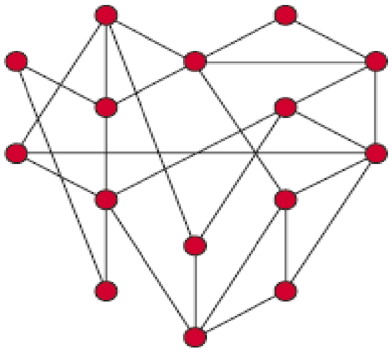
DEI
DEPARTAMENTO
DE ENGENHARIA INFORMÁTICA
TÉCNICO LISBOA

Random graphs

The null model of network science



Network Science, 2025/2026

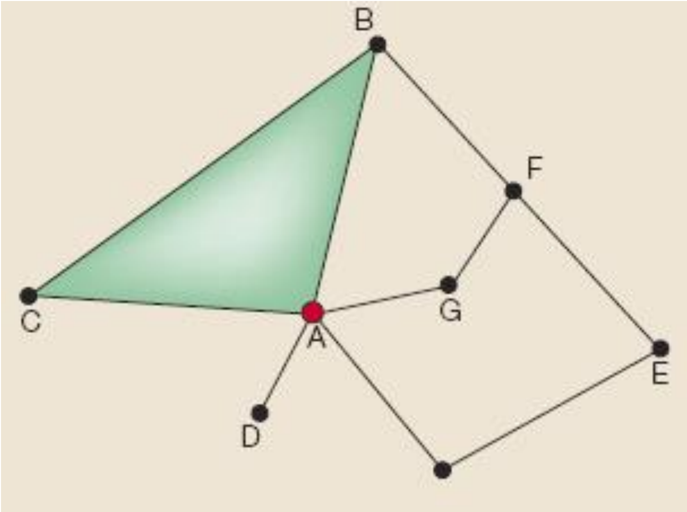
Practical class

Pick from the following options:

1. Try the exercises listed @ Problem set 1. You may take a look at the algorithms component (ex. 1) and check if you got the main ideas.
2. Try out Gephi (ex. 3 @ Problem Set 1) *OR*...
3. Start exploring a (more serious) network package associated with your favorite language. This may take a bit. Start as soon as possible (ex. 4 @ Problem Set 1).
4. Try to choose your Project and use it as test bed for (3). Let us know if you need any help.

Last class: characterization of networks

Average distances, diameter, shortest paths, transitivity, clustering, etc, etc.



Among those, We discussed 3:

- Degree distribution, *P(k)*
- Average path length, <L>
- Clustering Coefficient, C

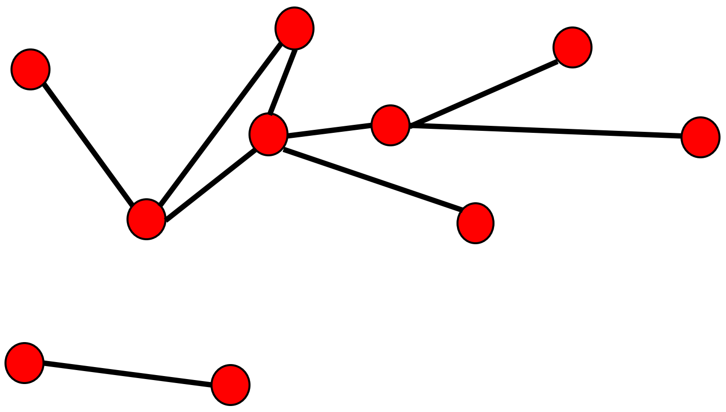
Degree distribution

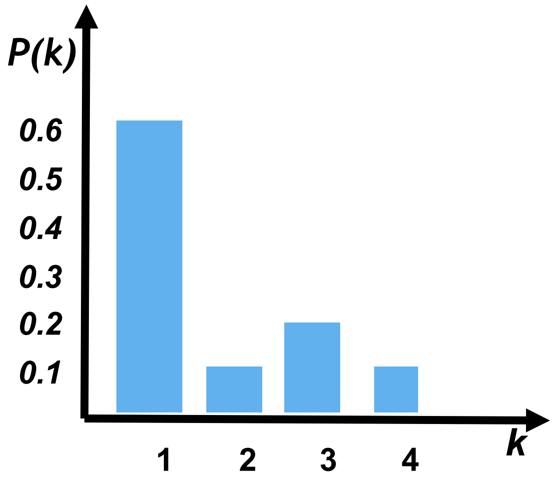
We have a sample of values $\mathbf{x_1, ..., x_N}$ (in our case degrees of nodes)

Distribution of **x** (a.k.a. PDF): probability that a randomly chosen value is **x**

$$P(x) = (\#\ \text{values}\ x) / N$$

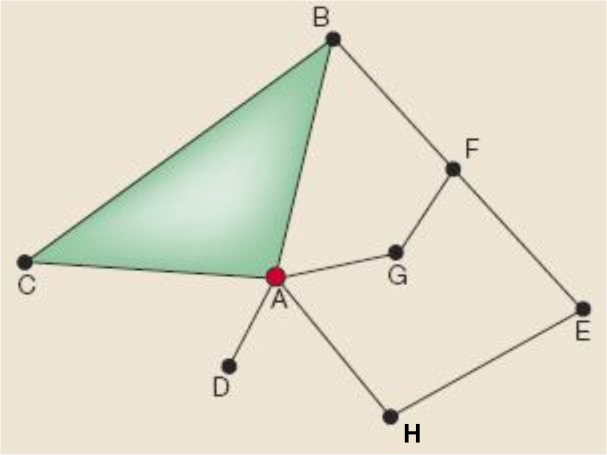
$$\Sigma_i P(x_i) = 1 \quad \text{always!}$$





Average Path Length (APL), diameter, etc

shortest path & average path length (APL) :



in the example above, there are various alternatives to join C & E; the shortest path d_{CE} is either [CBFE] or [CAHE], which have the same “length”, that is, the number of hops is 3.

the average path length $<d>$ is the average over all shortest paths between all pairs of n’s of the graph :

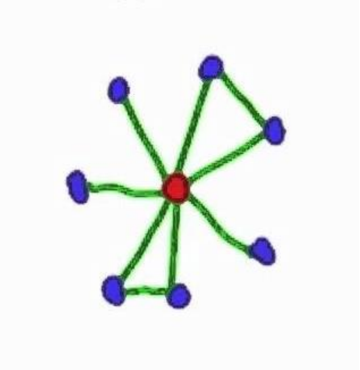
$$\langle d \rangle = \frac{1}{N(N-1)} \sum_{ik} d_{ik}$$

mind your friends’ friends

Clustering coeff.

People tend to have friends who are also friends with each other.
How do we measure this?

$$C_i = \frac{\text{\# edges among neigs.}}{\text{max \# edges among neigs.}} = \frac{e_i}{k_i(k_i - 1)/2}$$



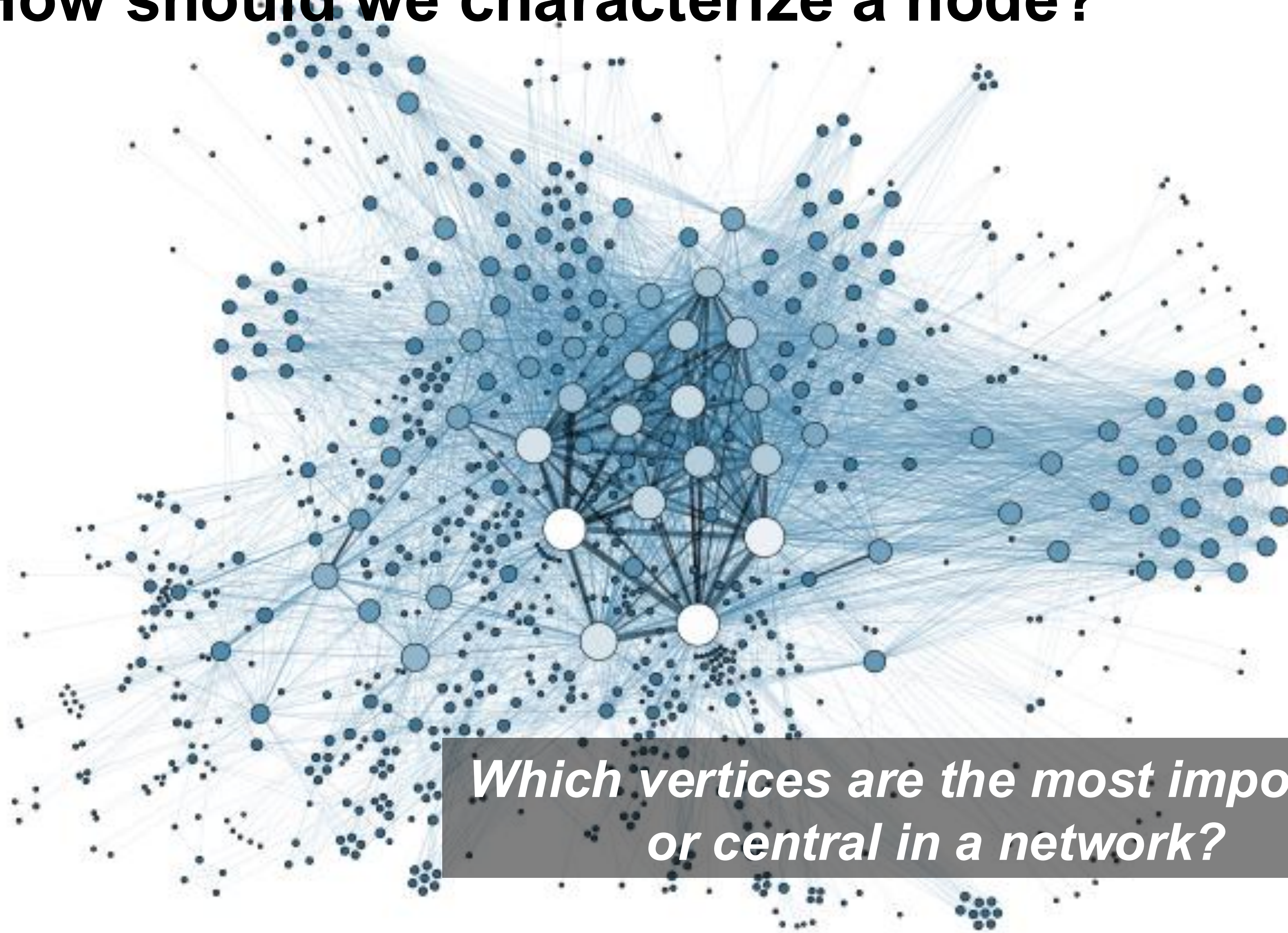
k_i=7, e_i=2, thus

$$C_i = \frac{2}{7(7-1)/2} = 0.095$$

The network clustering coefficient is the average of the clustering coefficients for all nodes

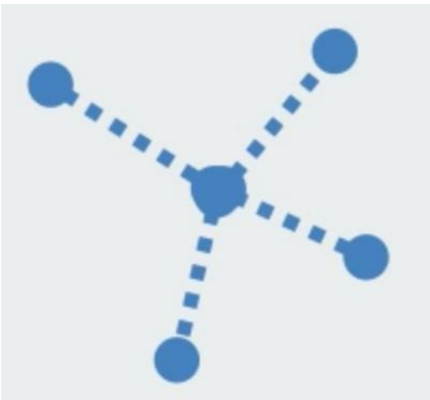
$$\langle C \rangle = \frac{1}{N} \sum_i C_i$$

How should we characterize a node?



Which vertices are the most important or central in a network?

Centrality or Importance **should depend on the context**



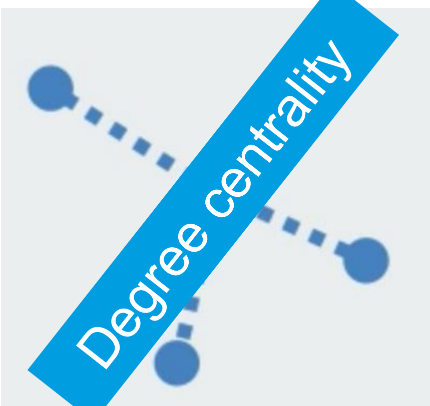
How popular you are

How prestigious your friends are

How close you are

How do you influence the flow of information, or bridge different parts of the network

Centrality or Importance should depend on the context



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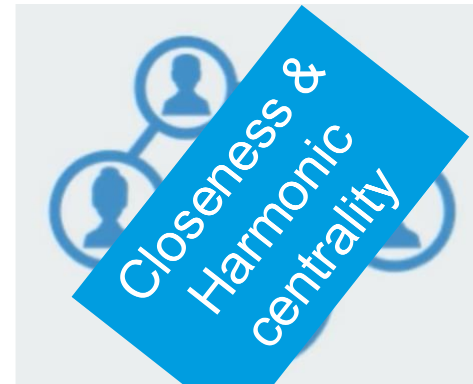
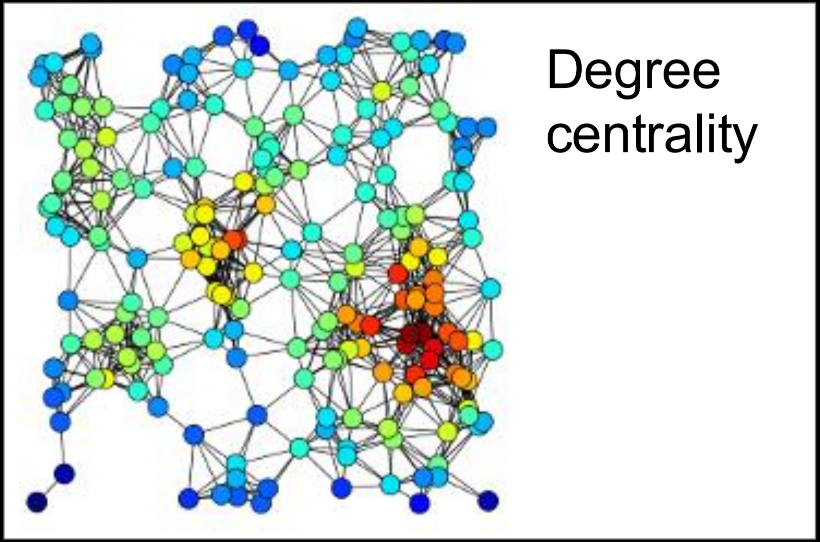
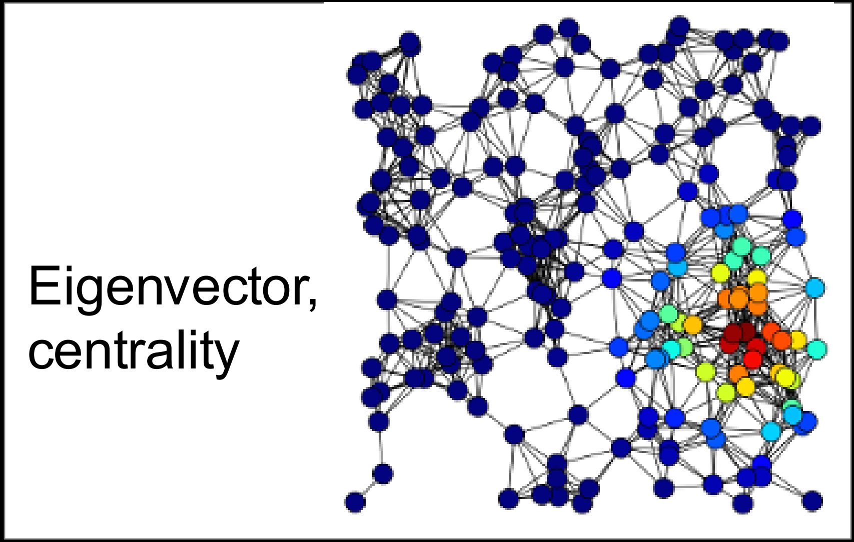


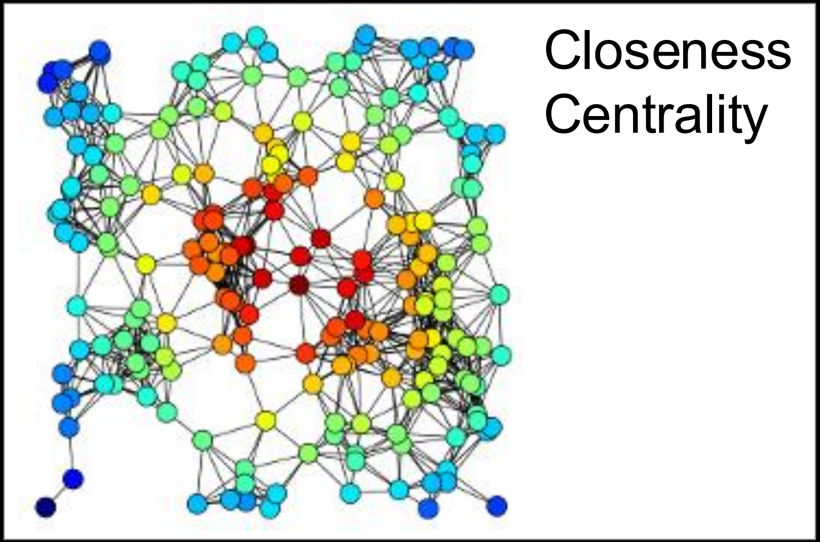
How close you are

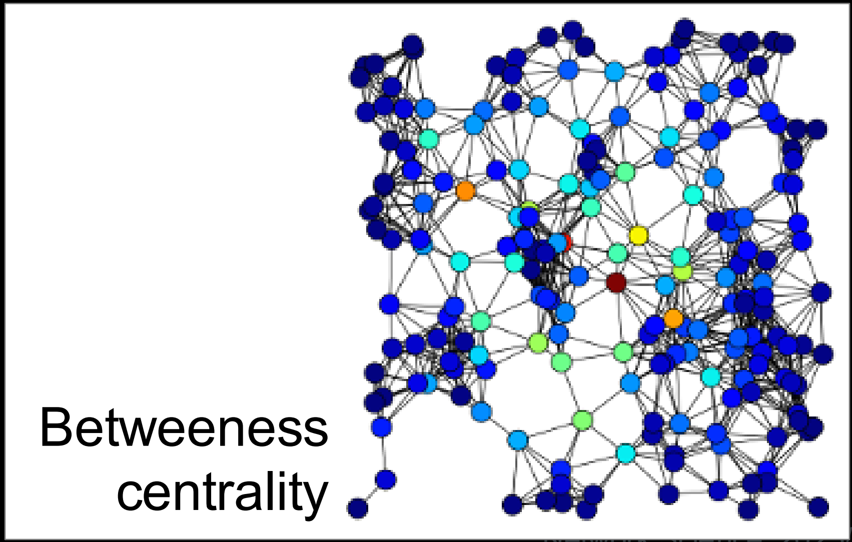
How do you influence the flow o rmation,
or bridge different parts of the network

Same network, different metrics, different concepts.





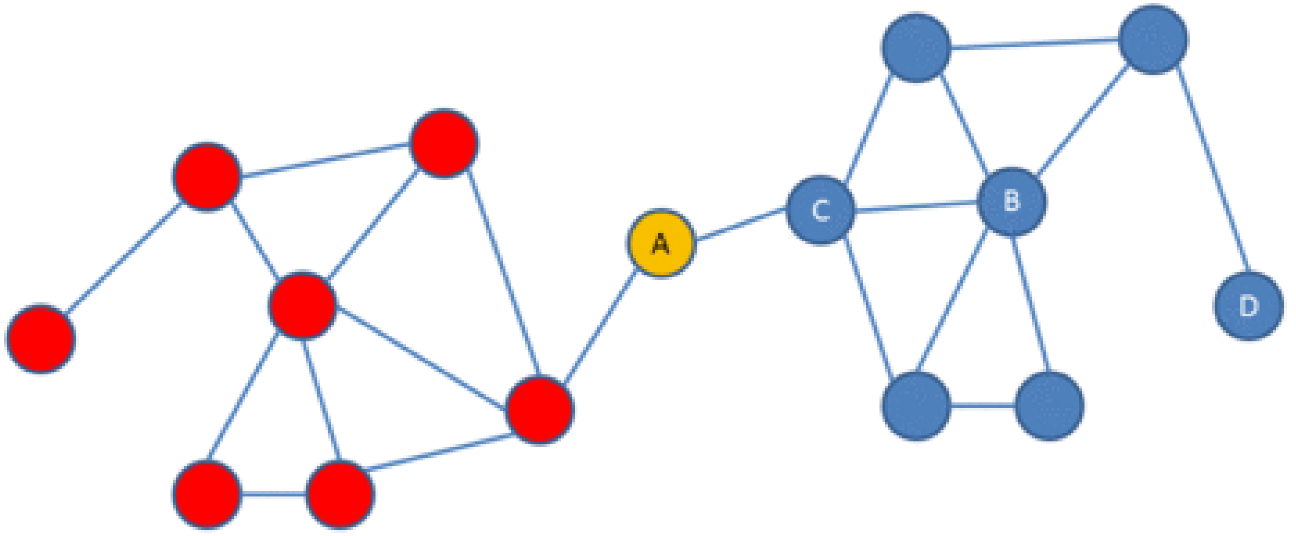




Betweenness centrality (C_B)

You're on everyone's way!

Measures the extent to which a vertex lies on paths between other vertices.



Betweenness centrality (C_B)

You're on everyone's way!

Number of shortest paths from node s to t that pass through i

$$C_B(i) = \sum_{s \neq t \neq i} \frac{\sigma_{st}(i)}{\sigma_{st}}$$

Total number of shortest paths from node s to t

Scales with the number of pairs

Betweenness centrality (C_B)

You're on everyone's way!

Number of shortest paths from node s to t that pass through i

$$C_B(i) = \frac{1}{N^2} \sum_{s \neq t \neq i} \frac{\sigma_{st}(i)}{\sigma_{st}}$$

One natural choice is to divide by the total number of ordered pairs which ~N^2

Total number of shortest paths from node s to t

Betweenness centrality (C_B)

You're on everyone's way!

- Betweenness is a measure of the influence a node has over the spread of information through the network.
- By counting only shortest paths, however, the conventional definition implicitly assumes that information spreads only along those shortest paths.

- **Alternative**: Random walks

Include contributions from all paths between nodes, not just the shortest by counting how often a node is traversed by a random walker.

Newman, M. E. (2005). A measure of betweenness centrality based on random walks. Social networks, 27(1), 39-54.

Betweenness centrality (C_B)

You're on everyone's way!

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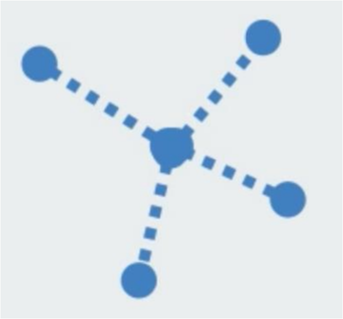
- **Alternative**: Random walks

There's also a close connection between random walks and the PageRank: PageRank is a way to organize random walk of various lengths (F Chung & W Zhao, *Fete of combinatorics and computer science*, 2010)

Newman, M. E. (2005). A measure of betweenness centrality based on random walks. Social networks, 27(1), 39-54.

Same network,



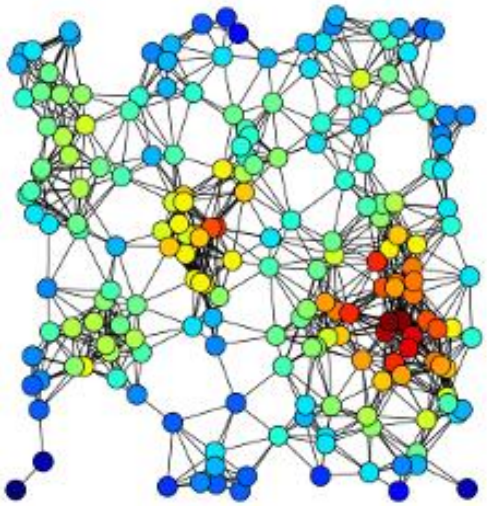
Degree centrality

Eigenvector, Katz, PageRank centralities

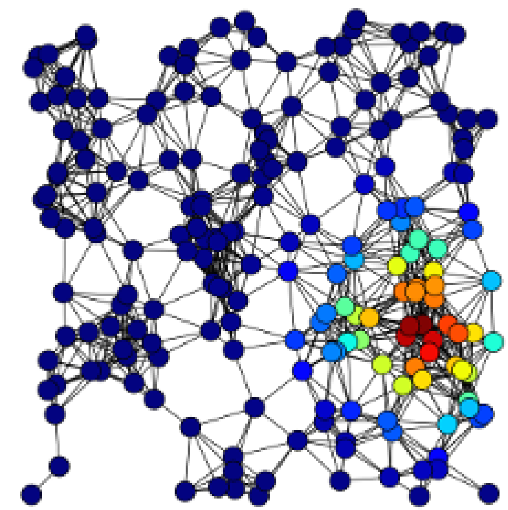
Closeness Centrality & Harmonic Centrality

Betweeness centrality

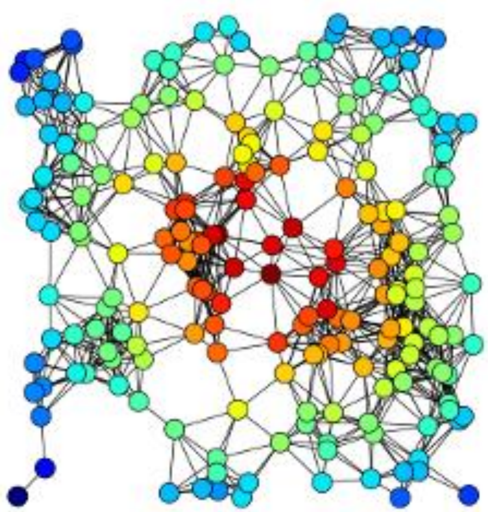
Same network



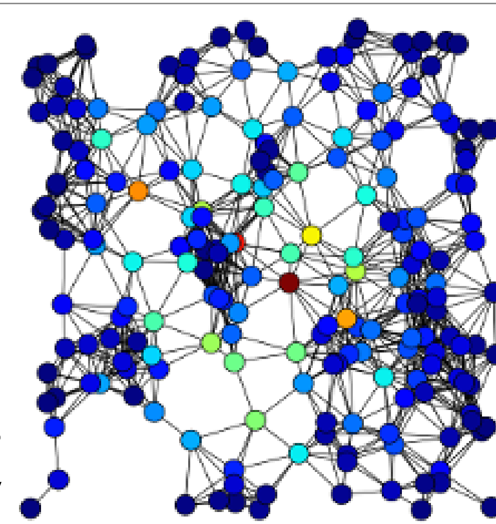
Degree centrality



Eigenvector, centrality

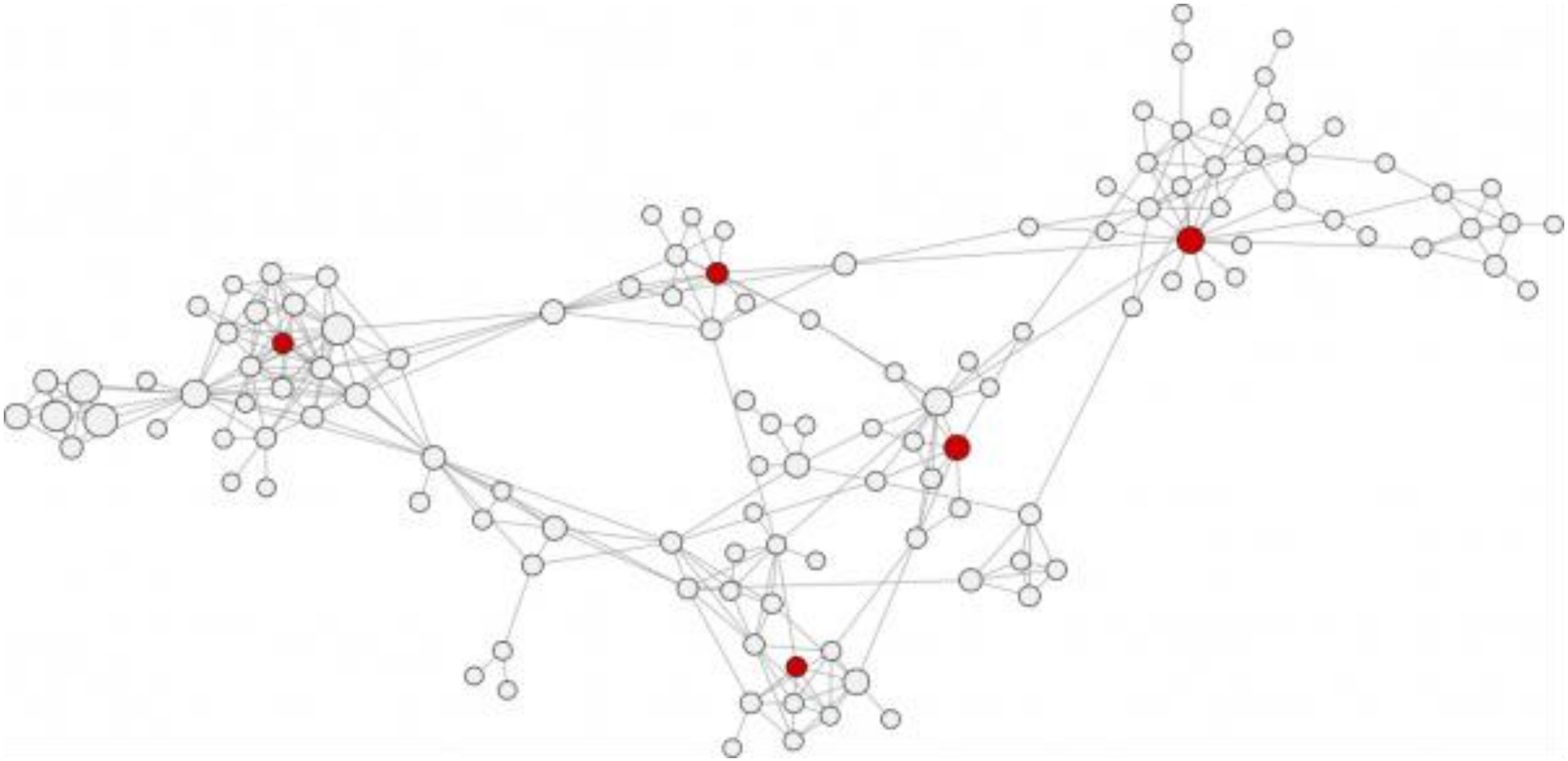


Closeness Centrality

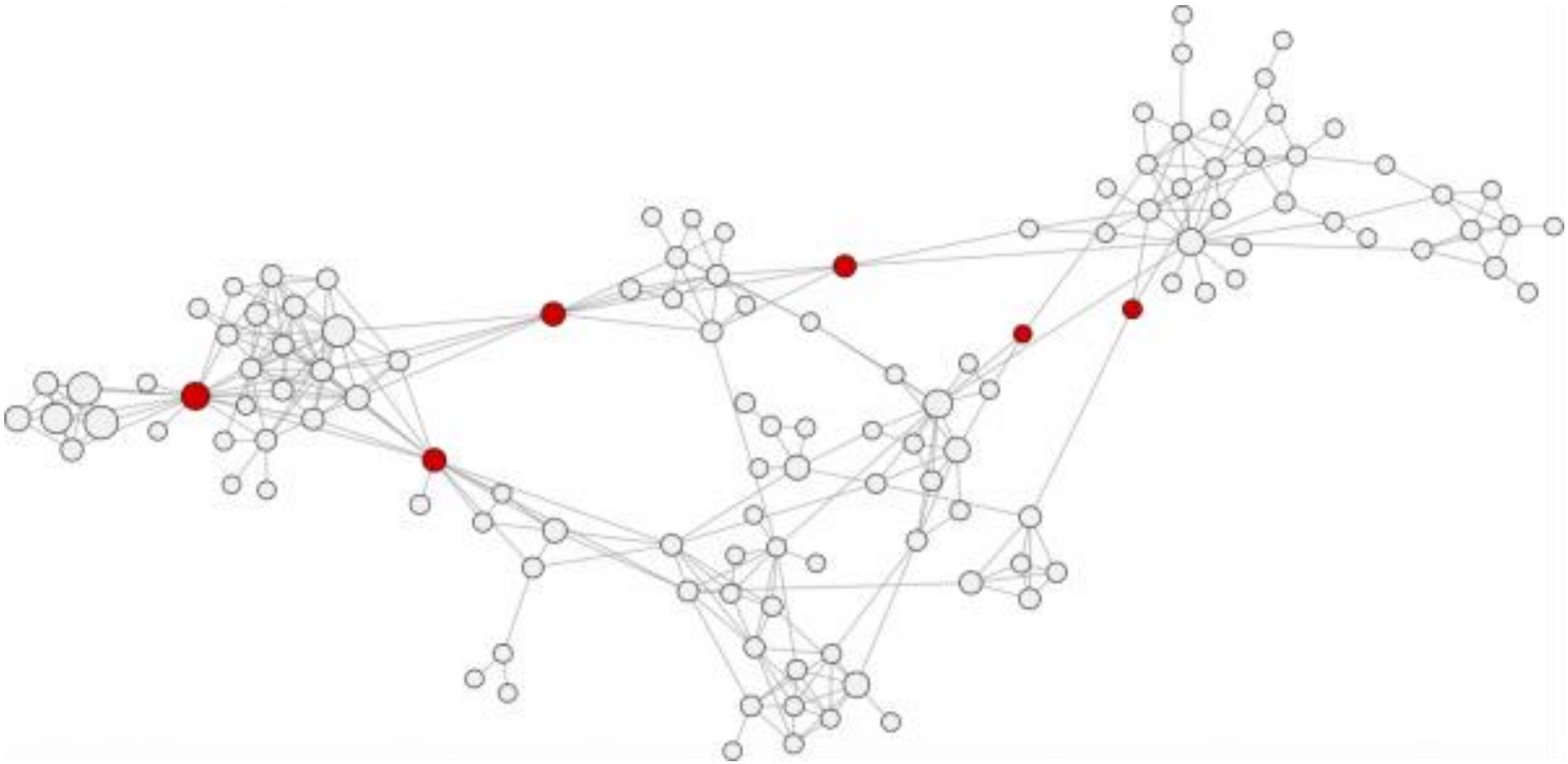


Betweeness centrality

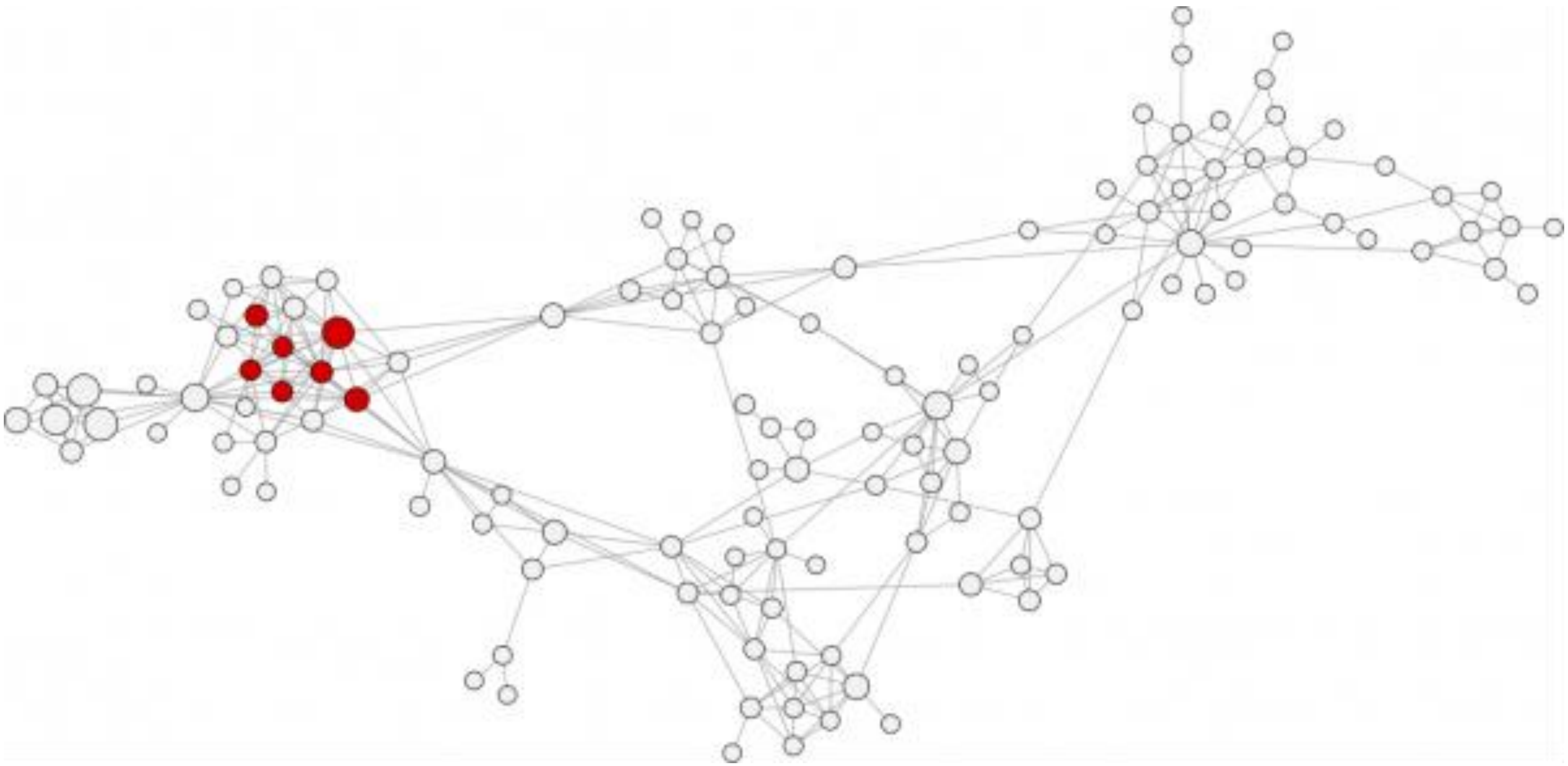
Closeness centrality



Betweenness centrality



Eigenvector centrality



Gephi

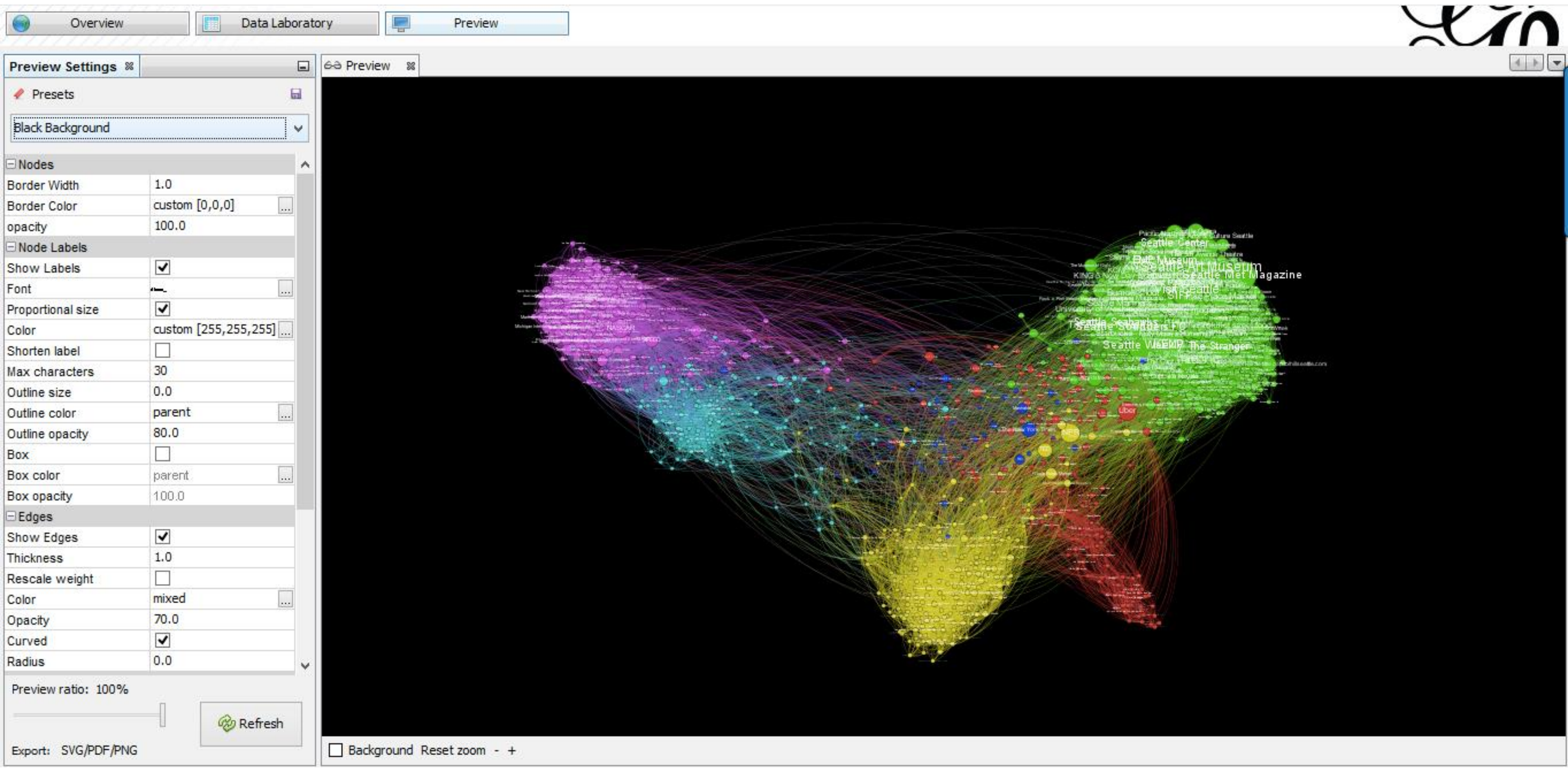
Boost

NetworkX
Network Analysis in Python

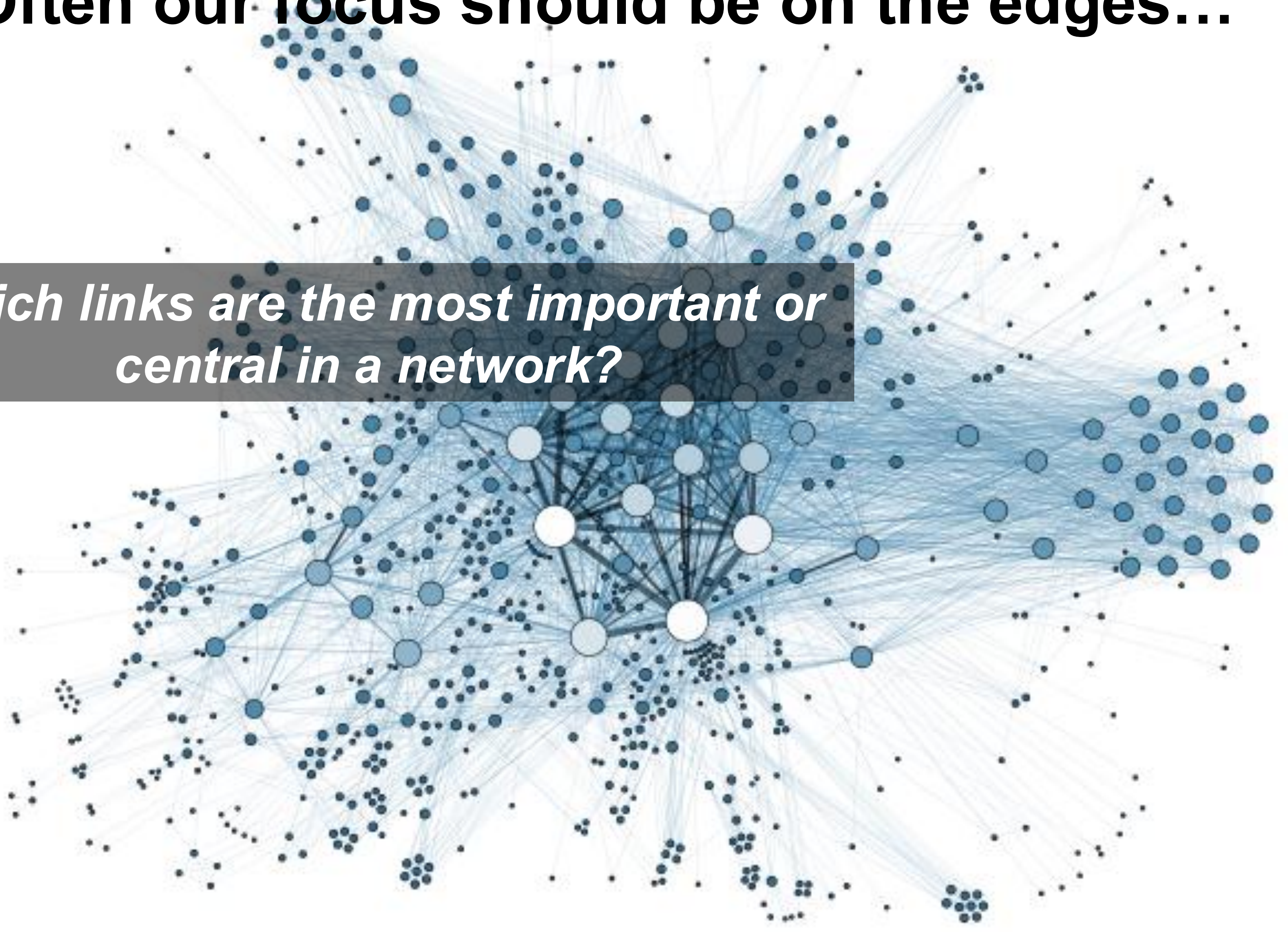
igraph

WebGraph
http://webgraph.di.unimi.it/

Mathematica



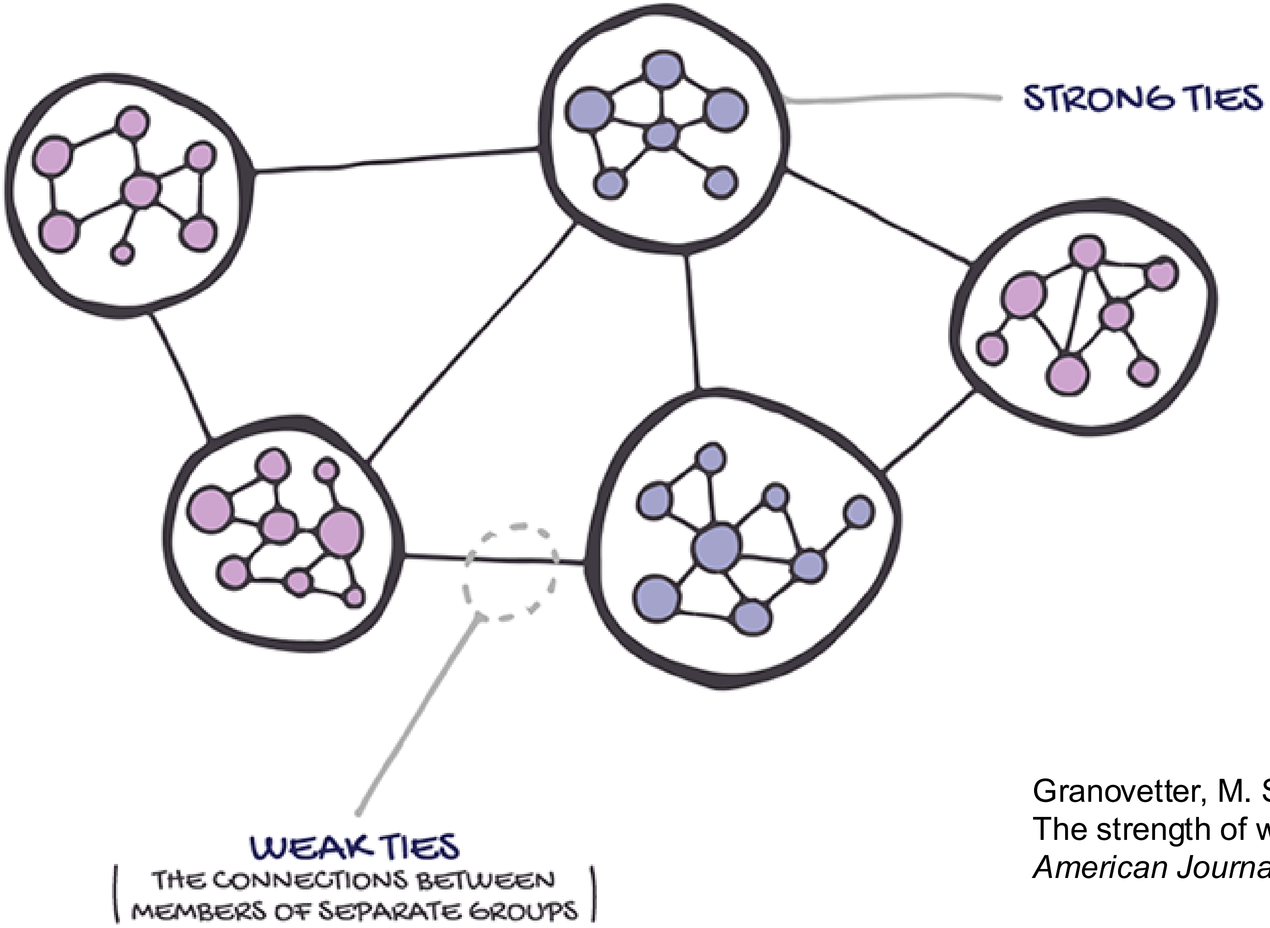
Often our focus should be on the edges…



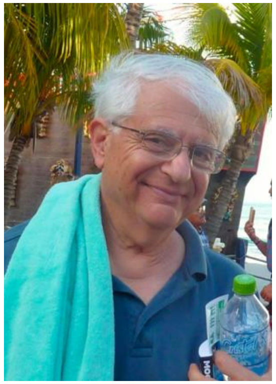
Which links are the most important or central in a network?

The strength of weak ties

90% got information about their current job not from the closest peers but from people that they do not know very well… their weak ties.



