

Randomized Algorithms Exercises

November 18, 2025

1 Monday

$$P\left(\bigcap_{i=1}^t A_i\right) = \prod_{i=1}^t P[A_i] \quad (1)$$

$$\forall J \subset [t], P\left[\bigcap_{j \in J} A_j\right] = \prod_{j \in J} P[A_j] \quad (2)$$

$$\forall J \subset [t], j \in [t] \setminus J, P[A_j \mid \bigcap_{i \in J} A_i] = P[A_j] \quad (3)$$

Question 1. Find an example of a set of events \mathcal{E} and a corresponding probability space such that Condition 1 holds but Condition 2 does not.

Question 2. Show that Conditions 2 and 3 are equivalent.

Question 3. For a graph G with $|E(G)| \geq 1$, show that there is a bipartite subgraph $H \subset G$ with strictly more than $|E(G)|/2$ edges.

Question 4. For every graph G with n vertices and minimum degree $\delta \geq 1$, show that there is a dominating set of size at most $n/2$.

2 Tuesday

Question 5 (Turán's Theorem). Let G be a graph with n vertices and no clique of size $r + 1$. Prove that it has at most $\left(\frac{r-1}{r}\right) \binom{n^2}{2}$ edges.

Question 6. Let G be a graph with n vertices and m edges and minimum degree at least 1. Show that there exists a bipartite subgraph of G with at least $m/2 + n/6$ edges.

Algorithm 1 Independent Set

- 1: $G_0 := G, i := 0, X_i := 0$
 - 2: Repeat until G_i empty:
 - 3: Pick a vertex $v \in V(G_i)$ with the smallest degree in G_i
 - 4: $X_{i+1} := X_i \cup v_i$
 - 5: $G_{i+1} := G_i - v_i - \text{neighbors of } v_i$
 - 6: $i := i + 1$
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Question 7. Show that Algorithm 1 returns an independent set of size at least $\sum_{v \in V(G)} \frac{1}{1+\deg(v)}$.

Question 8. Show that any k -uniform set system \mathcal{F} with less than 2^{k-1} sets is 2-colorable.

Question 9. Find a deterministic algorithm that determines a valid 2-coloring of the \mathcal{F} from Question 8 in polynomial time.