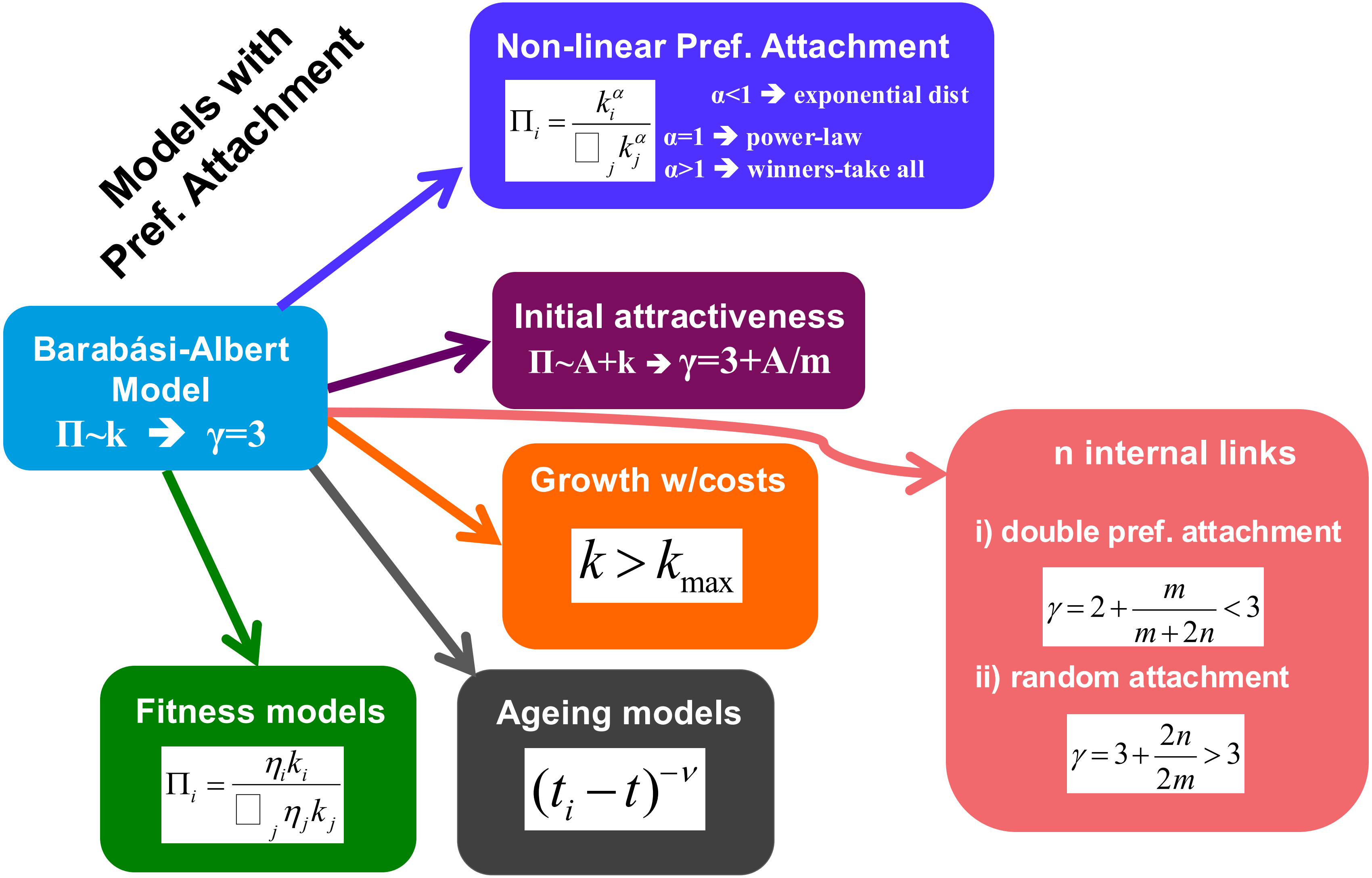
Fitness models

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

Ageing models

$$(t_i - t)^{-\nu}$$