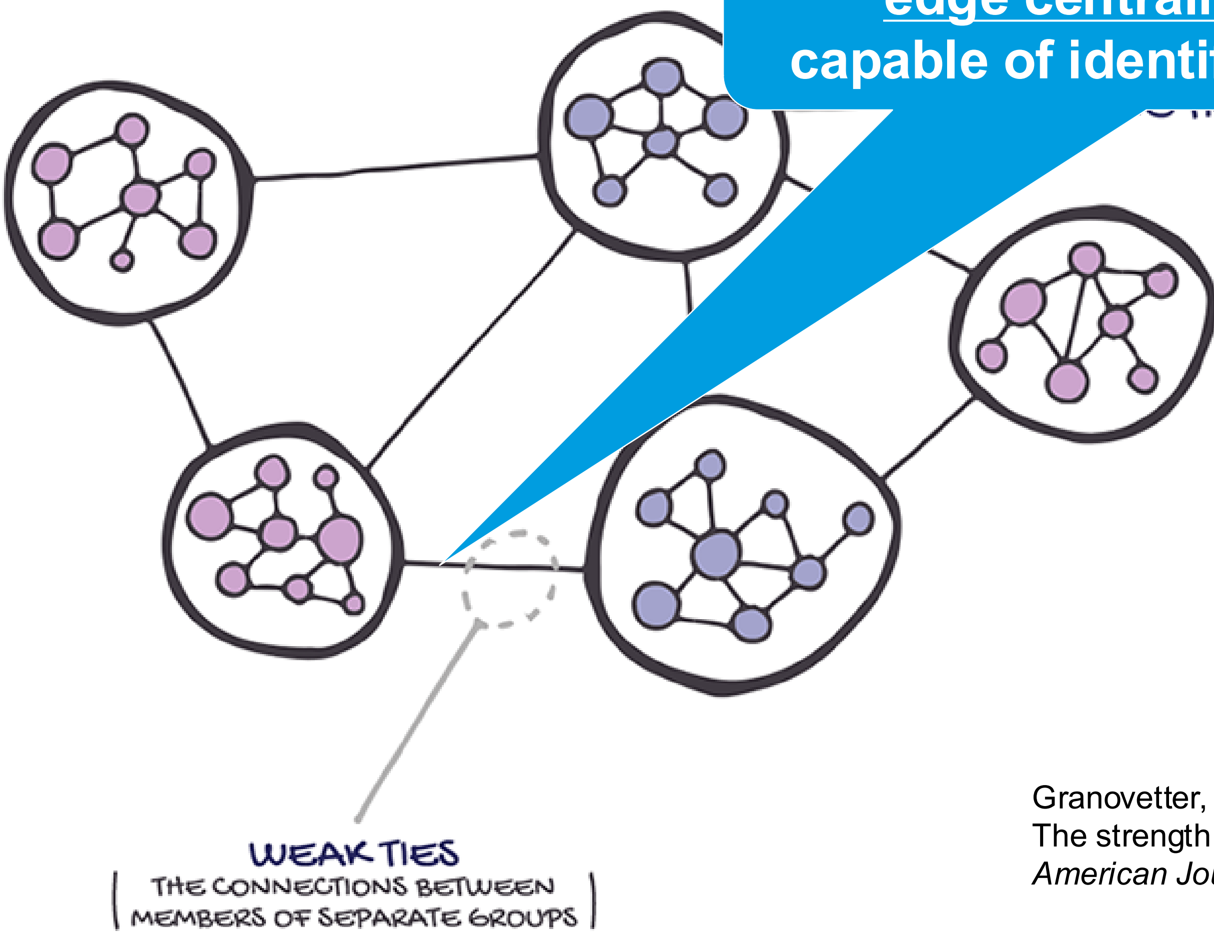
Granovetter, M. S. (1973).
The strength of weak ties.
American Journal of Sociology



The strength of weak ties

Weak ties enable reaching populations and audiences that are not accessible via strong ties.

Propose an
<u>edge centrality measure</u>
capable of identifying weak ties



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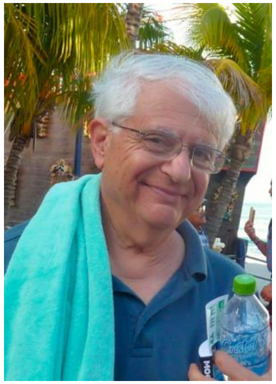
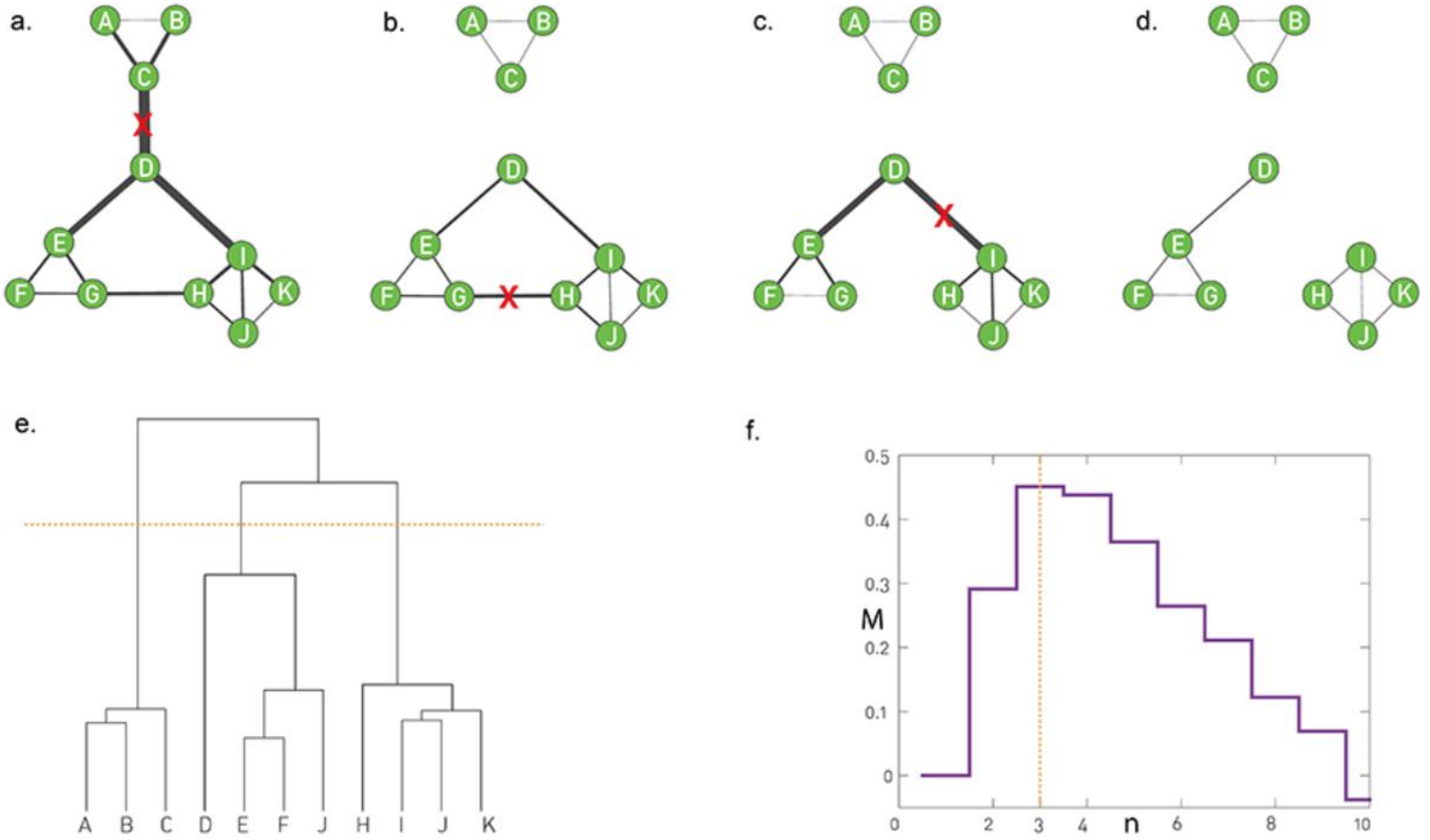


Weak ties & community finding

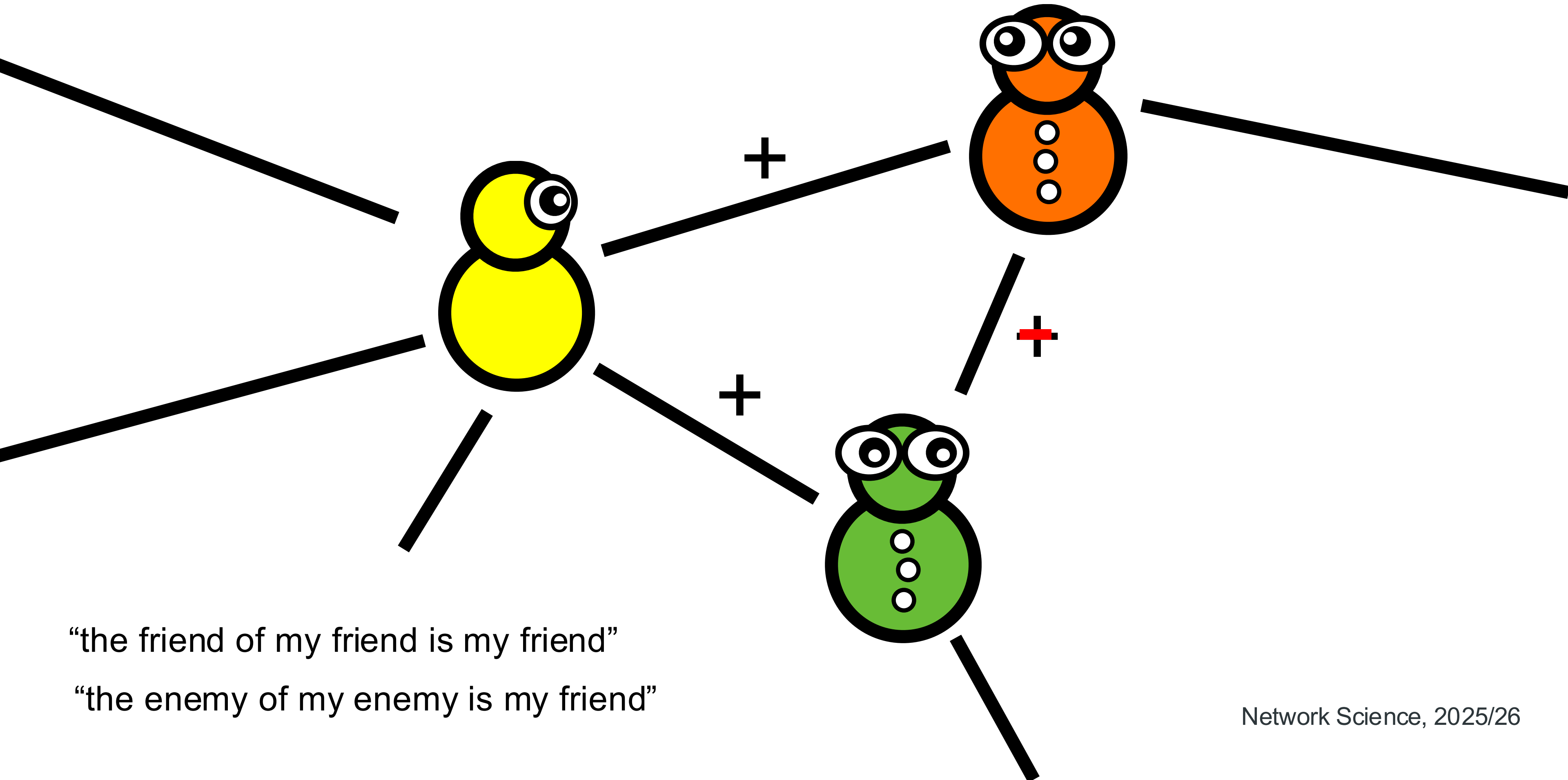
Girvan-Newman Community finding algorithm
(we will return to this later)



Other example…

Social balance theory

social networks often contain both friendly and unfriendly pairwise links between individual nodes



“the friend of my friend is my friend”

“the enemy of my enemy is my friend”

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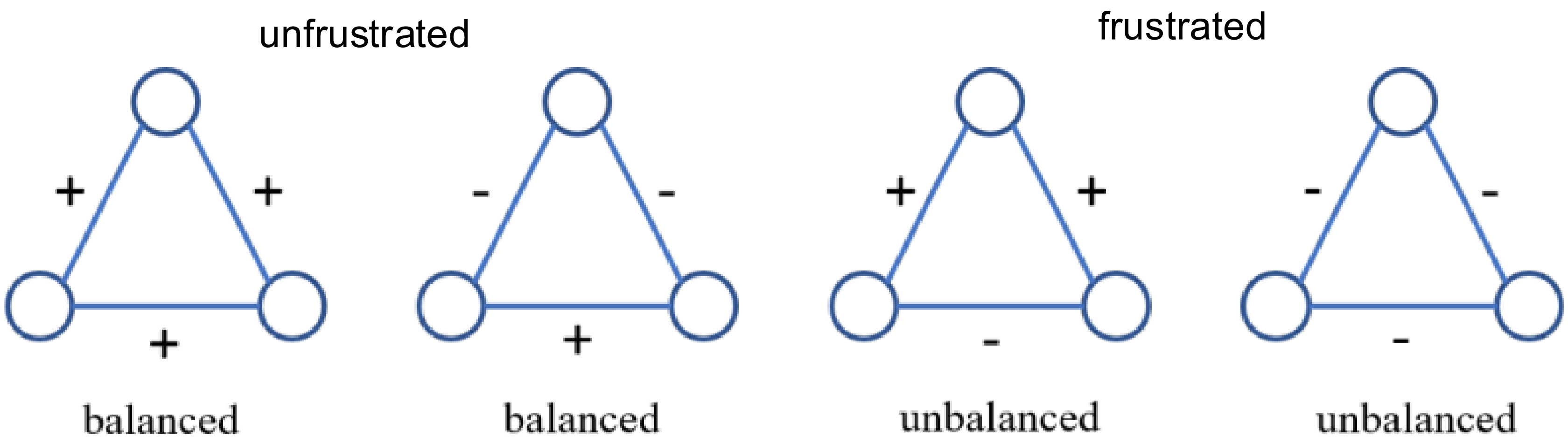

“the friend of my friend is my friend”

“the enemy of my enemy is my friend”

A triad is considered balanced if the product of the signs is positive

Other example…

Social balance theory

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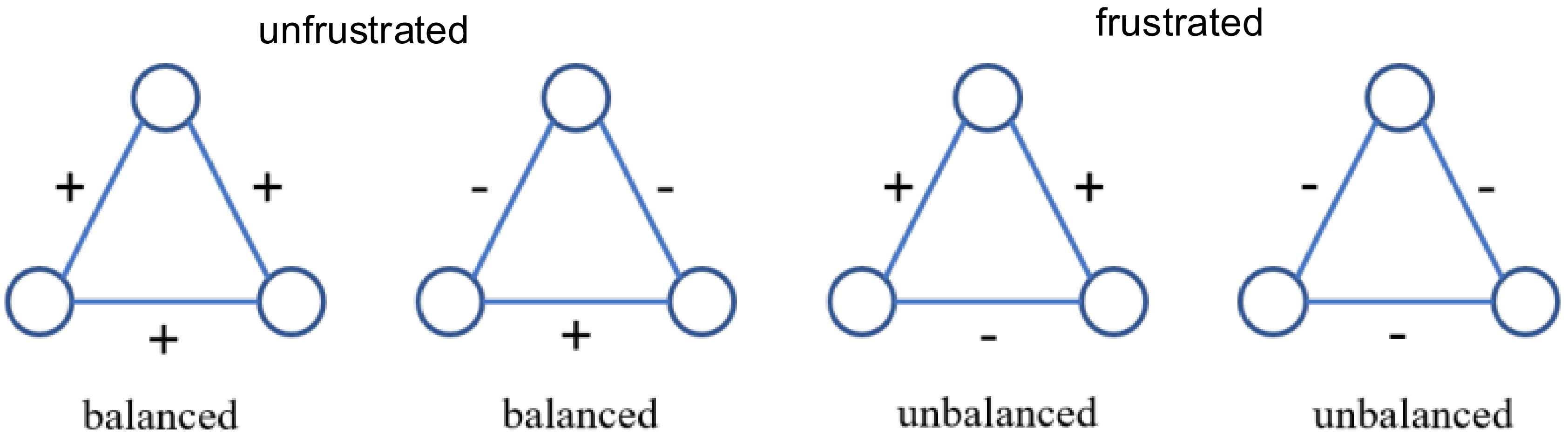


The social balance index of a network is the fraction of balanced triads in a network

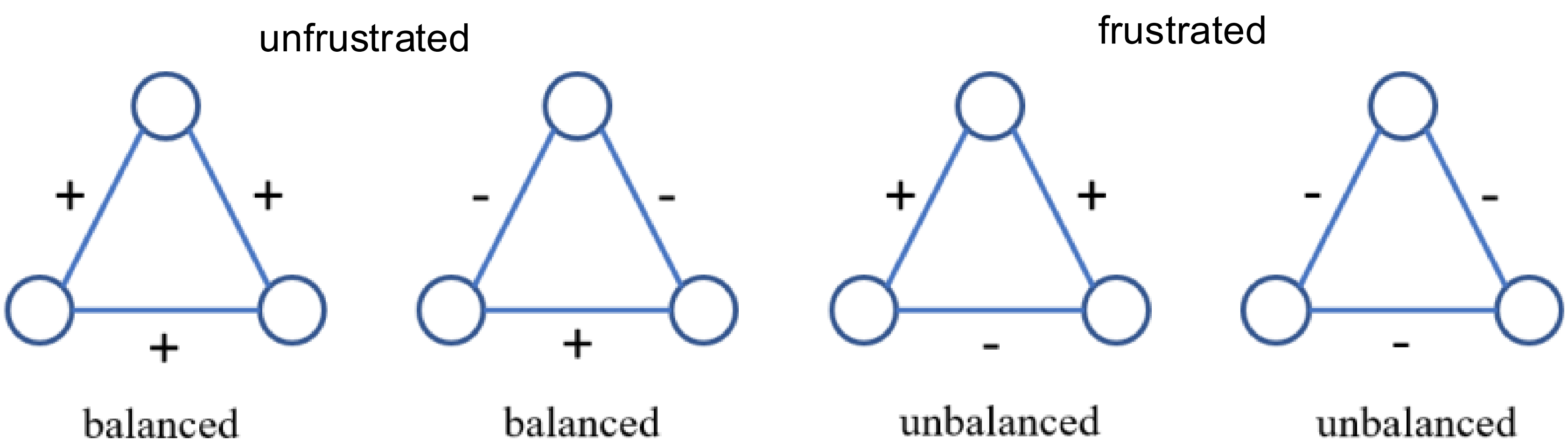
A network is balanced if each constituent triad is balanced

A seemingly more general definition of a balanced network is to require that each closed cycle is balanced

Other example…

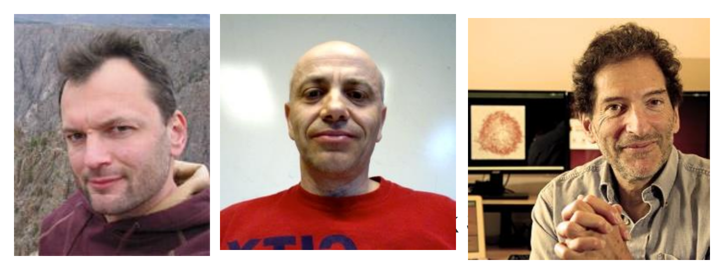
Social balance theory

social networks often contain both friendly and unfriendly pairwise links between individual nodes



The social balance index of a network is the fraction of balanced triads in a network

Possible project: Which type of simple social/network dynamics give rise to a balanced network?



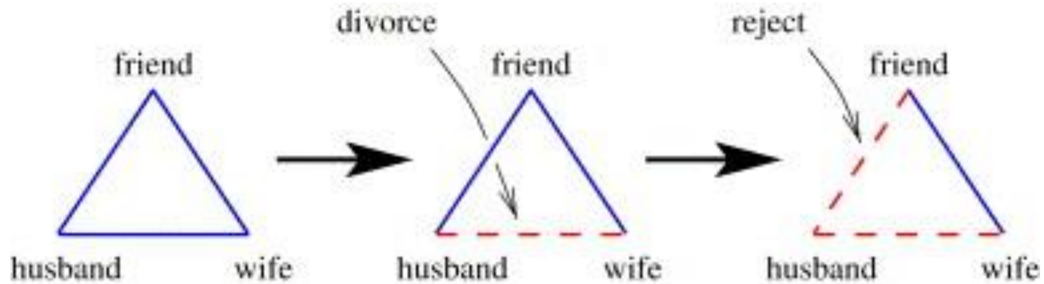
Tibor Antal, Pavel Krapivsky, and Sid Redner

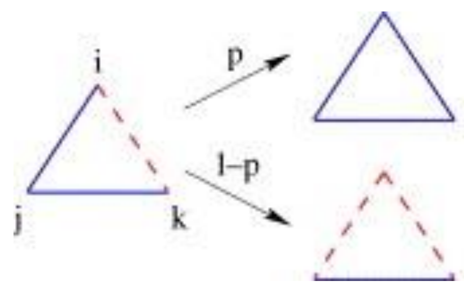
Antal, Krapivsky, Redner,
Dynamics of social balance on networks, Physical Review E, 2005
Social balance on networks: The dynamics of friendship and enmity, Physica D, 2006

Other example…

Social balance theory

Examples:





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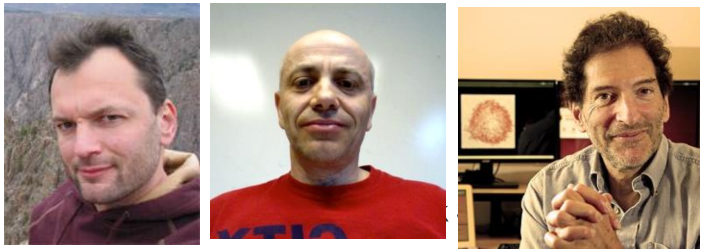
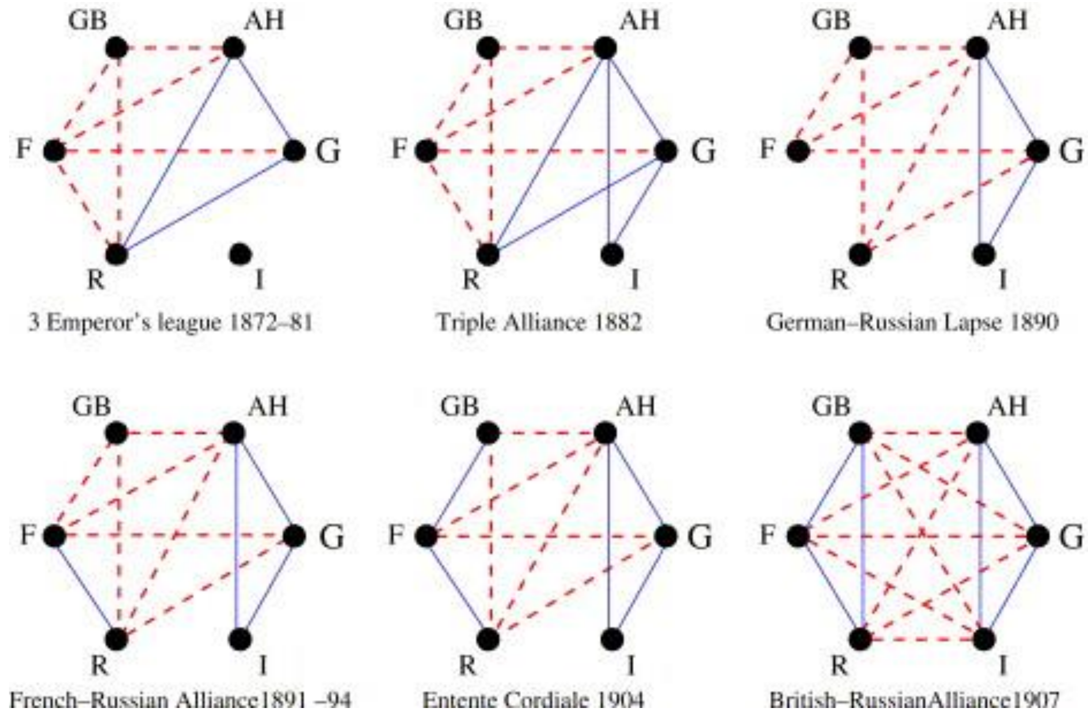


Tibor Antal, Pavel Krapivsky, and Sid Redner

Other example…

Social balance theory

Evolution of the major relationship changes between the protagonists of World War I from 1872–1907



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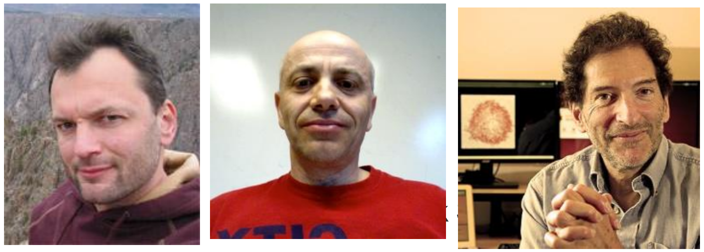

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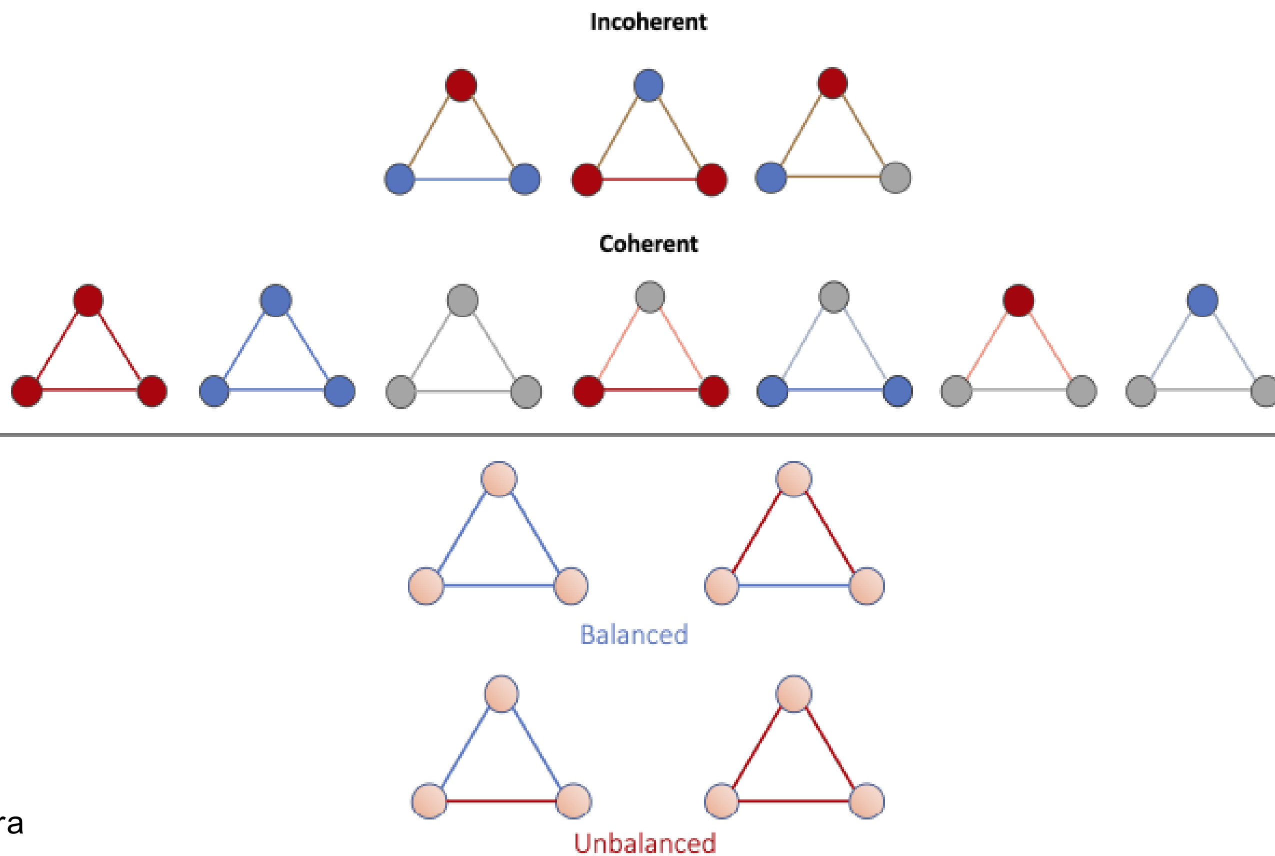
Other example…

Social balance theory & suicidal notes



Andreia Sofia Teixeira

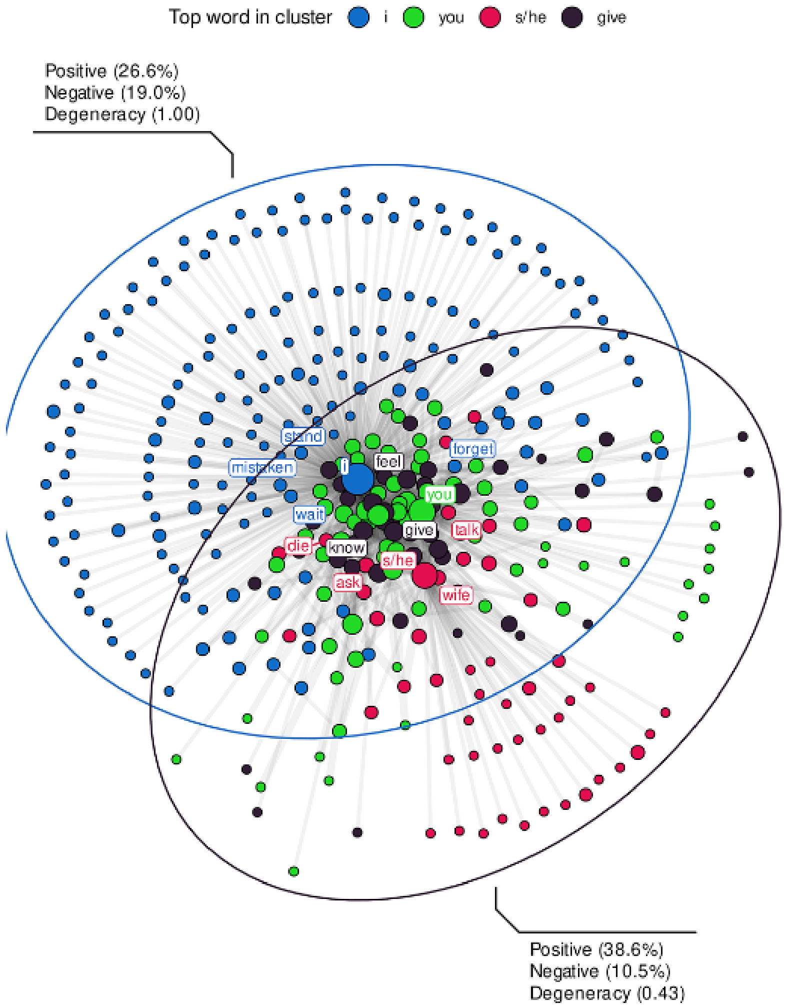
Figure 1: Comparison of emotional complexity in word networks (top) versus structural balance in social networks (bottom). Colours indicate positive (blue), negative (red) or neutral (gray) words (top) or social ties (bottom).

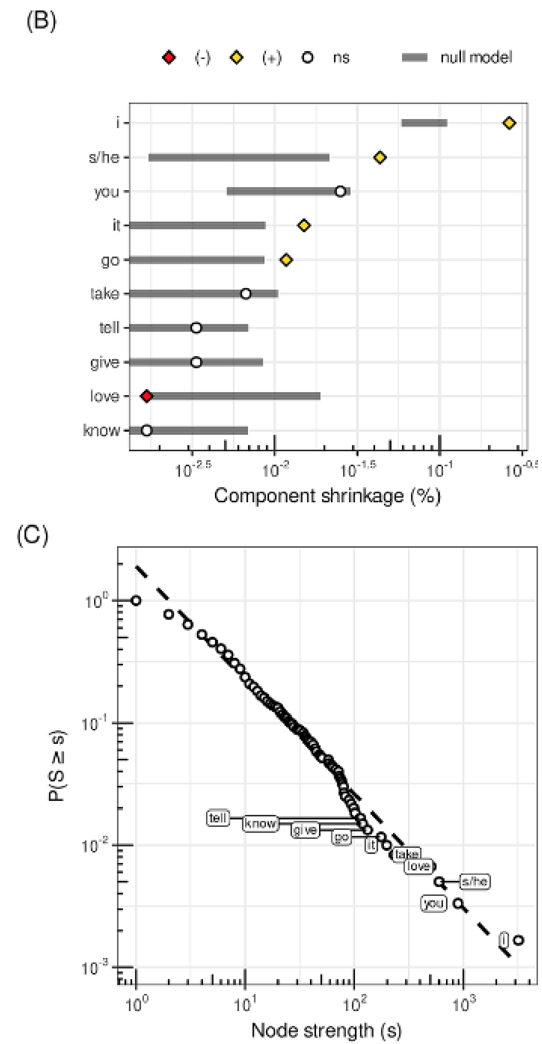
Teixeira, A. S., Talaga, S., Swanson, T. J., & Stella, M. (2021). Revealing semantic and emotional structure of suicide notes with cognitive network science. *Scientific Reports*.

Stella, M., Swanson, T., Hills, T. T., & Teixeira, A. S. (2021). Cognitive network science as a framework for detecting structural patterns and emotions in suicide letters.

Other example…

Social balance theory & suicidal notes





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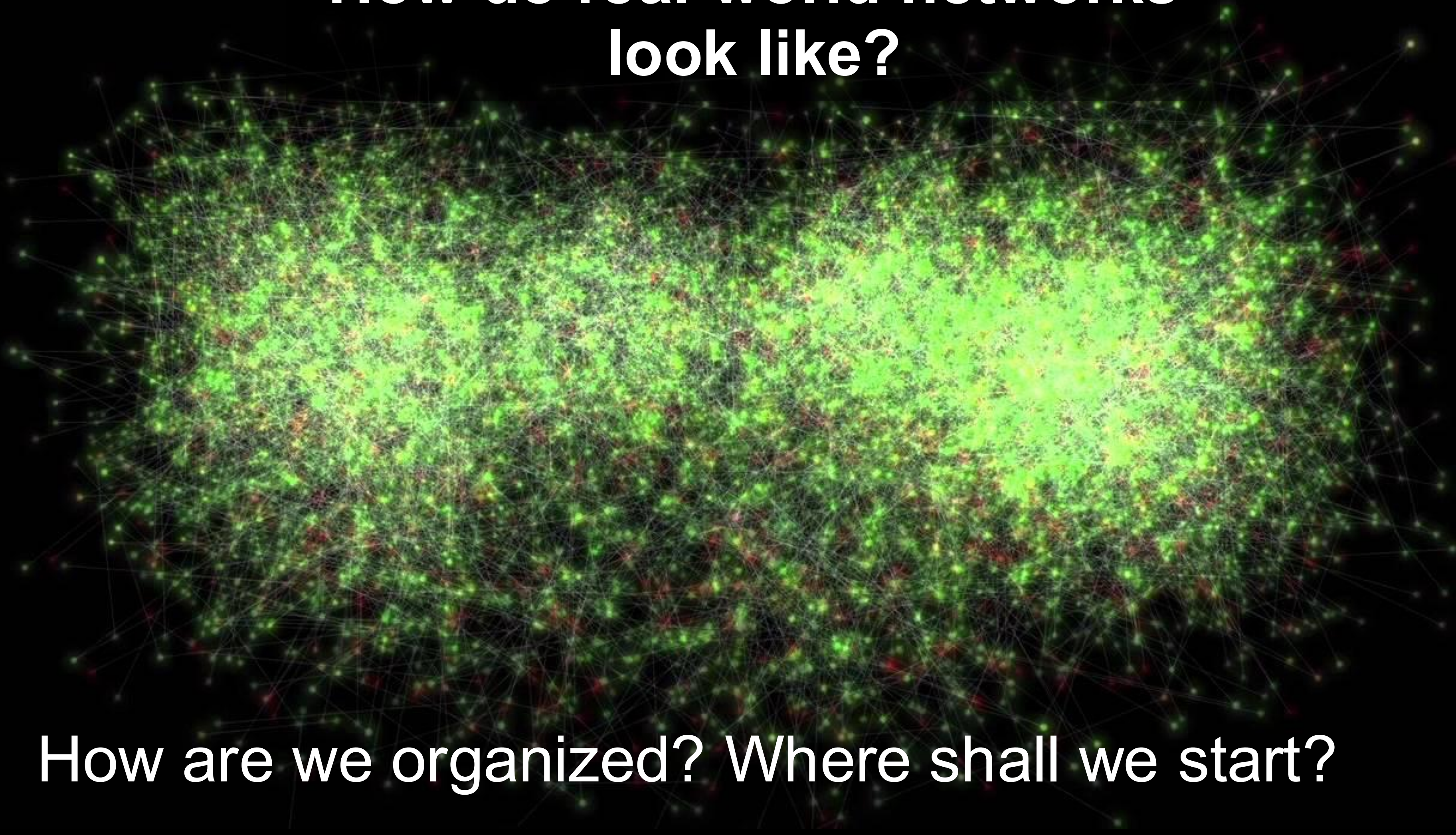
How do real-world networks look like?



How are we organized? Where shall we start?

The small world problem

Stanley Milgram (Harvard & Yale, Dep. Psychology, 1967)

Experimenter, 2015

The small-world experiment

Stanley Milgram (Harvard & Yale, Dep. Psychology, 1967)

- **Sources:** 296 letters originating from 296 different people from Omaha, Nebraska, were sent; the instructions in these letters were that they should be sent to whoever the recipient thought might be in better position to forward them to a given target person.

- **Targets:** Milgram chose a stock broker in Boston and a student in Sharon, Massachusetts as targets.

The small-world experiment

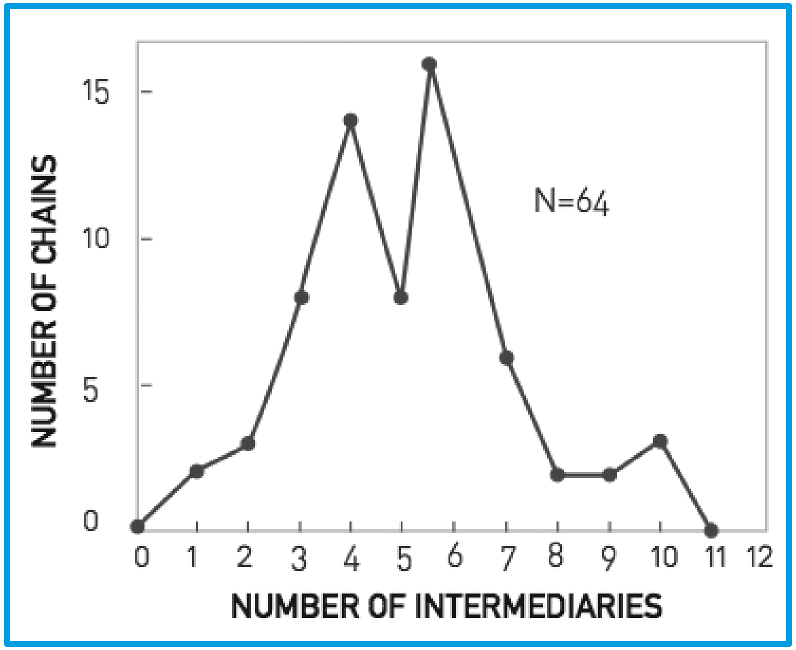
Stanley Milgram (Harvard & Yale, Dep. Psychology, 1967)

- Only 64 letters actually arrived; on average, ***each letter hopped 6 times before reaching the designated address***. Hence the designation "***6 degrees of separation***".

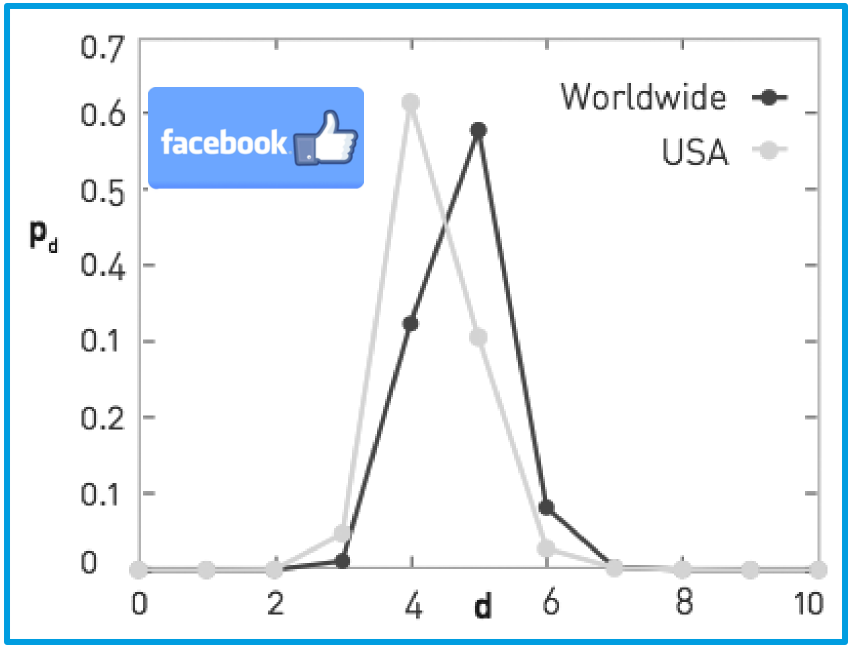
- This is the pioneering experiment on the famous issue of the small-world effect.

Small Worlds & the 6-degrees of separation

Milgram experiment

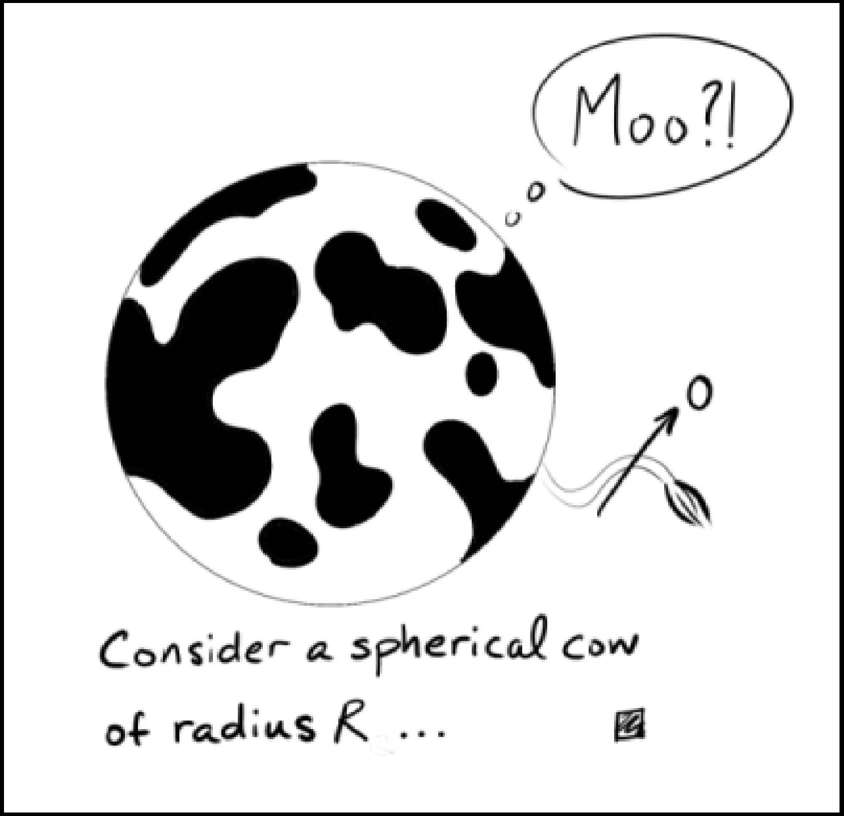


Facebook



Modeling small-world properties? Where shall we start?

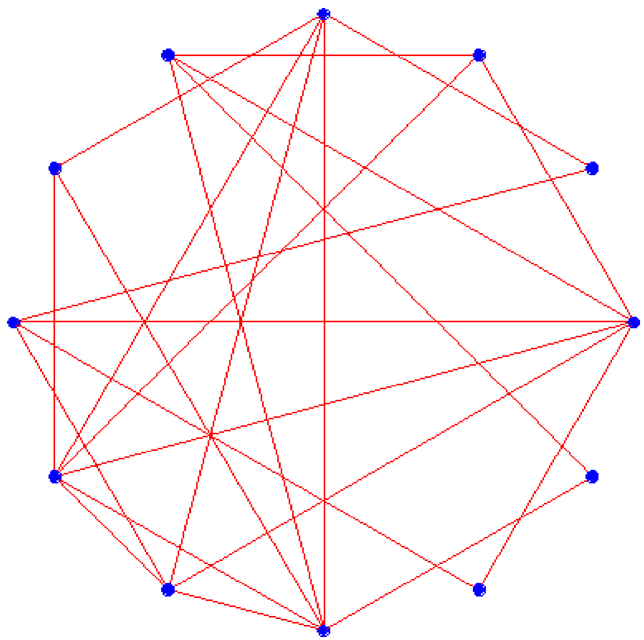
Modeling small-world properties? Where shall we start?



The simplest model one can think…

P. Erdős and A. Rényi, On Random Graphs, Publ. Math. 6, 290 (1959).

- A random network consists of N nodes where each pair of nodes is connected with probability ***p***.



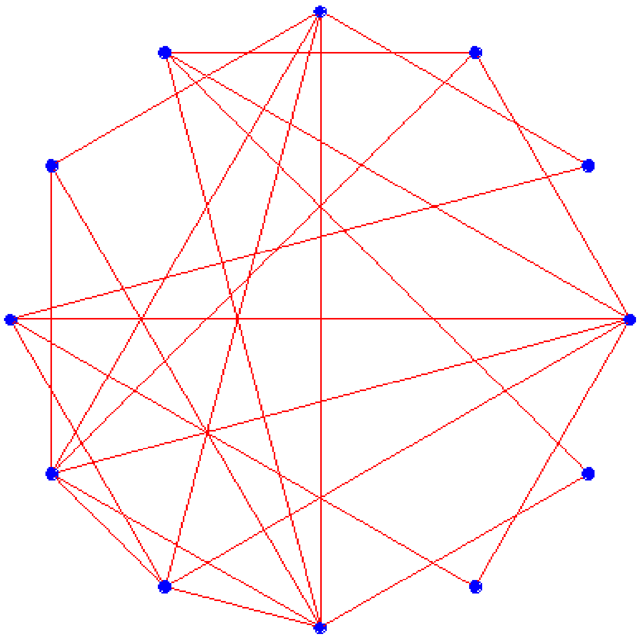
Random network model or the *Erdös and Renyi (ER) Model*

The null model of network science

The simplest model one can think…

P. Erdős and A. Rényi, On Random Graphs, Publ. Math. 6, 290 (1959).

- Number of links?
- Average degree?
- Degree distribution?
- Clustering Coefficient?
- Small-world effect?
- Maximum degree?
- Giant component?

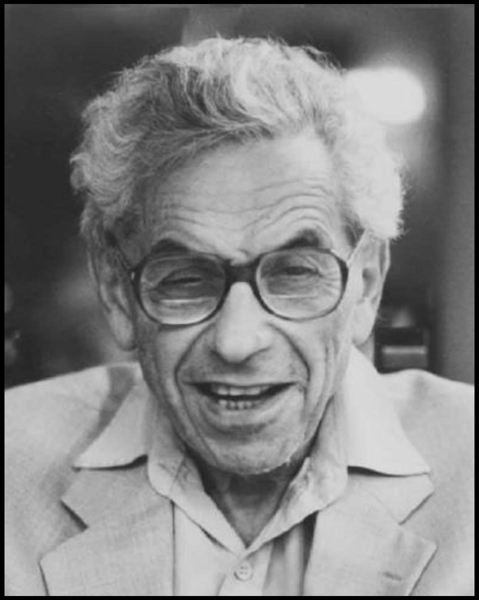


Random network model or the *Erdös and Renyi (ER) Model*

The null model of network science

Erdös & Rényi

"A persistent rumor has circulates in the USA: There are two intelligent races living on the surface of planet Earth: the standard people and the hungarians."
- Isaac Asimov



Pál Erdős (1913-1996)

Alfréd Rényi (1921-1970)

Erdös & Rényi

"A persistent rumor has circulates in the USA: There are two intelligent races living on the surface of planet Earth: the standard people and the hungarians."
- Isaac Asimov

John von Neumann, Paul Erdős, Eugene Wigner e Edward Teller, Leó Szilárd, Theodore von Kármán, Paul Halmos, George Polya e John G. Kemeny

Pál Erdös (1913–1996)

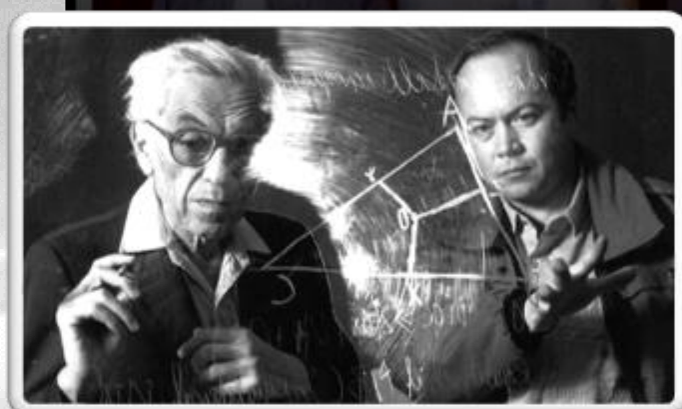
Erdős's hunt for interesting problems took him around the world.

