DEI
DEPARTAMENTO
DE ENGENHARIA INFORMÁTICA
TÉCNICO LISBOA

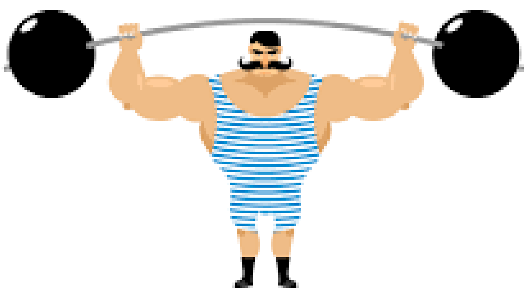
Resilience of complex networks and cascading effects

Network Science, 2025/2026

Does it really matter?

- Robustness in ***biology and medicine***: there are countless protein misfolding errors, missed cell reactions, and mutations which are neutral and others that lead to diseases.
- Stability of ***human societies and institutions***: social, economical and political networks are constantly being perturbed by wars, political and economical cycles, etc.
- ***Ecology and sustainability***: analyze the disruptive effect of climate & human activity in ecological networks.
- ***Engineering***: design communication systems, cars or airplanes which cope with occasional component failures.

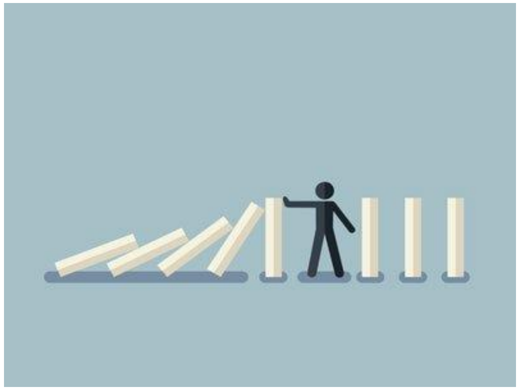
Synopsis



robustness

Random failures & attacks

Cascading effects

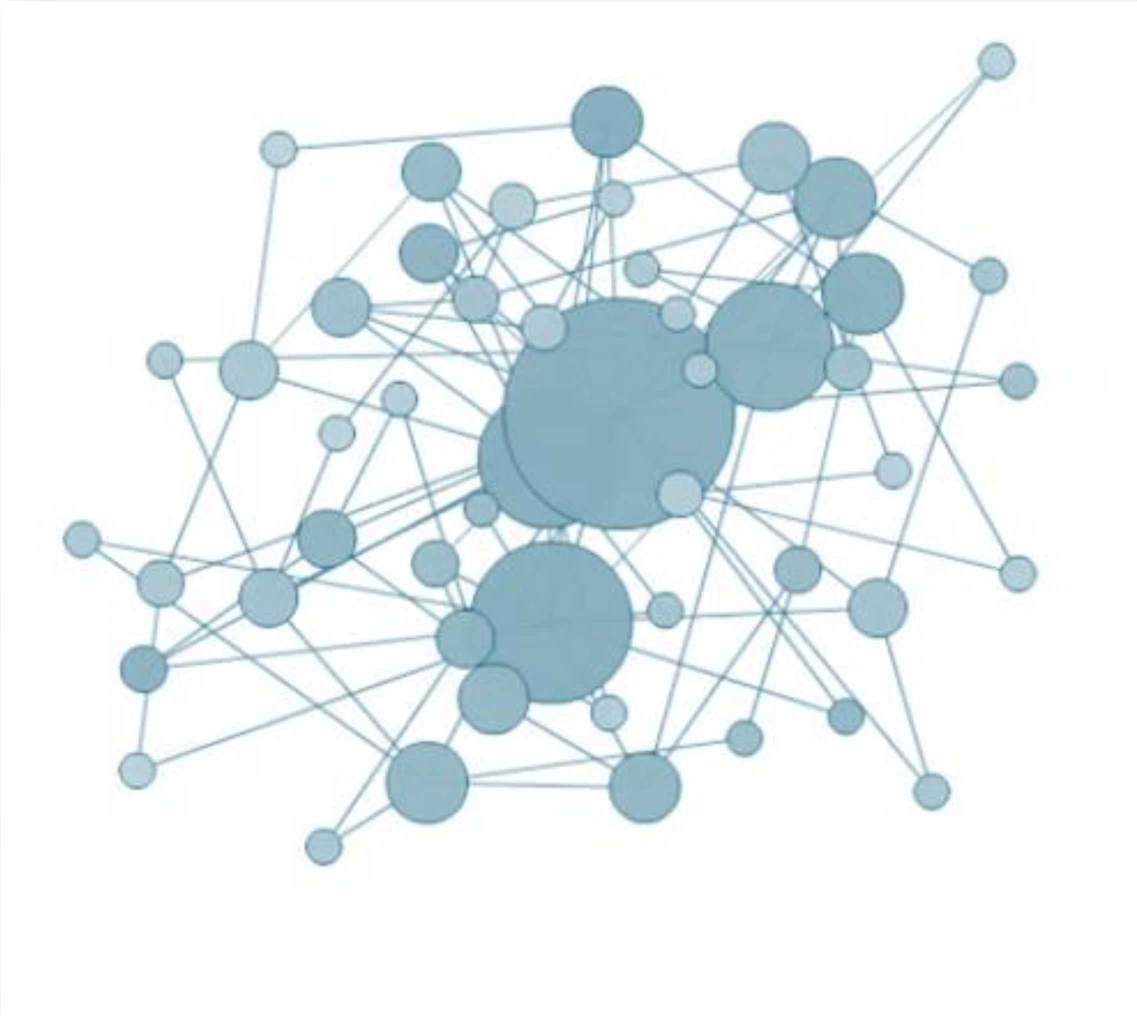


Building robustness

Modeling cascading failures

Complex Networks Under Node Failures

Randomly select and remove nodes, one by one.