Conclusion: Understanding topological variety



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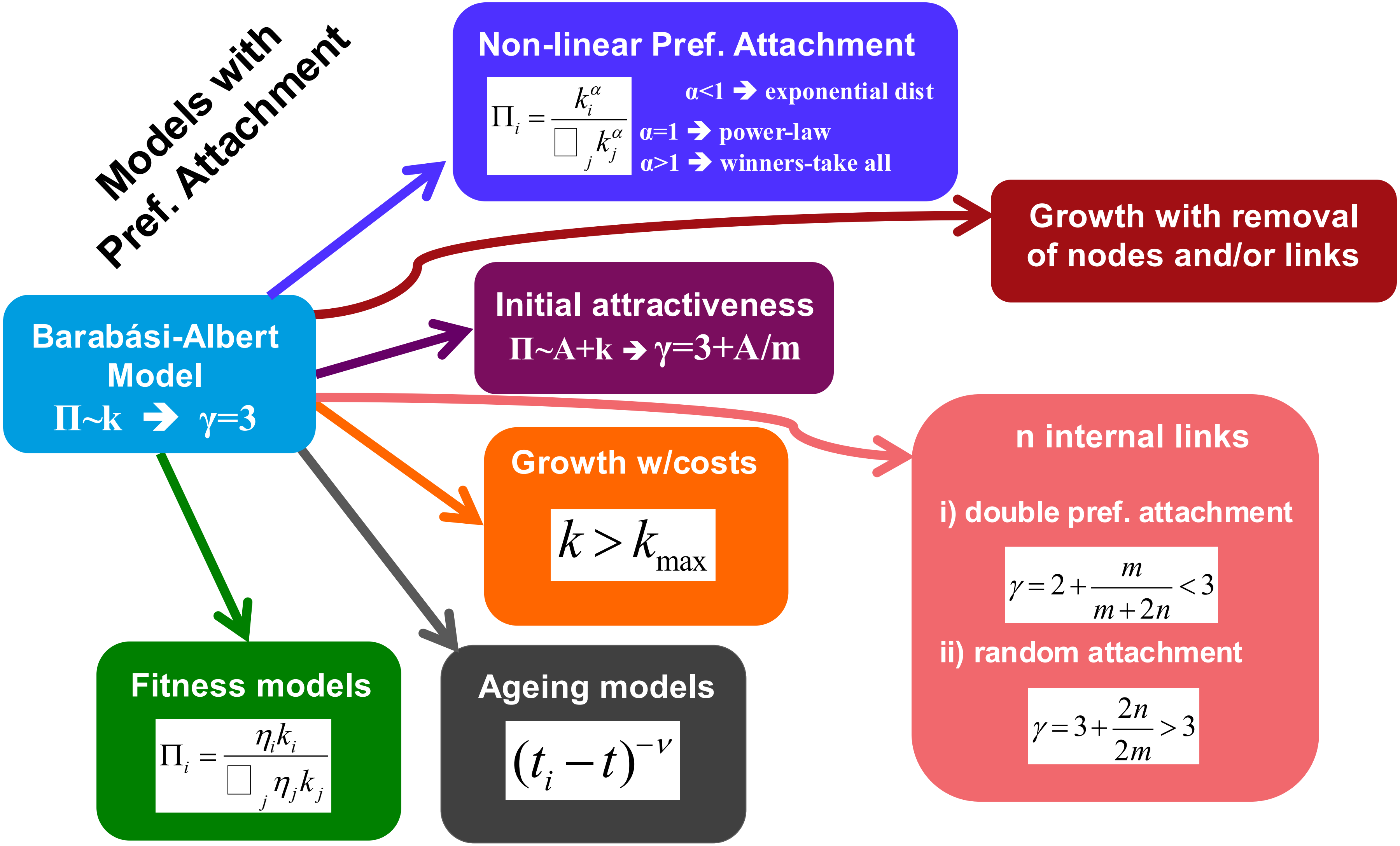
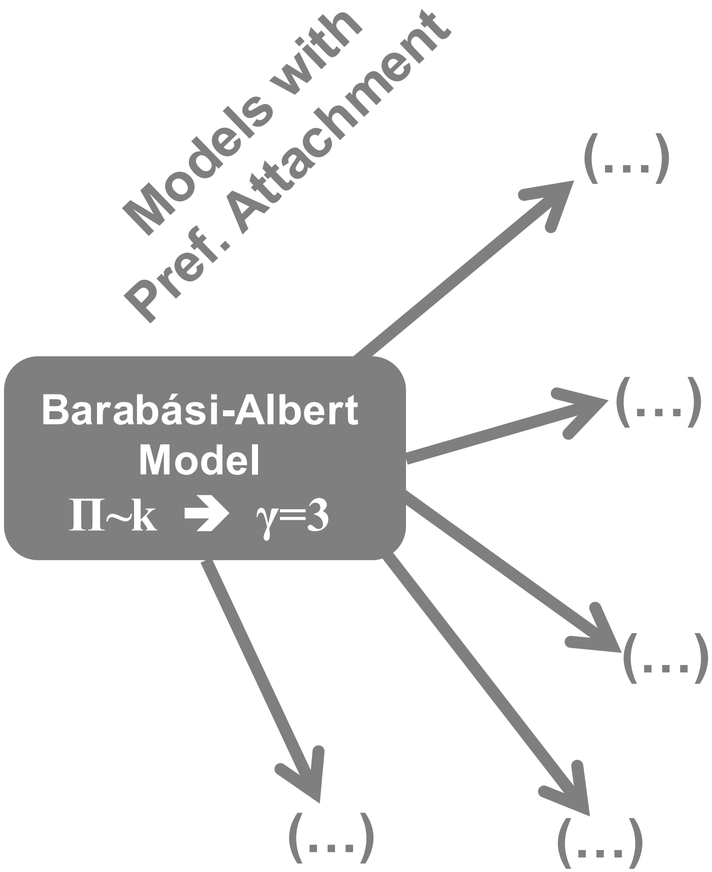