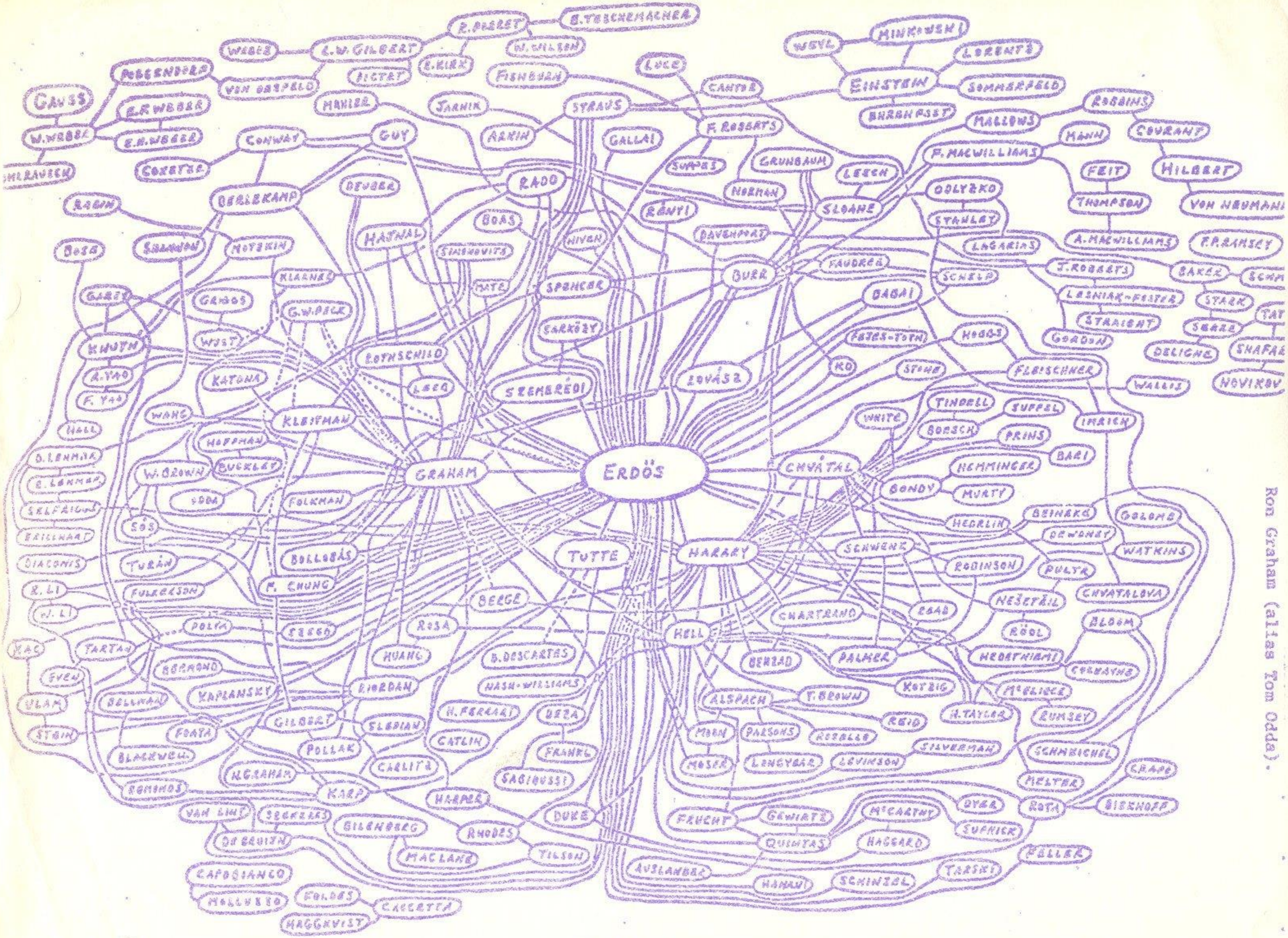
He was a mathematical nomad, living from two half-full suitcases without a chequebook or credit card.

He would turn up at colleagues' houses and making mathematics into a social affair.

This led to him to an astonishing productivity: he published 1,525 papers during his life, with over 500 co-authors!

Pál Erdös (1913–1996)





Ron Graham (alias Tom Odda).

Figure 1

To appear in Topics in Graph Theory (F. Harary, ed.) New York Academy of Sciences (1979).

Science through Erdös

	Average Erdös-number
Fields medal	3.36
Nobel Economics	4.11
Nobel Chemistry	5.48
Nobel Medicine	5.50
Nobel Physics	5.63

Back to random networks…

G(N,p) model

Each pair of N nodes is connected with probability p.

G(N,E) model

N nodes are connected by E randomly placed links.

P. Erdős and A. Rényi, On Random Graphs, Publ. Math. 6, 290 (1959).

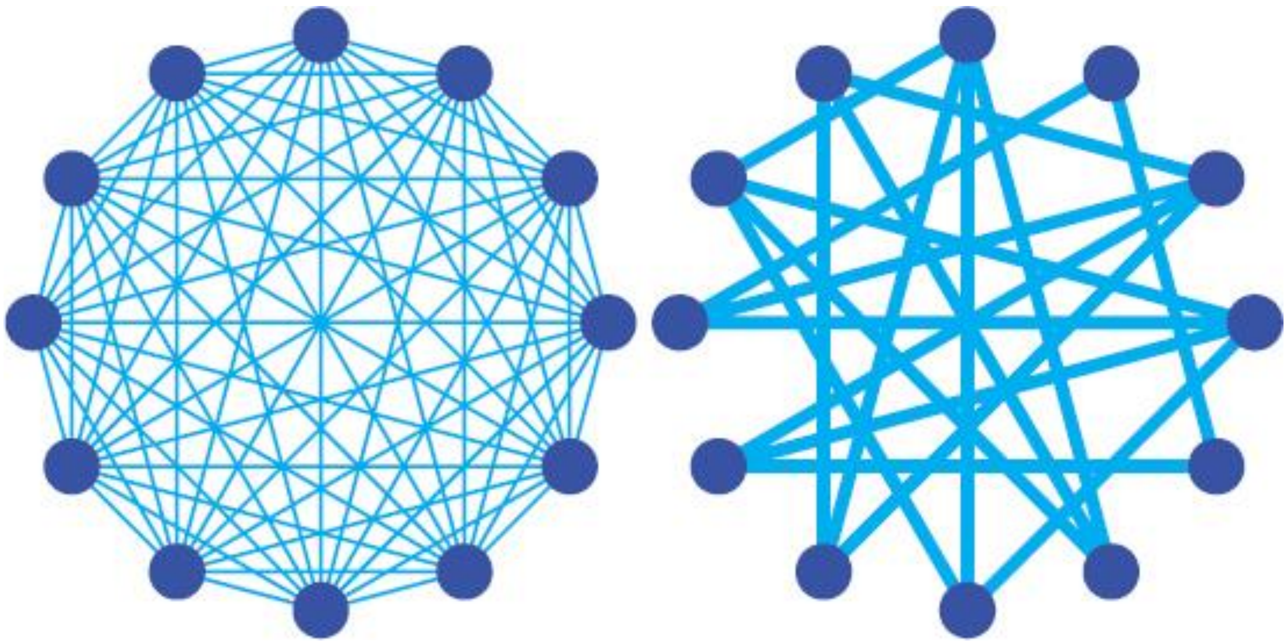
Are real-world networks random?

Model fits empirical data?

- Small world effect ??
- Degree distribution ??
- Maximum degree ??
- Clustering coeff. ??

Some simple and fun math

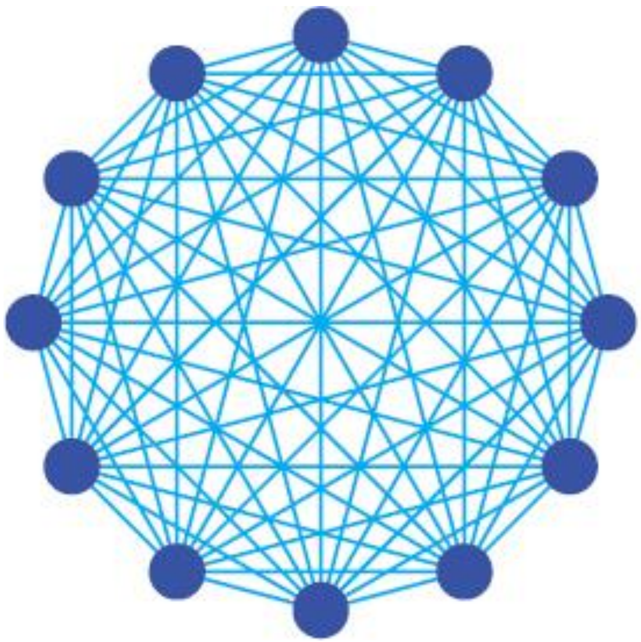
- What’s the average number of links of this network?
- …or, if you prefer, what’s the prob. of having ***E*** edges?



$\boldsymbol{p = 1}$ $\boldsymbol{0 < p < 1}$

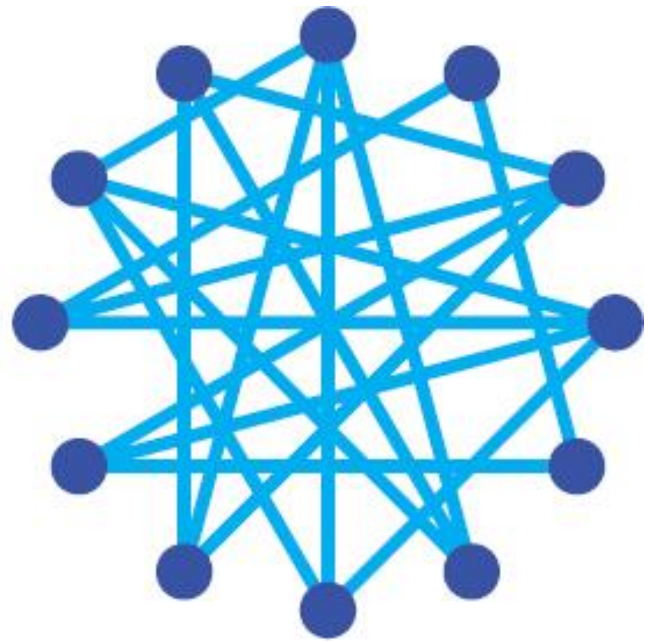
Some simple and fun math

- What's the average number of links of this network?
- …or, if you prefer, what's the prob. of having ***E*** edges?



$$E = \frac{N(N-1)}{2}$$

***p* = 1**



$$E = ??$$

0 < *p* < 1

Binomial distribution

- The binomial distribution describes the number of successes in ***n independent experiments*** with 2 possible outcomes, in which one occurs with prob. p and the other with 1-p.
- The probability of having x successes is given by:

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- With an average value of

$$\langle x \rangle = \sum_{x=0}^{n} x P(x) = np$$

Some simple and fun math

- What's the prob. of having ***E*** edges?

E_{max}

$$P(E) = \binom{\frac{N(N-1)}{2}}{E} p^E (1-p)^{\frac{N(N-1)}{2} - E}$$

- **Average number of edges** of a random graph

$$\langle E \rangle = \sum_{E=0}^{N(N-1)/2} E\,P(E) = \frac{N(N-1)}{2} p$$

Some simple and fun math

- What's the prob. of having ***E*** edges?

$$P(E) = \binom{\frac{N(N-1)}{2}}{E} p^E (1-p)^{\frac{N(N-1)}{2} - E}$$

E_{max}

- **Average degree** of a random graph

$$\langle k \rangle = \frac{2\langle E \rangle}{N} = \frac{2}{N}\frac{N(N-1)}{2} p = p(N-1)$$

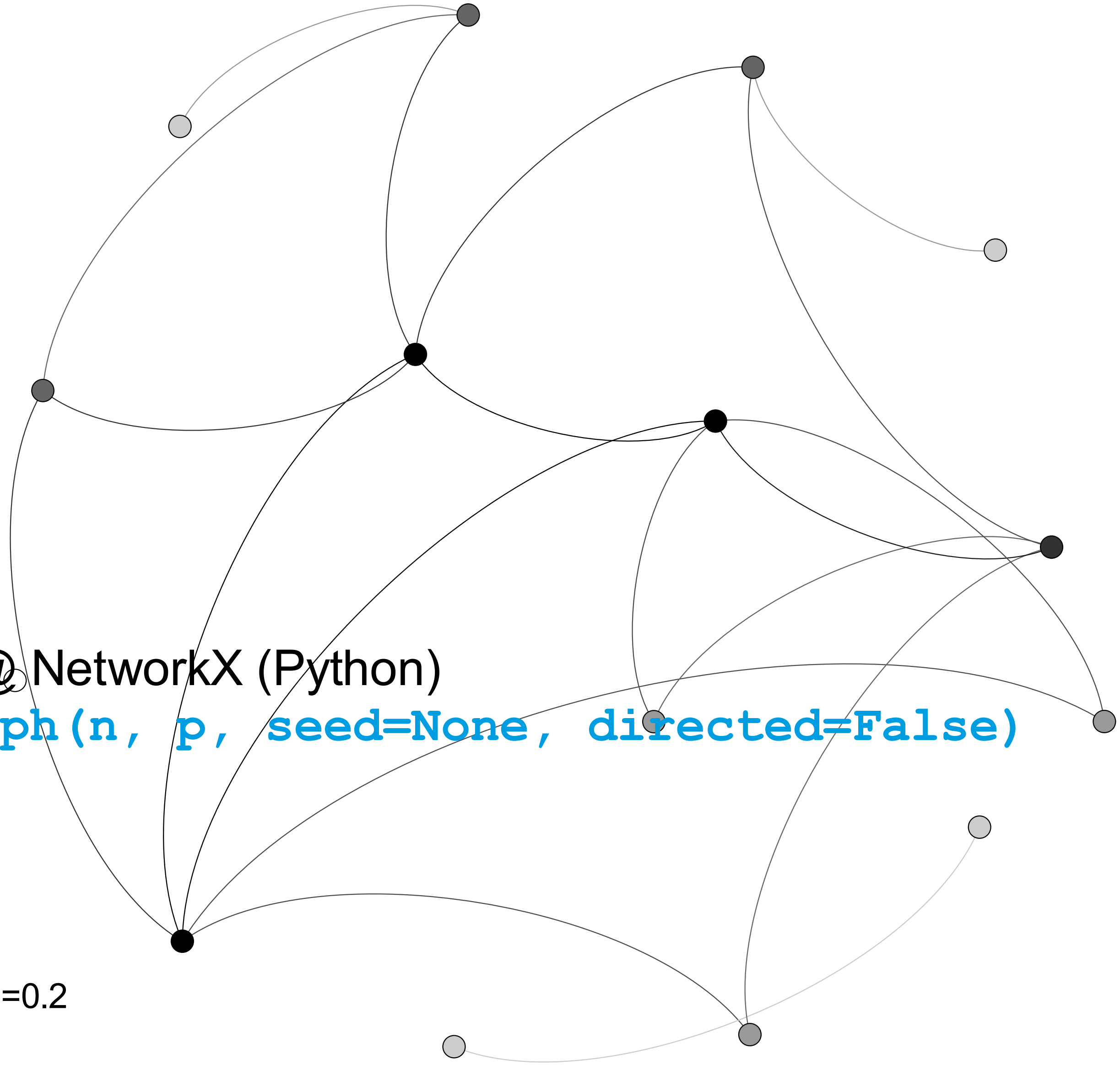
Average degree increases linearly with p

Try it!

NetworkX

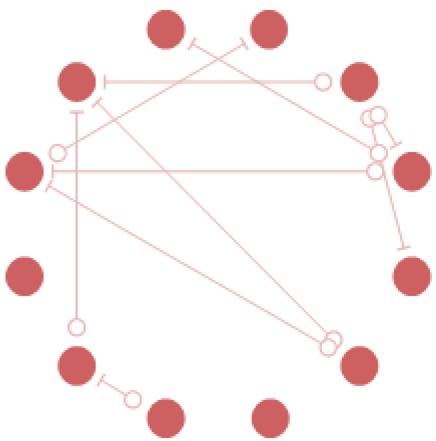
Random networks @ NetworkX (Python)

`erdos_renyi_graph(n, p, seed=None, directed=False)`

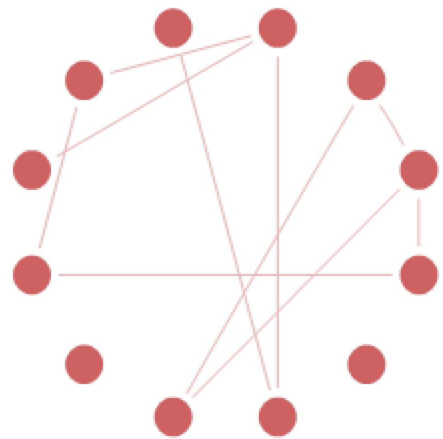


N=15, p=0.2

Random graphs are truly “random”. Try it!



<k>=1.6

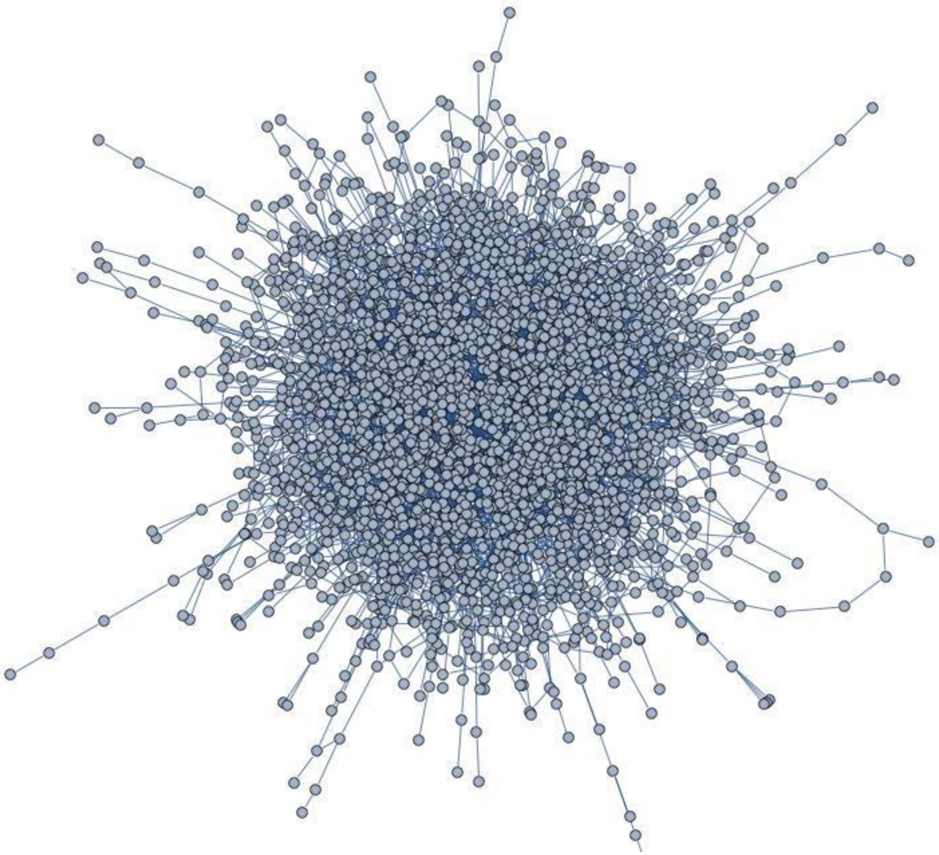


<k>=1.6

<k>=1.3

N=12, p=1/6, Expected <k> =1.833

Nice! What’s the degree distribution of a Erdos-Renye (ER) random network?



Degree distribution of a random graph

- The probability that a node i has exactly k partners is given by

$$P(k) = \quad p^k \quad \times \quad (1-p)^{N-1-k} \quad \times \quad \binom{N-1}{k}$$

Probability that k links are present × Probability that N-1-k links are NOT present × Number of ways one can arrange k links out of N-1 possible

Max degree = N-1

Degree distribution of a random graph

- The probability that a node i has exactly k partners is given by ***a binomial distribution***

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

Number of ways we can arrange k links out of N-1 possible

Probability that k links are present

Probability that N-1-k links are NOT present

Max number of links = N-1

Degree distribution of a random graph

- The probability that a node i has exactly k partners is given by ***a binomial distribution***

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

- Real networks are large and sparse *(i.e., <**k**> << **N**)*
- For large/sparse networks the binomial dist. is well approximated by ***a Poisson distribution***:

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

Degree distribution of a random graph

Small nets

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

N>>1

Large nets

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

How would you test the validity of these analytical insights?

Degree distribution of a random graph

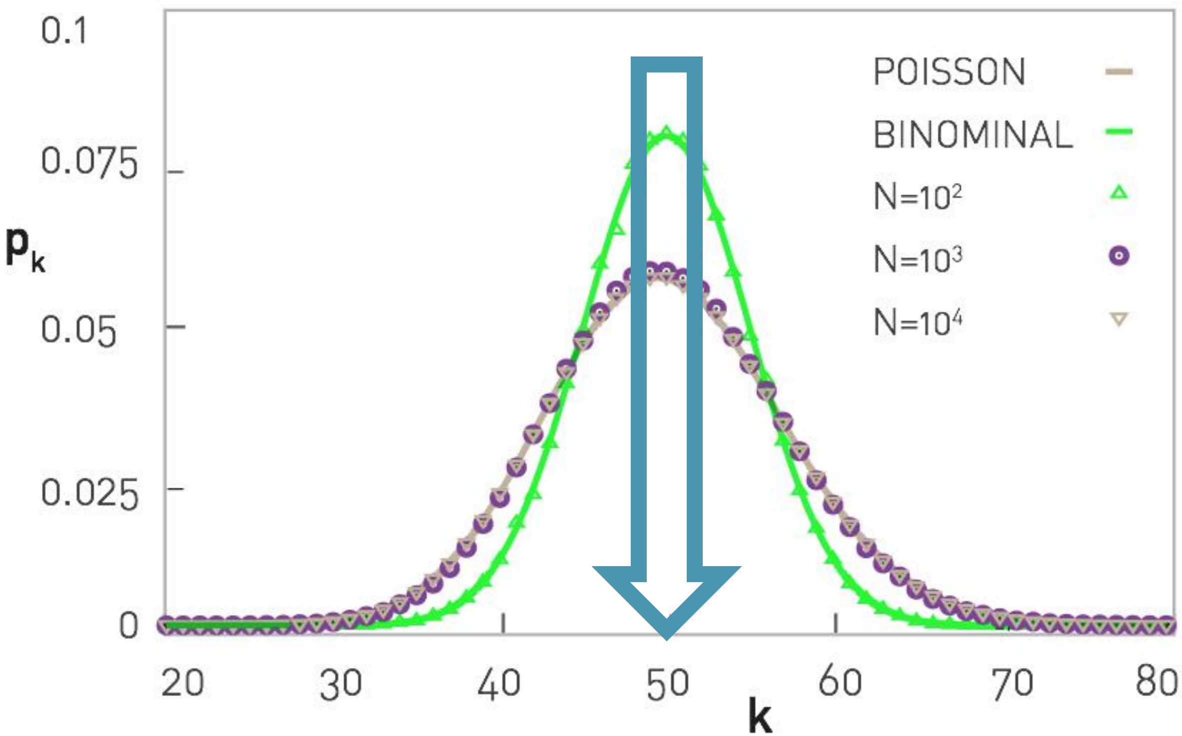
Small nets

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

N>>1

Large nets

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$



Analytical

Simulations

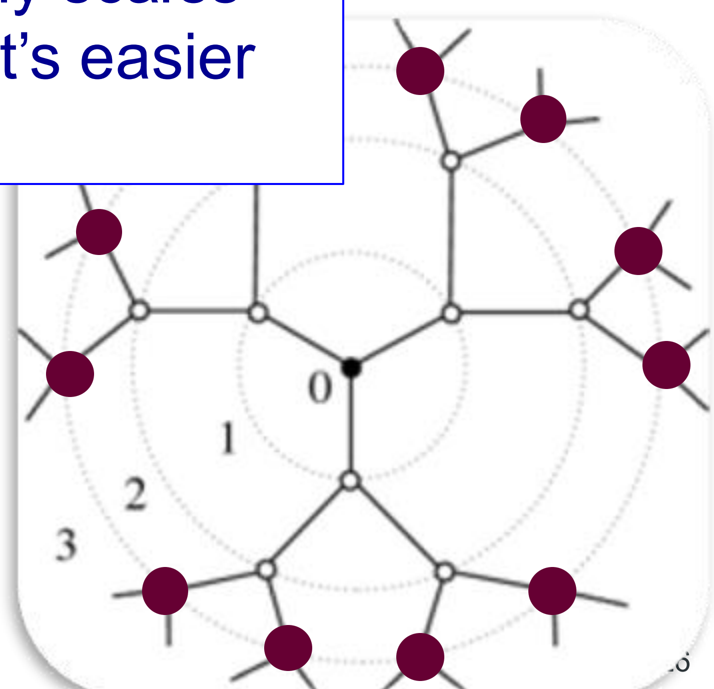
Is a random graph a small world?

- Can we estimate the ***average path length (APL, $<d>$)*** or the ***diameter ($<d_{max}>$)*** for random nets? Or, at least, how it scales with N?

The APL is trickier to computer analytically... Yet, the diameter usually scales with the APL, and it's easier to grasp!

- Assuming a lo… e, i.e., one may say that
  - 1 individual at distance d=0
  - k^1 individuals at d=1
  - k^2 individuals at d=2

    …
  - k^n individuals at d=n



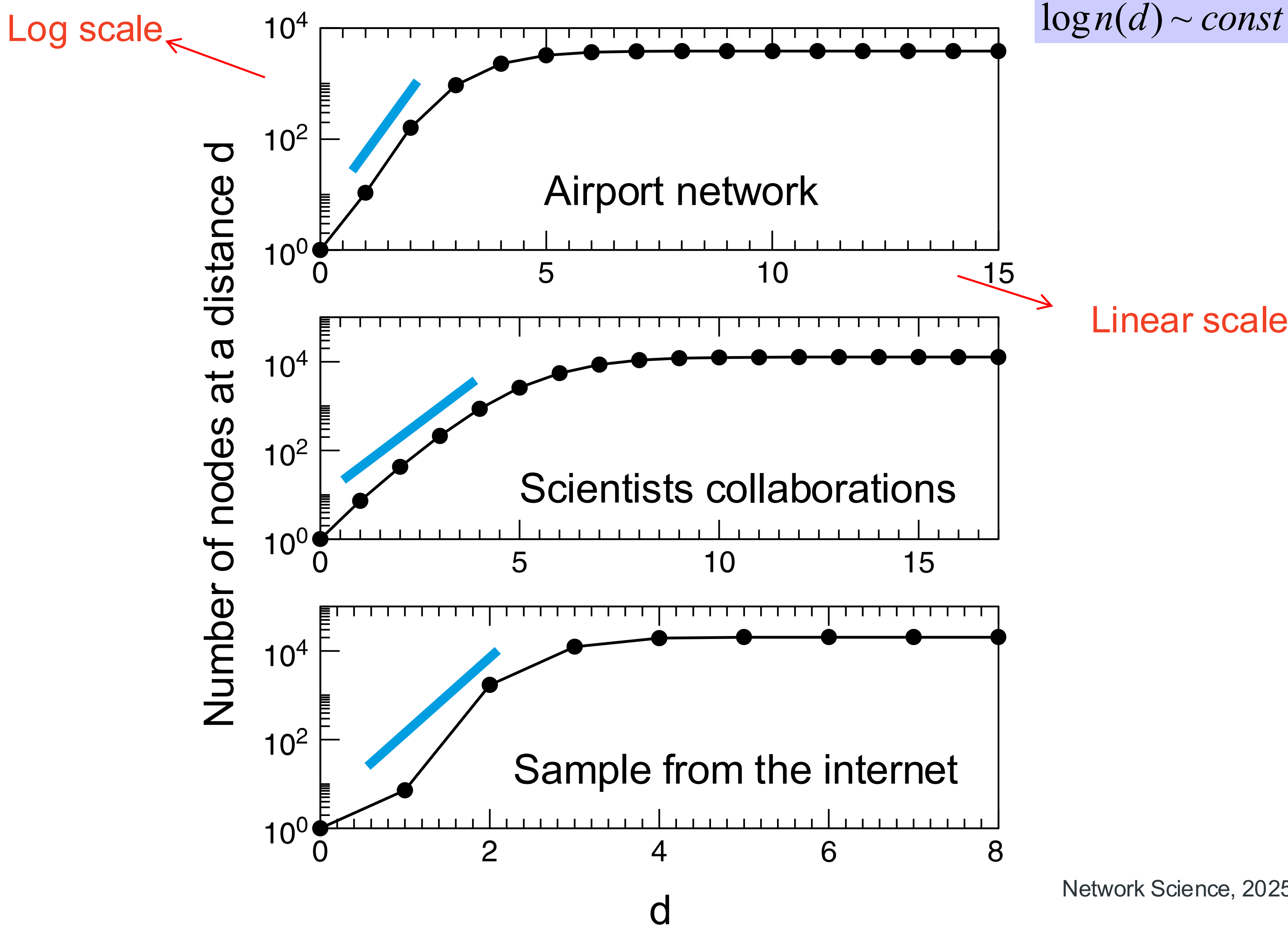
Number of vertices at distance d:

$$n(d) \sim k^d$$

$$\log n(d) \sim const \cdot d$$

Is it a good approximation? YES!!

$$n(d) \sim k^d$$
$$\log n(d) \sim const \cdot d$$



Small world?

- Can we estimate the APL or the diameter for random nets?

YES. Let's compute the number of nodes @ $d \leq d_{max}$

- Thus, the number of individuals at $\boldsymbol{d \leq n}$ is given by

$$N(d) = 1 + \langle k \rangle^1 + \langle k \rangle^2 + \ldots + \langle k \rangle^d = \frac{\langle k \rangle^{d+1} - 1}{\langle k \rangle - 1}$$

What we want to compute

- Networks are finite, thus $N(d_{max}) = N$

- Assuming

$$N(d_{max}) = \frac{\langle k \rangle^{d_{max}+1} - 1}{\langle k \rangle - 1} \approx \frac{\langle k \rangle^{d_{max}+1}}{\langle k \rangle} \approx N = \langle k \rangle^{d_{max}}$$

Small world?

- Can we estimate the APL or the diameter for random nets?

YES. Let's compute the number of nodes @ $d \leq d_{max}$

- Thus, the number of individuals at $d \leq n$ is given by

$$N(d) = 1 + \langle k \rangle^1 + \langle k \rangle^2 + \ldots + \langle k \rangle^n = \frac{\langle k \rangle^{d+1} - 1}{\langle k \rangle - 1}$$

- Networks are finite, thus $N(d_{max}) = N$

- We get

$$d_{max} \approx \frac{\ln N}{\ln \langle k \rangle}$$

Small world?

We reached to the diameter of a random network

$$d_{\max} \sim \frac{\ln N}{\ln\langle k\rangle}$$

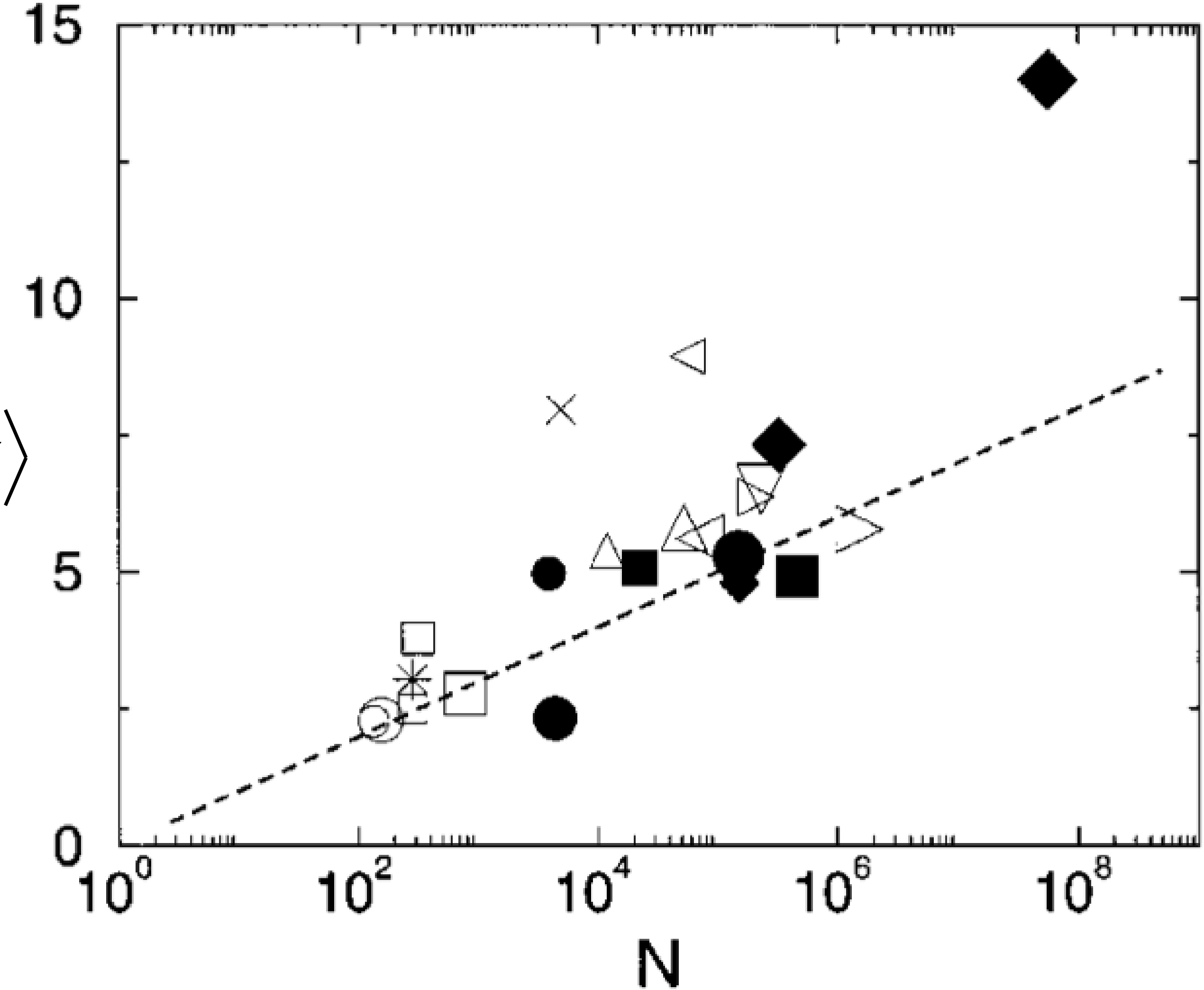
d_{max} is often dominated by the extreme parts, being subject to large fluctuations. The average path length <d> will not suffer from this, and is often well approximated by this scaling

$$APL = \langle d\rangle \sim \frac{\ln N}{\ln\langle k\rangle}$$

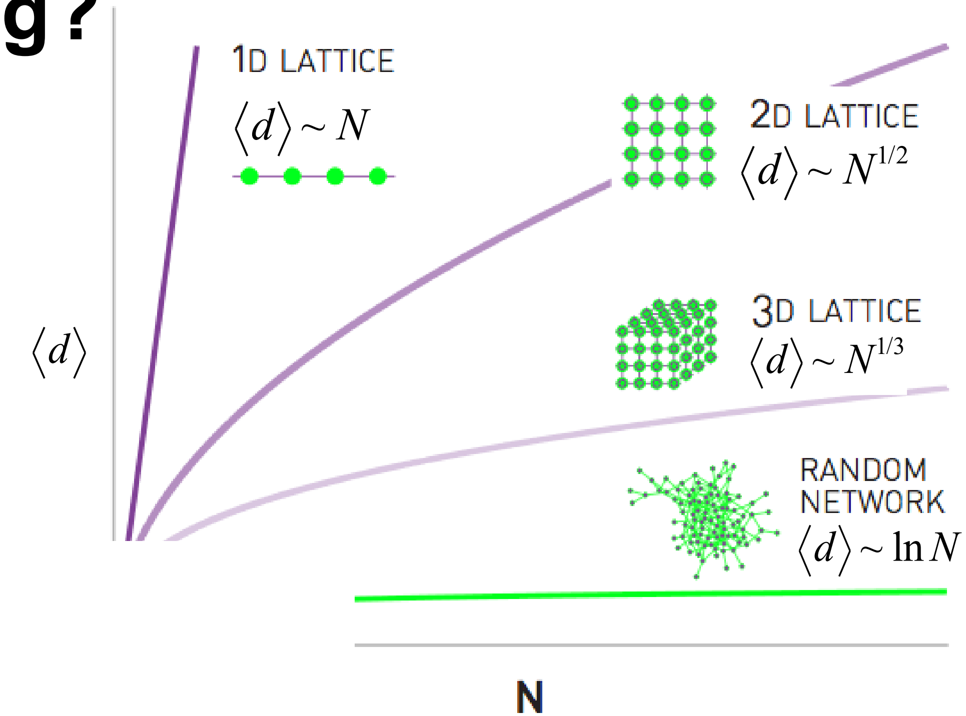
constant

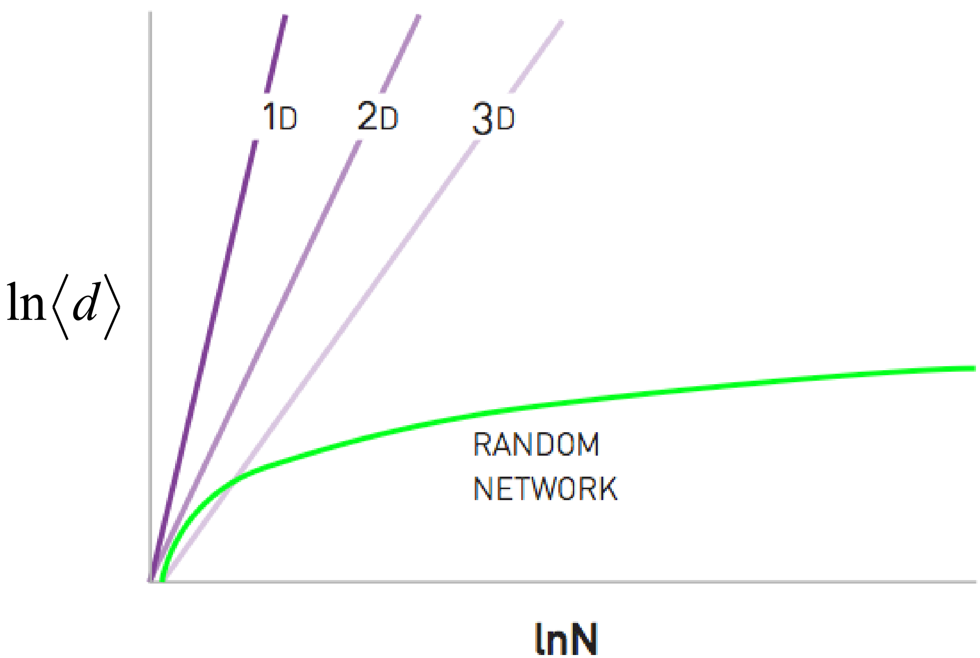
How does the APL scale with N? It works!!!

$$\langle d \rangle \ln \langle k \rangle$$



Why is it surprising?





This log dependence became the fingerprint of a small-world!

It works!!!

Empirical data

NETWORK	N	L	$\langle k \rangle$	$\langle d \rangle$	$\frac{\ln N}{\ln \langle k \rangle}$
Internet	192,244	609,066	6.34	6.98	6.58
WWW	325,729	1,497,134	4.60	11.27	8.31
Power Grid	4,941	6,594	2.67	18.99	8.66
Mobile Phone Calls	36,595	91,826	2.51	11.72	11.42
Email	57,194	103,731	1.81	5.88	18.4
Science Collaboration	23,133	93,439	8.08	5.35	4.81
Actor Network	702,388	29,397,908	83,71	3,91	3,04
Citation Network	449,673	4,707,958	10.43	11,21	5.55
E. Coli Metabolism	1,039	5,802	5.58	2.98	4.04
Protein Interactions	2,018	2,930	2.90	5.61	7.14

Random networks: Overview (1)

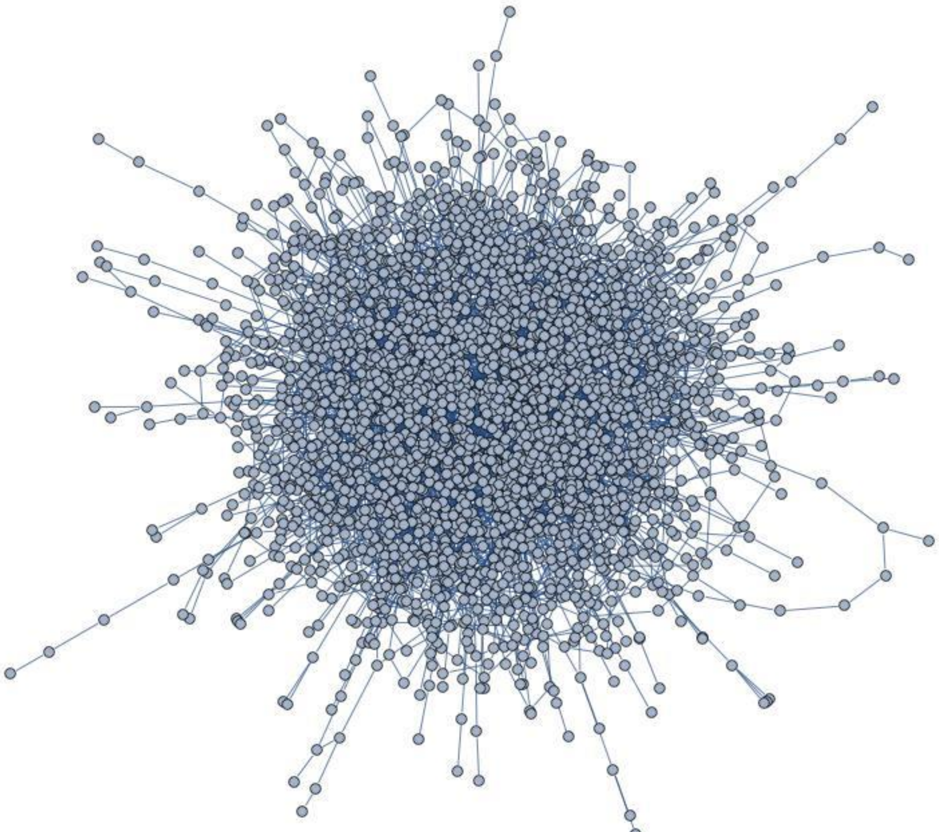
Model fits empirical data?

- Small world effect **YES!!**

Nice! Let's continue to compare the ER model with real networks…

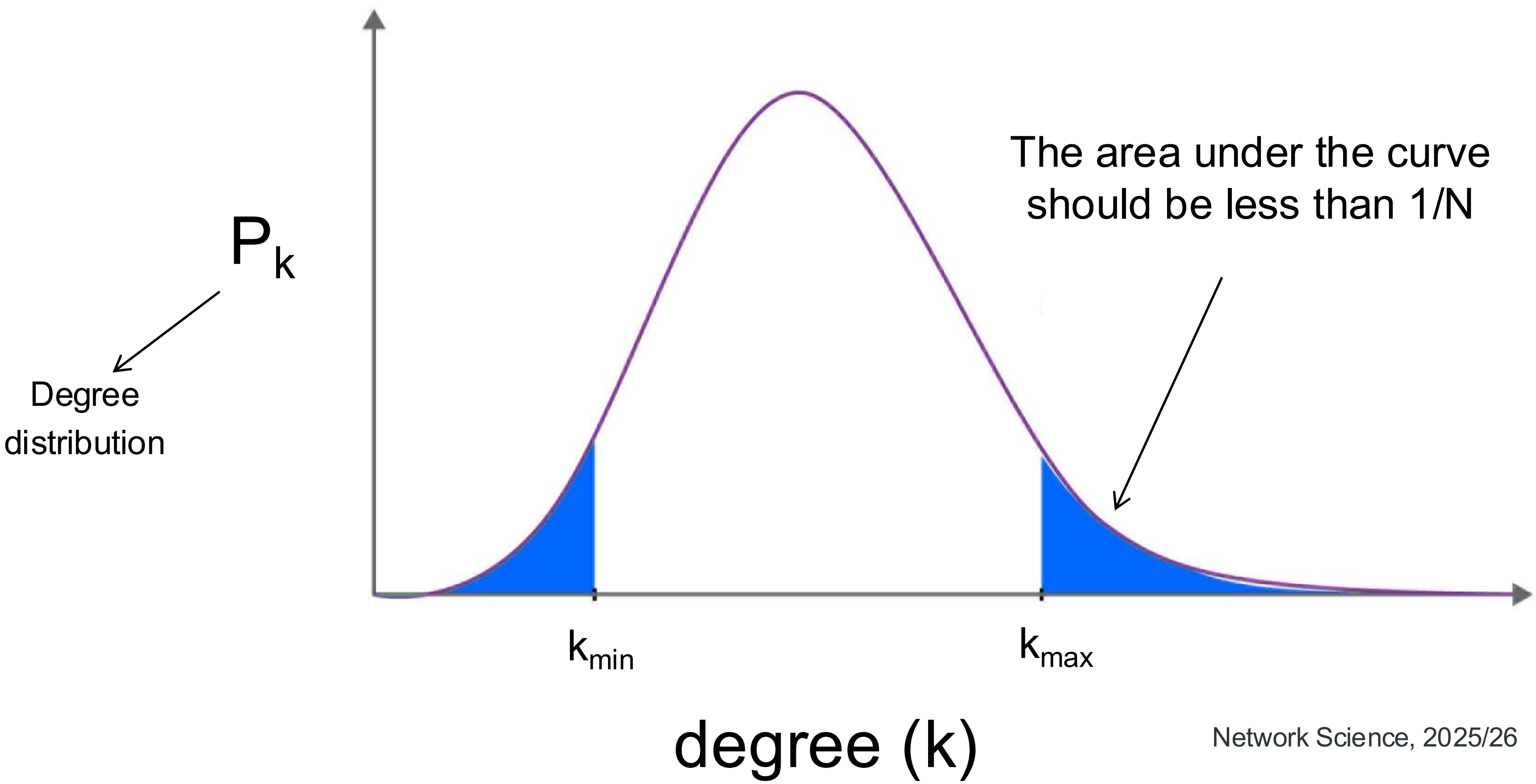
Nice! Let's continue to compare ER networks with real networks…

- Can we estimate the maximum and minimum degree of a random network?



Nice! Let's compare with real networks…

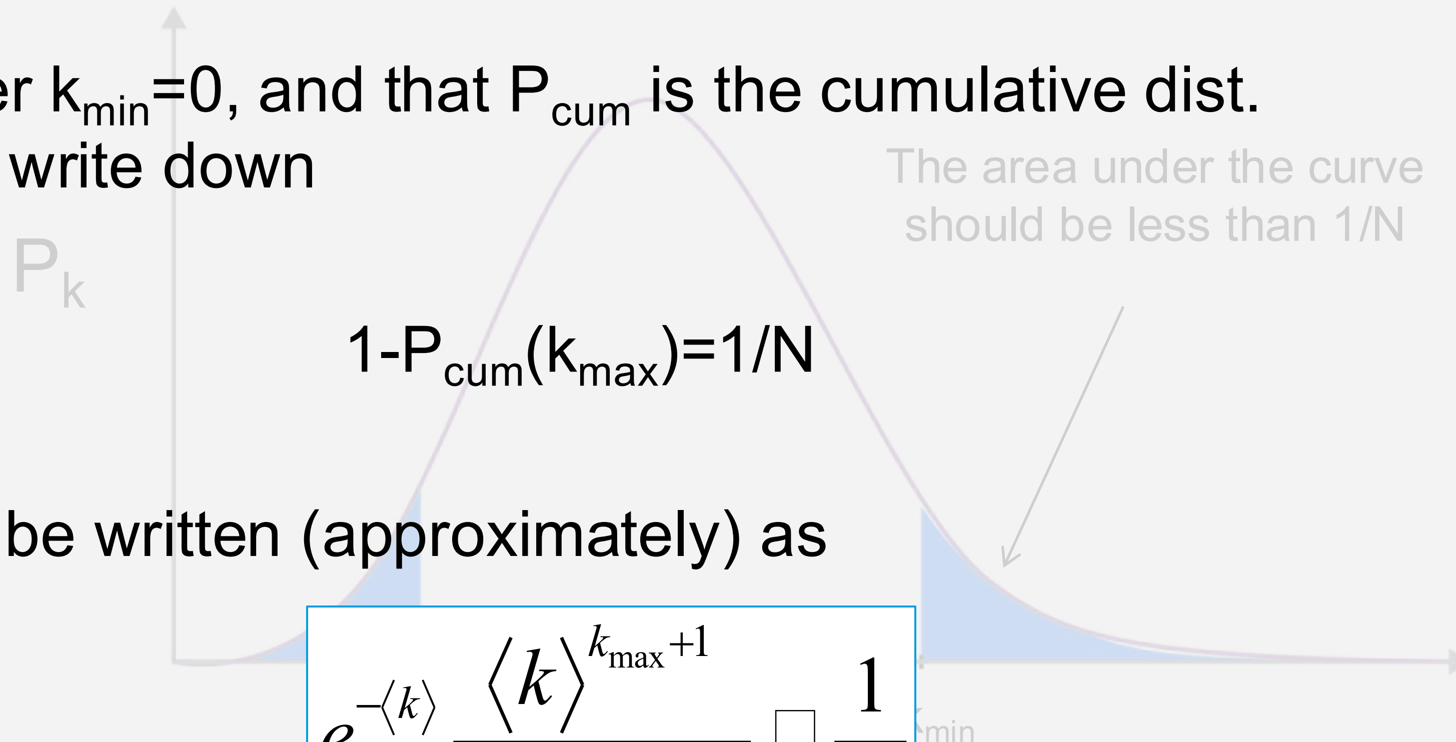
- **Can we estimate the maximum and minimum degree of a random network?**



Nice! Let's compare with real networks…

- **Can we estimate the maximum and minimum degree of a random network?**

- Consider k_{min}=0, and that P_{cum} is the cumulative dist. We can write down



$$1-P_{cum}(k_{max})=1/N$$

which can be written (approximately) as

$$e^{-\langle k\rangle}\frac{\langle k\rangle^{k_{max}+1}}{(k_{max}+1)!} \approx \frac{1}{N}$$

Nice! Let's compare with real networks…

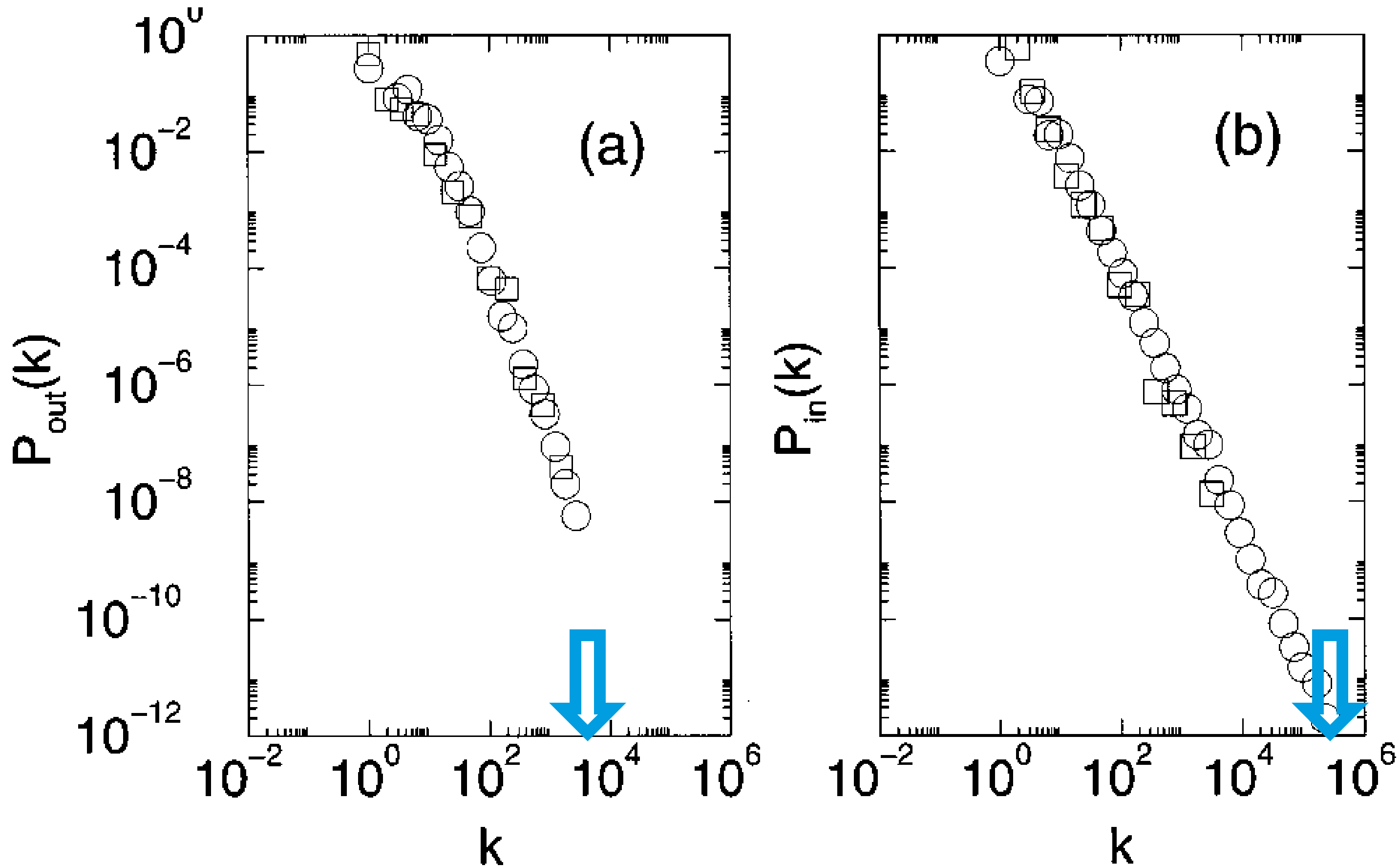
- What's the maximum and minimum degree of a random network?