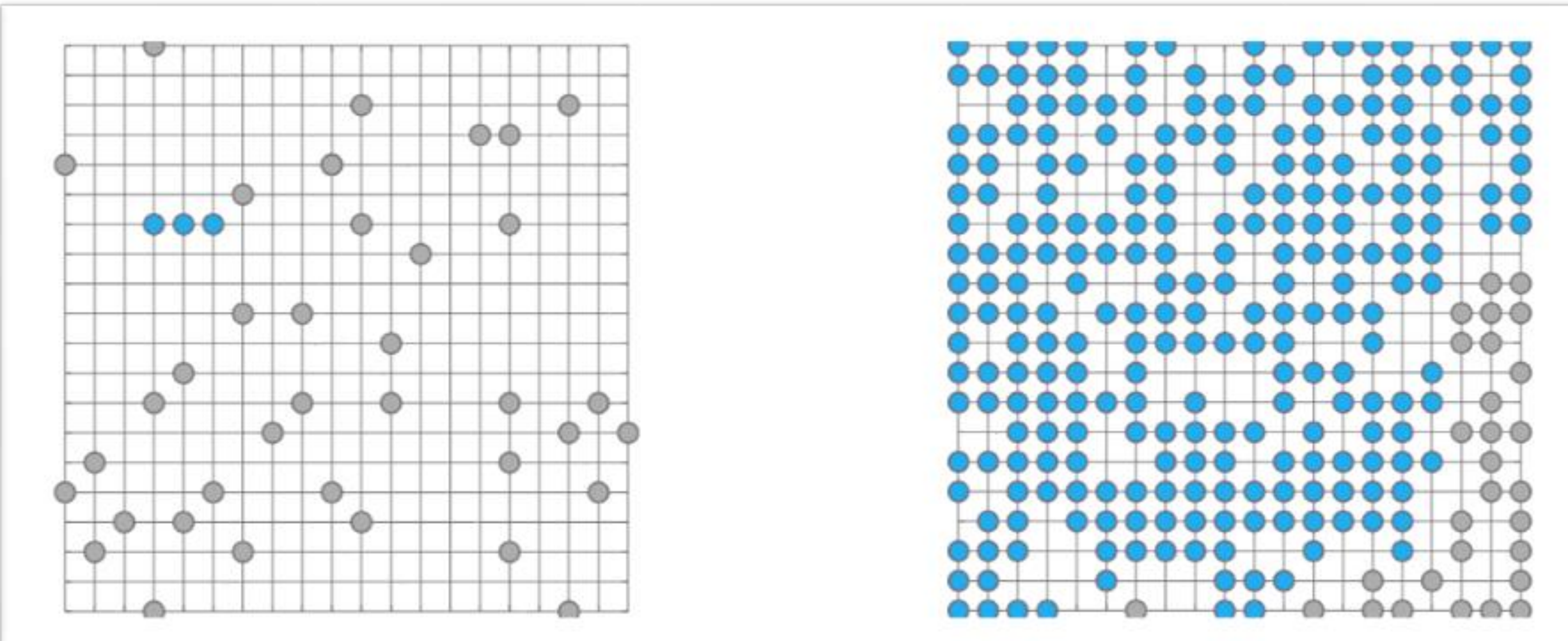
Example of a network under random node removal (video by Dashun Wang)

How many nodes do we have to delete to fragment the network into isolated components?