Conclusion: Understanding topological variety

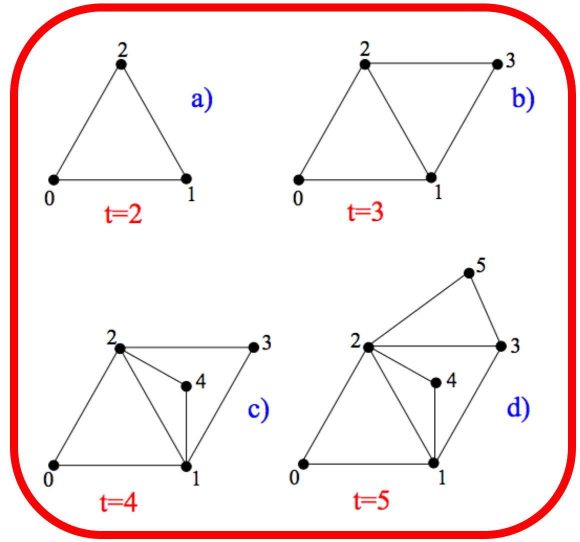


Models without Pref. Attachment?

Conclusion: Understanding topological variety

**Models without
Pref. Attachment**

Minimal or link-selection model
$\Pi \sim k$ ➔ $\gamma=3$



Conclusion: Understanding topological variety

Models with
Pref. Attachment

Barabási-Albert Model
$\Pi \sim k \rightarrow \gamma = 3$

(…)

(…)

(…)

Models without Pref. Attachment

Ranking models

$$\Pi_s = \frac{R_s^{-\alpha}}{\sum_j R_j^{-\alpha}}$$

$$P(k) \sim k^{-\left(1+\frac{1}{\alpha}\right)}$$

Minimal or link-selection model
$\Pi \sim k \rightarrow \gamma = 3$

Duplication models

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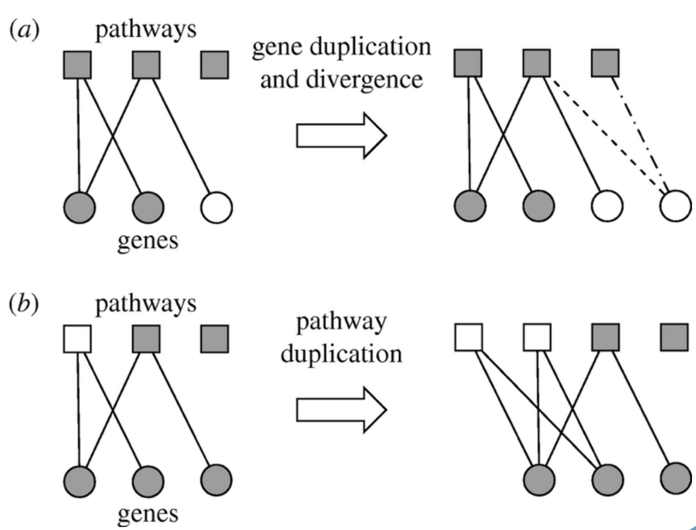

Conclusion: Understanding topological variety

Models with Pref. Attachment

Barabási-Albert Model
$\Pi \sim k$ ➔ $\gamma=3$

(…)

(…)

(…)

(…)

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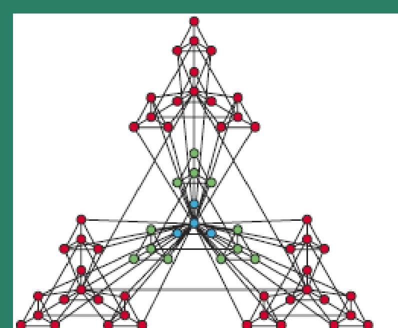
Hierarchical nets



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Conclusion: Understanding topological variety

Power-laws: BA-model, minimal model, etc, etc.