- Let's say we have ***$N=2x10^8$*** documents in the WWW (estimates indicate ***$<k>\sim4.60$***).
- What would be the **maximum degree of the WWW** if we had a random graph?

$k_{max}\sim21$… Weird!!

We are lacking hubs in ER networks!

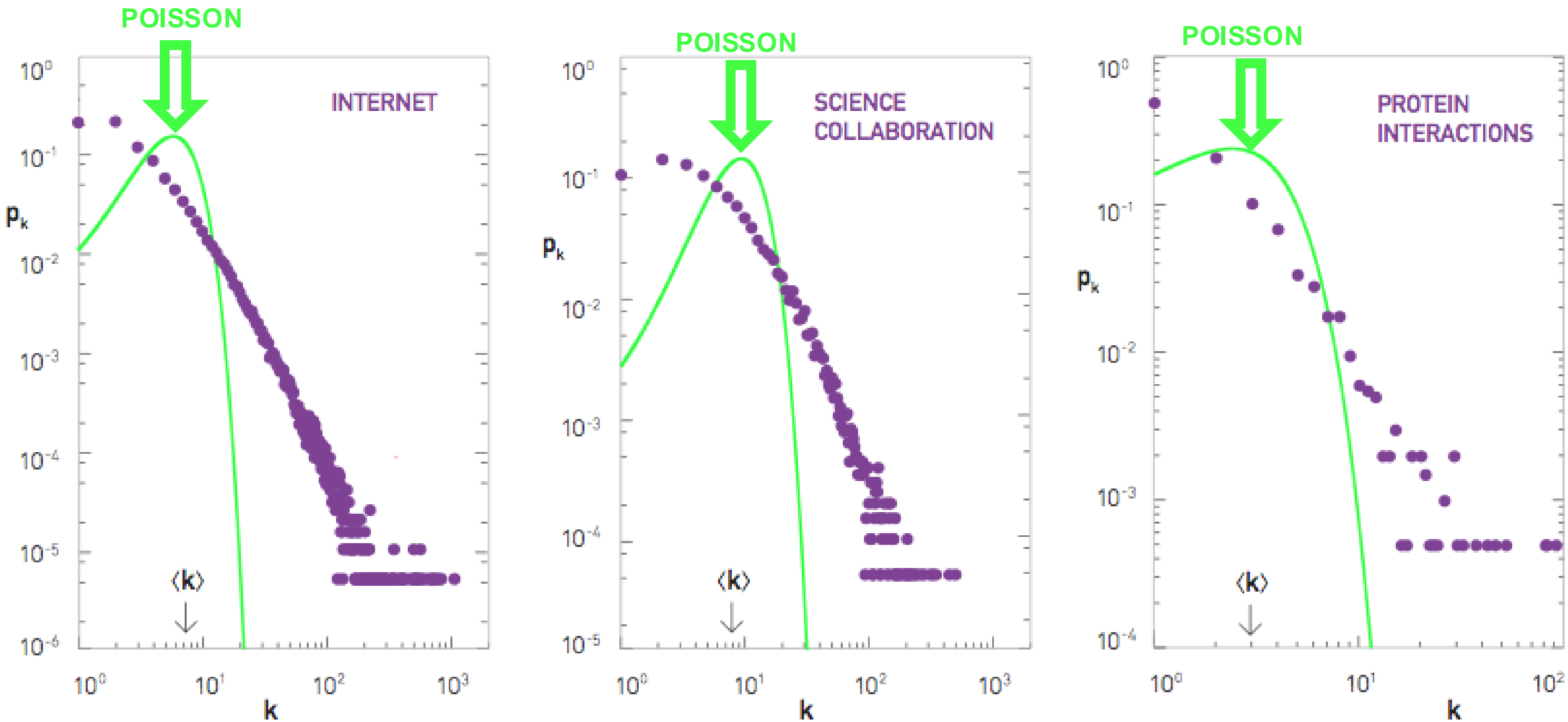
Degree distribution of the World-Wide-Web



Albert *et al.* (1999) (~326 thousand pages)

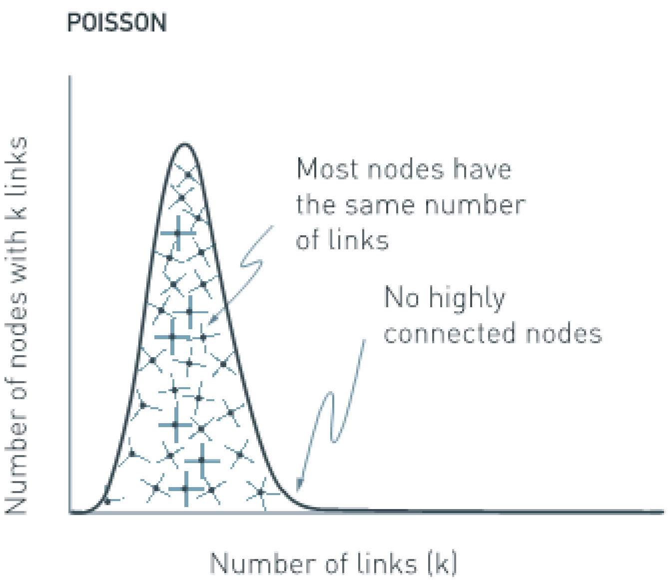
Broder et al. (2000) (~200 millions pages)

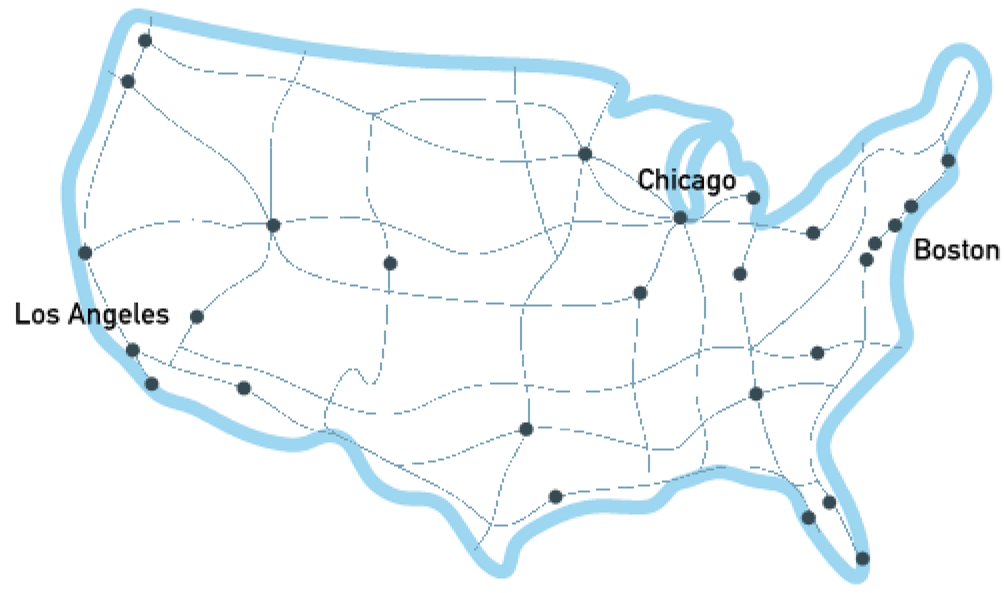
Real degree distributions



Real degree distributions

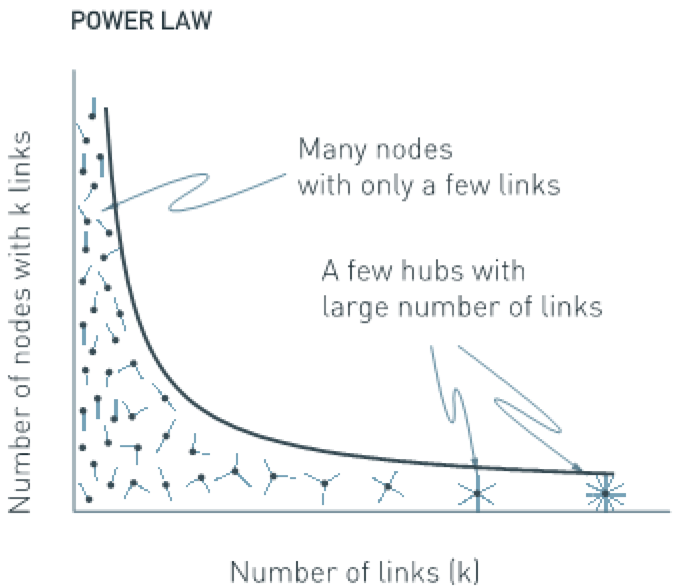
In some cases, random networks provide an accurate description

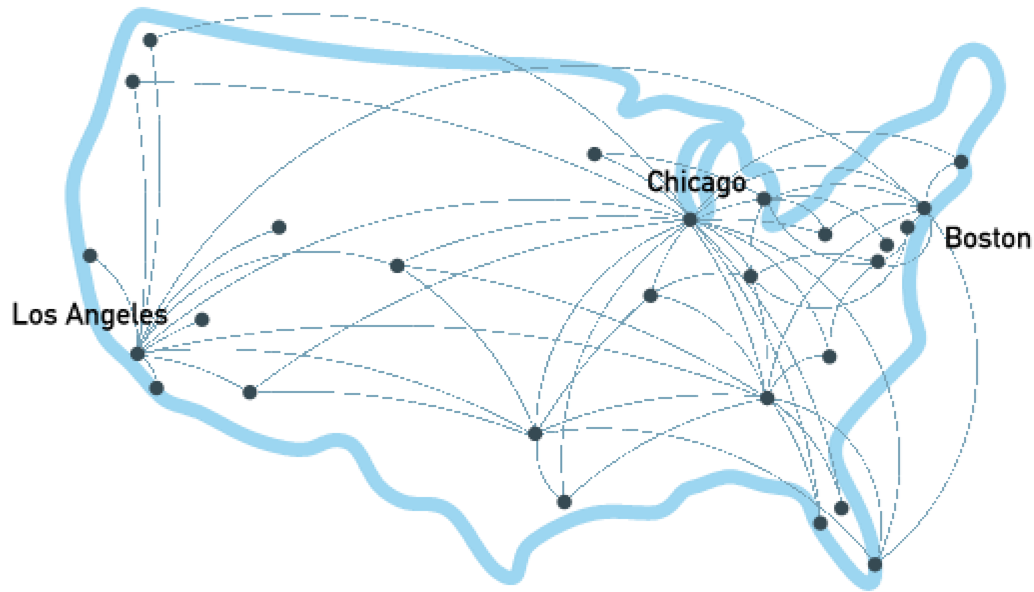




Real degree distributions

...but in many other cases it fails.



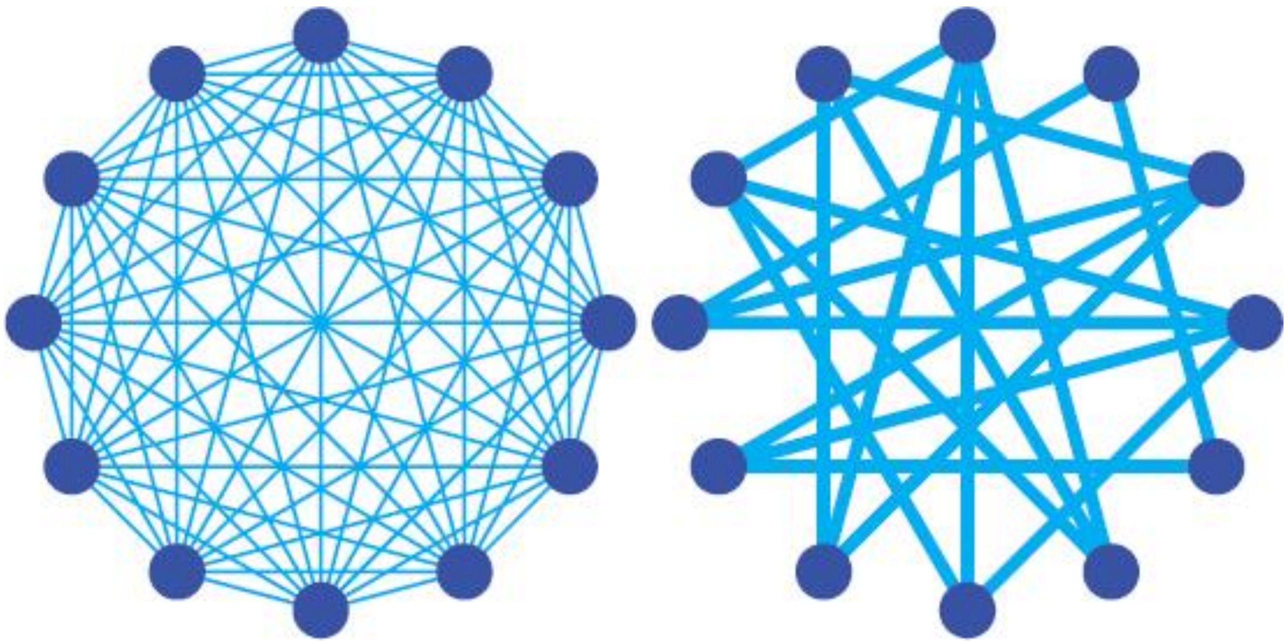


Random networks: Overview (2)

	Model fits empirical data?
• Small-world effect	**YES!**
• Degree distribution	**Sometimes. Mostly NO!**
• Maximum degree	**Sometimes. Mostly NO!**

Mind your friend's friends

- Can we compute the clustering coefficient for random nets?

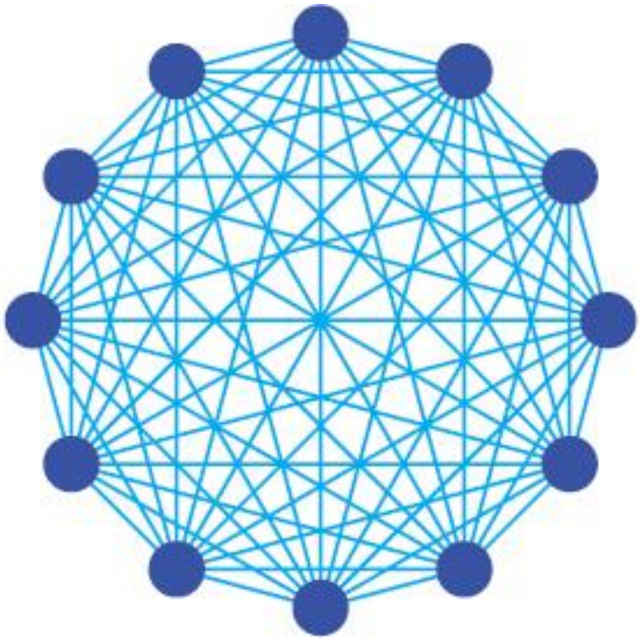


$$C_i = \frac{\text{\# edges among neigs.}}{\text{max \# edges among neigs.}} = \frac{e_i}{k_i(k_i - 1)/2}$$

Mind your friend's friends

- Can we compute the clustering coefficient for random nets?

complete graph



Average number of links e_i in the neighborhood of a node i with $\boldsymbol{k_i}$ partners?

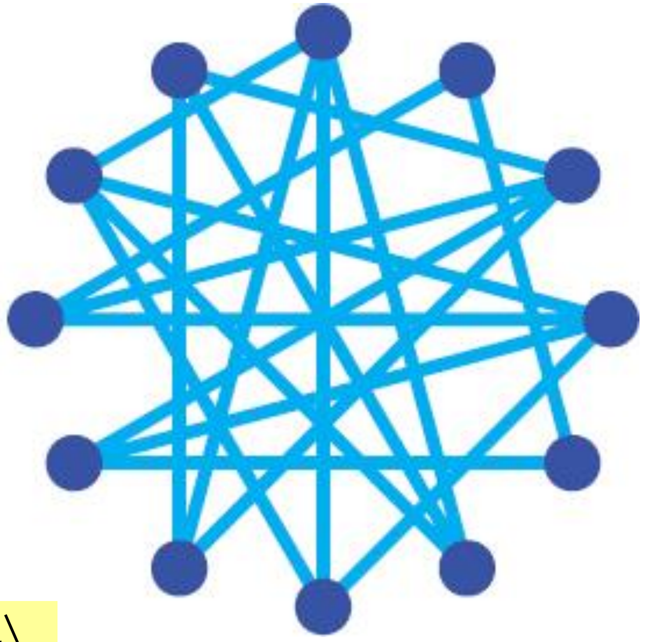
$$\frac{k_i(k_i-1)}{2}$$

Mind your friend's friends

- Can we compute the clustering coefficient for random nets?

Average number of links e_i in the neighborhood of a node i with k_i partners?

$$p \times \frac{k_i(k_i-1)}{2}$$



Thus, the clustering coefficient reads

$$C_i = \frac{e_i}{k_i(k_i-1)/2} = p\frac{k_i(k_i-1)/2}{k_i(k_i-1)/2} = p = \frac{\langle k \rangle}{N-1}$$

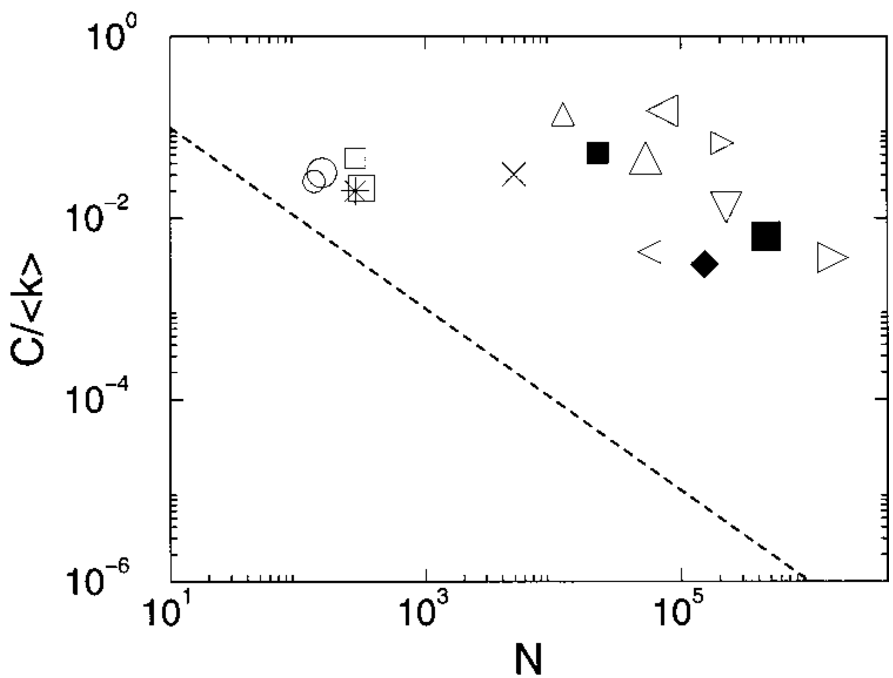
Mind your friend’s friends

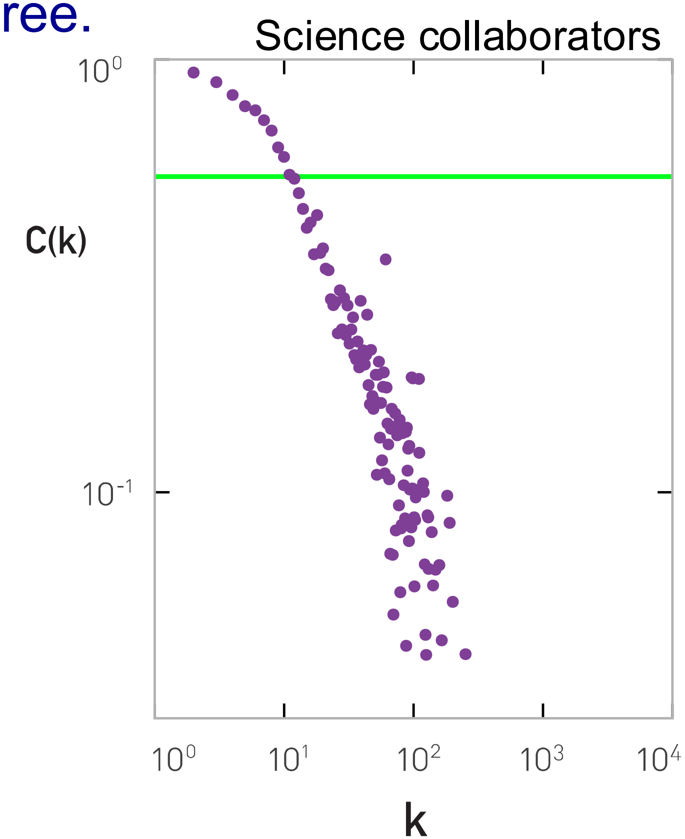
$$C_i \,\square\, \frac{\langle k \rangle}{N}$$

Predictions:

1. For a fixed <k>, larger the network, smaller the clustering coefficient.

2. Clustering coefficient is independent of the node’s degree.





Clustering coeff. of a random graph

Real world networks often show large Clustering Coeff.

Film actors	➔	C=0.2
Company directors	➔	C=0.59
Math co-authorship	➔	C=0.15
Physics co-authorship	➔	C=0.45
Physics co-authorship	➔	C=0.45
WWW	➔	C=0.11
Neural networks	➔	C=0.18

$$C = \frac{\langle k \rangle}{N} \quad ???$$

Conclusion:

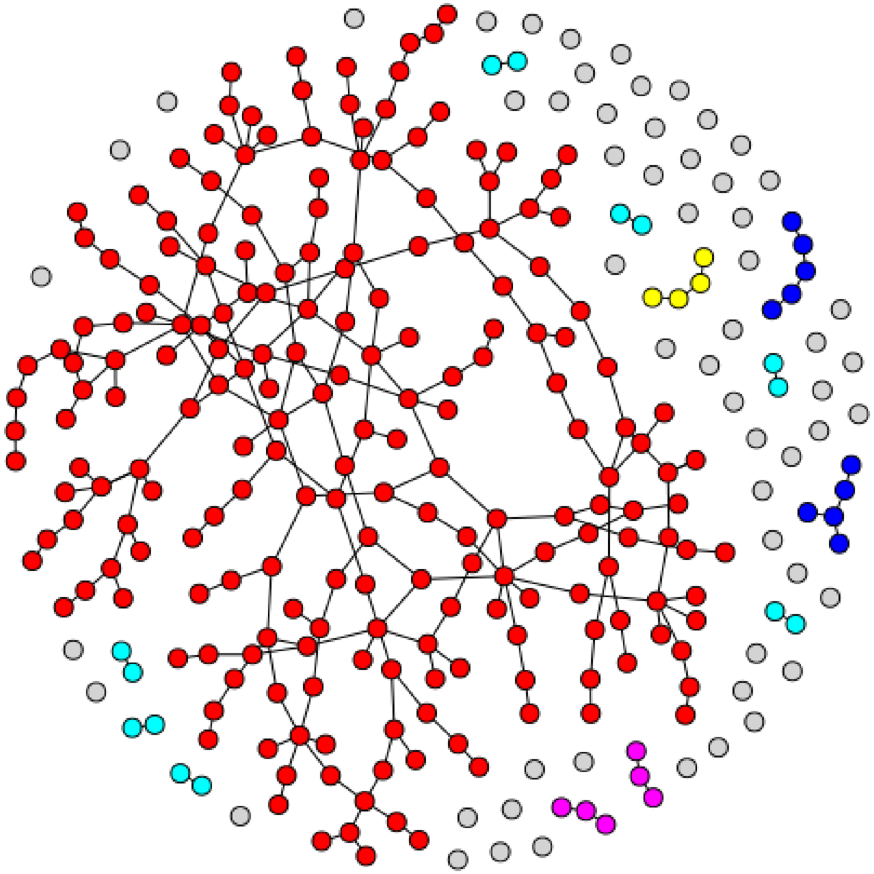
Are real-world networks random?

(Watts & Strogratz, Nature 1998)

	Model fits empirical data?
• Degree distribution	Sometimes. Mostly NO!
• Maximum degree	Sometimes. Mostly NO!
• Small world effect	YES!!
• Clustering coeff.	NO! CC is often high, scaling with the degree and not with N.

A last (& funny) message on random networks:

The bizarre world of the largest component

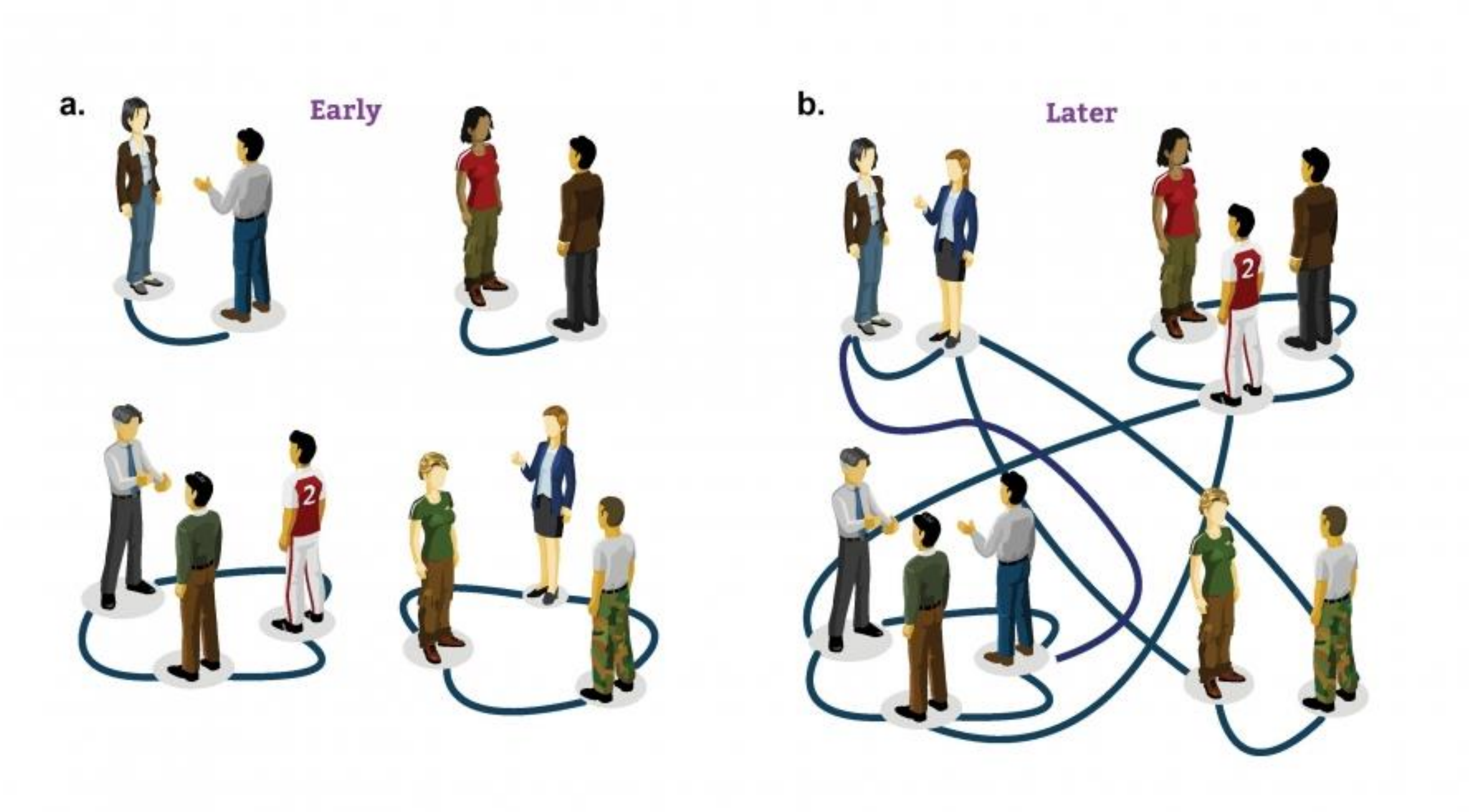


Last message on random networks:

The bizarre world of the largest component

Early on the guests form isolated groups.

As individuals mingle, changing groups, a connected network emerges



The bizarre world of the largest component

For p=0 all nodes are isolated

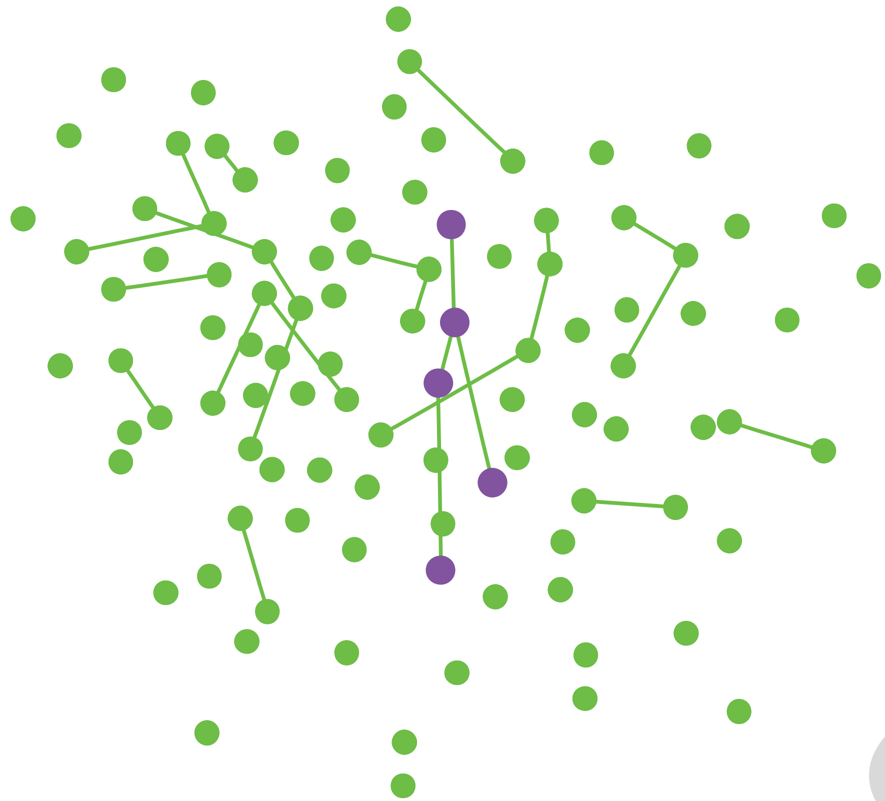
For p=1 we have a fully-connected graph

?

Average degree increases linearly with p

What happens with the size of the giant component?

The bizarre world of giant components

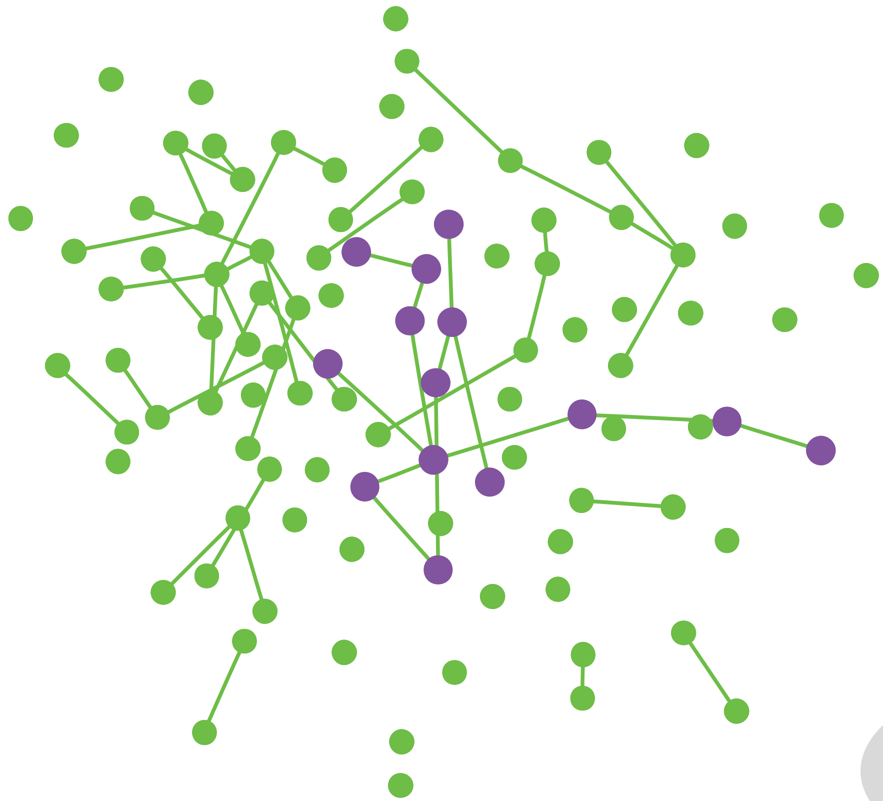


$p<1/N$

Subcritical Regime

$\langle k \rangle < 1$

The bizarre world of giant components

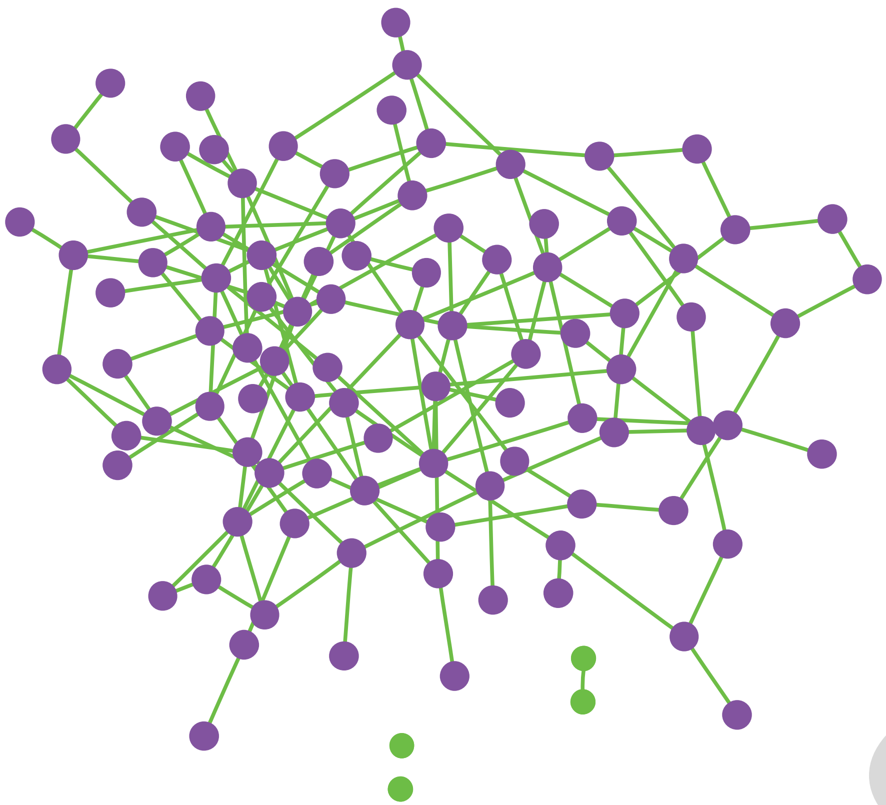


p=1/N

Critical Point

$\langle k \rangle = 1$

The bizarre world of giant components



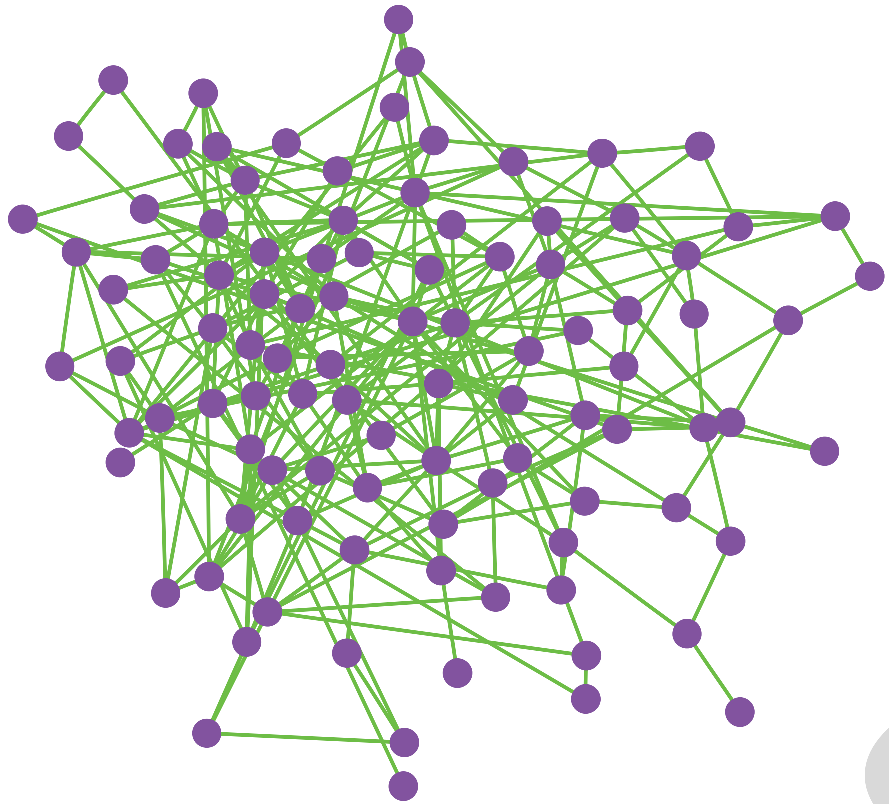
$p > 1/N$

Supercritical Regime

$\langle k \rangle > 1$

giant component coexists with many disconnected components

The bizarre world of giant components



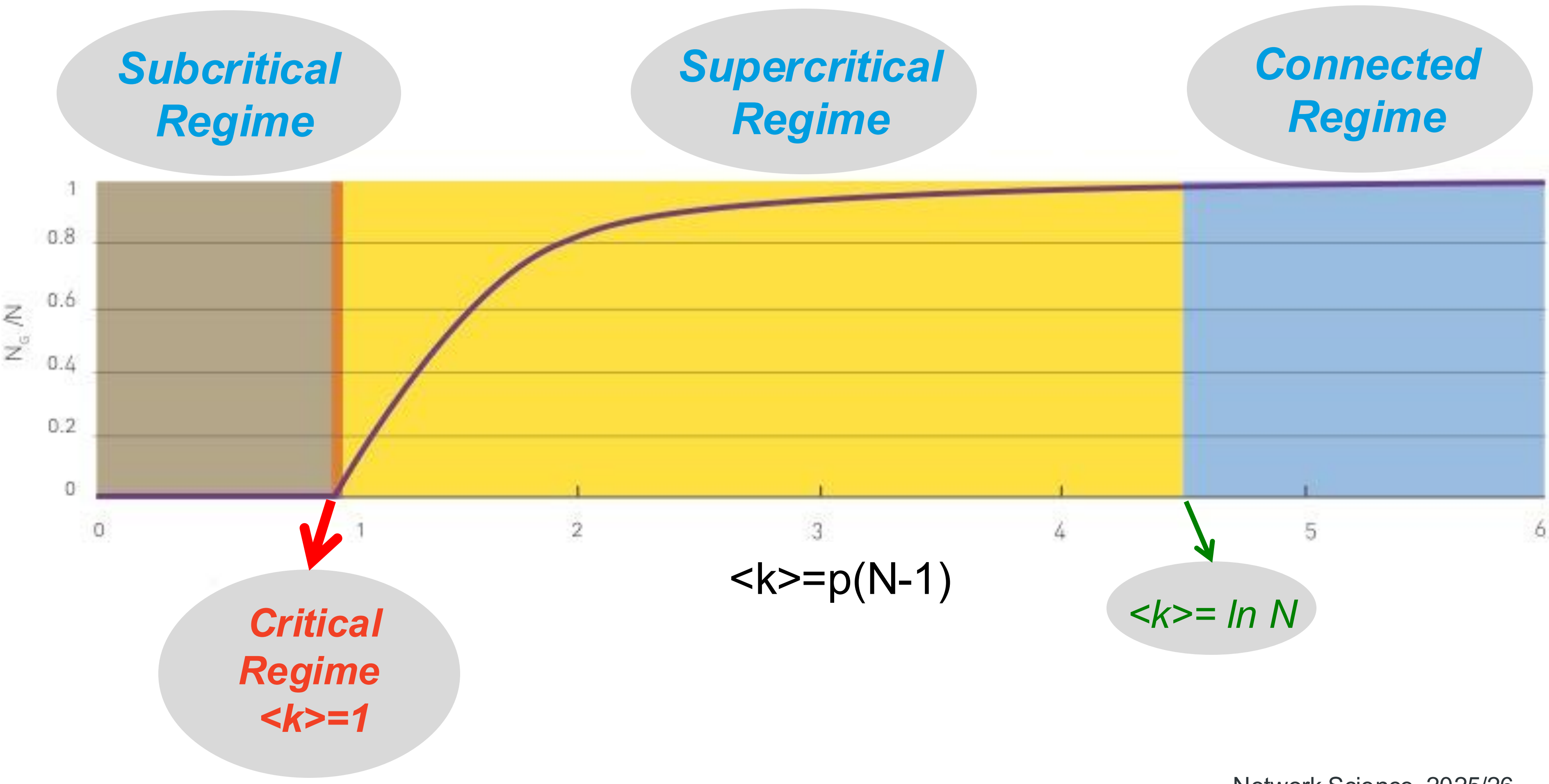
$Np > \ln N$

Connected Regime

$\langle k \rangle \gg \ln N$

Connected network!

The bizarre world of giant components



Two predictions

Two predictions of random network theory are of direct importance for real networks:

- Once the average degree exceeds <k> = 1, a giant component should emerge, with plenty of disconnected components. Hence only for ‹k› > 1 the nodes organize themselves into a recognizable network.

- For ‹k› > ln N all components are absorbed by the giant component, resulting in a single connected network.

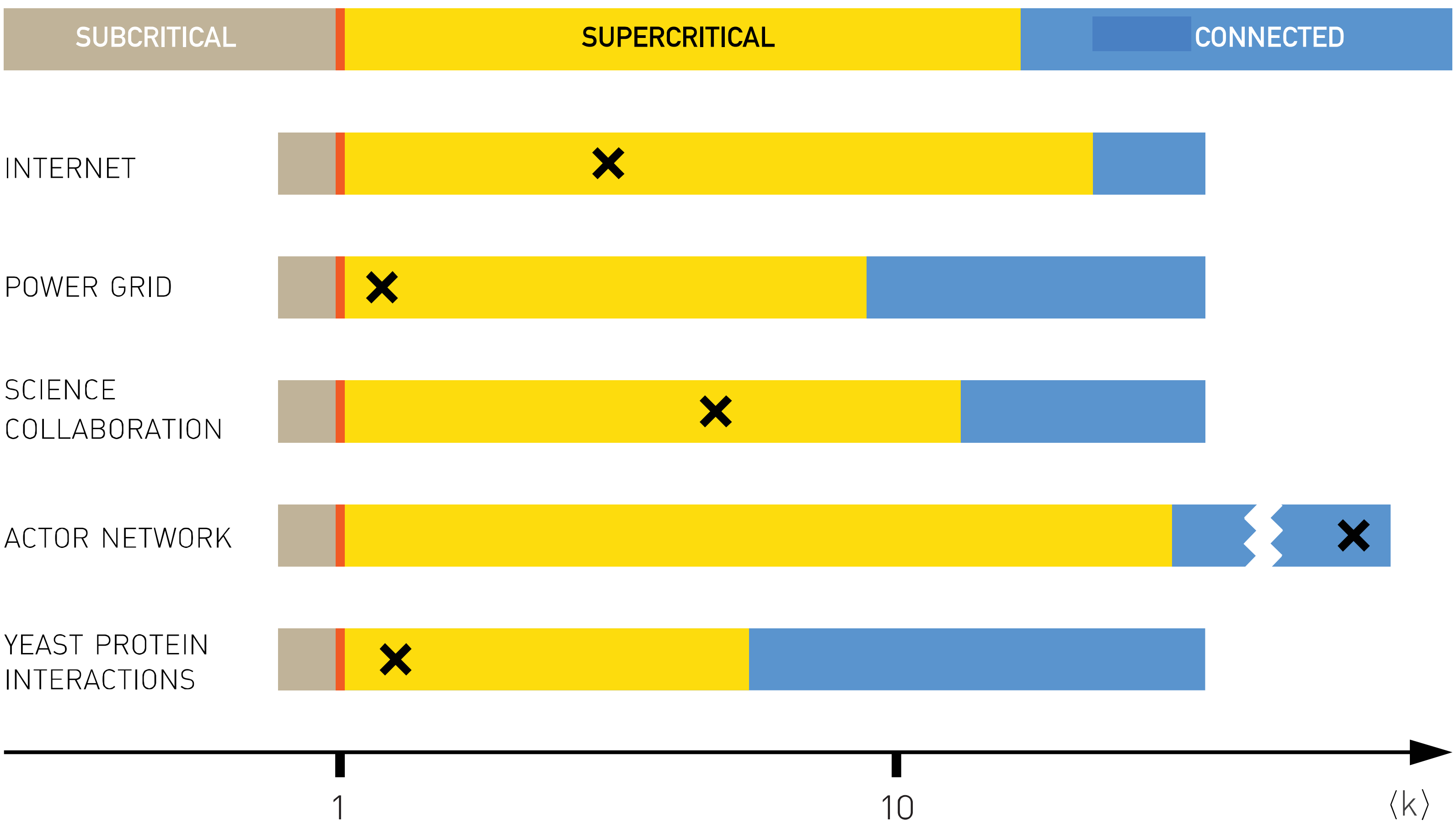
We have a problem!! Most networks show a single connected component. However, ‹k› is lower than *ln N*

Where should we place real networks?

Network	N	L	‹K›		lnN
Internet	192,244	609,066	6.34	<	12.17
Power Grid	4,941	6,594	2.67	<	8.51
Science Collaboration	23,133	94,437	8.08	<	10.05
Actor Network	702,388	29,397,908	83.71		13.46
Protein Interactions	2,018	2,930	2.90	<	7.61

Supercritical Regime

Where should we place real networks?



Where should we place real networks?

- Most real networks fall into the supercritical regime and, therefore, assuming that the world follows a ER graph, we should expect
  - **A giant component (correct!!)**
  - **Plenty of disconnected components (not necessarily!)**

In real-world networks, we can have connected networks for very low average degrees... We are still missing something!