Robustness toolkit: percolation theory

Example: site percolation in 2D

- Let's place pebbles with probability p at each intersection

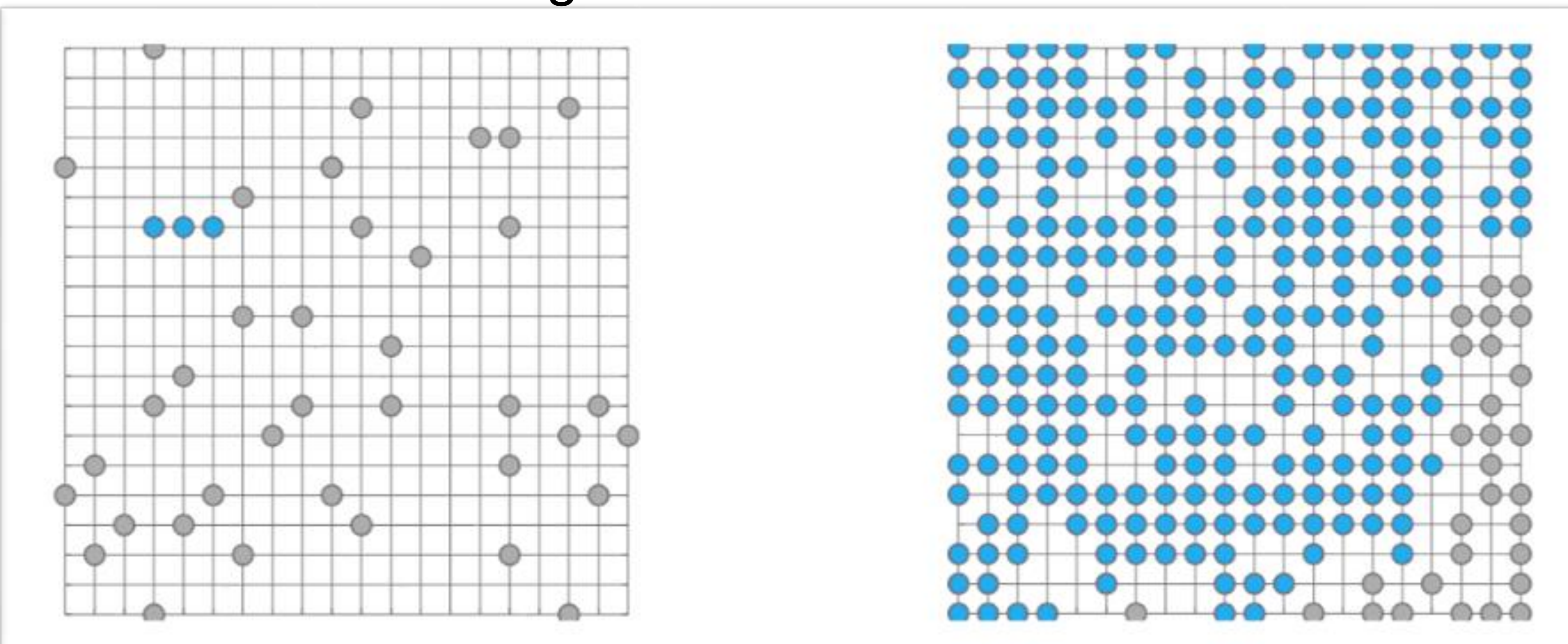


p=0.1 p=0.7

Robustness toolkit: percolation theory

Example: site percolation in 2D

- What's the expected size of the largest cluster?
- What's the average cluster size?



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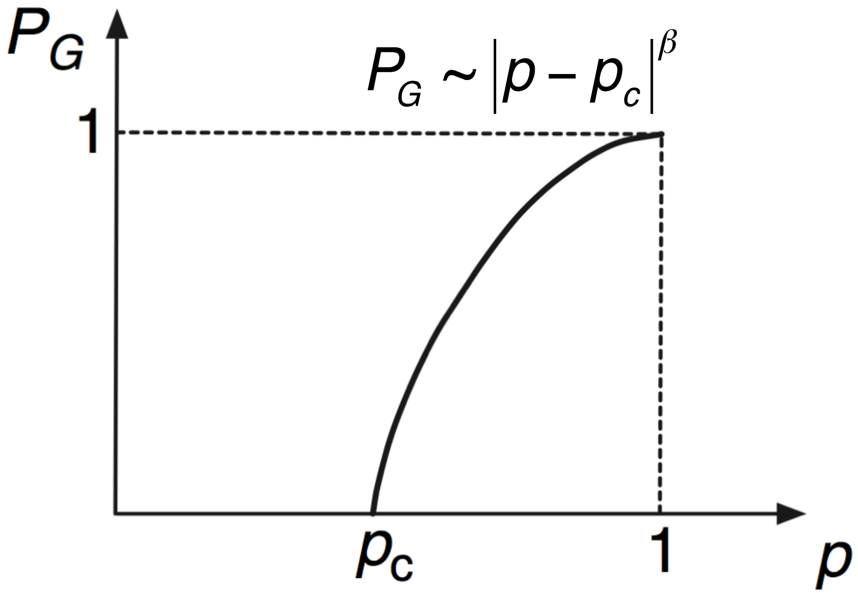