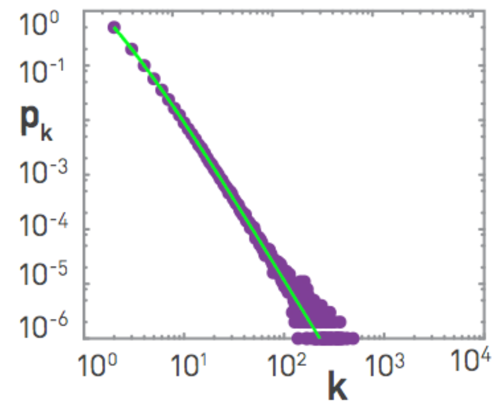
Stretched exponentials: If preferential attachment is sublinear you get a stretched exponential.

Fitness-induced corrections: Ranking models, Fitness models, initial attractiveness model.

Small-degree saturations: Initial attractiveness adds a random component to preferential attachment, particularly for low degrees.

High degree cutoffs: Node and link removal, costs and cutoffs, and node ageing, can induce high-degree cutoffs.

Hierarchical structure & power-law dep. in clustering: Minimal/link-selection model, duplication models and hierarchical networks model.



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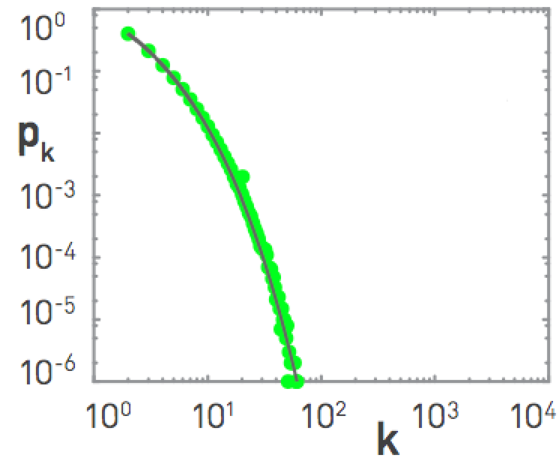

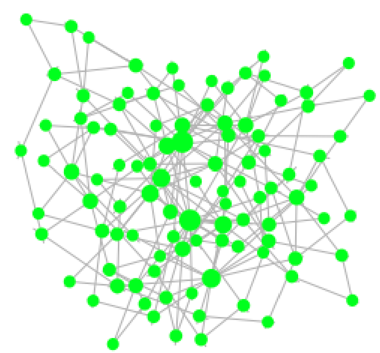

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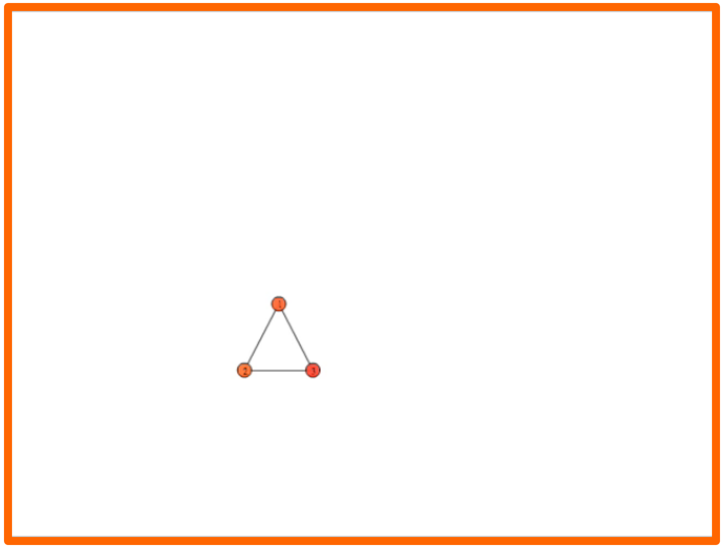