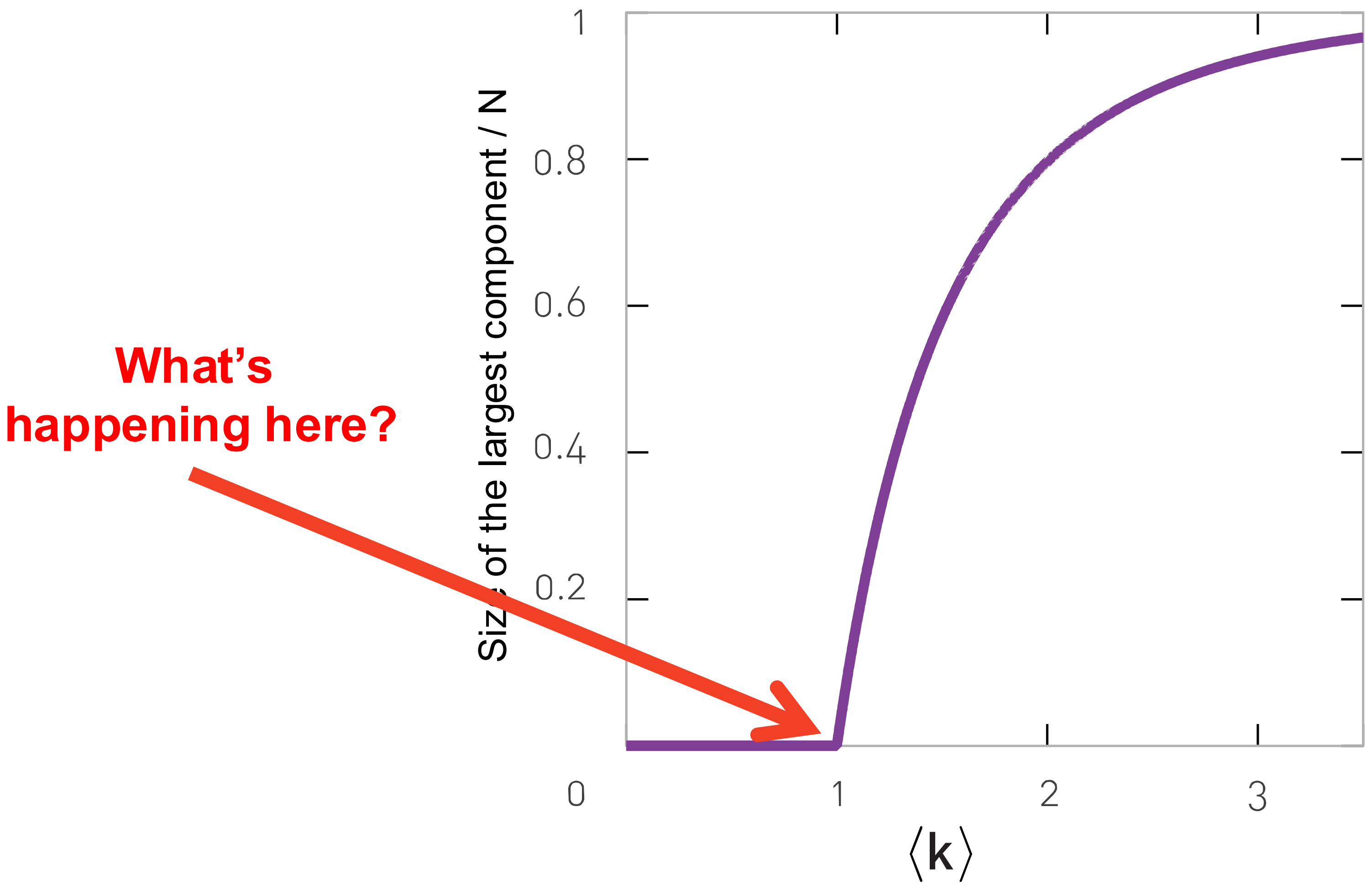
Today's conclusions on random graphs

	Model fits empirical data?
• Degree distribution	Sometimes. Mostly NO!
• Maximum degree	Sometimes. Mostly NO!
• Clustering coeff.	NO! CC is often high, scaling with the degree and not with N.
• Small world effect	YES!!

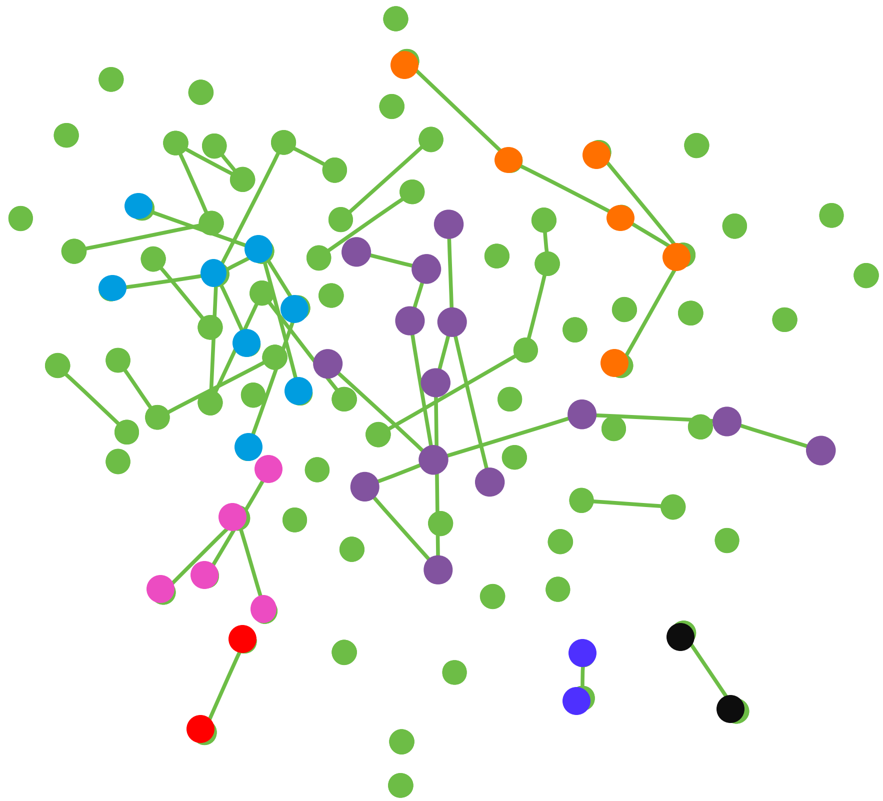
- We need an additional mechanism to guarantee a single connected component for sparse graphs

Let me return
to such ”critical” transition...

Critical transitions?

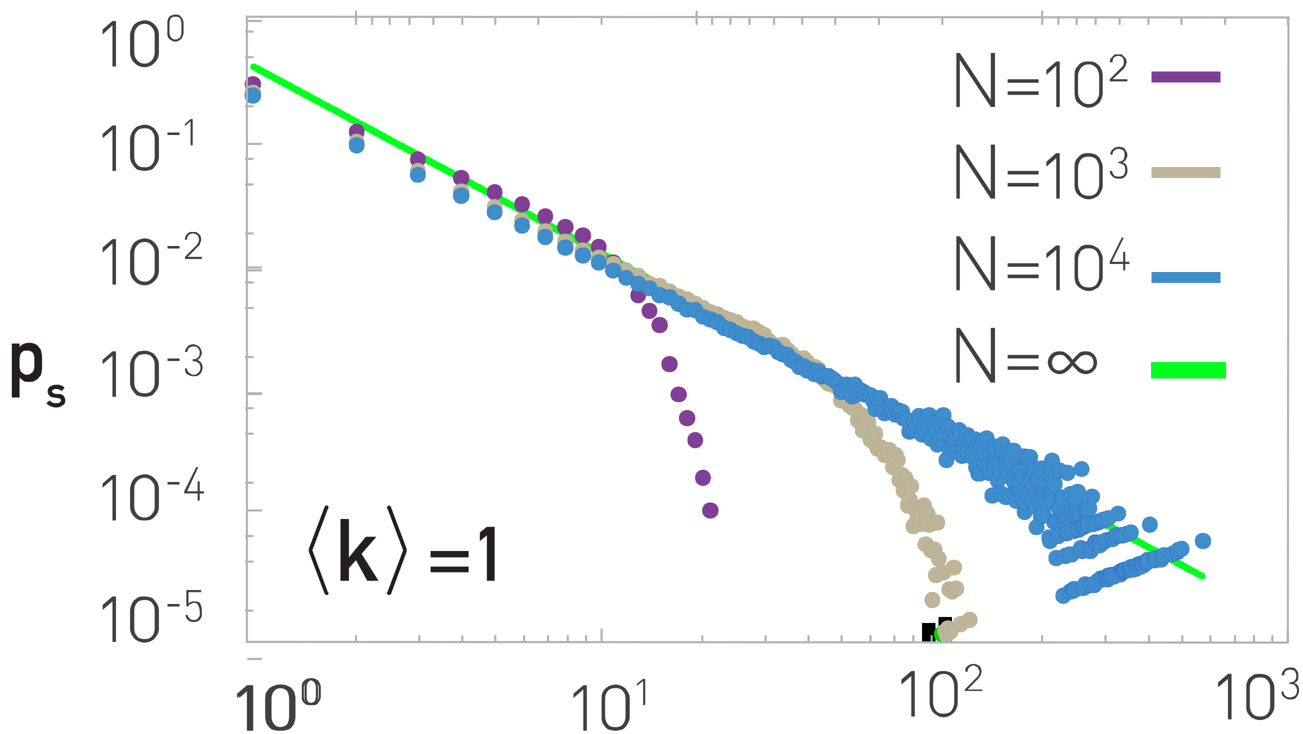


Critical transitions?



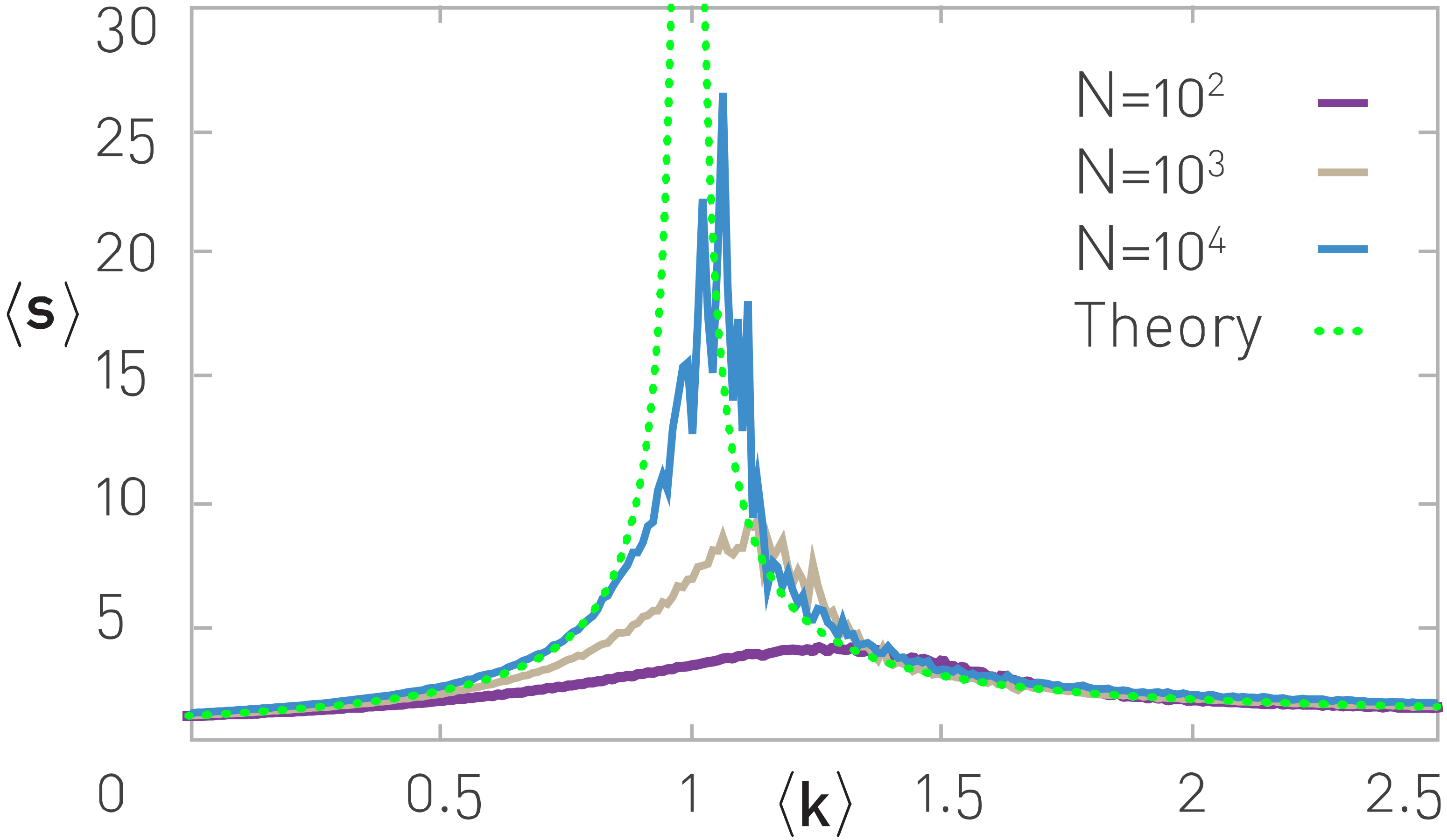
$\langle k \rangle = 1$

Component size dist. @ transition



$$p_s \sim s^{-3/2}$$

Average size of components @ transition

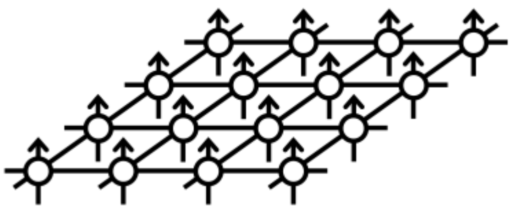


$$\langle s \rangle = \frac{1}{1 - \langle k \rangle + \langle k \rangle N_G / N}$$

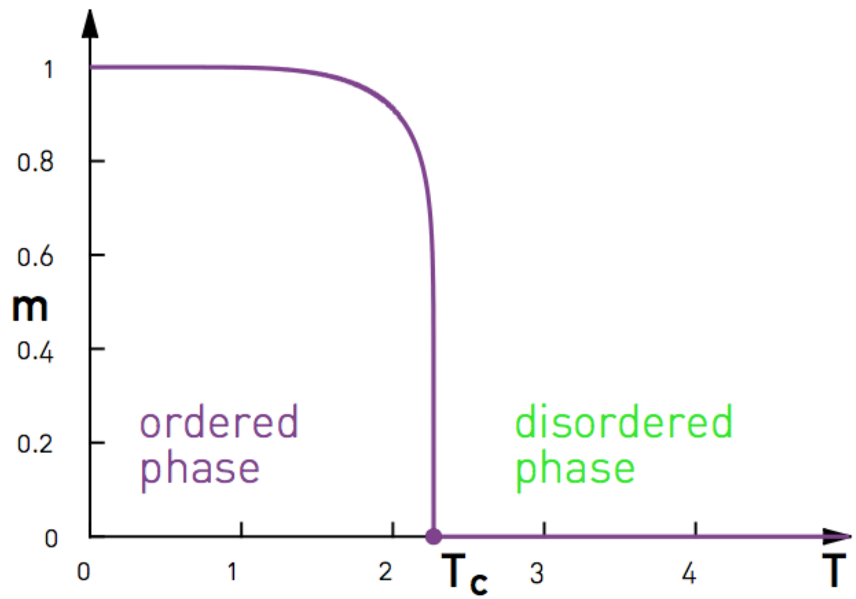
Phase/critical transitions everywhere

In ferromagnetic materials the magnetic moments of the individual atoms (spins) can point in two different directions.

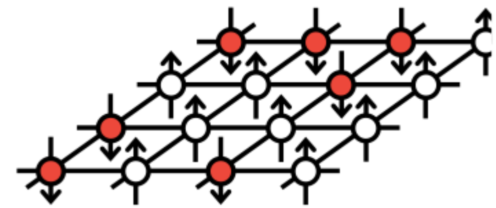
At low temperatures all spins point in the same direction



ordered phase



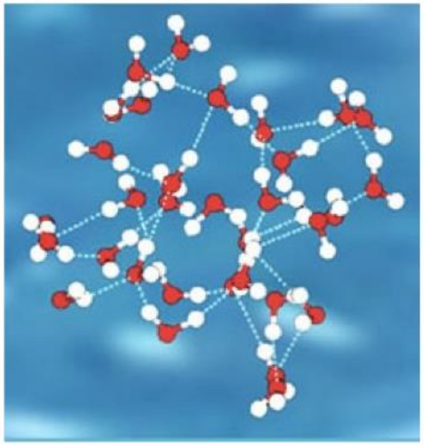
At high temperatures they choose randomly their direction.
Here m (magnetization = sum of ups – downs) is zero.

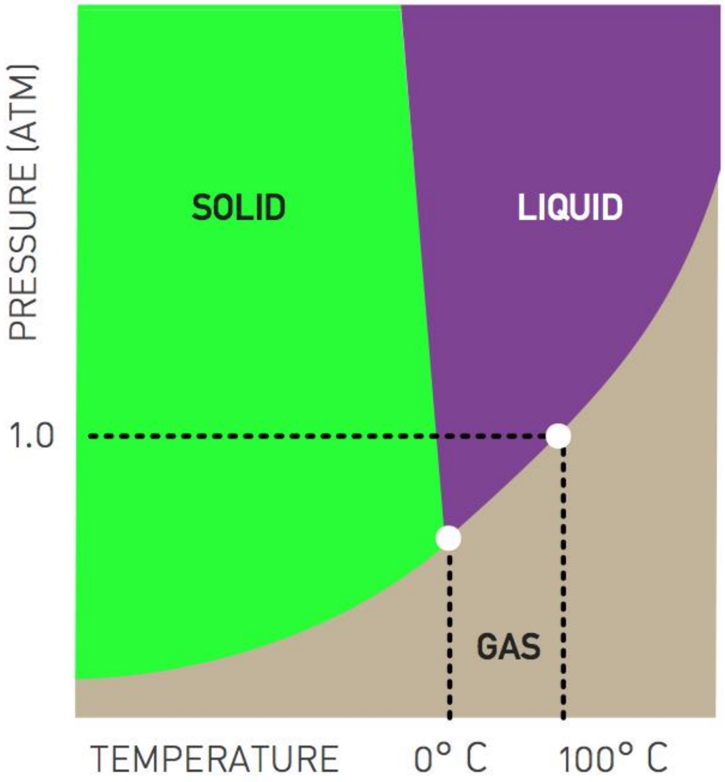


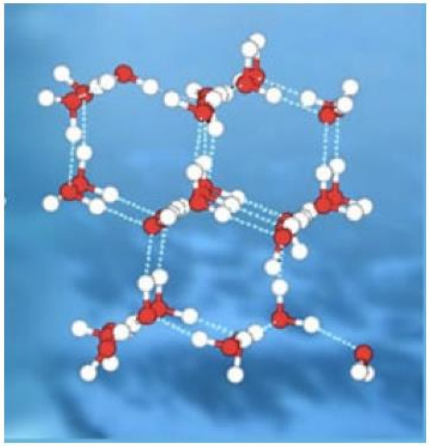
disordered phase

At a critical temperature T_c the system undergoes a phase transition just like we had in random graphs!

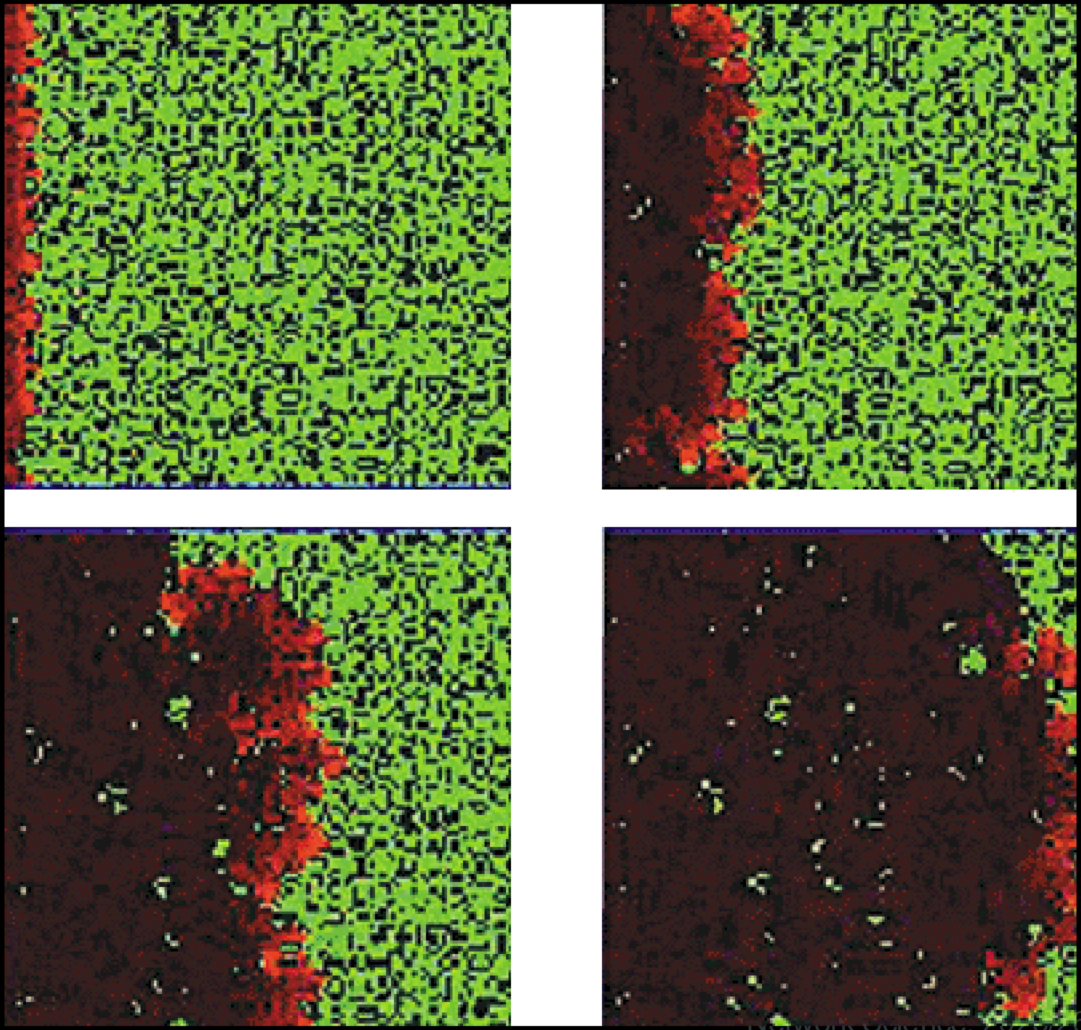
Phase/critical transitions everywhere







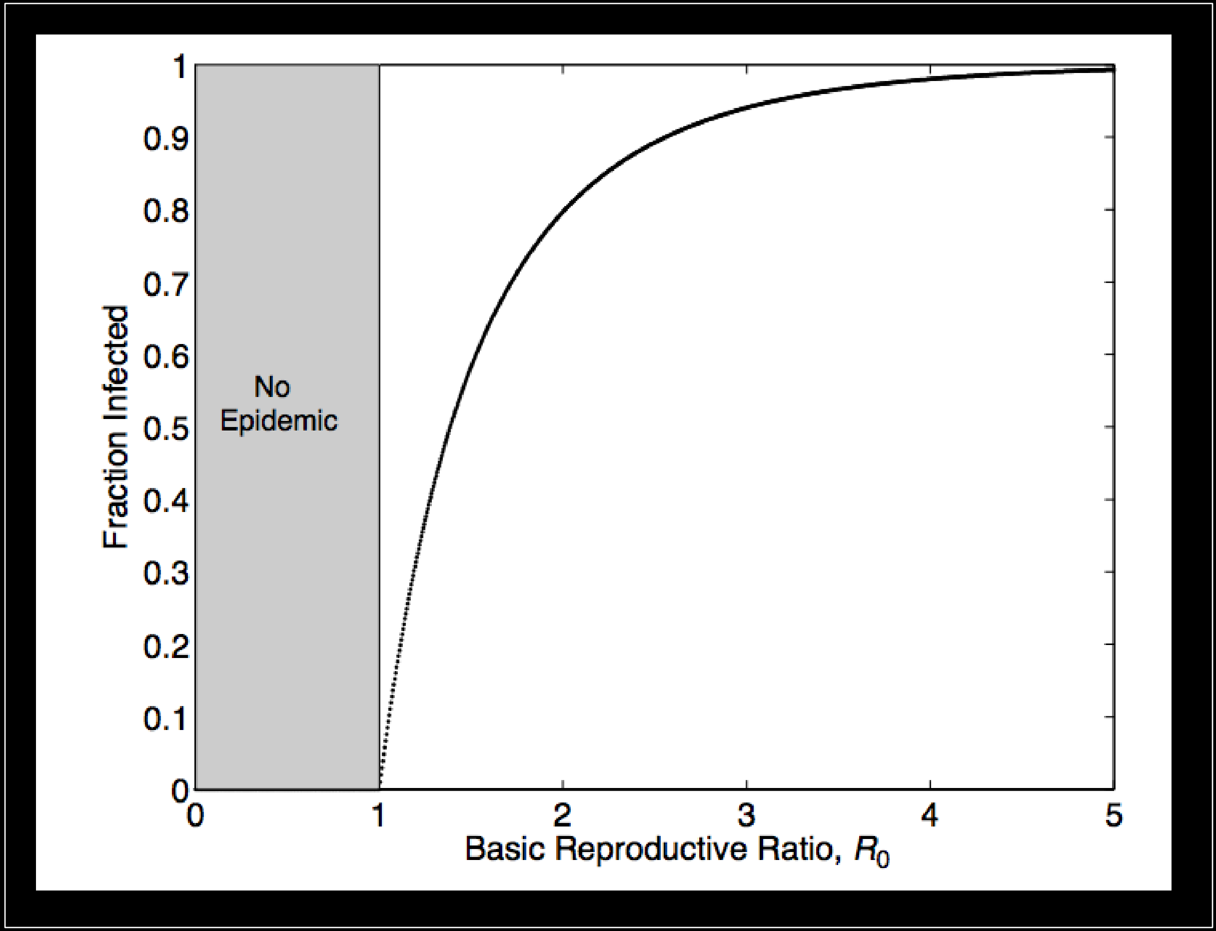
Phase/critical transitions everywhere



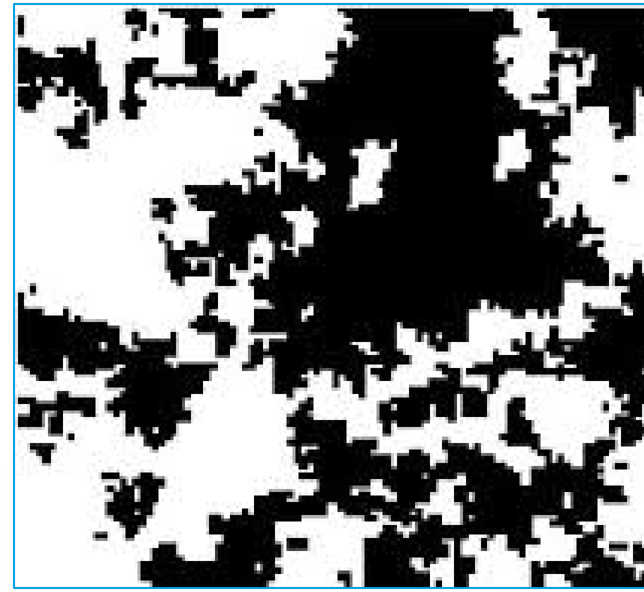
Phase/critical transitions everywhere

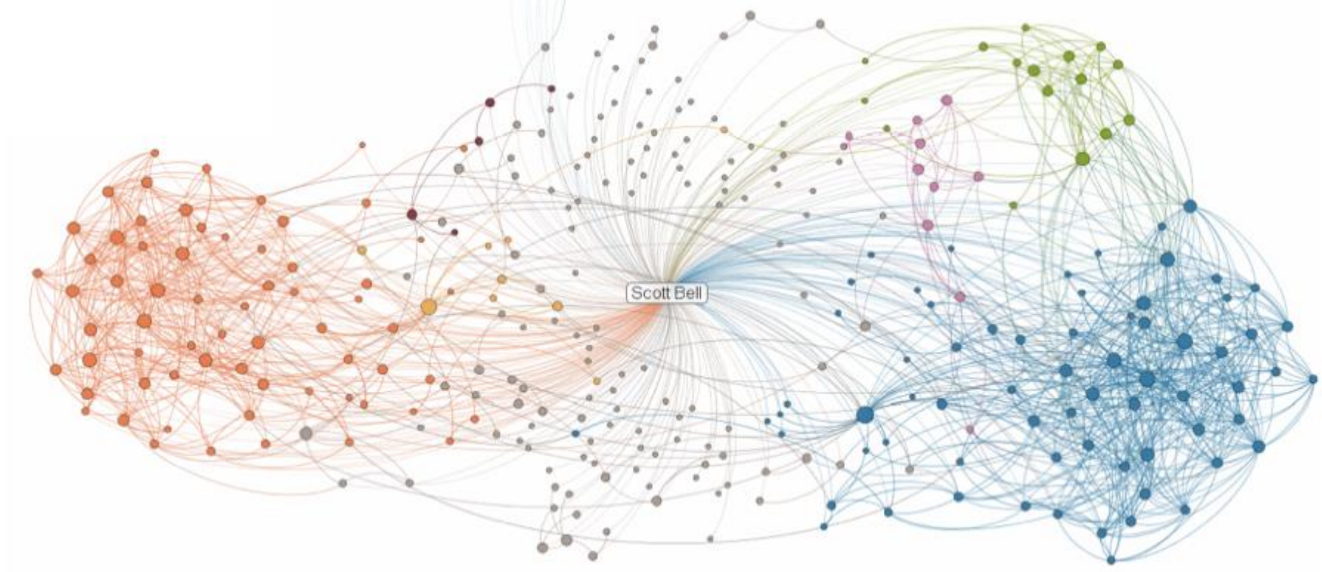
Disease	Transmission	R_0
Measles	Airborne	12-18
Pertussis	Airborne droplet	12-17
Diphtheria	Saliva	6-7
Smallpox	Social contact	5-7
Polio	Fecal-oral route	5-7
Rubella	Airborne droplet	5-7
Mumps	Airborne droplet	4-7
HIV/AIDS	Sexual contact	2-5
SARS	Airborne droplet	2-5
Influenza (1918 pandemic strain)	Airborne droplet	2-3

Phase/critical transitions everywhere

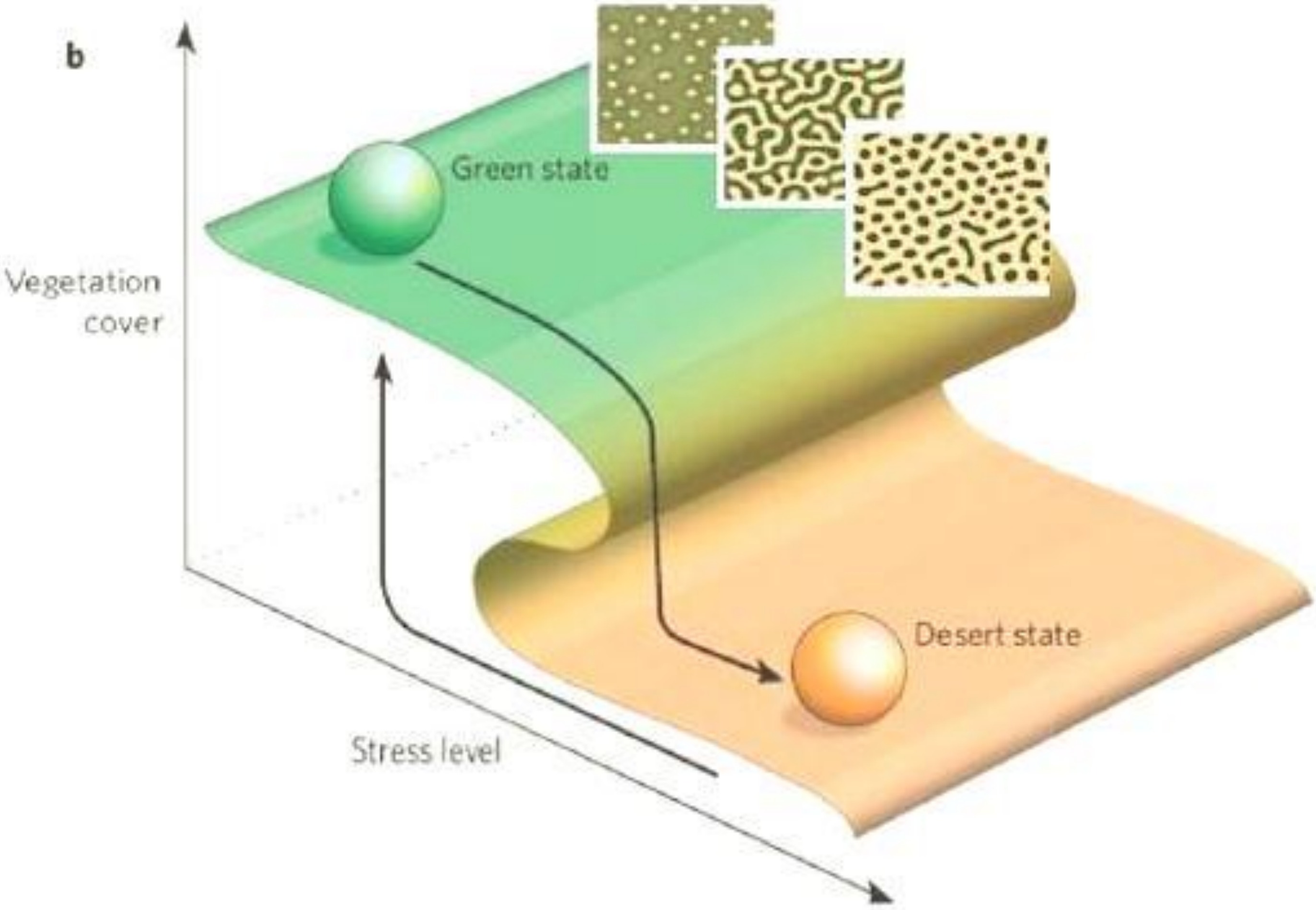


Phase/critical transitions everywhere

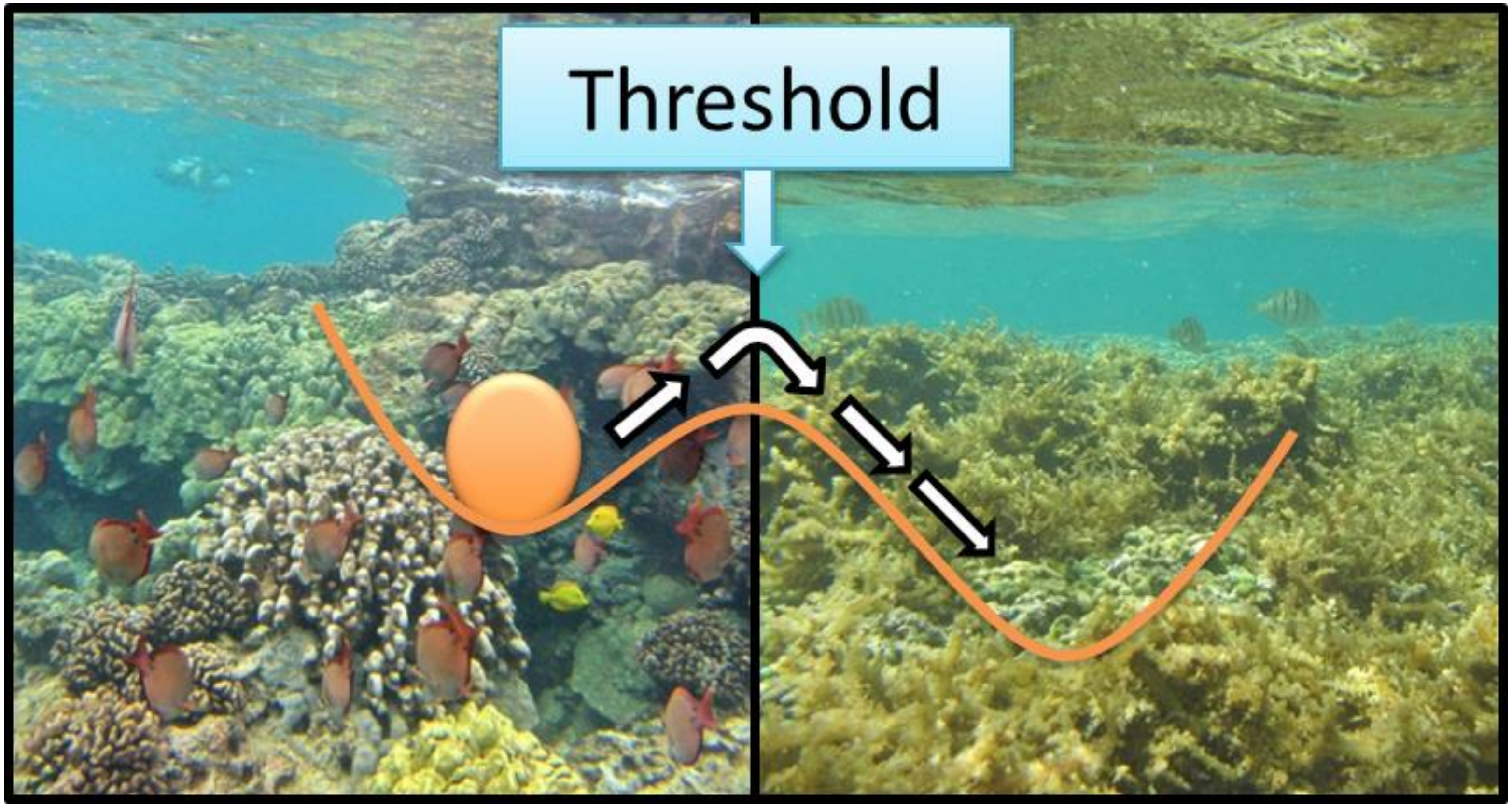




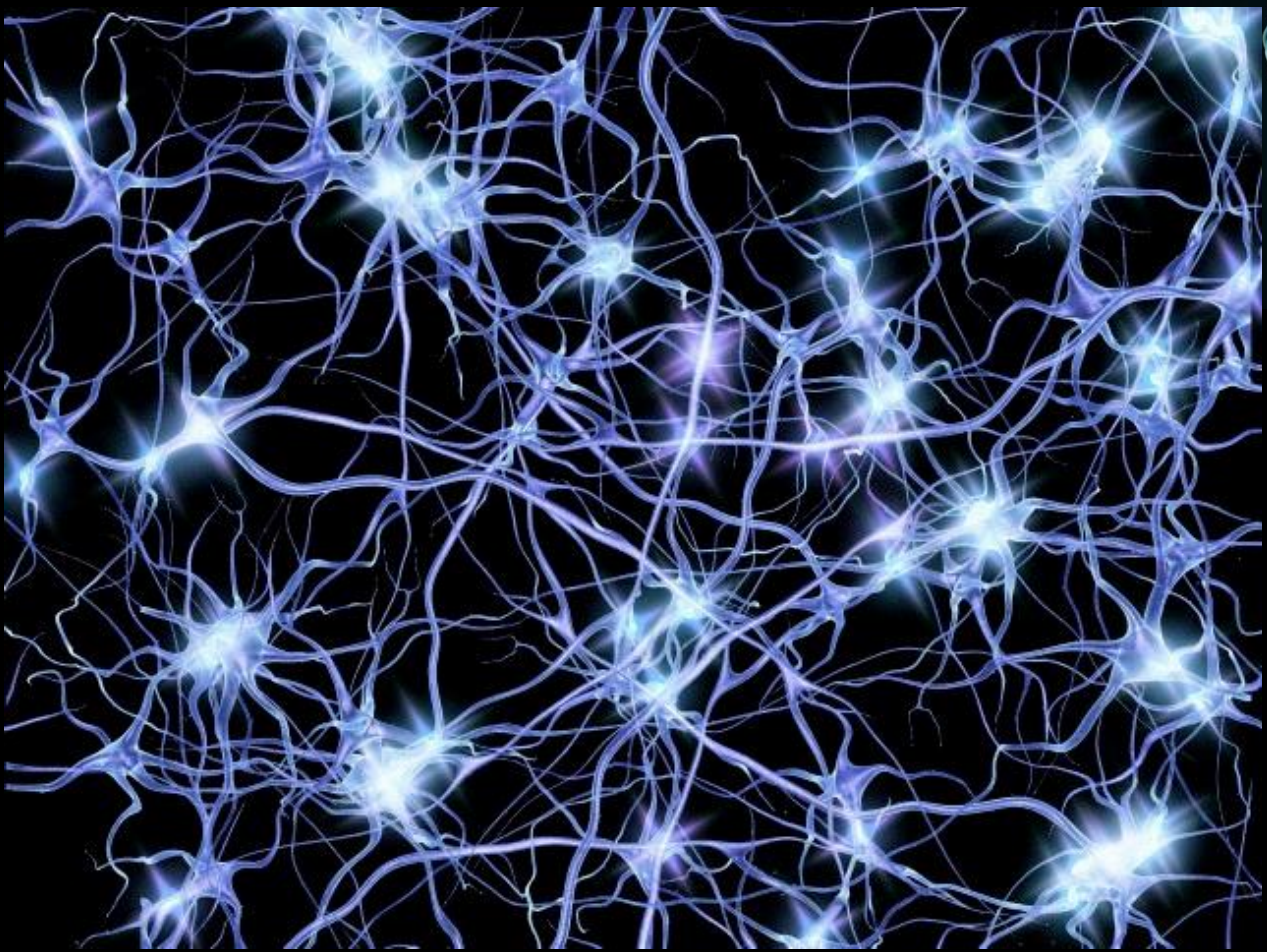
Regime shifts and tipping points



Regime shifts and tipping points



Phase transitions in the brain



Percolation in technological networks

The northeast blackout, 2003

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What's next: Networks' Revolutions

Watts & Strogatz, Nature 98

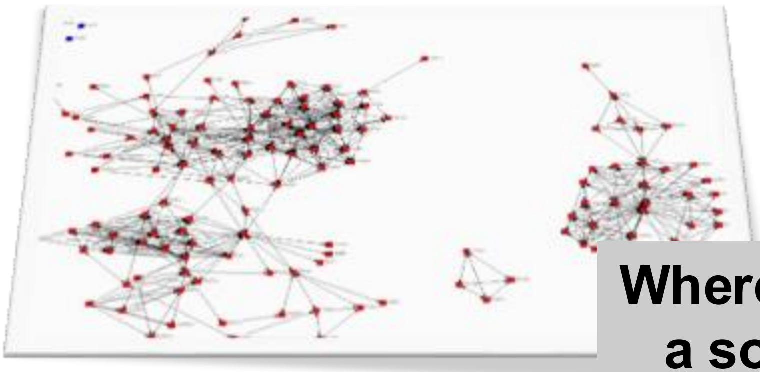
Conclusions so far:

Are real-world networks random?

	Model fits empirical data?
• Degree distribution	**Sometimes. Mostly NO!**
• Maximum degree	**Sometimes. Mostly NO!**
• Small world effect	**YES!!**
• Clustering coeff.	**NO! CC is often high, scaling with the degree and not with N.**

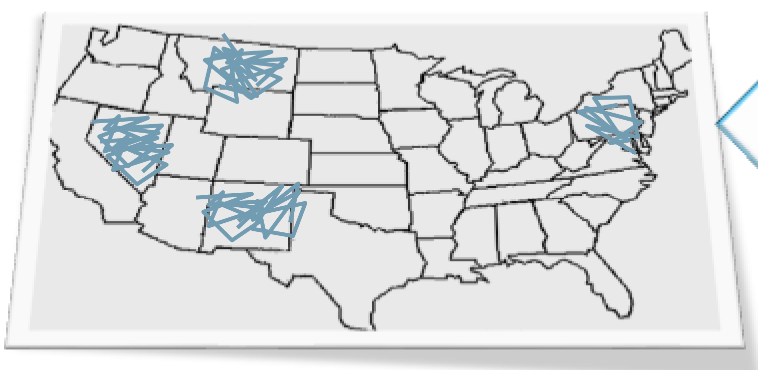
Clustering *vs* Randomness

Clustering implies locality



Randomness enables shortcuts

Where should we put a social network?



Locally Structured

Random

Could a network which is so strongly locally structured be at the same time a small world?

Merging structure and randomness

Nature 393 (1998) 440

Collective dynamics of 'small-world' networks

Duncan J. Watts* & Steven H. Strogatz

Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, New York 14853, USA

Networks of coupled dynamical systems have been used to model biological oscillators[1–4], Josephson junction arrays[5,6], excitable media[7], neural networks[8–10], spatial games[11], genetic control networks[12] and many other self-organizing systems. Ordinarily, the connection topology is assumed to be either completely regular or completely random. But many biological, technological and social networks lie somewhere between these two extremes.

works that can be tuned
tworks 'rewired' to intro-
/e find that these systems
lattices, yet have small
m graphs. We call them
with the small-world
x degrees of separation[15]).
enorhabditis elegans, the
es, and the collaboration
e small-world networks.
ll-world coupling display
mputational power, and
ous diseases spread more
easily in small-world networks than in regular lattices.

Nature 393 (1998) 409

news and views

It's a small world

James J. Collins and Carson C. Chow

The concept of Six Degrees of Separation has been formalized in so-called 'small-world networks'. The principles involved could be of use in settings as diverse as improving networks of cellular phones and understanding the spread of infections.

length is short, scaling logarithmically with the size of the network.

What Watts and Strogatz[5] do is to shift gradually from a regular network to a random network by increasing the probability of making random connections from 0 to 1 (see Fig. 1, page 441). They then measure the characteristic path length and the amount of clustering of the network as a function of the amount of randomness. They find that path length and clustering depend differently on the amount of randomness in the network. The characteristic path length drops quickly,

Merging structure and randomness

The Watts-Strogatz Small World model (Nature '98)

Watts & Strogatz invented a very simple model (***1 parameter!***) which interpolates between regular and random graphs.

Duncan J. Watts
Microsoft Research

Steven H. Strogatz
Cornell Univ.

Watts, Duncan J., and Steven H. Strogatz. "Collective dynamics of 'small-world' networks." Nature 393, no. 6684 (1998): 440.