Probability for a node to belong to the largest cluster

Robustness toolkit: percolation theory

Example: site percolation in 2D

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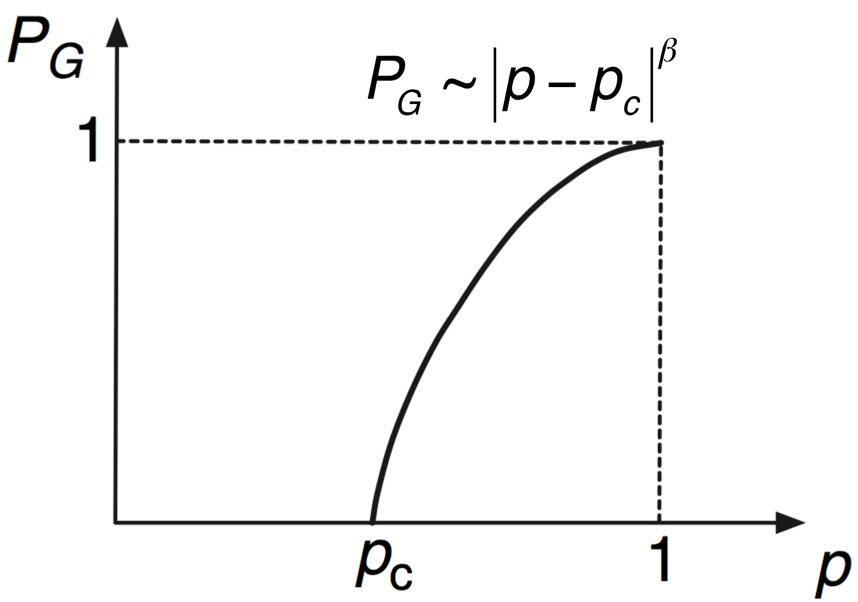
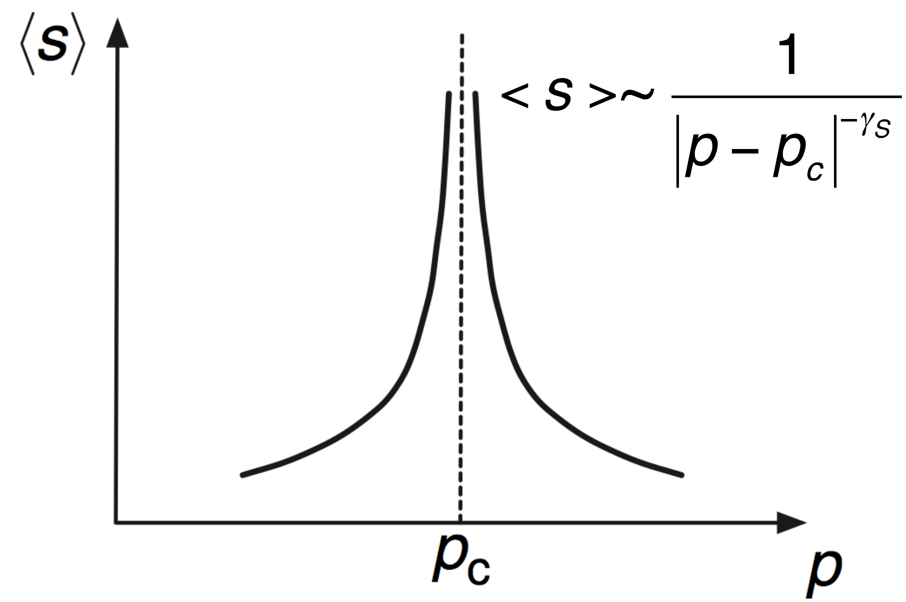


Probability for a node to belong to the largest cluster



Average cluster size
(apart from the largest)

Robustness toolkit: percolation theory

Example: site percolation in 2D

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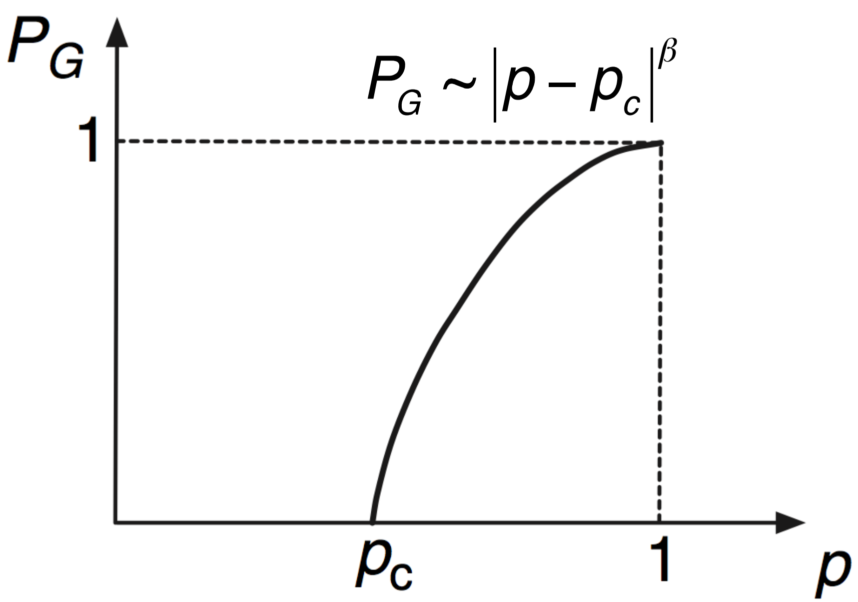
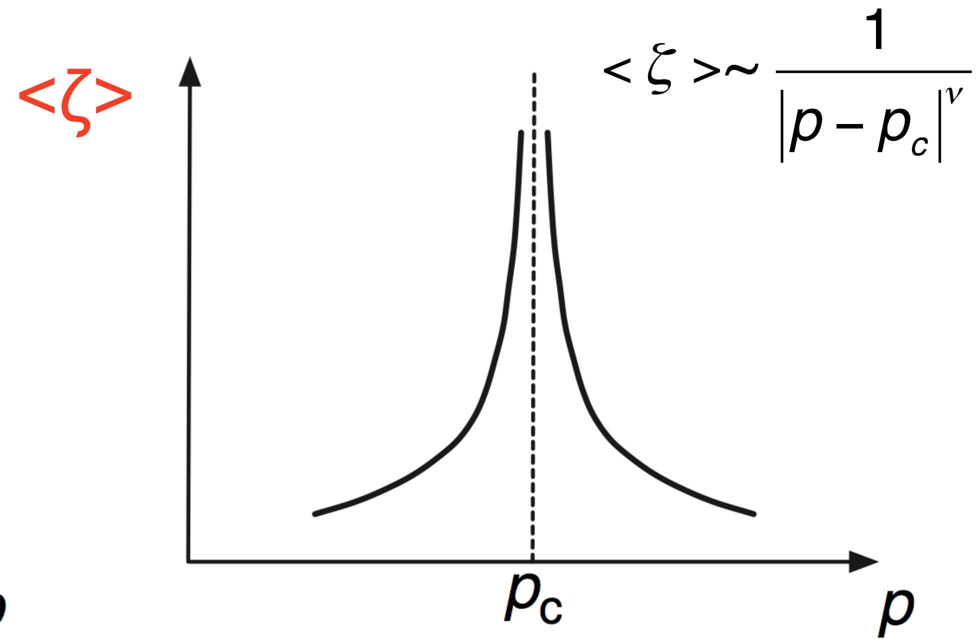


