

# Randomized Algorithms Exercises

November 18, 2025

## 1 Monday

$$P\left(\bigcap_{i=1}^t A_i\right) = \prod_{i=1}^t P[A_i] \quad (1)$$

$$\forall J \subset [t], P\left[\bigcap_{j \in J} A_j\right] = \prod_{j \in J} P[A_j] \quad (2)$$

$$\forall J \subset [t], j \in [t] \setminus J, P[A_j \mid \bigcap_{i \in J} A_i] = P[A_j] \quad (3)$$

**Question 1.** Find an example of a set of events  $\mathcal{E}$  and a corresponding probability space such that Condition 1 holds but Condition 2 does not.

**Question 2.** Show that Conditions 2 and 3 are equivalent.

**Question 3.** For a graph  $G$  with  $|E(G)| \geq 1$ , show that there is a bipartite subgraph  $H \subset G$  with strictly more than  $|E(G)|/2$  edges.

**Question 4.** For every graph  $G$  with  $n$  vertices and minimum degree  $\delta \geq 1$ , show that there is a dominating set of size at most  $n/2$ .

## 2 Tuesday

**Question 5** (Turán's Theorem). Let  $G$  be a graph with  $n$  vertices and no clique of size  $r + 1$ . Prove that it has at most  $\binom{r-1}{r} \binom{n^2}{2}$  edges.

**Question 6.** Let  $G$  be a graph with  $n$  vertices and  $m$  edges and minimum degree at least 1. Show that there exists a bipartite subgraph of  $G$  with at least  $m/2 + n/6$  edges.

---

**Algorithm 1** Independent Set

---

- 1:  $G_0 := G, i := 0, X_i := \emptyset$
  - 2: Repeat until  $G_i$  empty:
    - 3: Pick a vertex  $v \in V(G_i)$  with the smallest degree in  $G_i$
    - 4:  $X_{i+1} := X_i \cup v$
    - 5:  $G_{i+1} := G_i - v - \text{neighbors of } v$
    - 6:  $i := i + 1$
- 

**Question 7.** Show that Algorithm 1 returns an independent set of size at least  $\sum_{v \in V(G)} \frac{1}{1 + \deg(v)}$ .

**Question 8.** Show that any  $k$ -uniform set system  $\mathcal{F}$  with less than  $2^{k-1}$  sets is 2-colorable.

**Question 9.** Find a deterministic algorithm that determines a valid 2-coloring of the  $\mathcal{F}$  from Question 8 in polynomial time.