Merging structure and randomness

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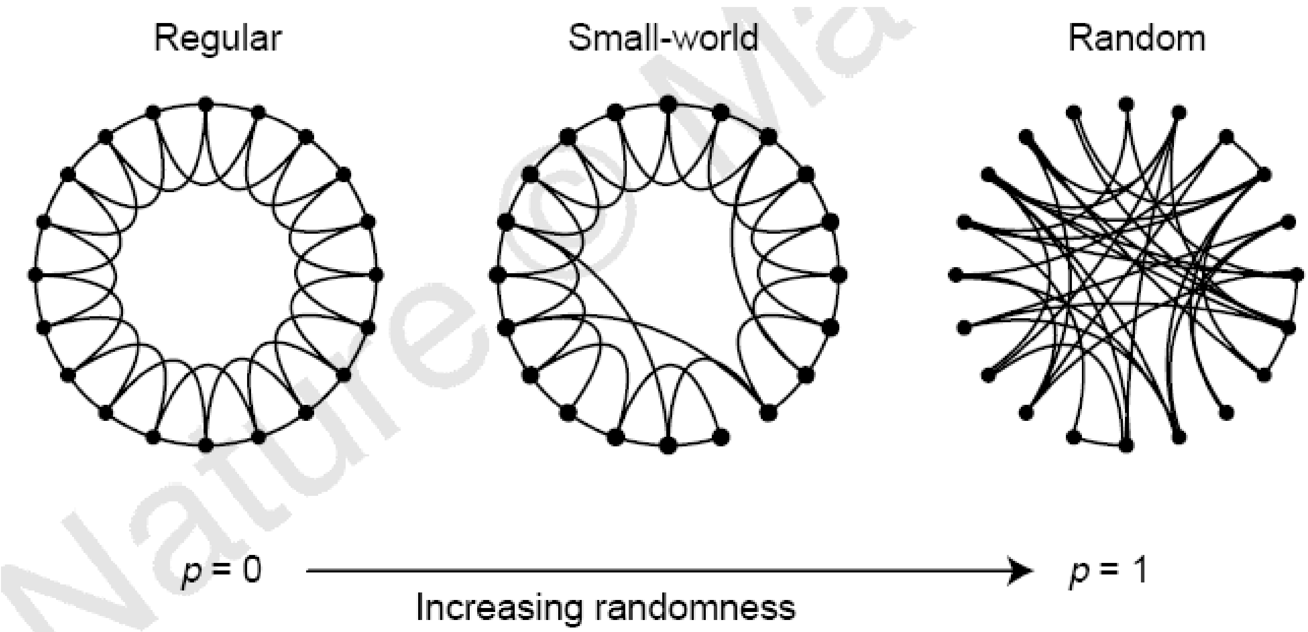
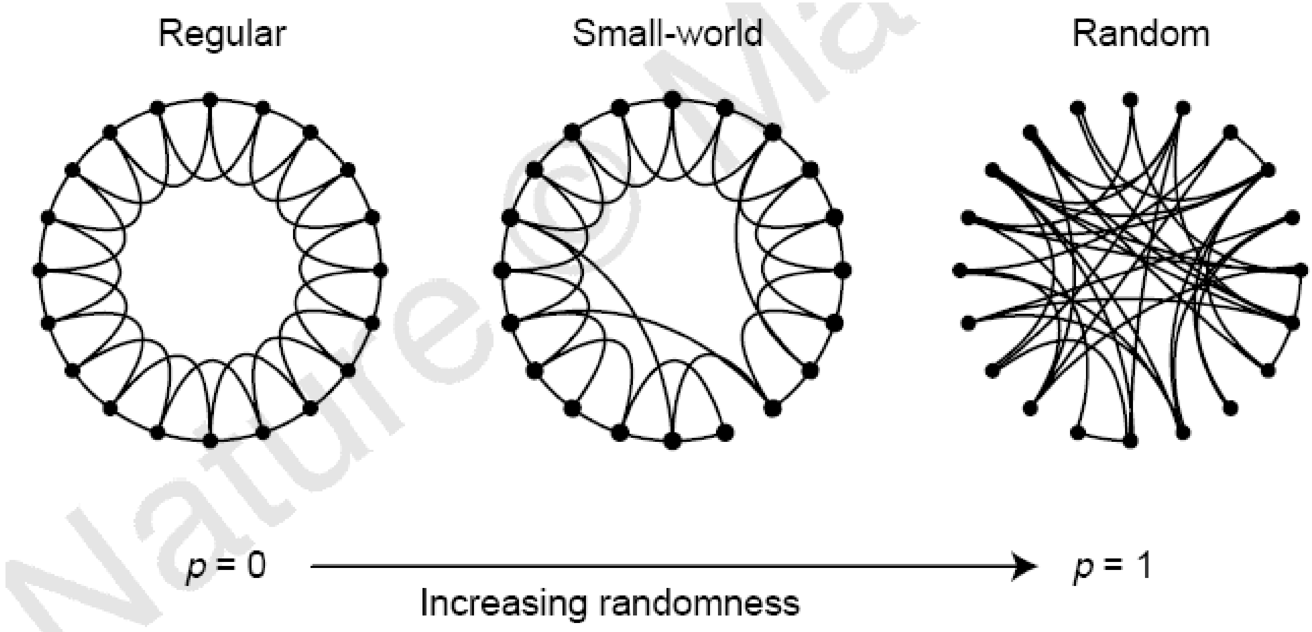


Merging structure and randomness

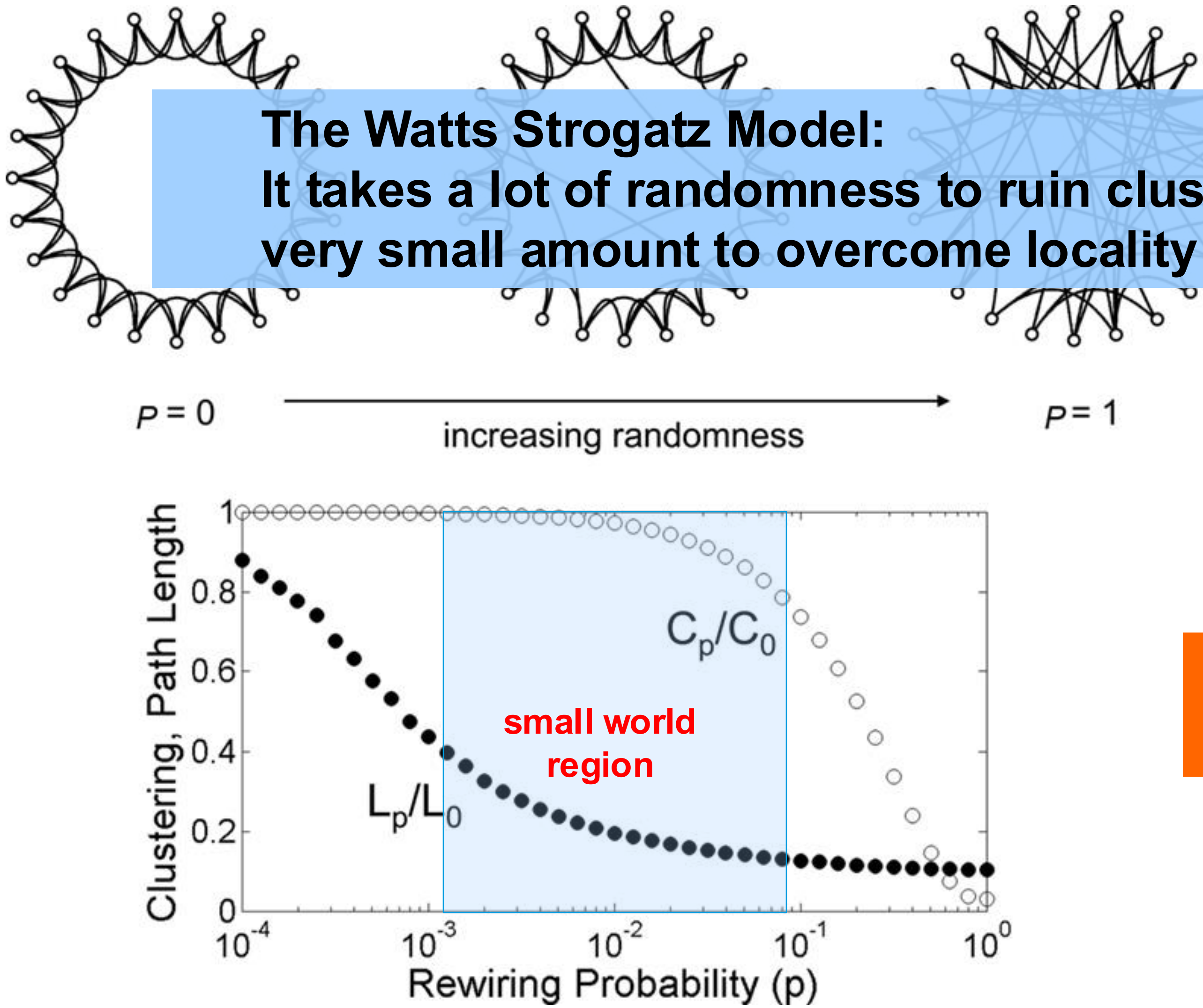
The Watts-Strogatz Small World model (Nature ‘98)

recipe : start from a regular graph (left); choose a circulating direction (say, clockwise); each edge one encounters is randomly re-directed with a probability ***p***; no repeated edges are allowed; stop when reaching the starting point;



Merging structure and randomness

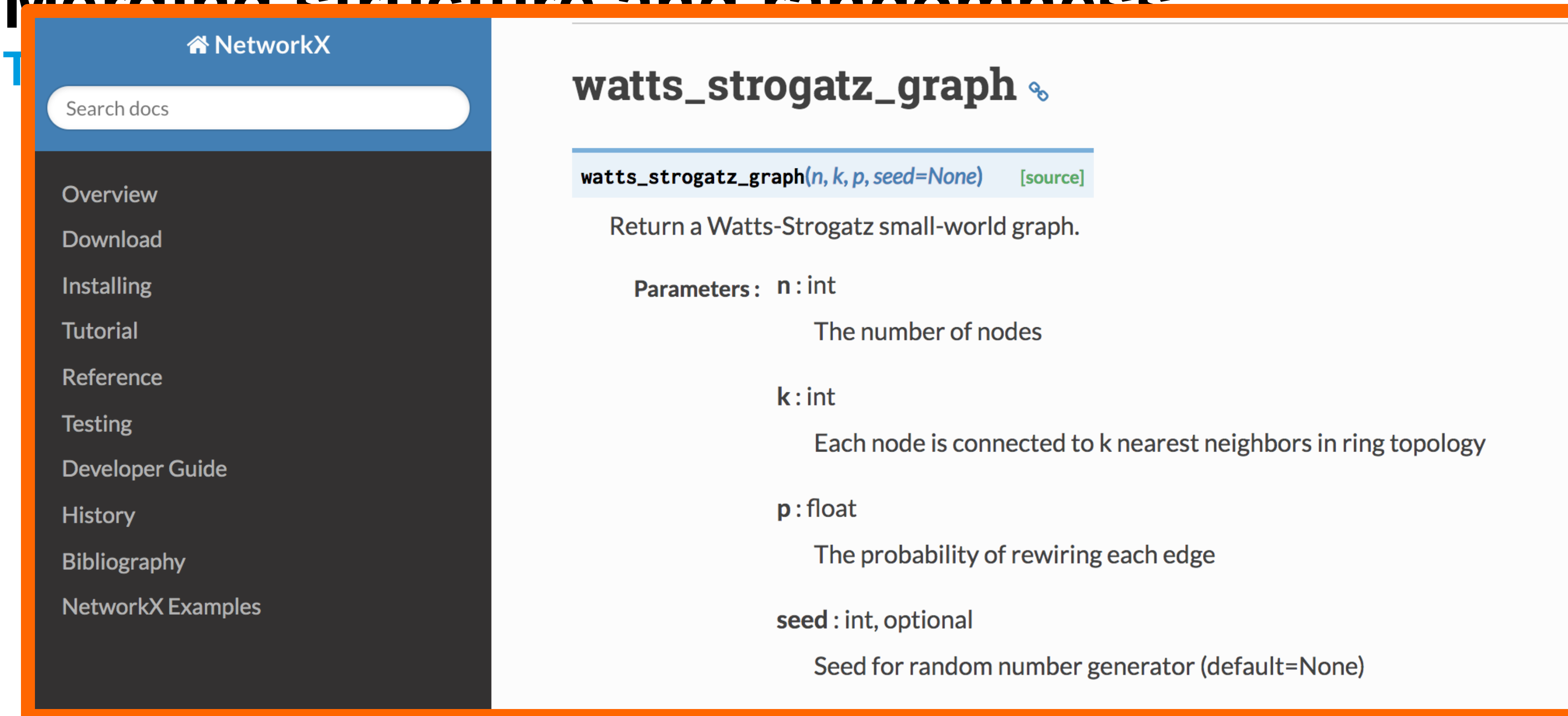
The Watts-Strogatz Small World model (Nature '98)



The Watts Strogatz Model:
It takes a lot of randomness to ruin clustering, but a very small amount to overcome locality

1st Challenge: Can you reproduce this result?

Merging structure and randomness


NetworkX

Search docs

- Overview
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- Installing
- Tutorial
- Reference
- Testing
- Developer Guide
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- Bibliography
- NetworkX Examples

watts_strogatz_graph

`watts_strogatz_graph(n, k, p, seed=None)` [source]

Return a Watts-Strogatz small-world graph.

Parameters : **n** : int

The number of nodes

k : int

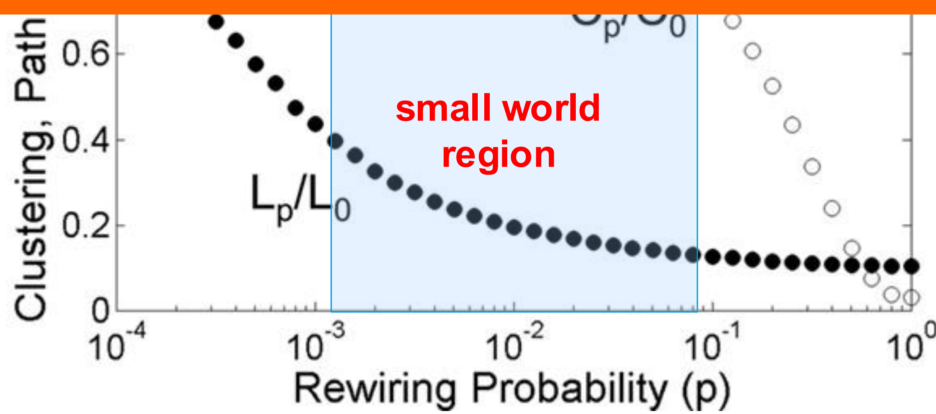
Each node is connected to k nearest neighbors in ring topology

p : float

The probability of rewiring each edge

seed : int, optional

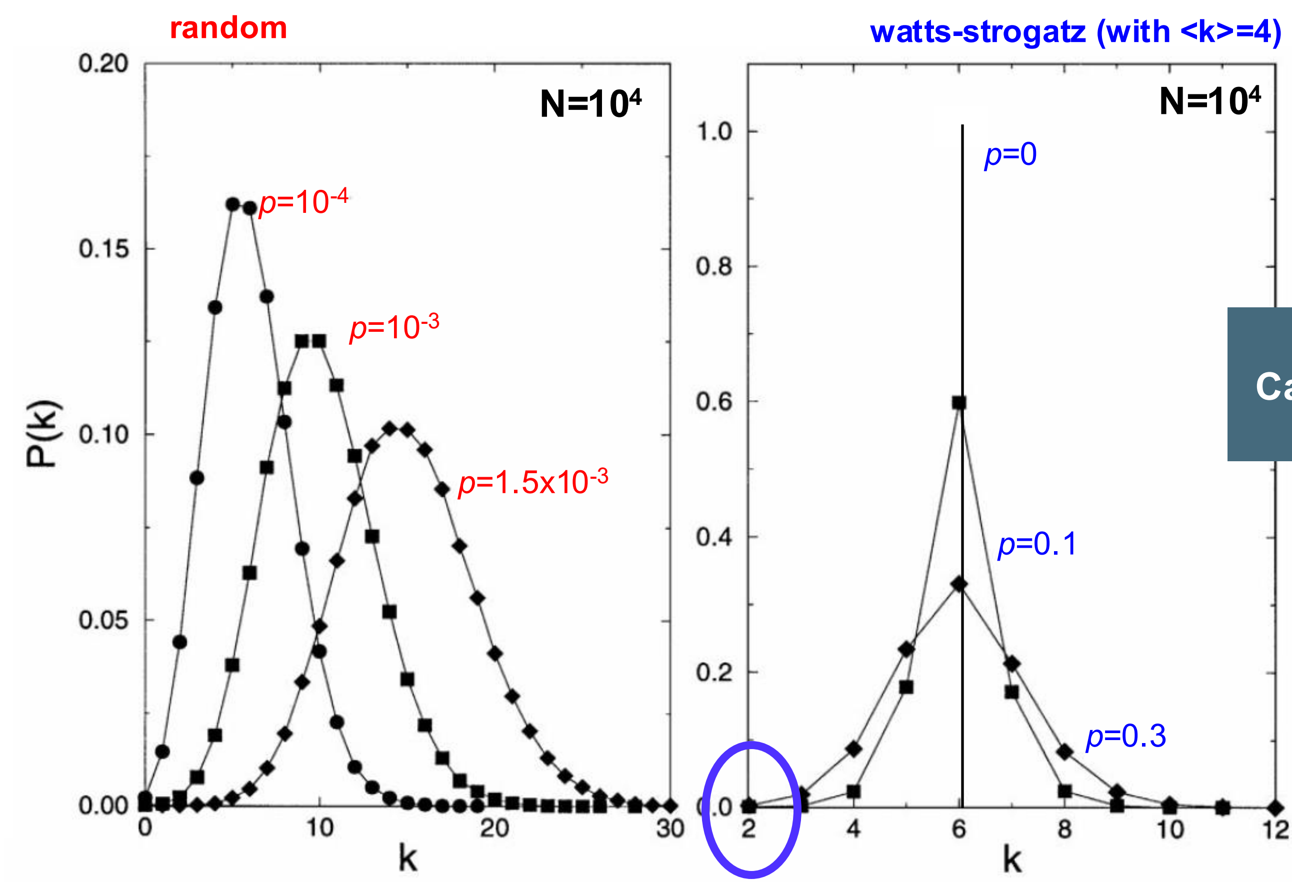
Seed for random number generator (default=None)



1st Challenge: Can you reproduce this result?

Merging structure and randomness

The Watts-Strogatz Small World model (Nature '98)



2^{nd} Challenge: Can you reproduce this result?

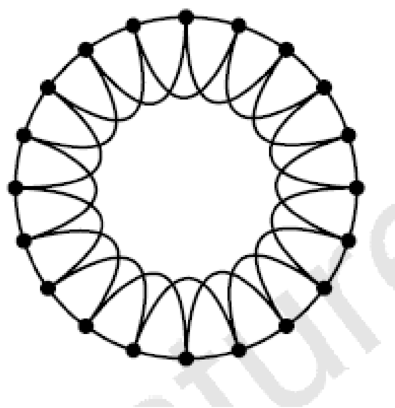
The degree distribution of the Watts–Strogatz model can also be computed analytically. Please check Barrat and Weigt, 2000.

Homogeneous small worlds

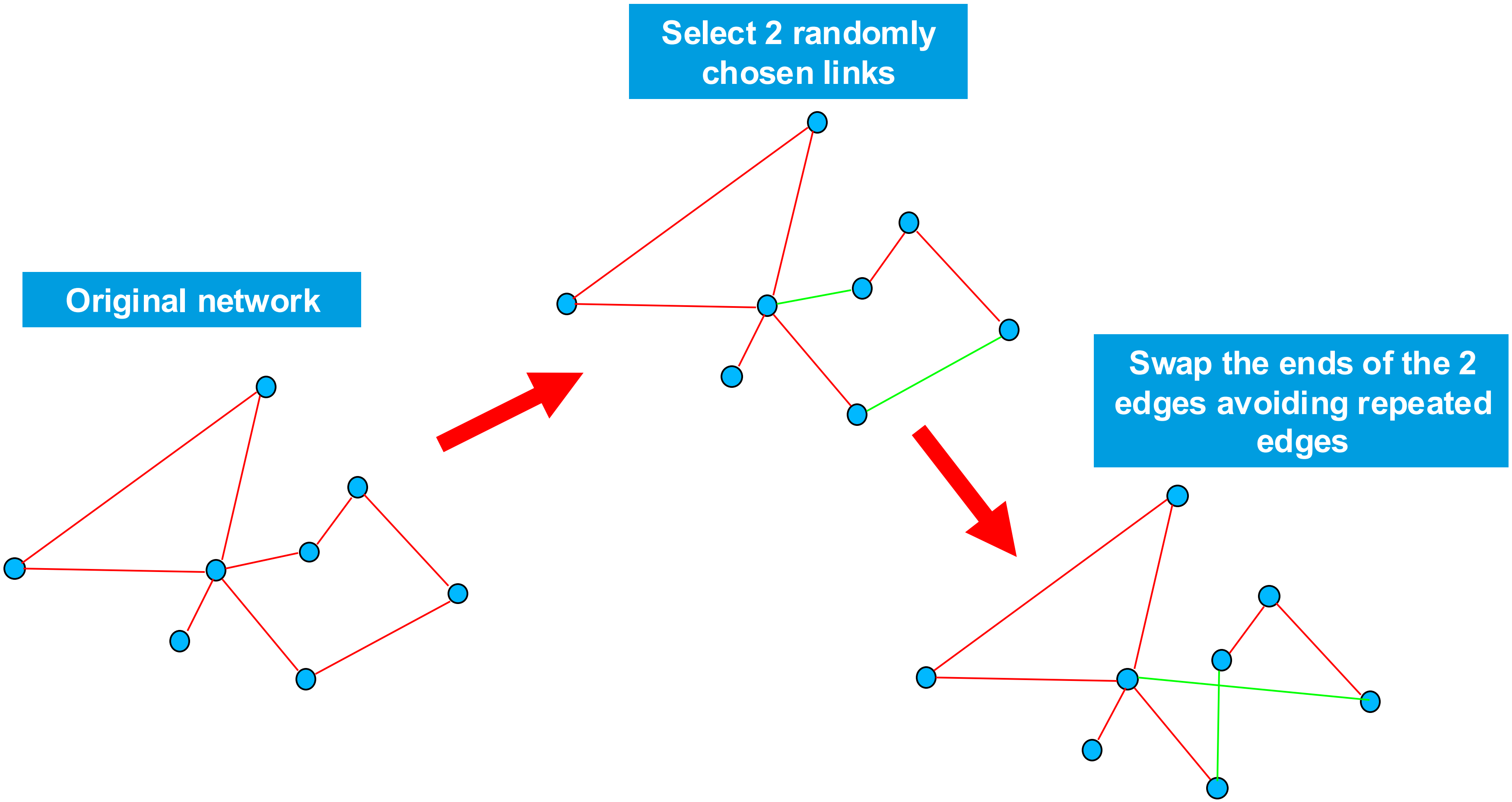
Challenge:

Imagine that you would like to test Small-world effects, yet keeping the degree distribution constant.

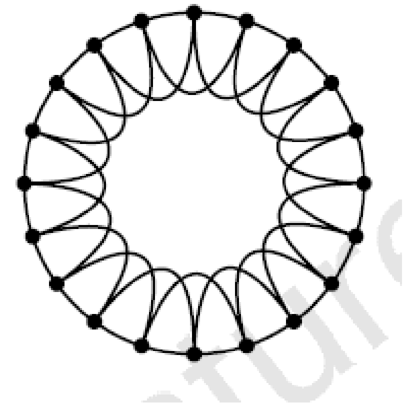
How can we adapt the WS model such we randomize a network without altering the degree of each node?



Degree-preserving randomization



Homogeneous small worlds

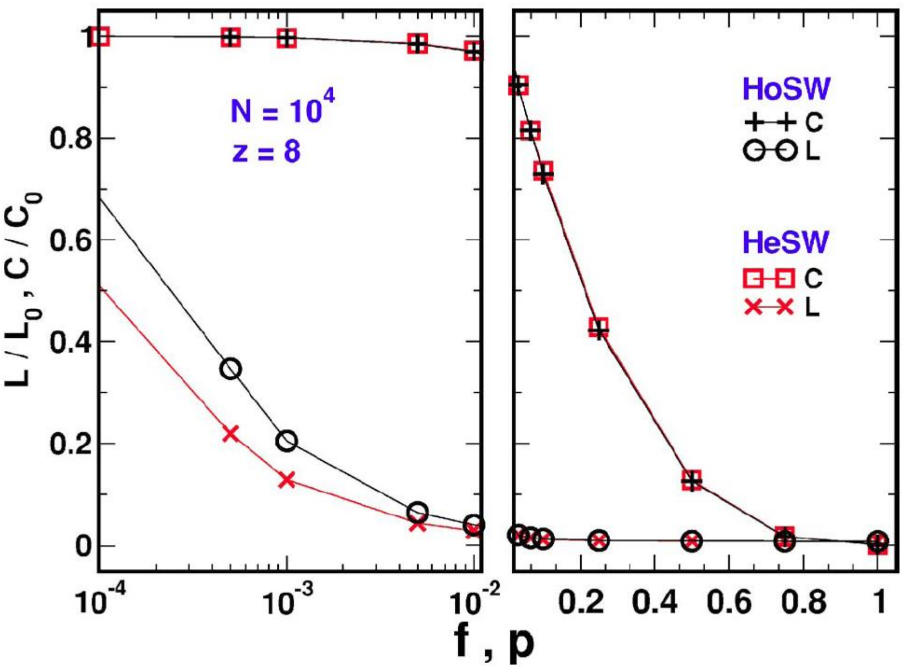


The following model generates small-world graphs with $P(k) = \delta(k - z)$

recipe : start with a regular graph;
repeat the following 2-step procedure until you have $f\,E$ edges rewired:

i) chose 2 e's at random not used before in ii)
ii) swap the ends of the two e's avoiding repeated e's;

similar to WS graphs, a 1-parameter model - f : we generate a $\langle C(f)\rangle$ & a $\langle L(f)\rangle$; what's their behavior ?



we can introduce small-world effects without introducing heterogeneity in the graph

Santos, Rodrigues, Pacheco, Phys Rev E (2006)

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2^{nd} network revolution

Barabási, Albert, Jeong, Dorogovtsev, Mendes, Havlin, Cohen…
(>1999)

Scale-free networks

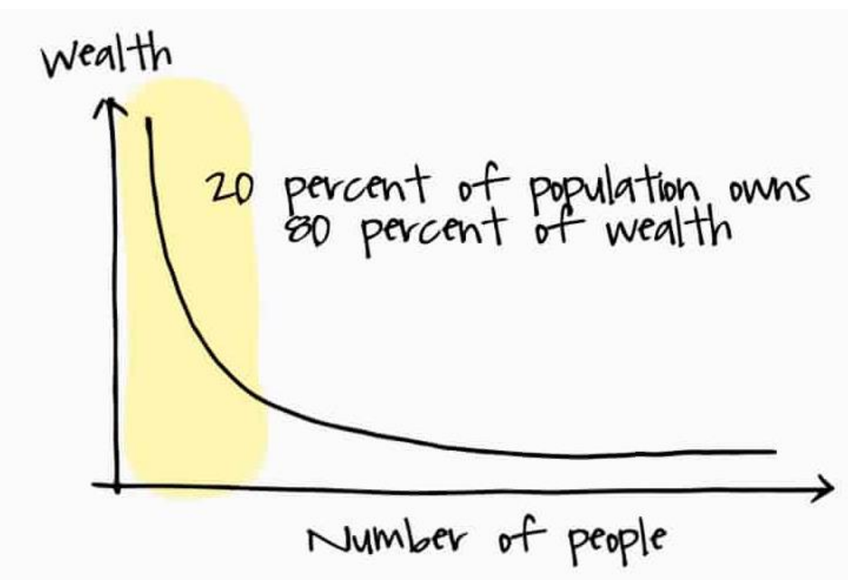
- Watts and Strogratz explained a key point in real-world networks… but missed a key one.

Emergence of Scaling in Random Networks
Albert-László Barabási, *et al.*
Science **286**, 509 (1999);
DOI: 10.1126/science.286.5439.509

Emergence of Scaling in Random Networks

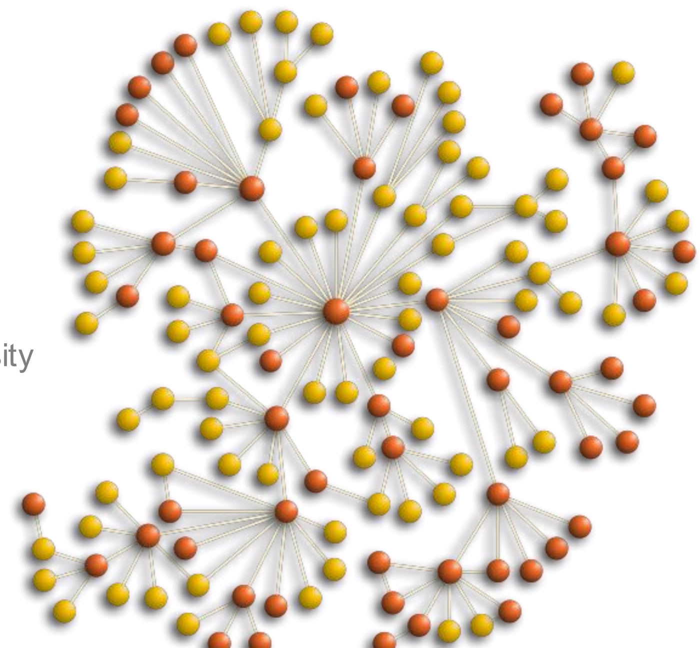
Albert-László Barabási* and Réka Albert

Systems as diverse as genetic networks or the World Wide Web are best described as networks with complex topology. A common property of many large networks is that the vertex connectivities follow a scale-free power-law distribution. This feature was found to be a consequence of two generic mechanisms: (i) networks expand continuously by the addition of new vertices, and (ii) new vertices attach preferentially to sites that are already well connected. A model based on these two ingredients reproduces the observed stationary scale-free distributions, which indicates that the development of large networks is governed by robust self-organizing phenomena that go beyond the particulars of the individual systems.

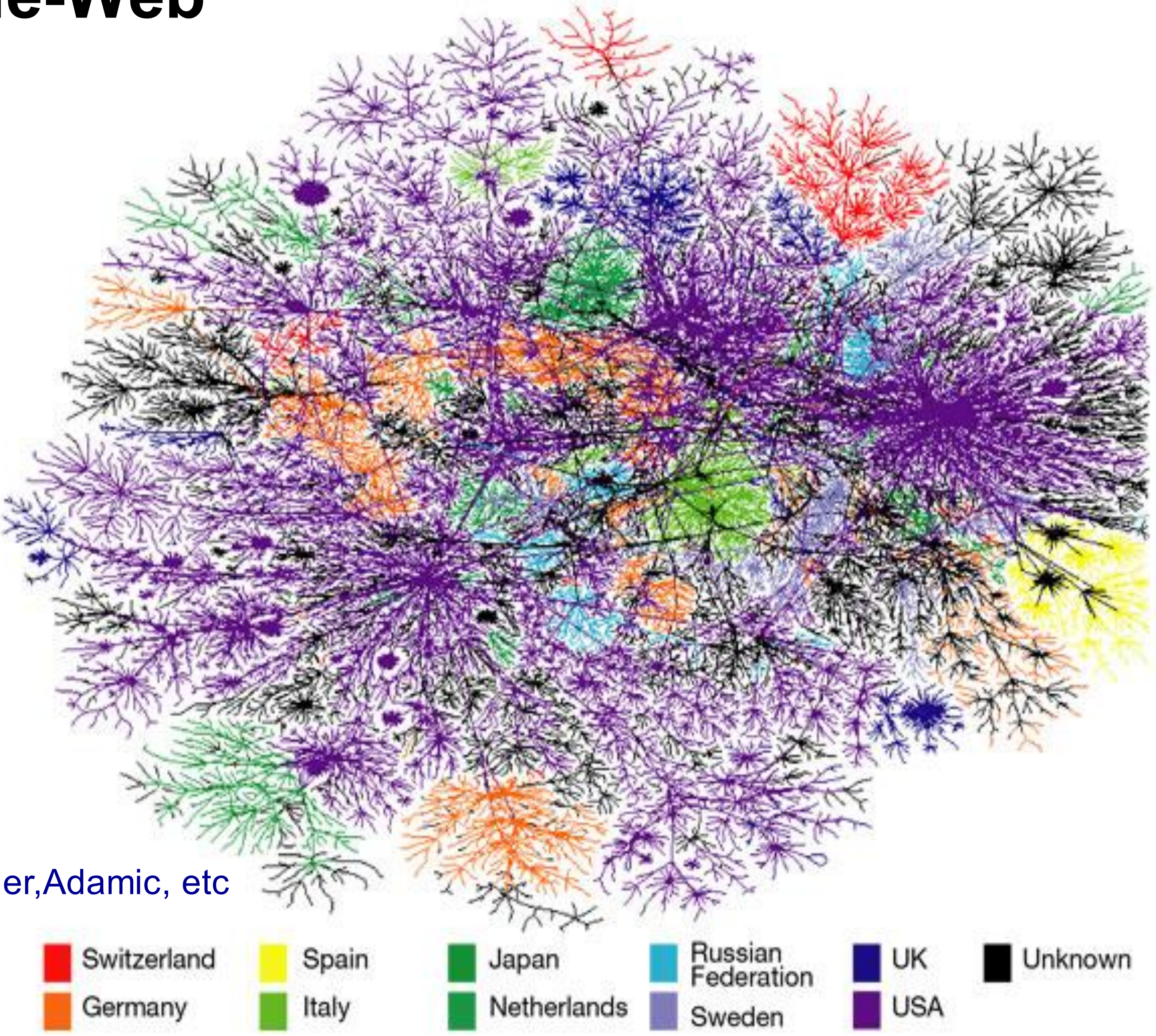


Reka Albert
Penn State University

Albert-László Barabási
Northeastern University



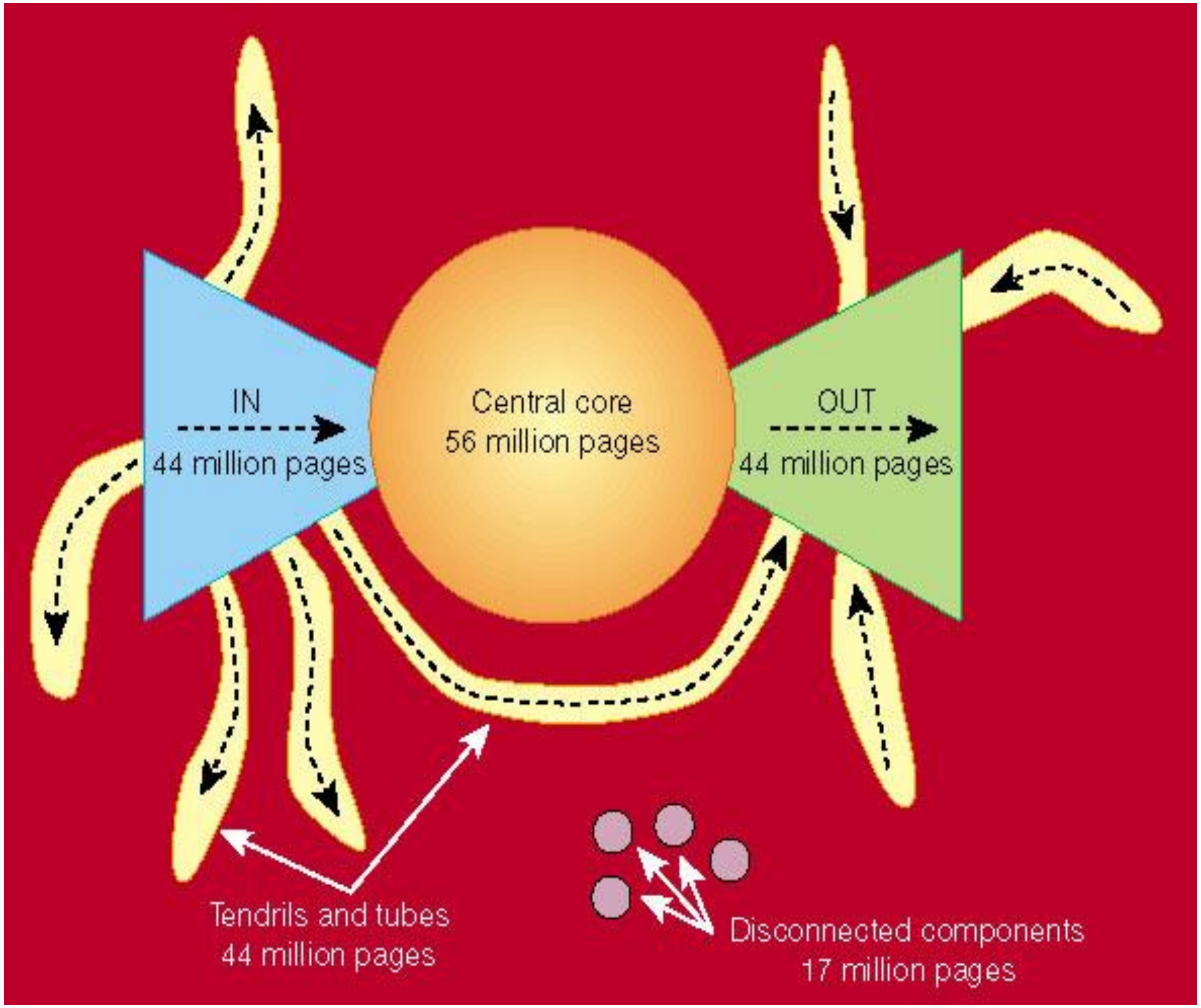
World-Wide-Web

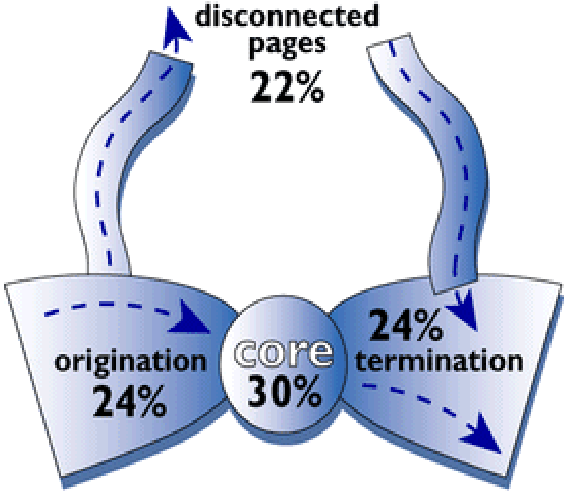


Kleinberg, Barabasi, Broder,Adamic, etc

World-Wide-Web is a "bow-tie"

200 million pages / 1500 million hyperlinks



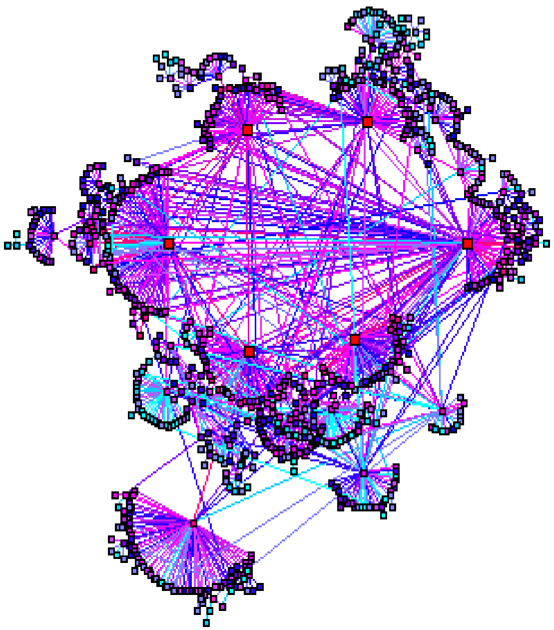


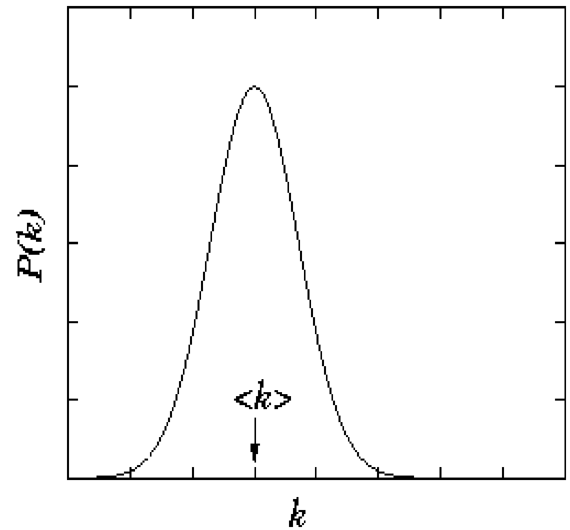
World-Wide-Web

Nodes: **WWW documents**
Links: **URL links**

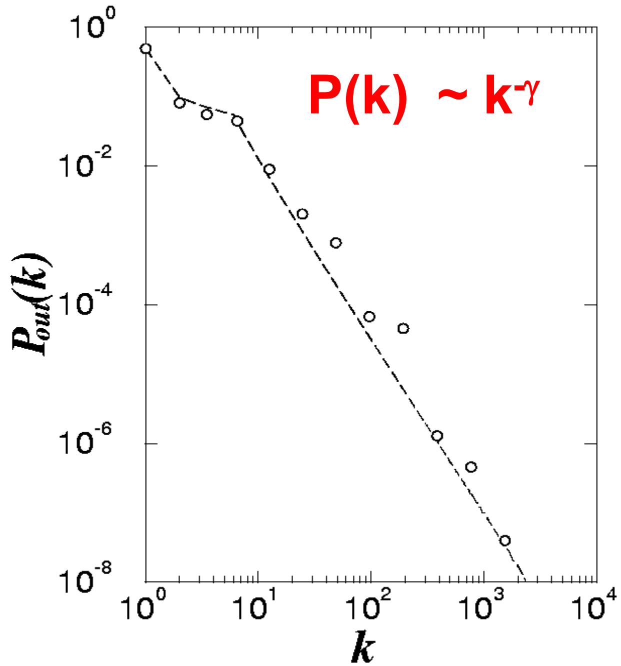
Over 3 billion documents

ROBOT: collects all URL's found in a document and follows them recursively





Expected

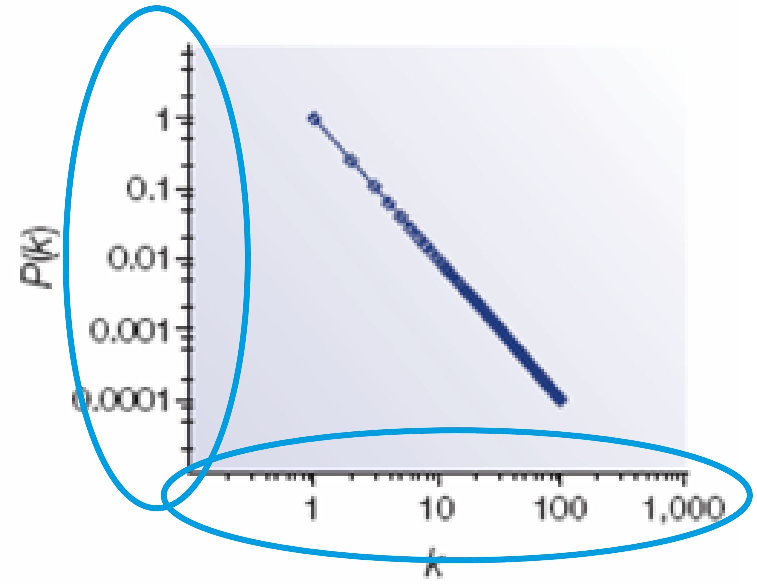


Found

R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).

Power-law degree distributions

$$P_k \sim k^{-\gamma}$$

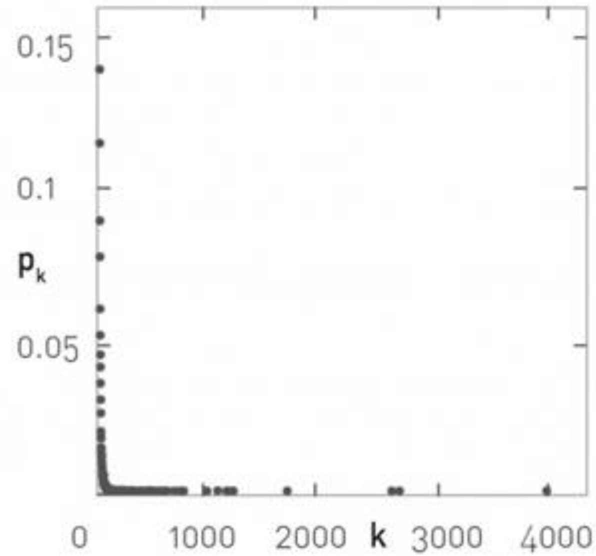


$$\ln P_k \sim \ln k^{-\gamma}$$

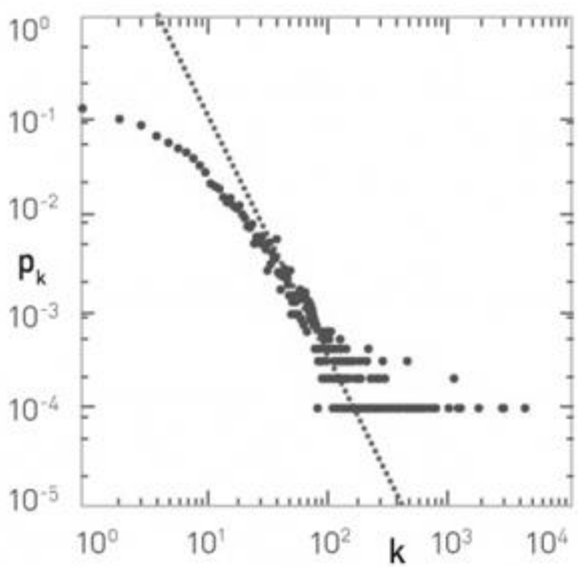
$$\ln P_k \sim -\gamma \ln k$$

Plotting degree distributions

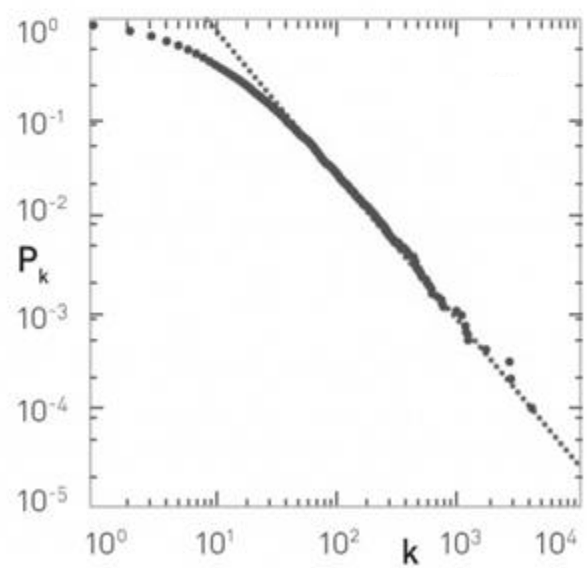
Linear scale



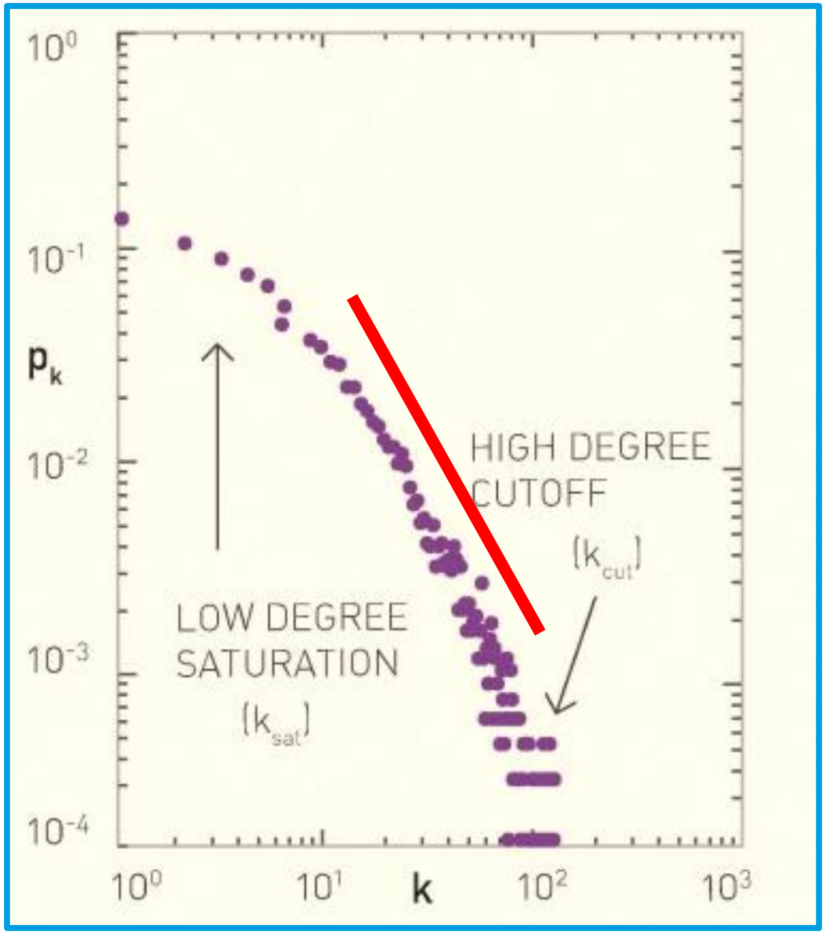
log-log scale



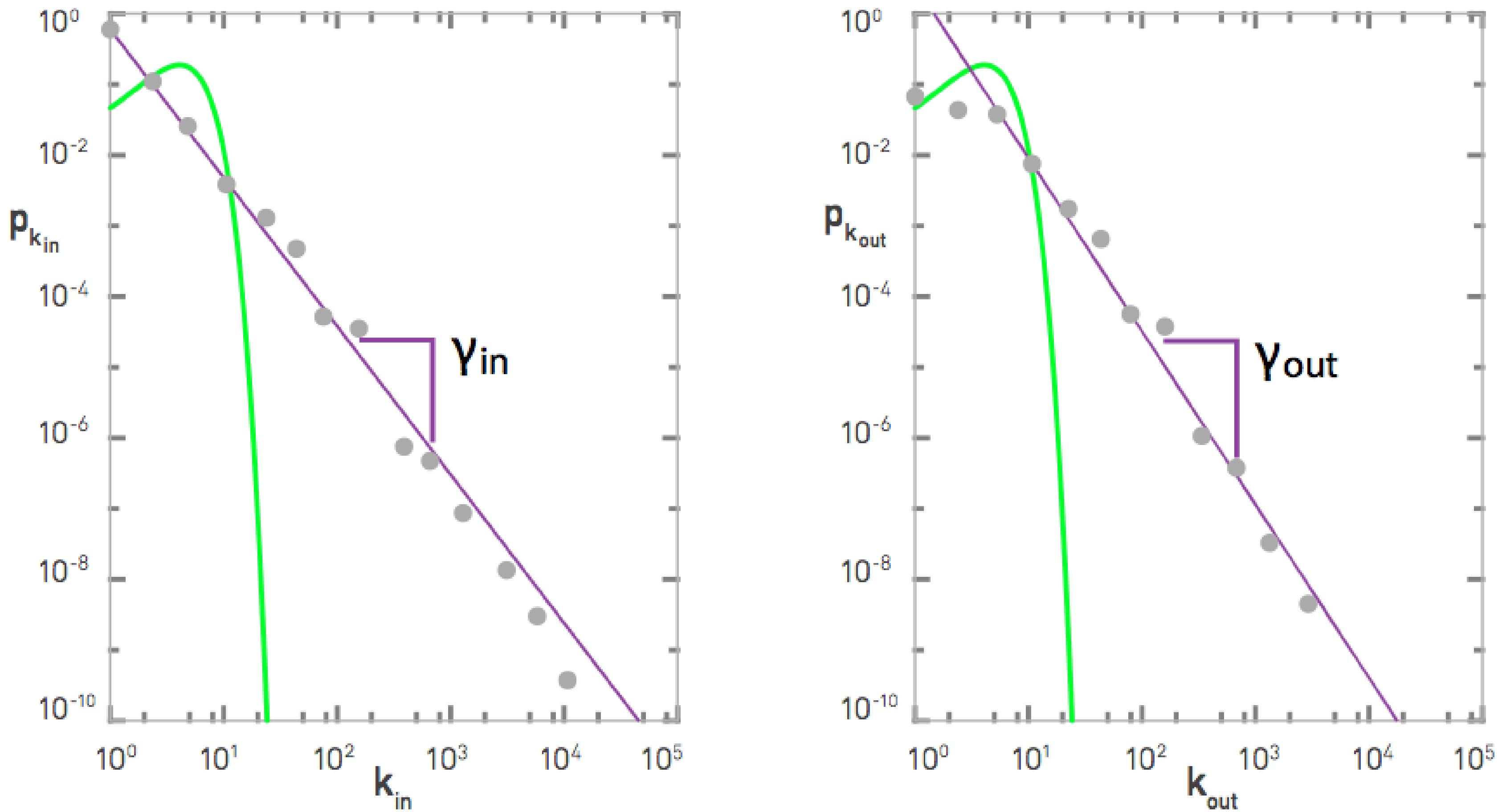
cumulative



Plotting degree distributions

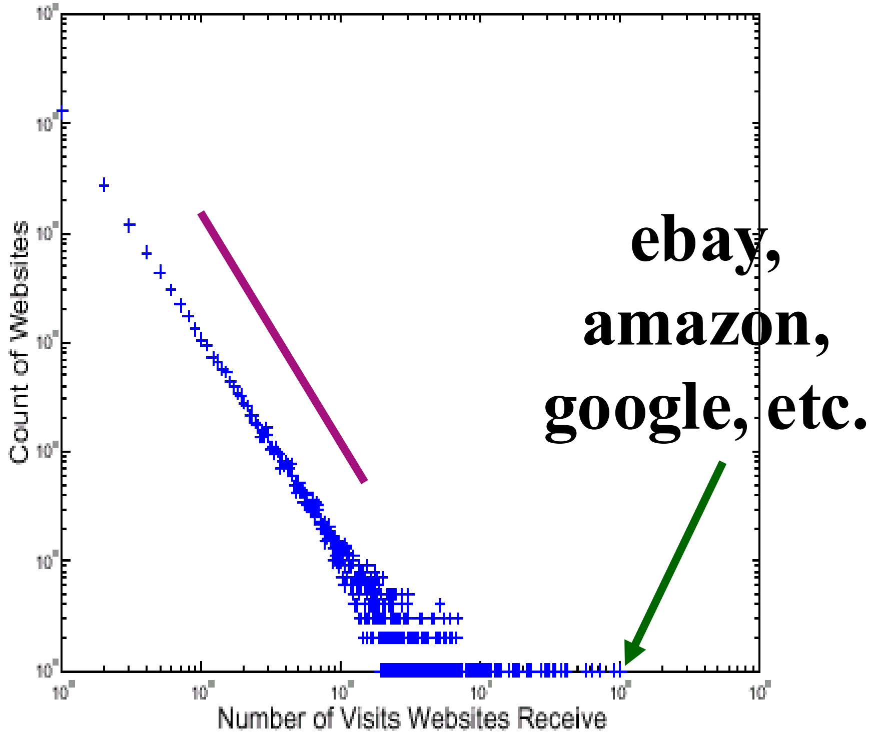


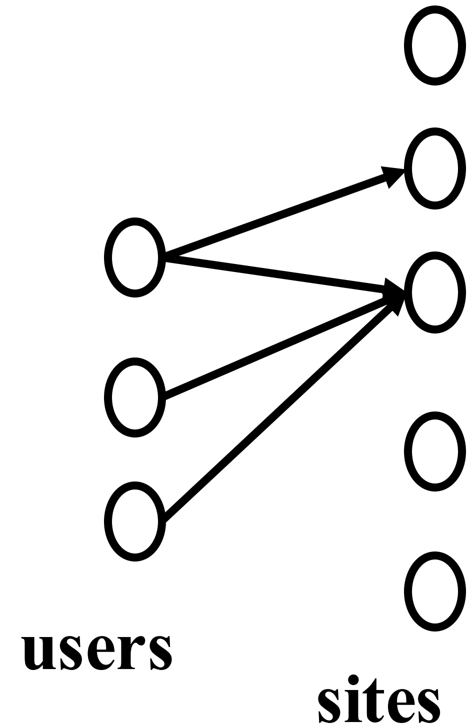
World-Wide-Web as a directed graph:
in & out degree distributions