Probability for a node to belong to the largest cluster



Similarly the correlation length ζ (mean distance between 2 pebbles of the same cluster) also diverges for $p=p_c$

Robustness toolkit: percolation theory

Example: site percolation in 2D

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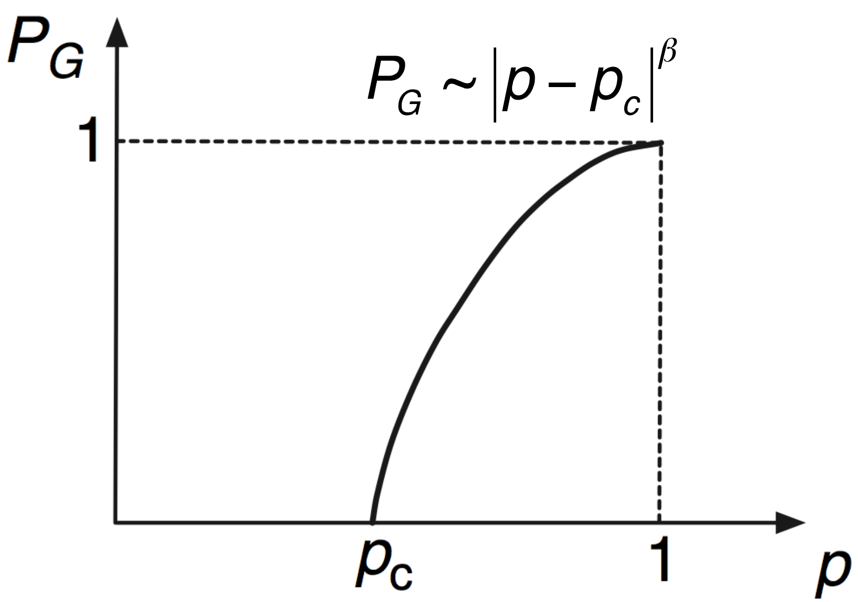
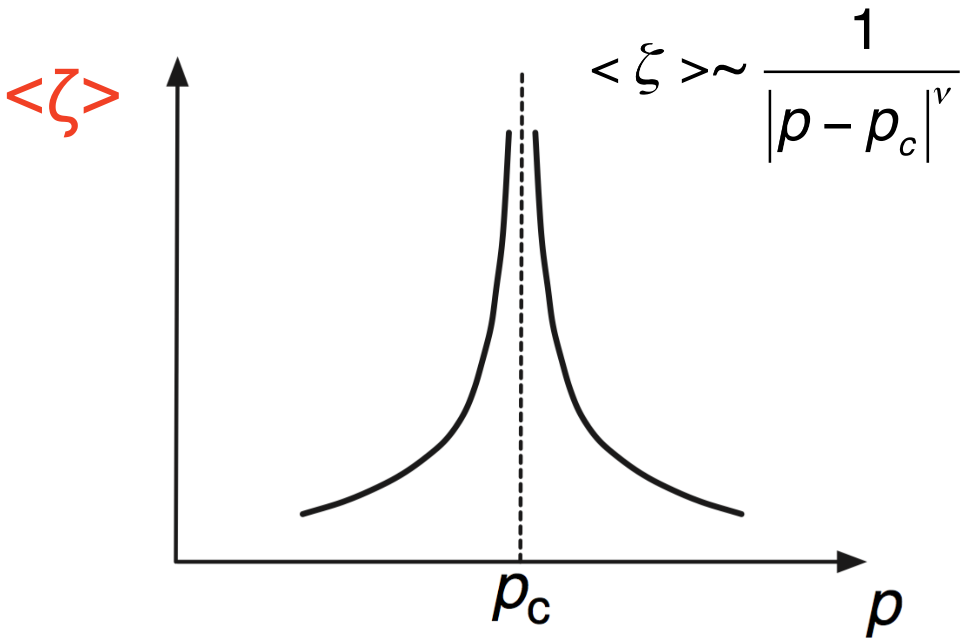