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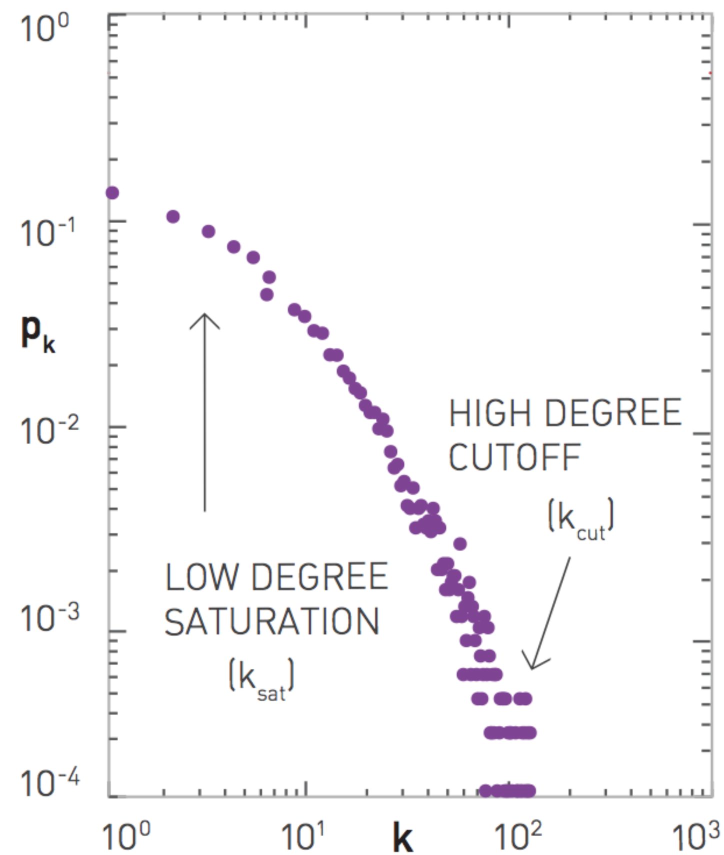

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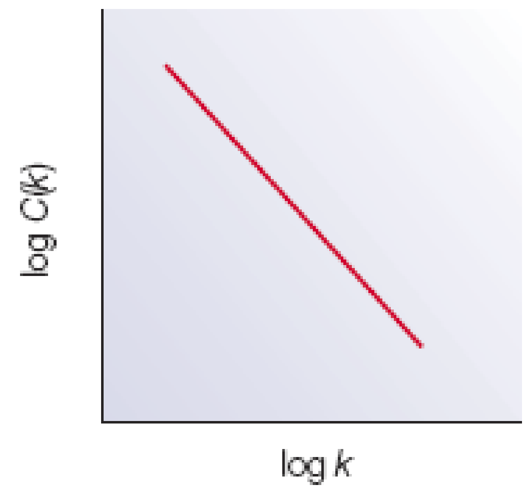

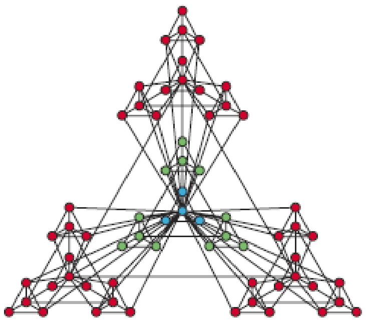

Classes of models in network science

- ***Static & generative models.*** ER model, Watts-Strogratz model, Configuration Model, etc.

- ***Evolving network models.*** BA model, Initial attractiveness model, fitness model, internal links model, node deletion model, accelerated model, aging model, costs model, minimal model, ranking model, duplication model, hierarchical networks model, etc.

Which models or principles should I consider to justify each case:

1) Network with a power-law degree distribution with $\gamma=3$.
2) Network with a power-law degree distribution with $\gamma=3$ with a significant exponential cutoff for large $k \sim k_{max}$.
3) Network with a power-law degree distribution with $\gamma=3$ and large clustering coefficient.
4) Power-law degree distribution with $\gamma<3.0$.
5) Power-law degree distribution with $\gamma>3.0$.
6) A network with exponential degree distribution.
7) A power-law degree distribution with saturation for low k.

Solutions:

1) BA model
2) BA model with costs or cutoffs
3) DMS minimal model
4) BA model with internal links chosen through preferential attachment
5) BA model with internal links chosen randomly (preferential attachment with initial attractiveness will also work)

Next step? Beyond degree distributions

Assortativity in complex networks