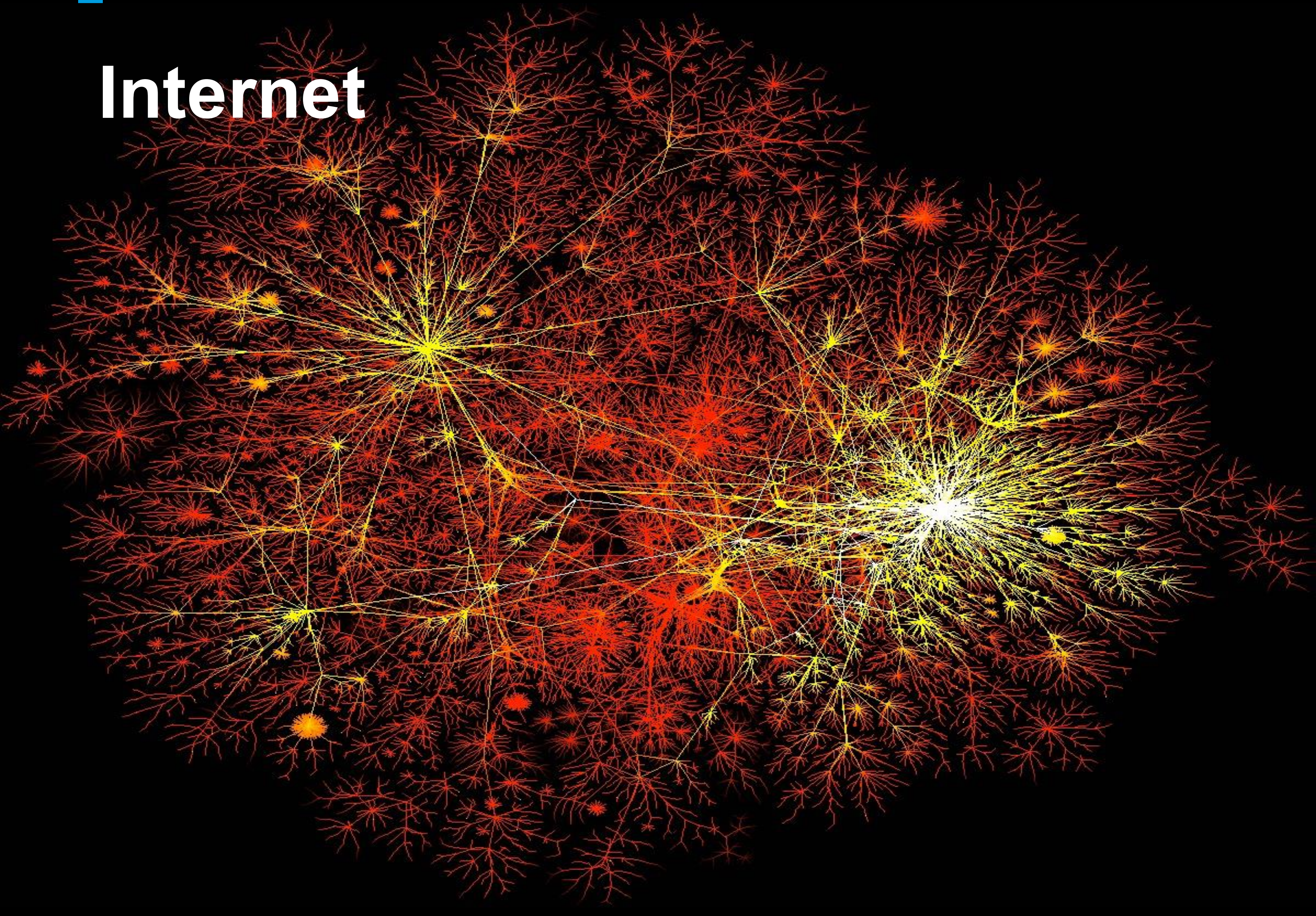
Power-laws are everywhere in the WWW

Web Site Traffic





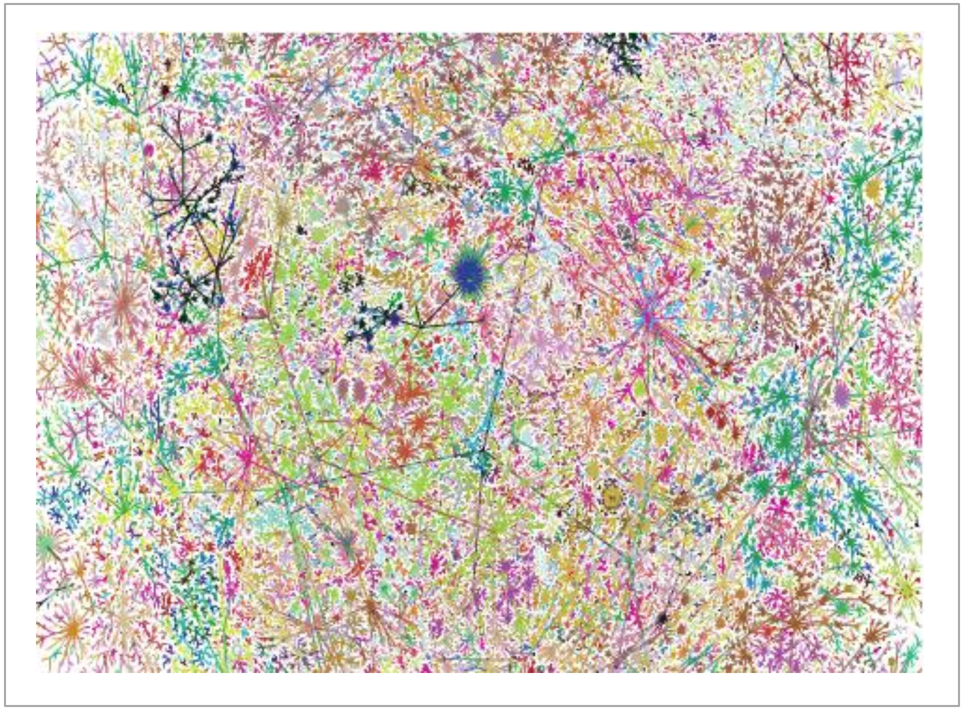
Internet

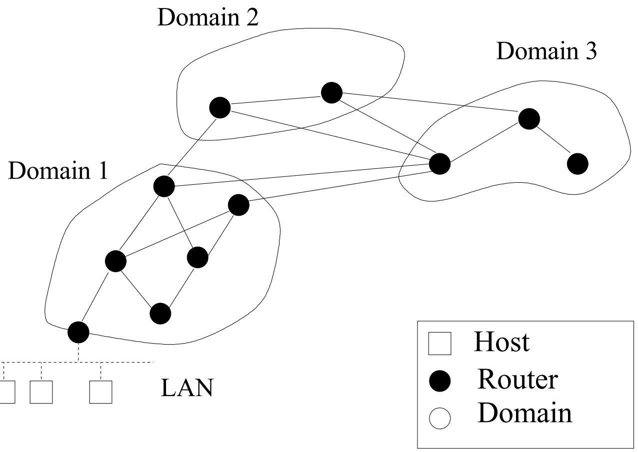


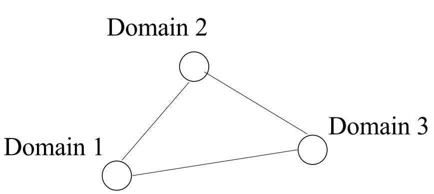
Internet

Internet (Faloutsos et al, 1999)
Nodes = computers, and edges physical connections among them … quite difficult

2 levels
Nodes = routers or
Nodes = domains



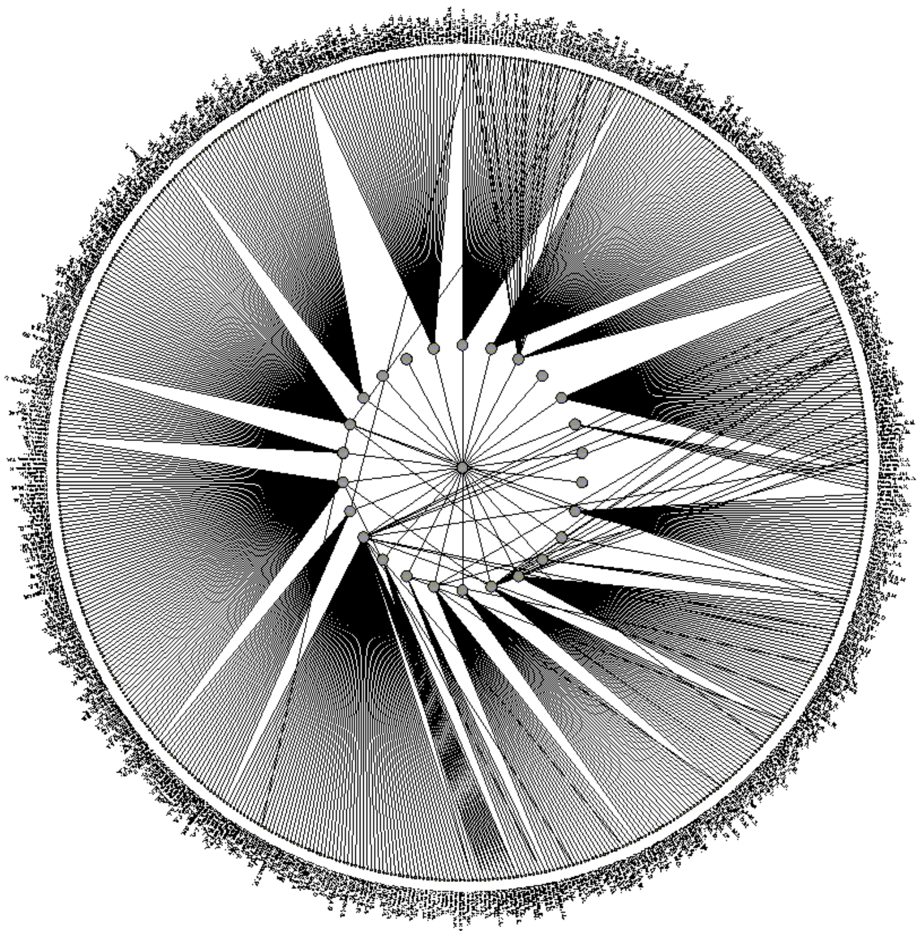


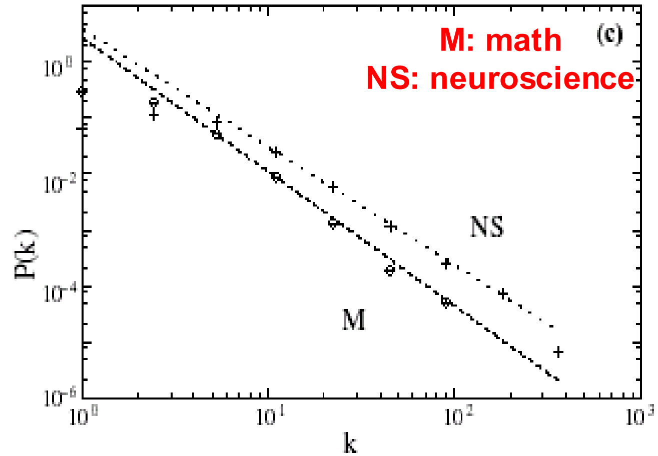


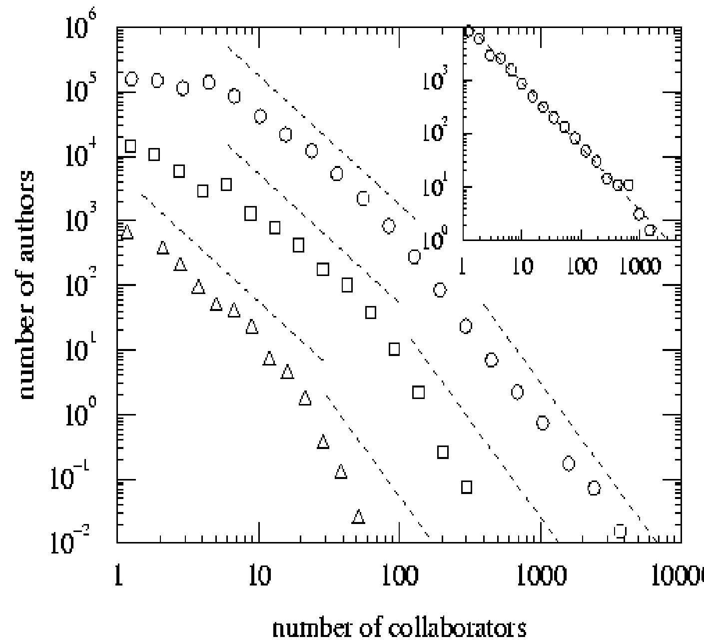
Both showed a scale-free degree dist.

Science collaborations

Nodes: scientist (authors)
Links: joint publication





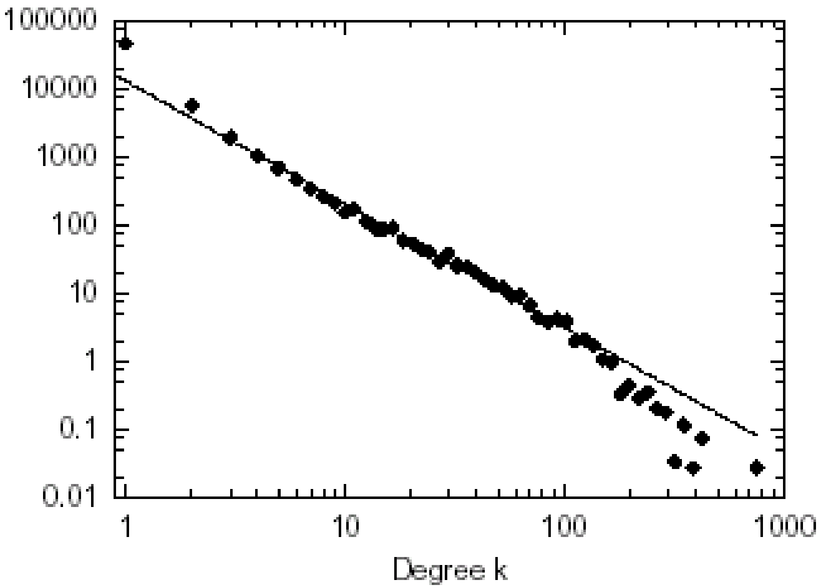


(Newman, 2000, Barabasi et al 2001)

Online communities

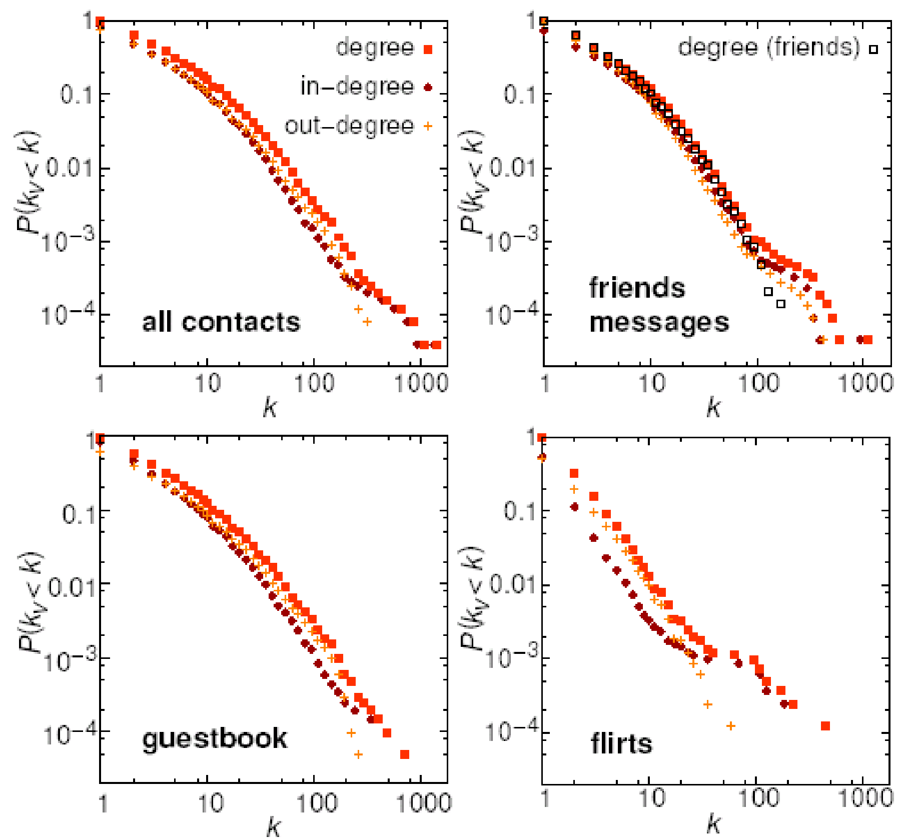
Nodes: online user
Links: email contact

Kiel University log files
112 days, N=59,912 nodes



Ebel, Mielsch, Bornholdtz, PRE 2002.

Pussokram.com online community;
512 days, 25,000 users.

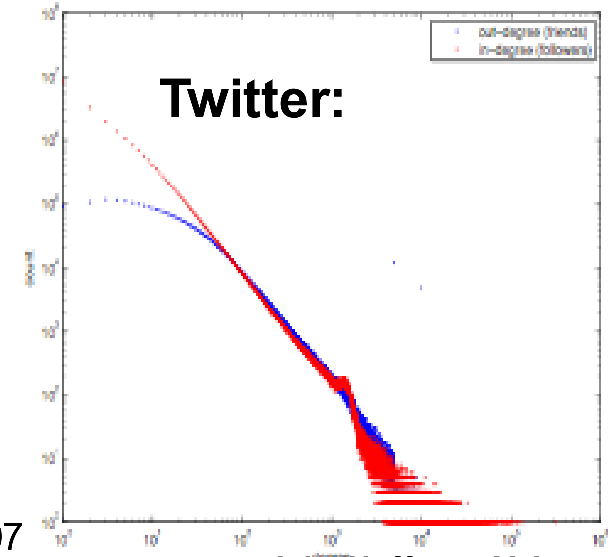


Holme, Edling, Liljeros, 2002.

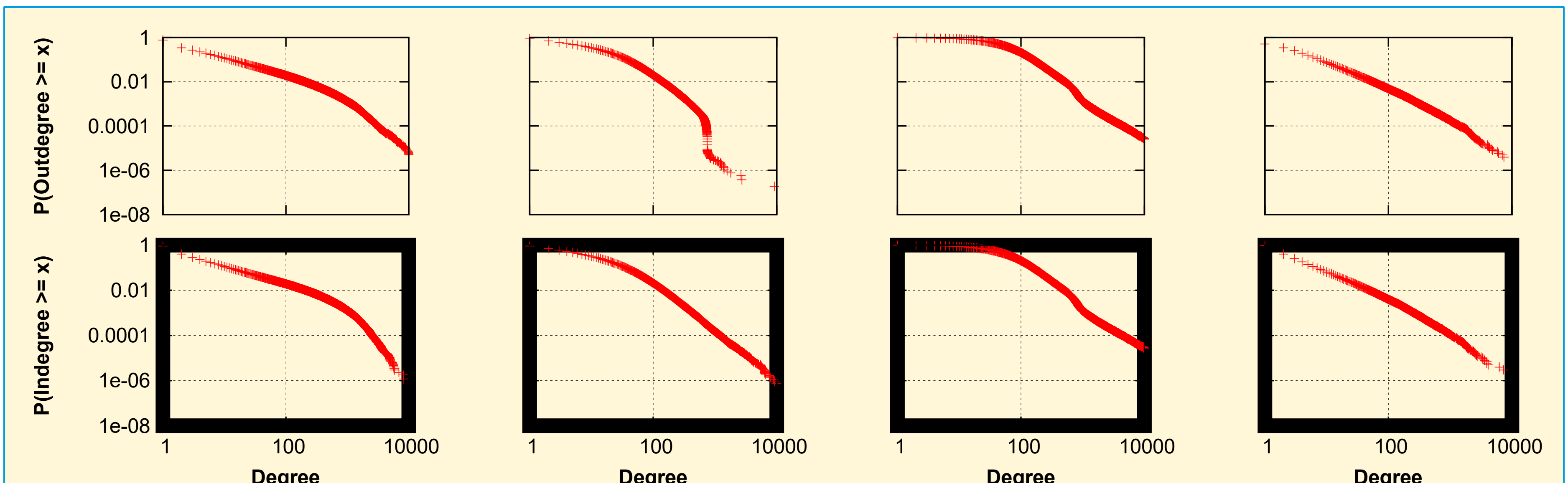
Online communities

All distribtions show a fat-tail behavior:
you have degrees spreading several orders of magnitude.

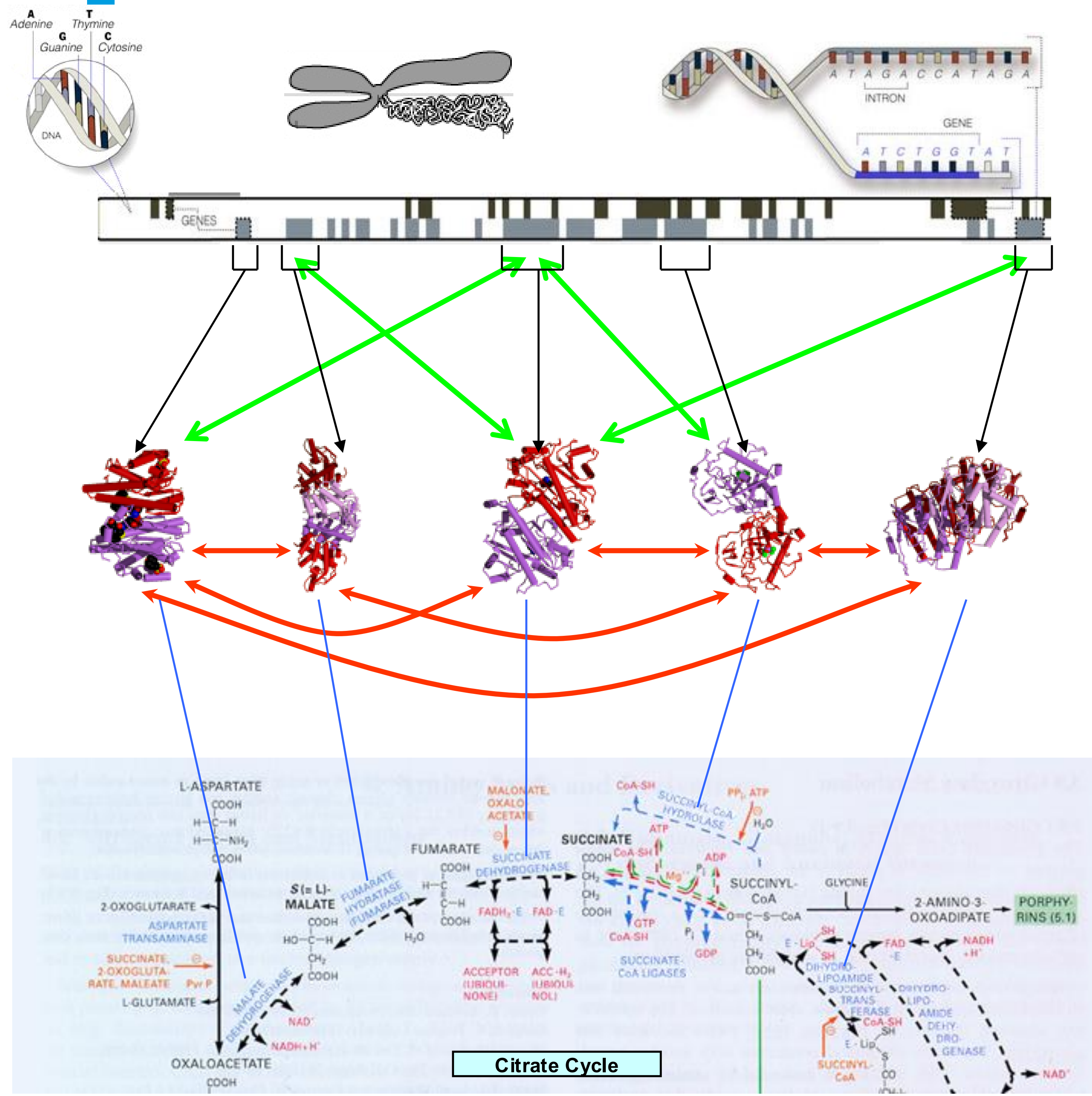
Mislove et al., Measurement and Analysis of Online Social Networks, IMC'2007



Jake Hoffman, Yahoo,



(a) Flickr (b) LiveJournal (c) Orkut (d) YouTube



GENOME

protein-gene interactions

PROTEOME

protein-protein interactions

METABOLISM

Bio-chemical reactions

@ our website…

NATURE REVIEWS | GENETICS

NETWORK BIOLOGY: UNDERSTANDING THE CELL'S FUNCTIONAL ORGANIZATION

Albert-László Barabási & Zoltán N. Oltvai‡*

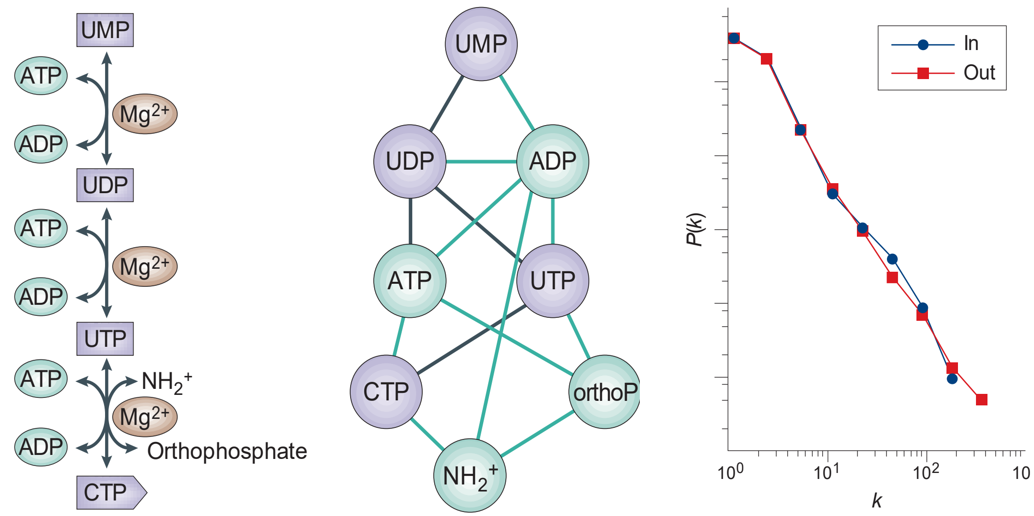
NATURE REVIEWS | GENETICS

Network medicine: a network-based approach to human disease

Albert-László Barabási‡§, Natali Gulbahce*‡|| and Joseph Loscalzo§*

Biological networks: Metabolic networks

Jeong, Albert, Oltvai, Barabási, Fell, Wagner, etc (after 2000).



Nodes : metabolites
Edges : enzyme-catalyzed biochemical reactions

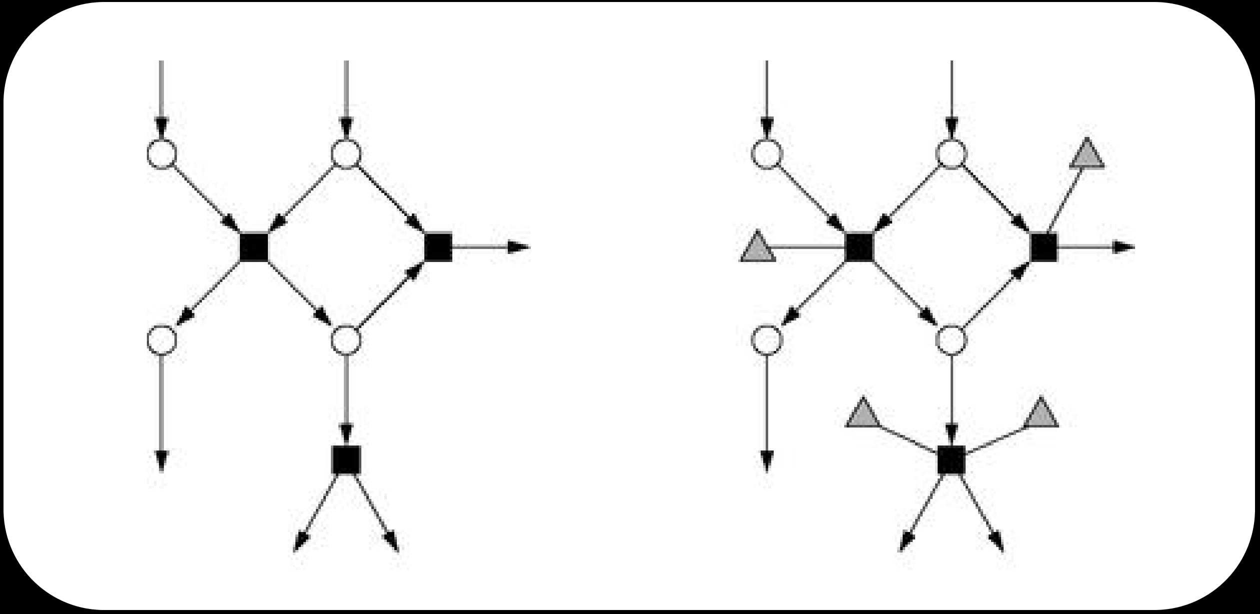
Again, this is a directed graph, as many reactions are irreversible.

The *in* (*out*) degree denotes the number of reactions that produce (consume) a given metabolite.

The analysis of these nets for 43 organisms indicates that "cellular biochemical nets" are ***scale-free*** (hubs = water, ATP, etc).

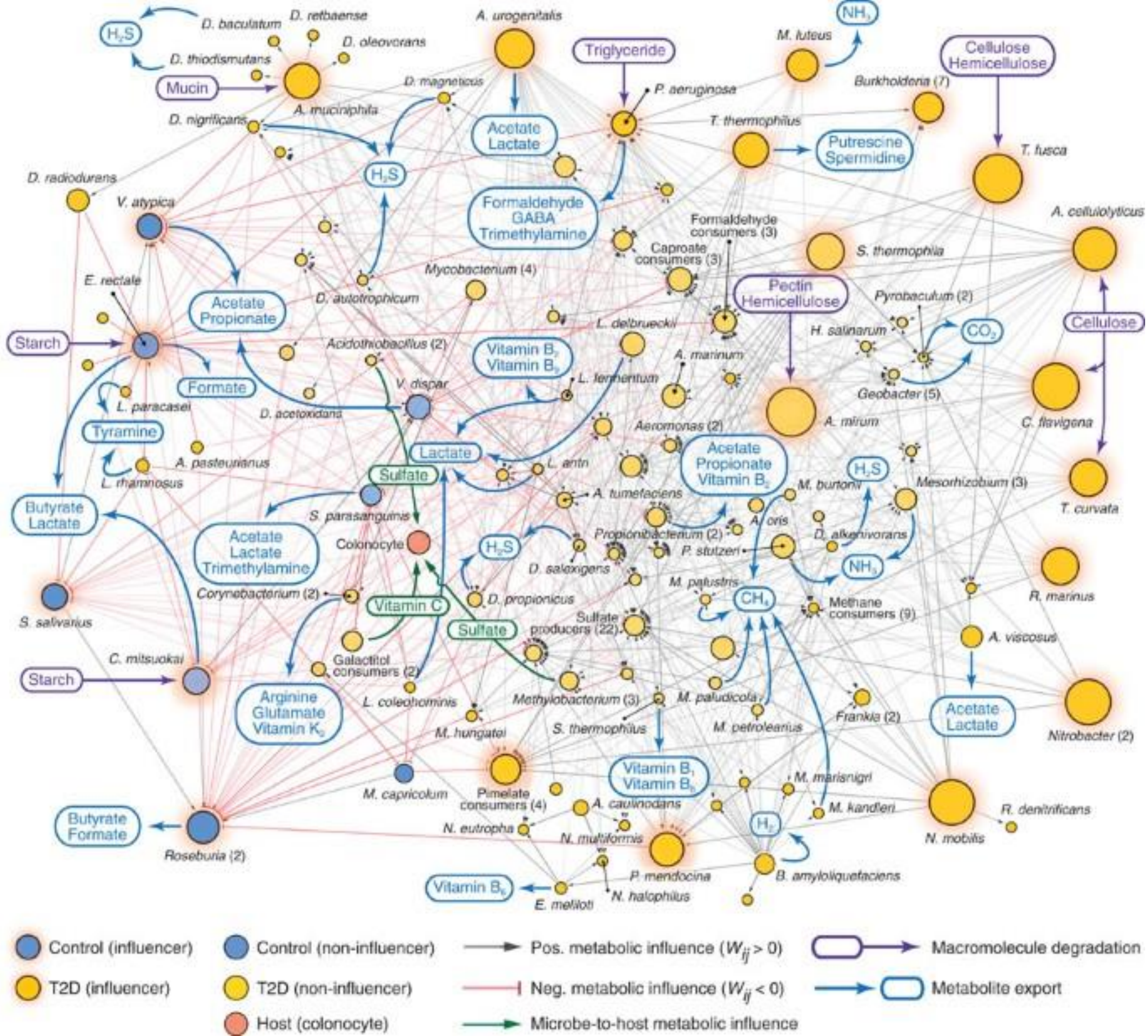
Biological networks: Metabolic networks

Jeong, Albert, Oltvai, Barabási, Fell, Wagner, etc (after 2000).



Bipartide representation (LEFT): metabolites (circles) and reactions (squares) and directed edges indicating which metabolites are substrates (inputs) and products (outputs) of which reactions.

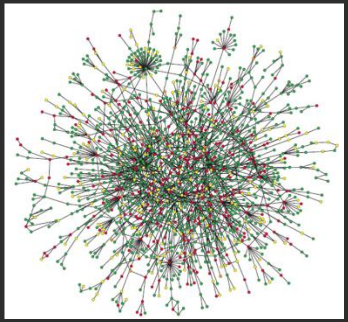
Tripartide representation (RIGHT): A third type of vertex (triangles) can be introduced to represent enzymes, with undirected edges linking them to the reactions they catalyze. The resulting network is a mixed directed/undirected tripartite network.



D. baculatum
D. retbaense
A. urogenitalis
M. luteus
NH_3
H_2S
D. oleovorans
D. thiodismutans
Mucin
A. muciniphila
Triglyceride
P. aeruginosa
Cellulose Hemicellulose
Burkholderia (7)
D. magneticus
D. nigrificans
Acetate Lactate
T. thermophilus
Putrescine Spermidine
T. fusca
H_2S
D. radiodurans
Formaldehyde GABA Trimethylamine
V. atypica
Formaldehyde consumers (3)
Caproate consumers (3)
A. cellulolyticus
S. thermophila
Mycobacterium (4)
E. rectale
D. autotrophicum
Pectin Hemicellulose
Pyrobaculum (2)
Acetate Propionate
L. delbrueckii
H. salinarum
CO_2
Cellulose
Starch
Acidothiobacillus (2)
Vitamin B_2 Vitamin B_3
A. marinum
Formate
V. dispar
L. fermentum
Geobacter (5)
L. paracasei
D. acetoxidans
A. mirum
C. flavigena
Tyramine
Aeromonas (2)
A. pasteurianus
Lactate
Sulfate
L. antri
Acetate Propionate Vitamin B_2
H_2S
Mesorhizobium (3)
L. rhamnosus
A. tumefaciens
M. burtonii
T. curvata
Butyrate Lactate
S. parasanguinis
A. oris
Acetate Lactate Trimethylamine
Colonocyte
Propionibacterium (2)
D. alkenivorans
H_2S
P. stutzeri
D. salexigens
NH_3
R. marinus
Corynebacterium (2)
M. palustris
CH_4
S. salivarius
Vitamin C
D. propionicus
Methane consumers (9)
Sulfate
Sulfate producers (22)
A. viscosus
C. mitsuokai
Starch
Galactitol consumers (2)
Methylobacterium (3)
M. paludicola
Arginine Glutamate Vitamin K_2
L. coleohominis
Frankia (2)
Acetate Lactate
Nitrobacter (2)
M. hungatei
S. thermophilus
M. petrolearius
Vitamin B_1 Vitamin B_6
M. marisnigri
M. capricolum
Pimelate consumers (4)
A. caulinodans
M. kandleri
R. denitrificans
Butyrate Formate
N. eutropha
H_2
N. multiformis
N. mobilis
Rosebura (2)
P. mendocina
B. amyloliquefaciens
Vitamin B_6
E. meliloti
N. halophilus
Control (influencer)
Control (non-influencer)
Pos. metabolic influence ($W_{ij} > 0$)
Macromolecule degradation
T2D (influencer)
T2D (non-influencer)
Neg. metabolic influence ($W_{ij} < 0$)
Metabolite export
Host (colonocyte)
Microbe-to-host metabolic influence

Biological networks: Protein-interaction networks

Uetz et al, Ito et al, Giot et al, Li et al, Jeong et al, Sneppen, Wagner, etc (after 2000).



Nodes : proteins
Edges : experimental evidence that they bind creating protein complexes

Biological networks: genetic regulatory nets

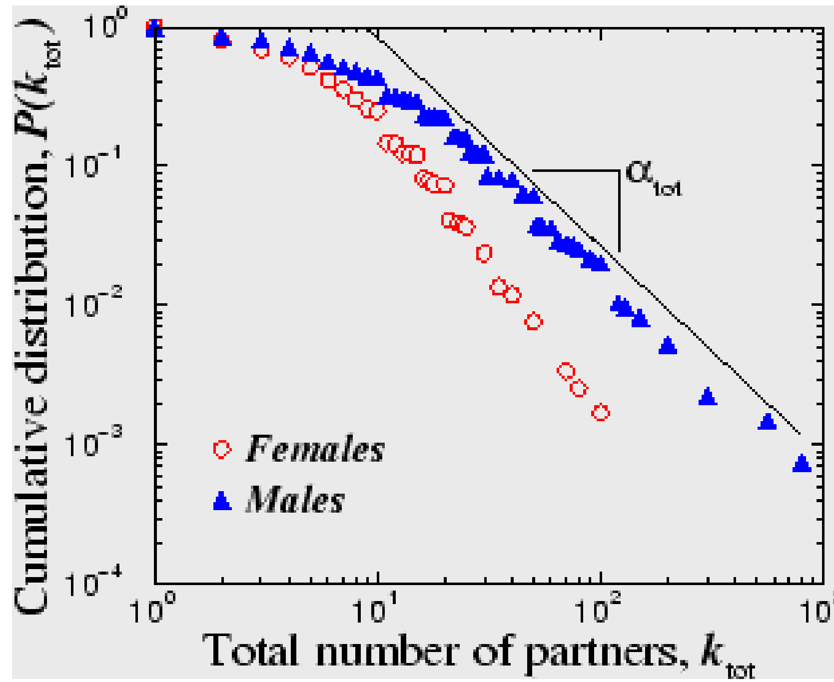
Agrawal 2002. Shaw 2003. Provero 2002, Farkas 2003.

Ex:
Nodes : DNA segments
Edges : indirect interaction through RNA and protein expression products

➡ Plethora of Modelling challenges (ex: Boolean nets, evolutionary networks) and application of “bioinformatics” techniques, DB, etc.

Swedish sexual interaction network

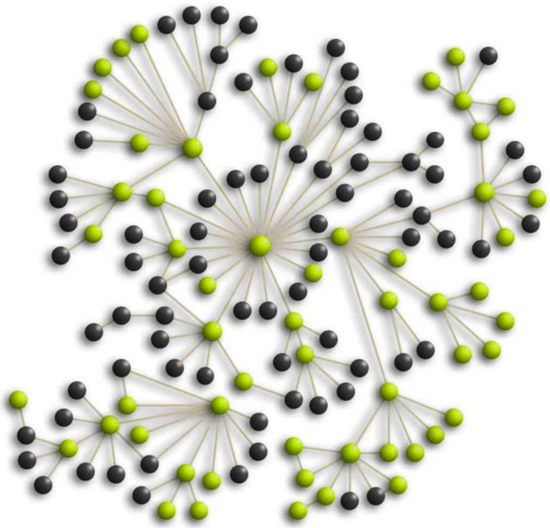
Nodes: people (Females; Males)
Links: sexual relationships



4781 Swedes; 18-74;
59% response rate.

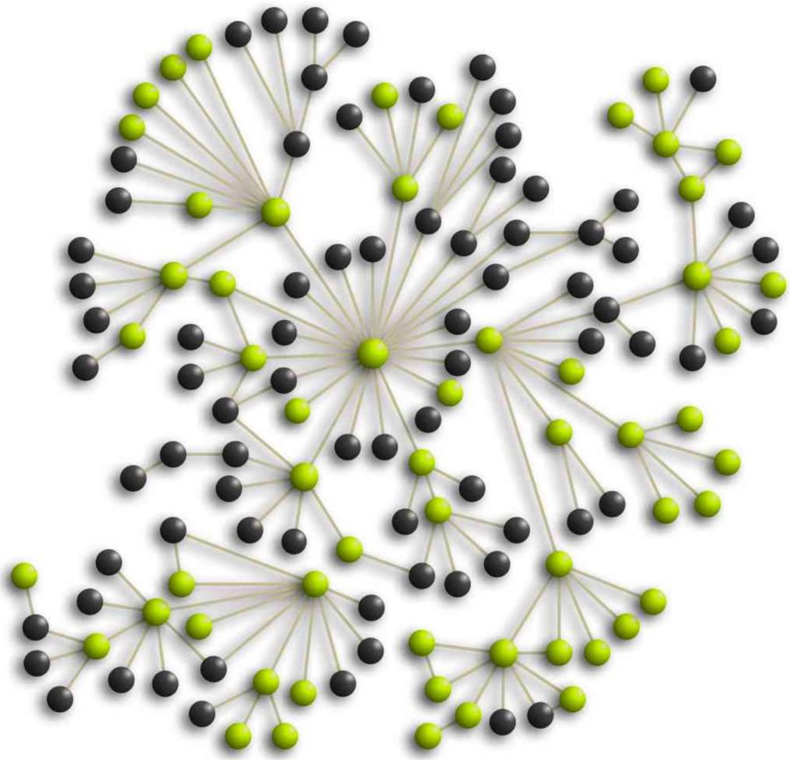
Liljeros et al. Nature 2001

Many real-world nets are scale-free

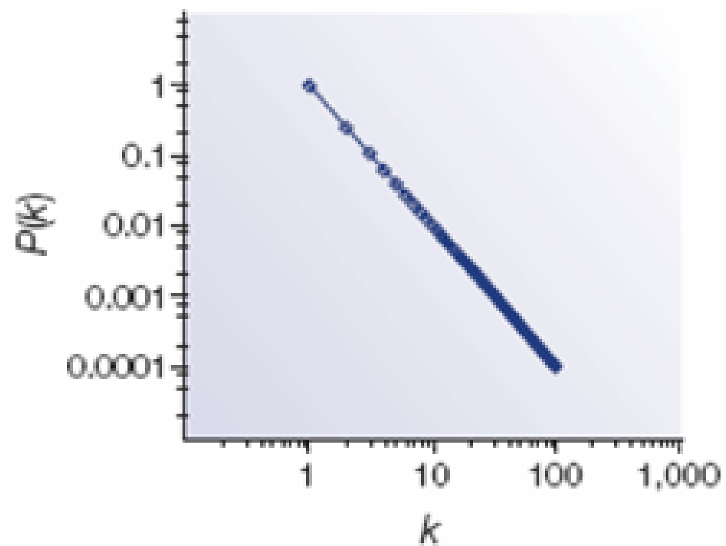


WWW, Internet (routers and domains), electronic circuits, computer software, movie actors, coauthorship networks, sexual web, instant messaging, email web, citations, phone calls, metabolic, protein interaction, protein domains, brain function web, linguistic networks, comic book characters, international trade, bank system, encryption trust net, energy landscapes, astrophysical network…

Universality? I will get back to this point...



$$P_k \sim k^{-\gamma}$$



WWW	actors	citations	sex	cellular	phones	linguistics
$\gamma = 2.1$	$\gamma = 2.3$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 2.1$	$\gamma = 2.1$	$\gamma = 2.8$

A closer look at power-law distributions

discrete representation: $P_k = Ck^{-\gamma}$ with C given by

$$\sum_{k_{\min}}^{\infty} P_k = 1 \Rightarrow C = \frac{1}{\sum_{k_{\min}}^{\infty} k^{-\gamma}} = \frac{1}{\varsigma(\gamma)}$$

Riemann-zeta function

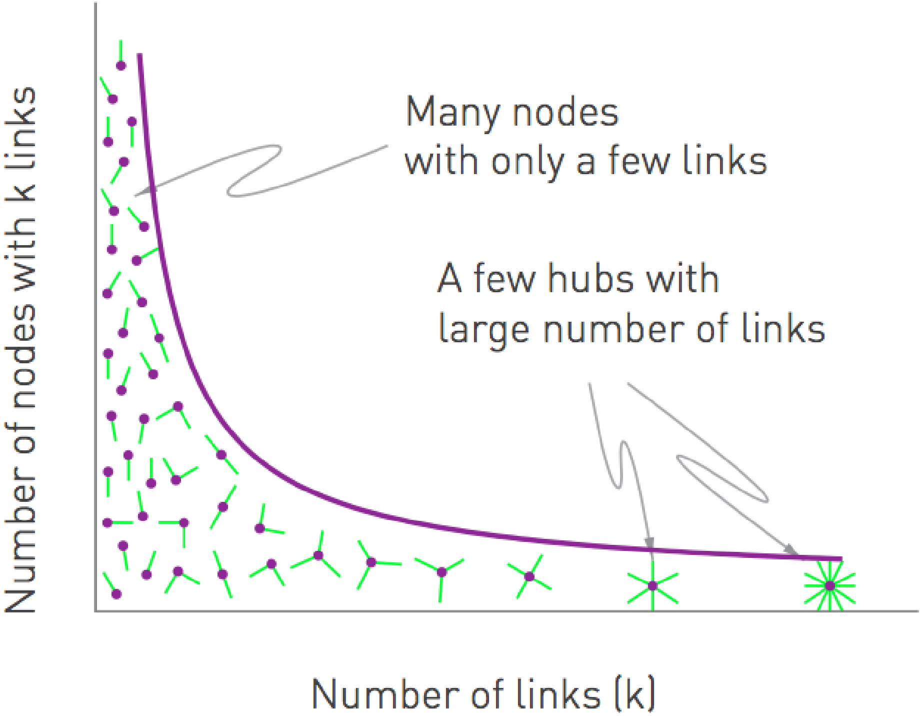
continuum description: $P_k = Ck^{-\gamma}$ with C given by

$$C = \frac{1}{\int_{K_{\min}}^{\infty} k^{-\gamma}\, dk} = (\gamma - 1)k_{\min}^{\gamma-1}$$

Largest hub

- Just for the fun of it: Imagine that we have a **scale-free network** with the size and average degree of the WWW ($N=10^{12}$, $\langle k \rangle=4.6$ and $\gamma=2.1$). What's the probability of having a node with a degree $k \geq 100$?

$$P_k = \left[(\gamma - 1) k_{\min}^{\gamma-1} \right] k^{-\gamma}$$

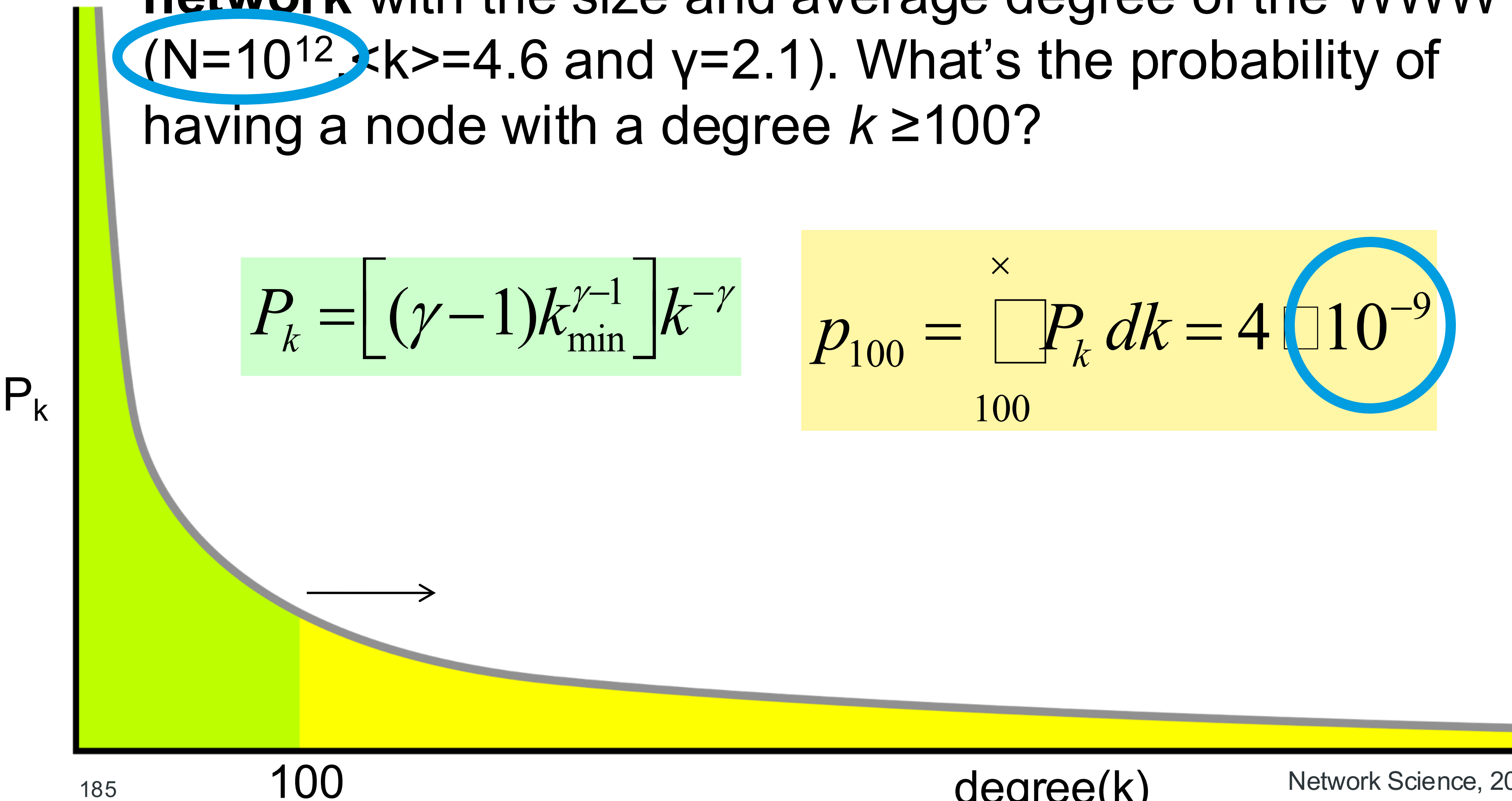


Largest hub

- Just for the fun of it: Imagine that we have a **scale-free network** with the size and average degree of the WWW ($N=10^{12}$, $\langle k\rangle=4.6$ and $\gamma=2.1$). What's the probability of having a node with a degree $k \geq 100$?

$$P_k = \left[(\gamma-1)k_{\min}^{\gamma-1}\right]k^{-\gamma}$$

$$p_{100} = \int_{100}^{\infty} P_k \, dk = 4 \cdot 10^{-9}$$



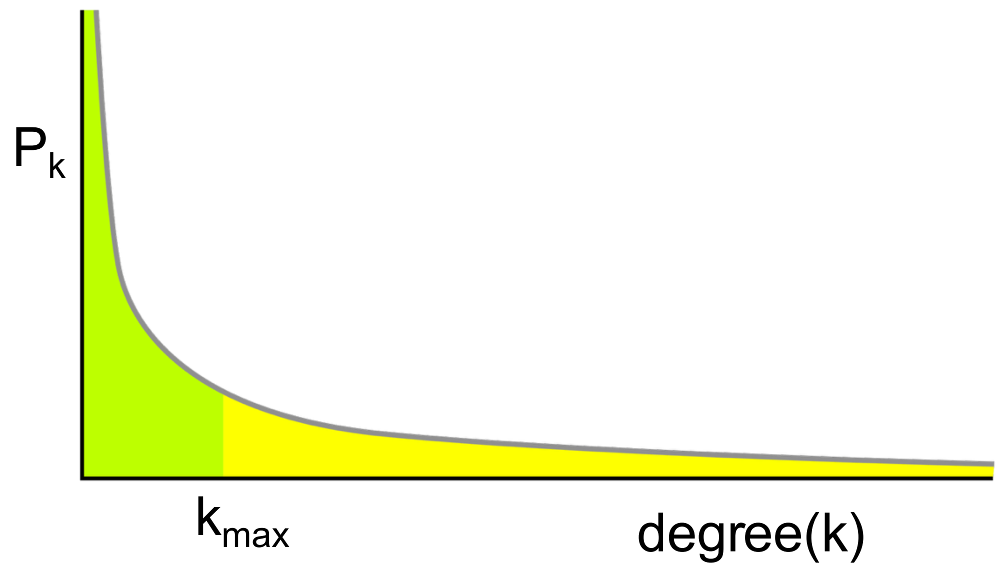
Can we estimate the largest hub?