Probability for a node to belong to the largest cluster



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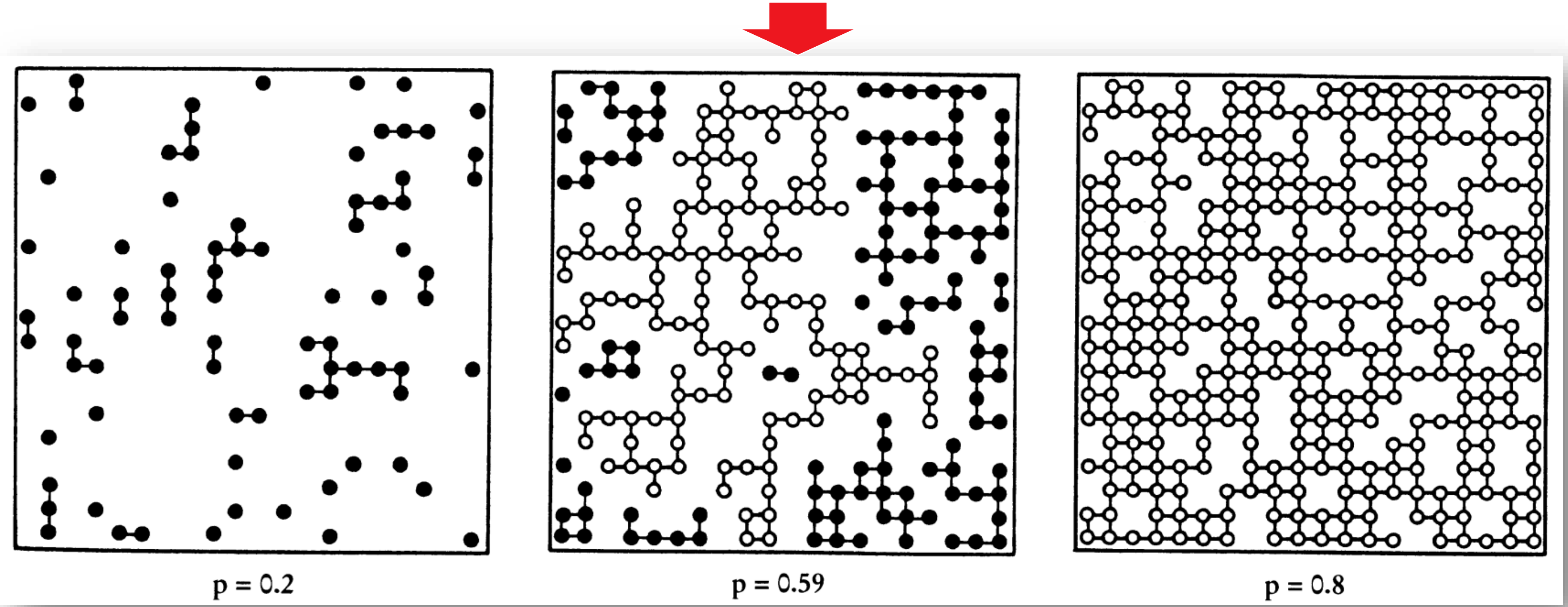
Robustness toolkit: percolation theory

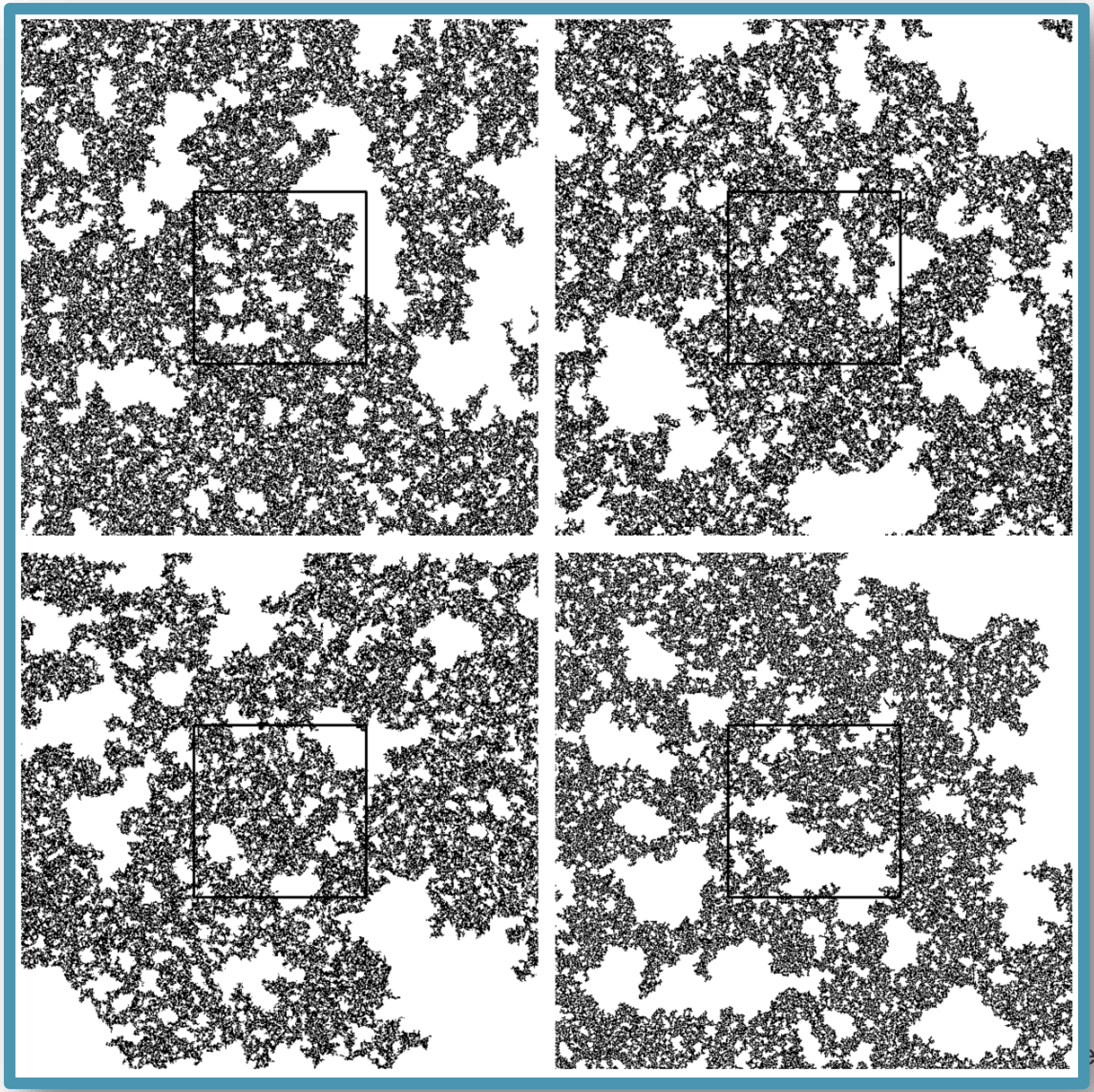
Example: site percolation in 2D

Maximal correlation length ζ
(mean distance between 2 pebbles of
the same cluster)



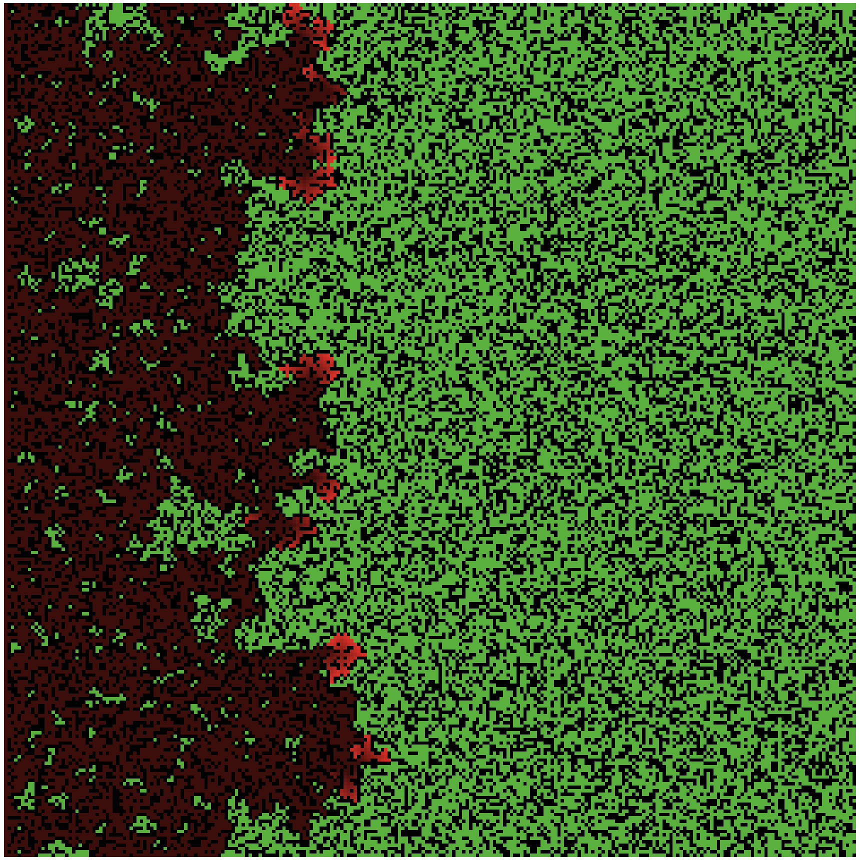
Robustness toolkit: percolation theory

Example: site percolation in 2D



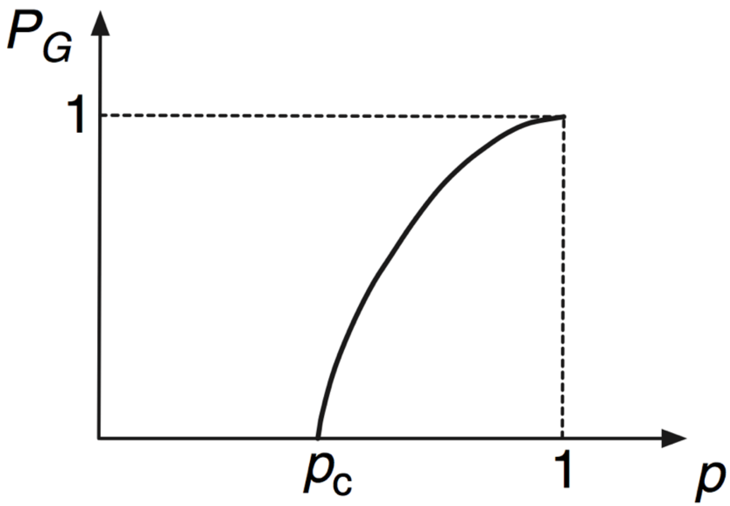
Percolation and simulation of forest fires