- We have the degree dist:

$$P_k = (\gamma - 1) k_{\min}^{\gamma-1} k^{-\gamma}$$

- k_{max} will be given by

$$\int_{k_{max}}^{\infty} P_k \, dk < \frac{1}{N}$$

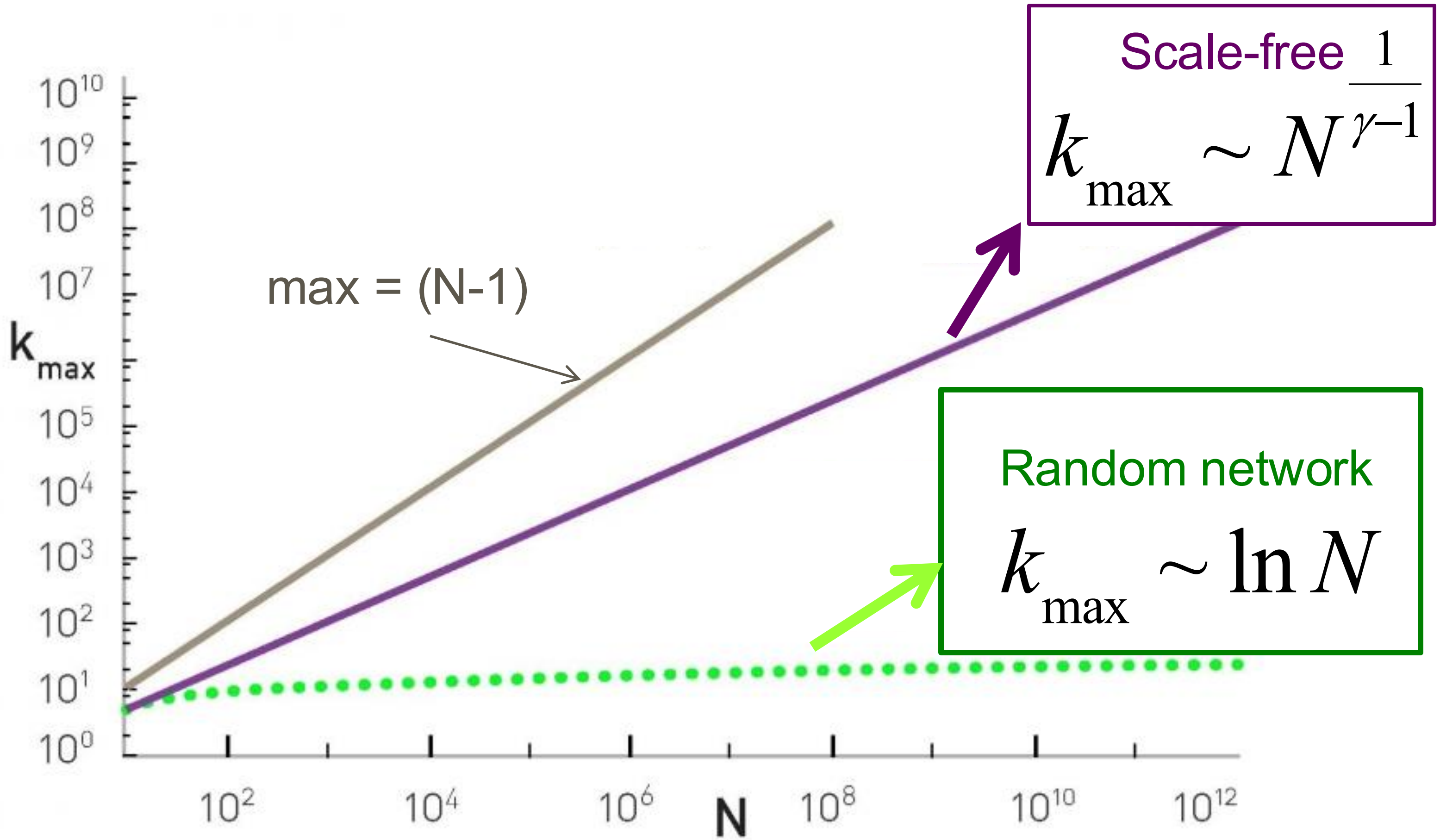


- Which yields

$$k_{max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

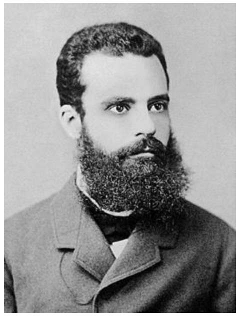
- Example: WWW gives k_{max}=95000

Can we estimate the largest hub?



Revisiting Pareto

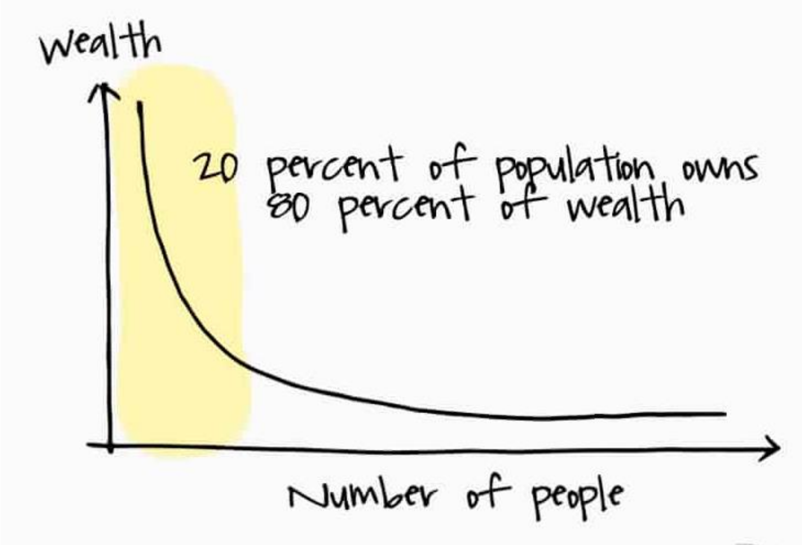
Vilfred Pareto
(1848-1923)



- Power-law distributions are also called Pareto distributions.
- Power-laws is the principle behind the 80/20 rule: *Roughly 80 percent of money is earned by only 20 percent of the population*.

Distribution of world GDP

Quintile of population	Income
Richest 20%	82.70%
Second 20%	11.75%
Third 20%	2.30%
Fourth 20%	1.85%
Poorest 20%	1.40%



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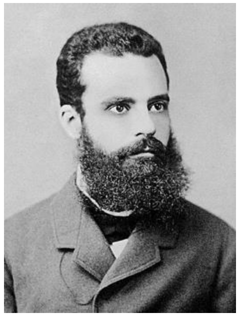
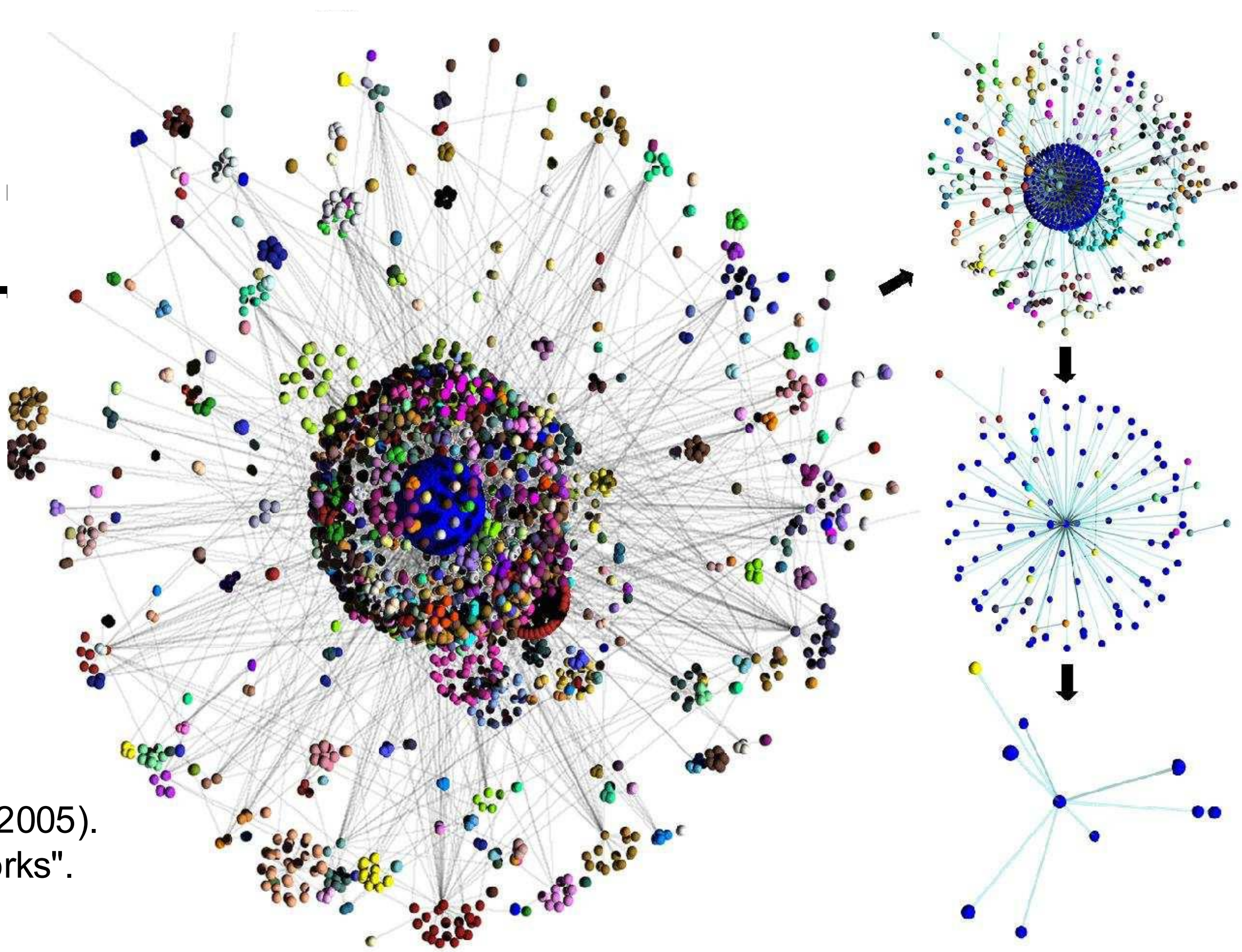


- Power-law distributions are also called Pareto distributions.
- Power-laws is the principle behind the 80/20 rule: *Roughly 80 percent of money is earned by only 20 percent of the population*.
  - WWW: 80% of the links point to 15% of the pages.
  - Citations: 80% of all citations point to 38% of scientists
  - Hollywood: 80% of all links connect 30% of actors.
  - USA: 1% of the population earns 15% of the total US income.

Scale-free? What does it mean?

1st explanation: Scale invariance

Scale-invariance, self-similarity, etc.



Song, C.; Havlin, S.; Makse, H.(2005). "Self-similarity of complex networks". Nature 433 (7024): 392–395.

Scale-free? What does it mean?

1st explanation: Scale invariance

Scale-invariance?

$$P(k) = k^{-\gamma}$$

$$P(ck) = (ck)^{-\gamma} = Const.P(k) \propto P(k)$$

- Scaling the argument ***k*** by a constant factor ***c*** causes only a proportionate scaling of the function itself.
- In other words, all power laws (with a given exponent) are equivalent up to constant factors, since each is simply a scaled version of the others.

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

- ***n^{th} moments of a distribution***

$$\langle k^n \rangle = \sum_{k_{min}}^{\infty} k^n P_k \approx \int_{k_{min}}^{\infty} k^n P_k \, dk$$

- n=0 sums to one.
- n=1 gives the ***average*** degree
- n=2 helps us to calculate the ***variance***
- n=3 determines the ***skewness***…

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2$$

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

- For Scale-free networks we have

$$\langle k^n \rangle \sim \int_{k_{\min}}^{k_{\max}} k^n P_k \, dk = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n-\gamma+1}$$

- n=0 sums to one.
- n=1 gives the ***average*** degree
- n=2 helps us to calculate the ***variance***
- n=3 determines the ***skewness***

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

Scales with N

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

- For Scale-free networks we have

$$\langle k^n \rangle \approx \int_{k_{\min}}^{k_{\max}} k^n P_k \, dk = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n-\gamma+1}$$

Do not scale with N

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

Scales with N

- n=0 sums to one.
- n=1 gives the **average** degree
- n=2 helps us to calculate the **variance**
- n=3 determines the **skewness**

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

For large N, it diverges for **γ<2**

- For Scale-free networks we have (for large N)

$$\langle k^n \rangle \xrightarrow{N\to\infty} N^{\frac{n-\gamma+1}{\gamma-1}}$$

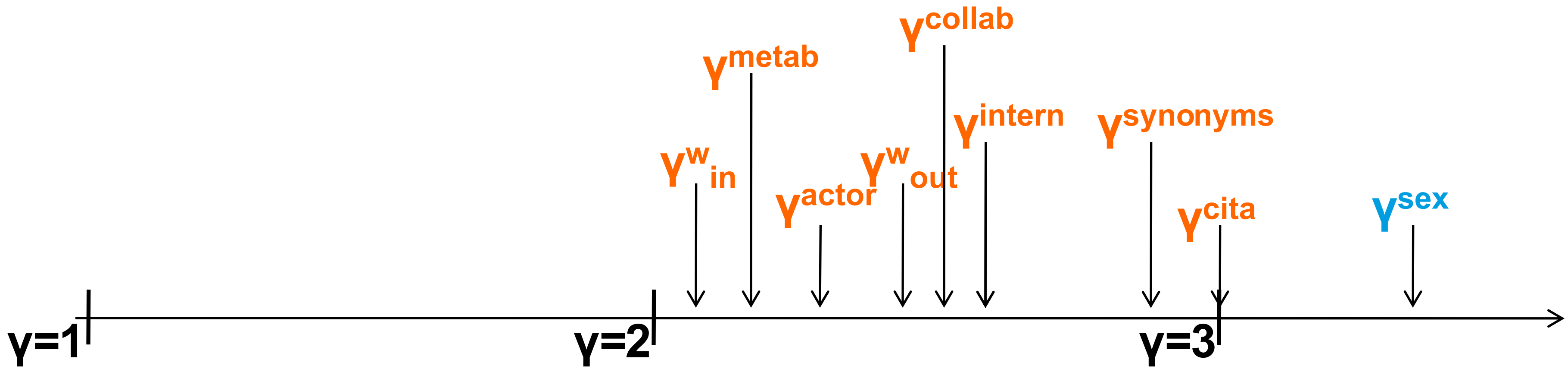
$$\langle k \rangle \sim N^{\frac{2-\gamma}{\gamma-1}}$$

$$\langle k^2 \rangle \sim N^{\frac{3-\gamma}{\gamma-1}}$$

Diverges for **γ<3**

- n=0 sums to one.
- n=1 gives the ***average*** degree
- n=2 helps us to calculate the ***variance***
- n=3 determines the ***skewness***

Scale-free? What does it mean?



$\langle k^2 \rangle$ diverges | $\langle k^2 \rangle$ finite

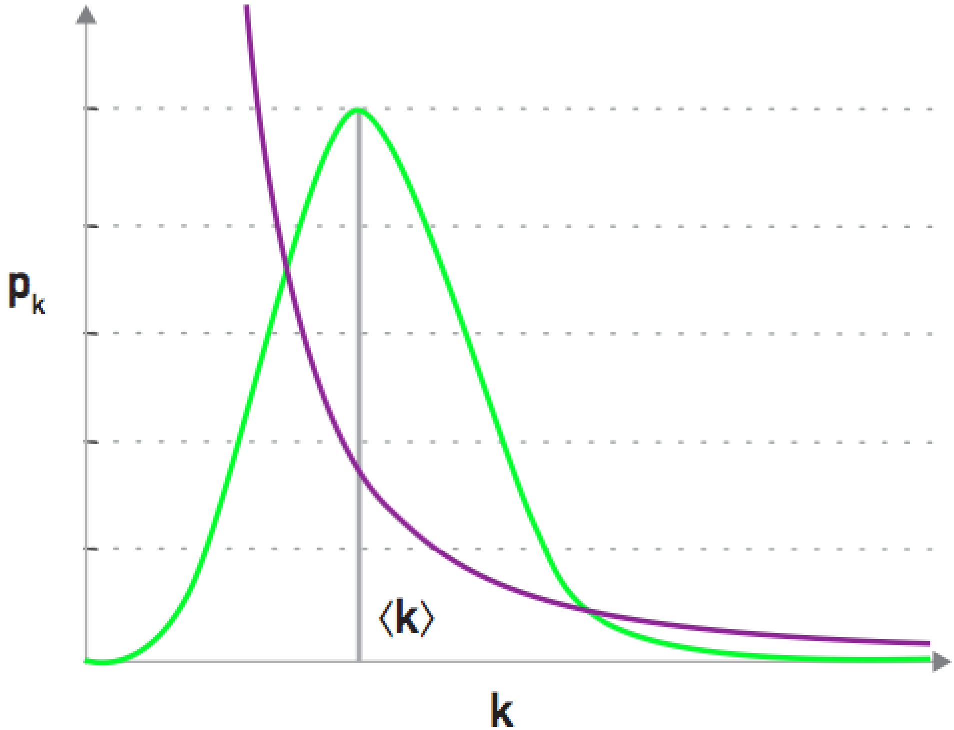
$\langle k \rangle$ diverges | $\langle k \rangle$ finite

Most networks are in a regime in which the variance diverges for large N ☺

$$k = \langle k \rangle \pm \times$$

For large N, average values are not meaningful, as fluctuations are too large!

Scale-free? What does it mean?

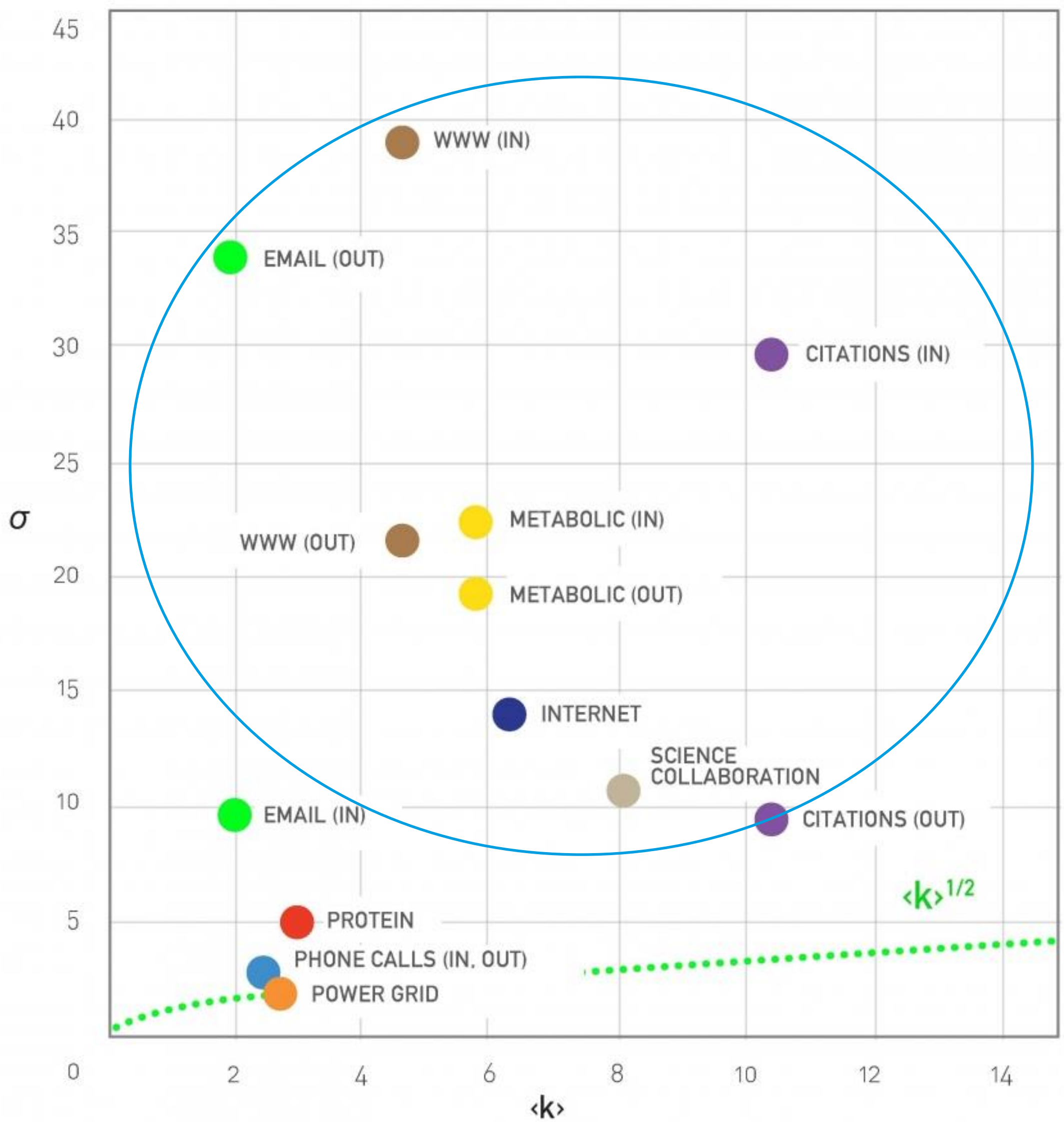


Random Network
Randomly chosen node: $k = \langle k \rangle \pm \langle k \rangle^{1/2}$
Scale: $\langle k \rangle$

Scale-Free Network
Randomly chosen node: $k = \langle k \rangle \pm \infty$
Scale: none

Scale-free? Networks are finite, yet…

Standard deviation is very large in real networks



Do hubs affect the small world property?

Let’s compute the average path length (APL) for a scale-free network...

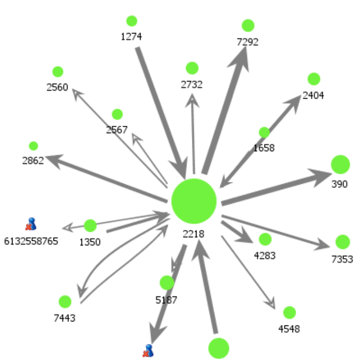
Do we live in a ultra-small-world?

$$APL = \langle L \rangle \sim \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

Anomalous regime (γ≤2)

$$k_{max} = k_{min} N^{\frac{1}{\gamma-1}}$$

K_{max} grows faster than *N*



const.	$\gamma=2$
$\ln \ln N$	$2<\gamma<3$
$\dfrac{\ln N}{\ln \ln N}$	$\gamma=3$
$\ln N$	$\gamma>3$

Ultra-small world regime

$$APL = \langle L \rangle \square \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

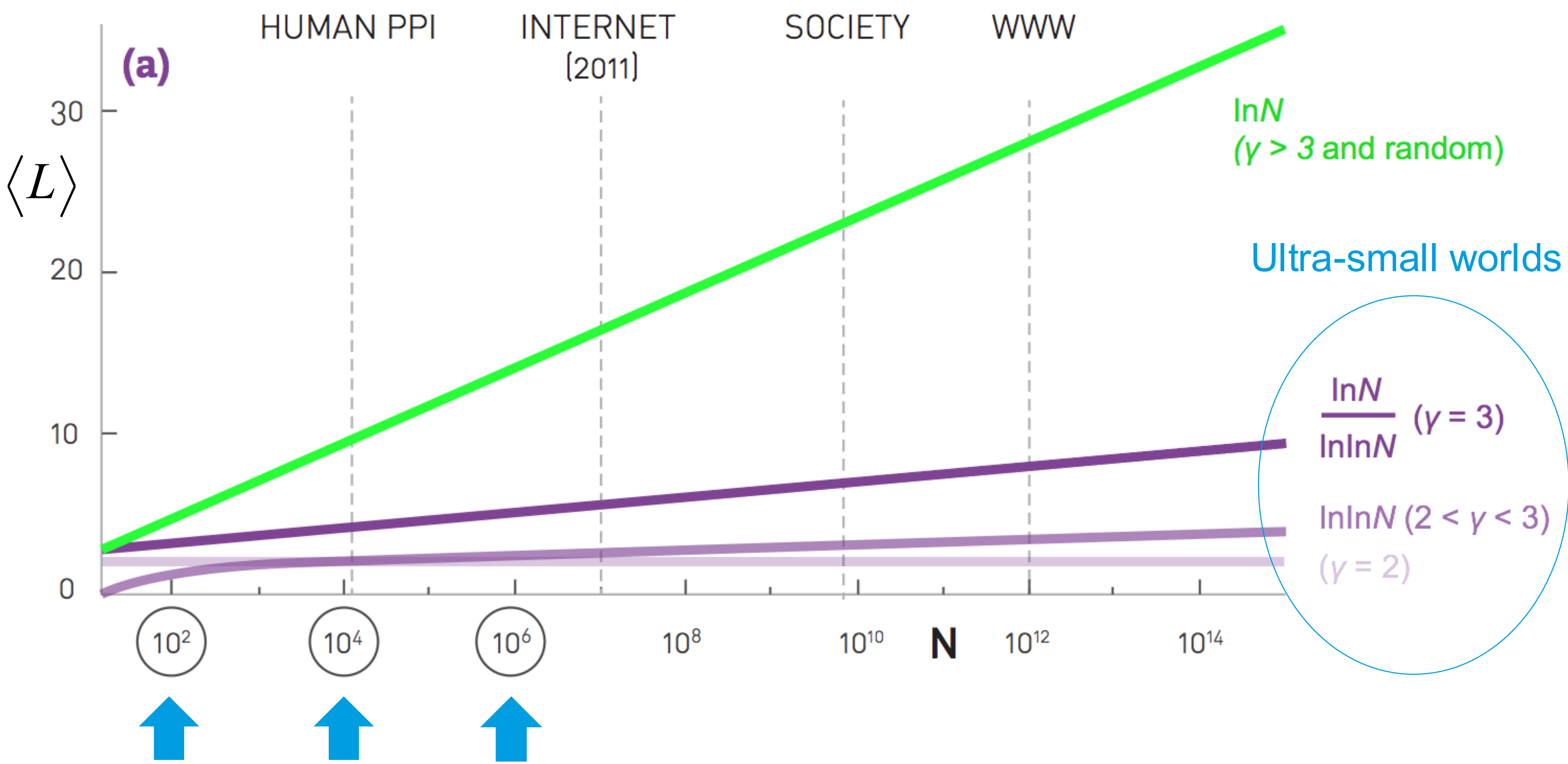
Critical regime

$$APL = \langle L \rangle \square \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

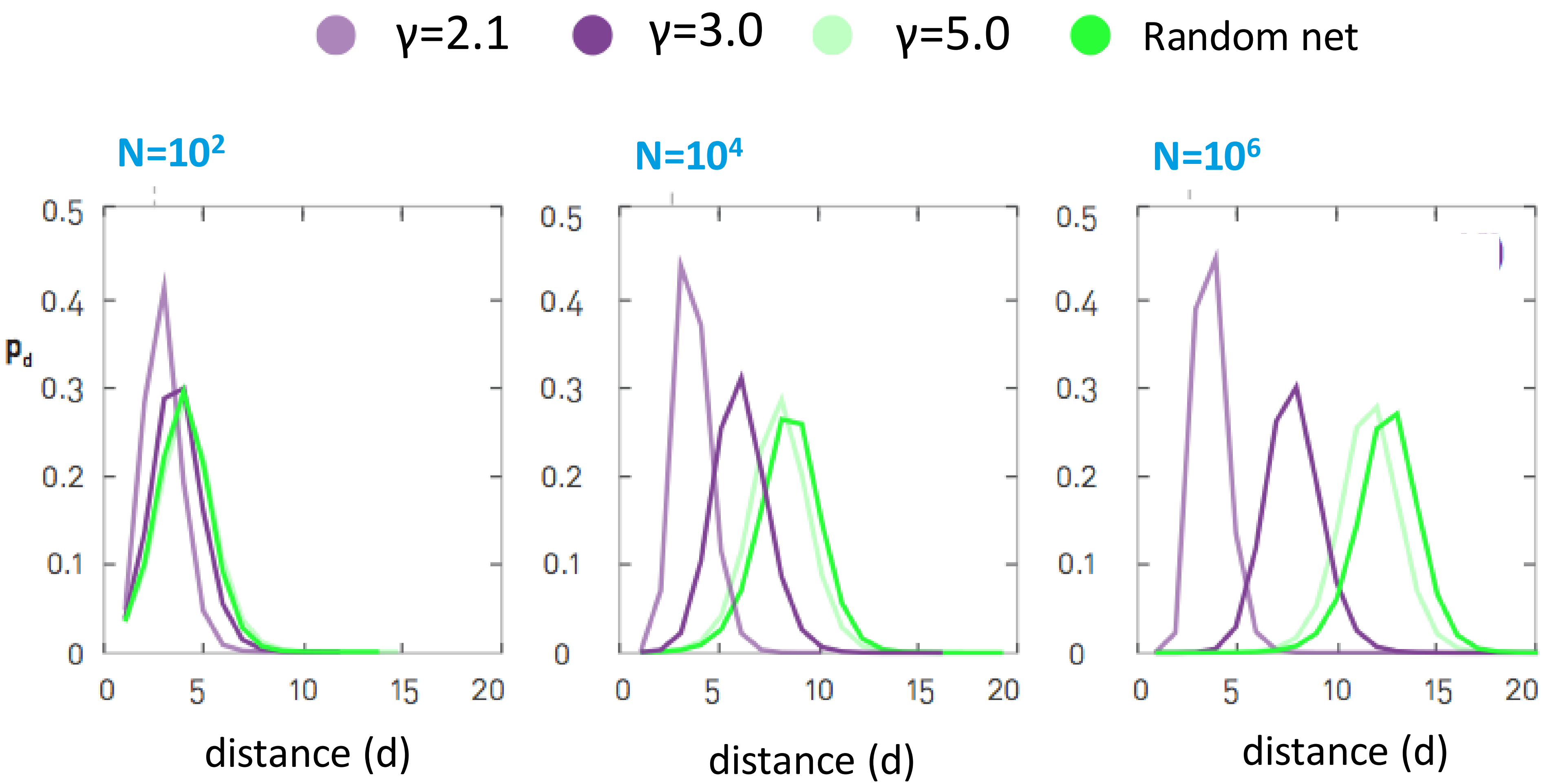
Small-world regime (same as random graphs)

$$APL = \langle L \rangle \, \square \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

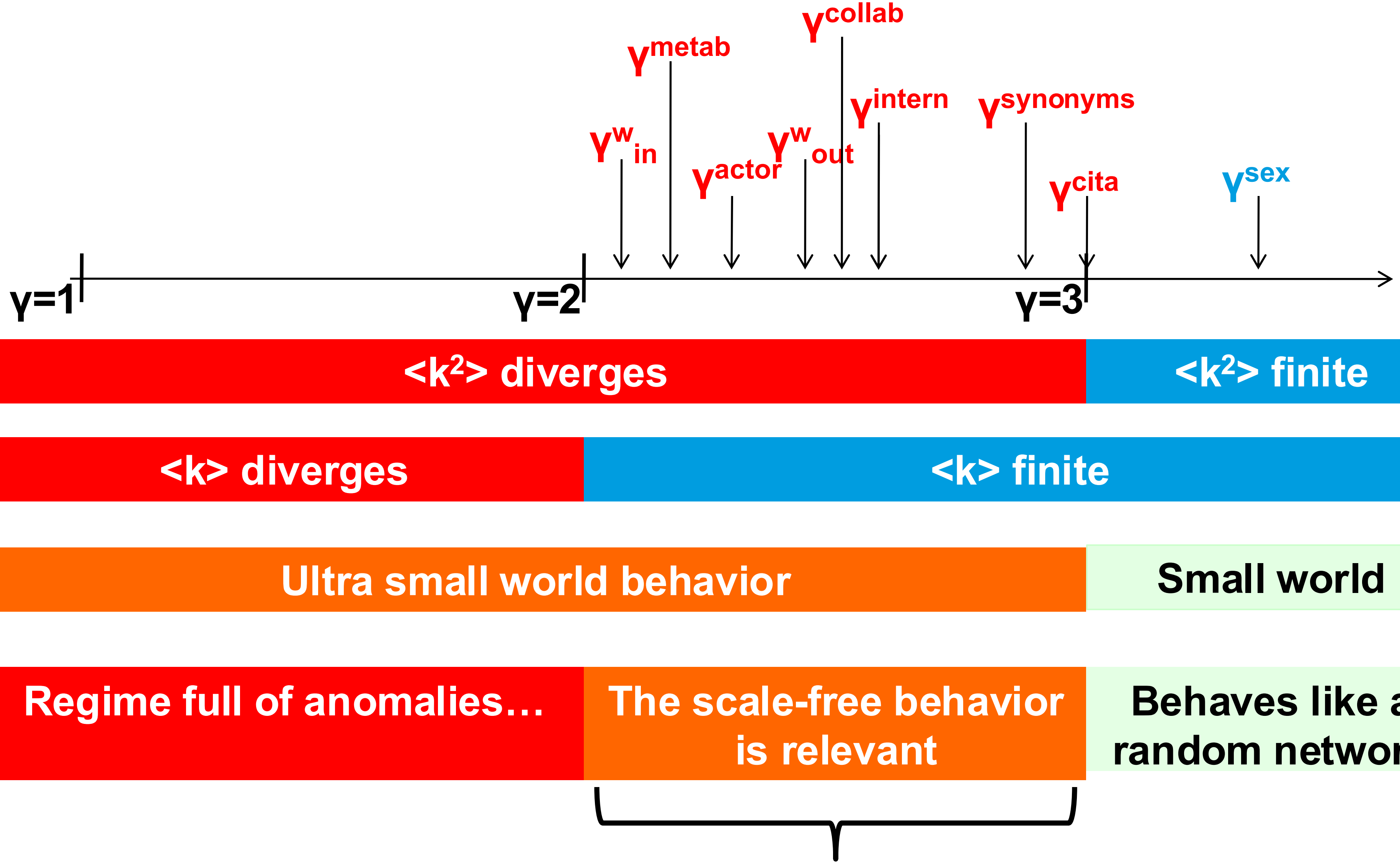
Do we live in a ultra-small-world?



Do we live in a ultra-small-world?



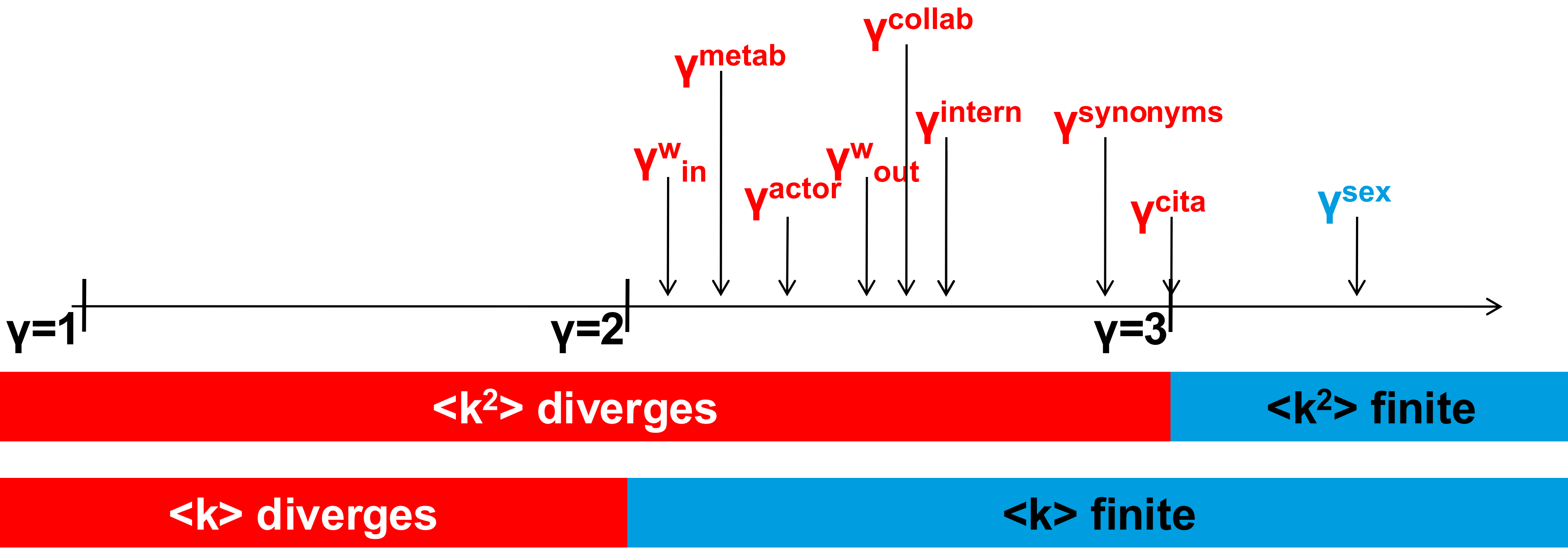
The universe of scale-free networks



Why are most exponents in this regime?

A world of magic exponents

Can you find a good argument which justifies this picture?

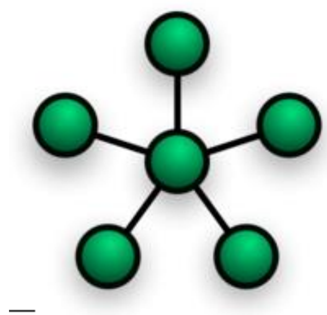


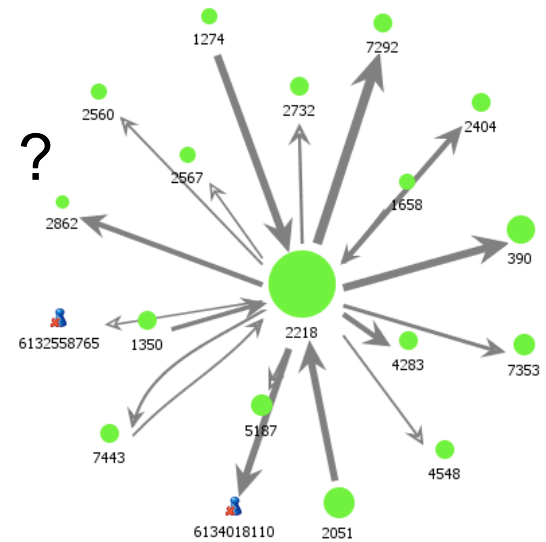
Magic exponents?

- Why is it hard to find networks with γ≤2 ?

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

K_{max} grows faster than *N*





- Why is it hard to find networks with γ>>3 ?

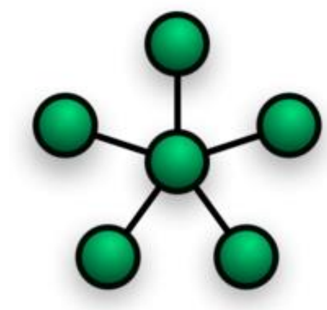
Magic exponents?

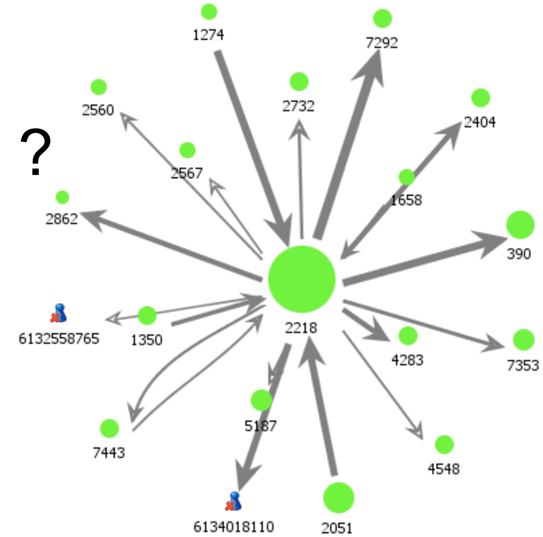
- Why is it hard to find networks with $\gamma \leq 2$?

$$k_{max} = k_{min} N^{\frac{1}{\gamma - 1}}$$

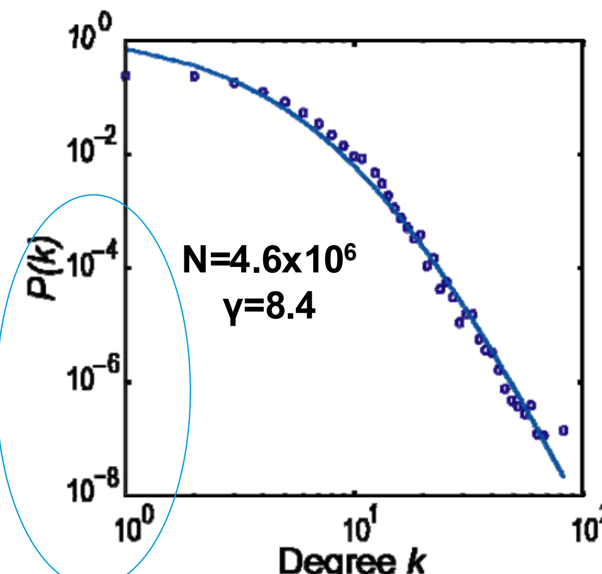
K_{max} grows faster than *N*

- **Why is it hard to find networks with $\gamma >> 3$?**





Mobile Call Network

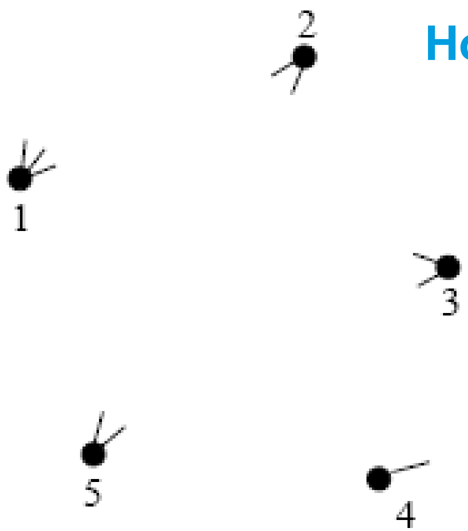


Onella et al. PNAS 2007

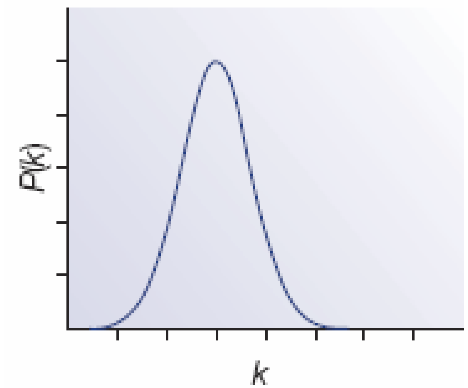
Let’s return to one of the challenges discussed in the last class

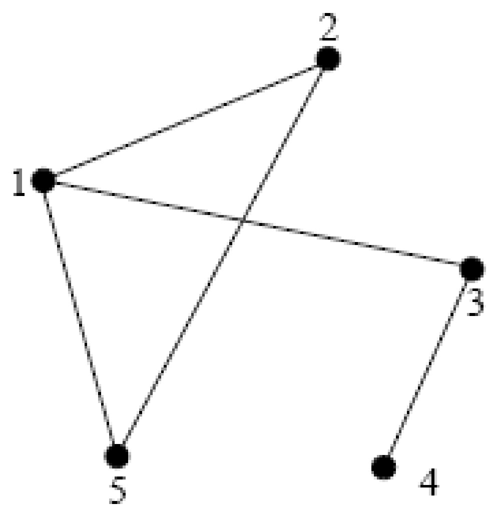
Challenge:
Can you imagine an algorithm capable of creating a random graph with an arbitrary degree distribution?

Configuration model



How to generate a graph compatible with a given $P(k)$



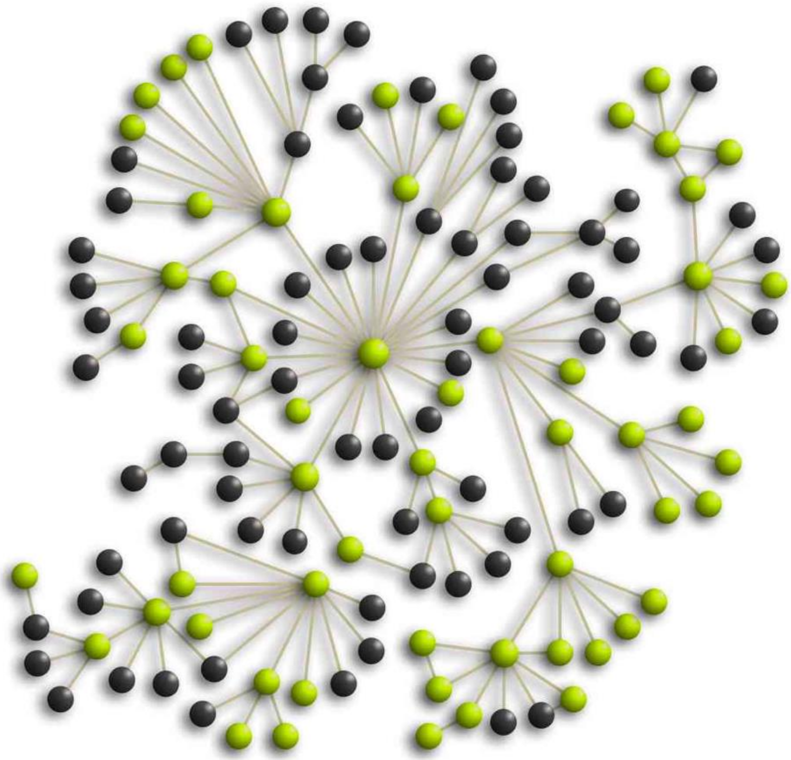


1. **Create a histogram of $P(k)$ (discretization) the sum of which gives N<k>**
2. **add stubs to n's according to histogram**
3. **connect the stubs at random.**
4. **this leads to a random graph with a pre-defined $P(k)$.**

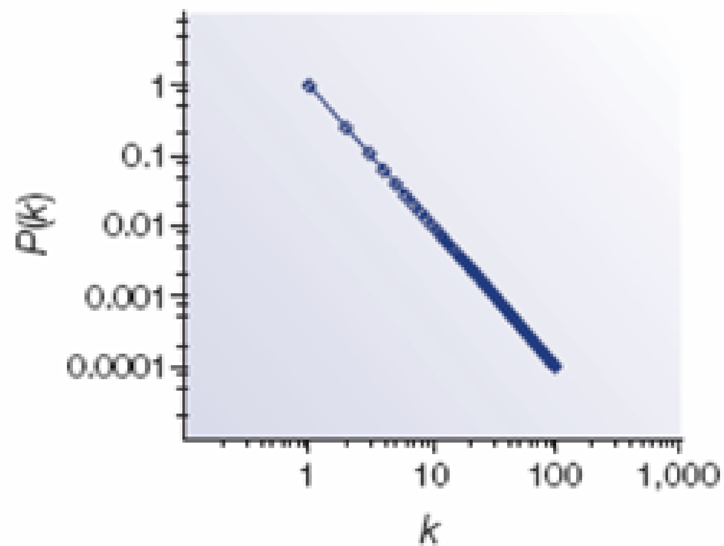
2^{nd} challenge of the day

Challenge 5:
Can you create an algorithm capable of increasing its clustering coefficient without changing its degree distribution?

Next challenge: Universality?



$$P_k \sim k^{-\gamma}$$



WWW	actors	citations	sex	cellular	phones	linguistics
$\gamma = 2.1$	$\gamma = 2.3$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 2.1$	$\gamma = 2.1$	$\gamma = 2.8$