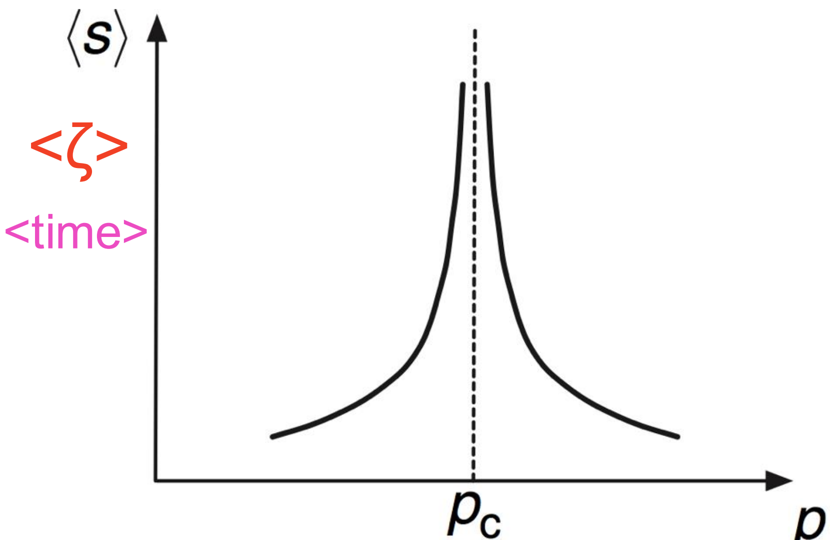
- Imagine a forest made of trees and empty patches.
- If a tree catches a fire, it ignites the neighboring trees with a probability p.
- Each tree burns for a limited number time-steps (e.g., 1)
- The fire continues until there has non-burning neighbor.
- If we ignite one tree what fraction of the forest will burn down? How long does it takes?

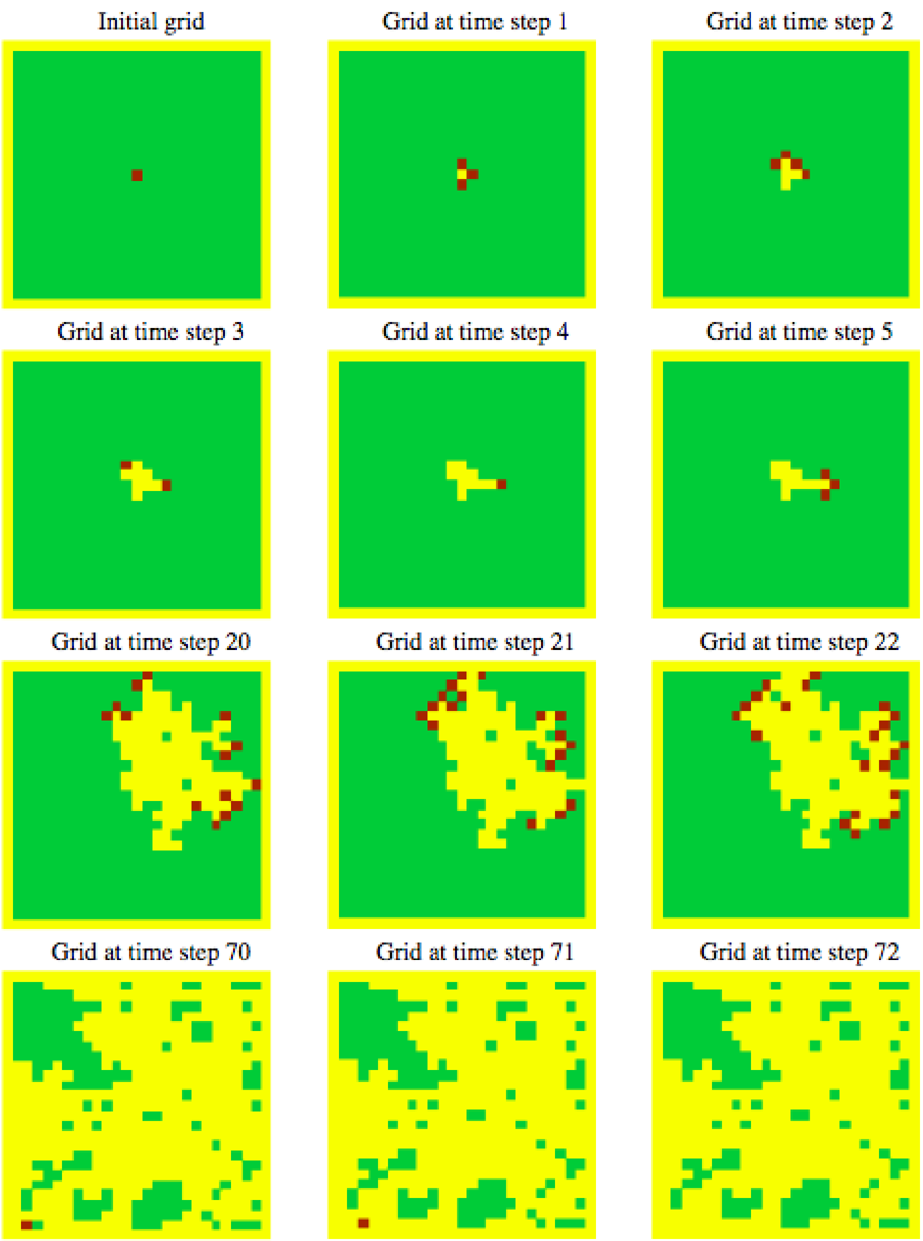


Try it with NetLogo: http://ccl.northwestern.edu/netlogo/models/Fire

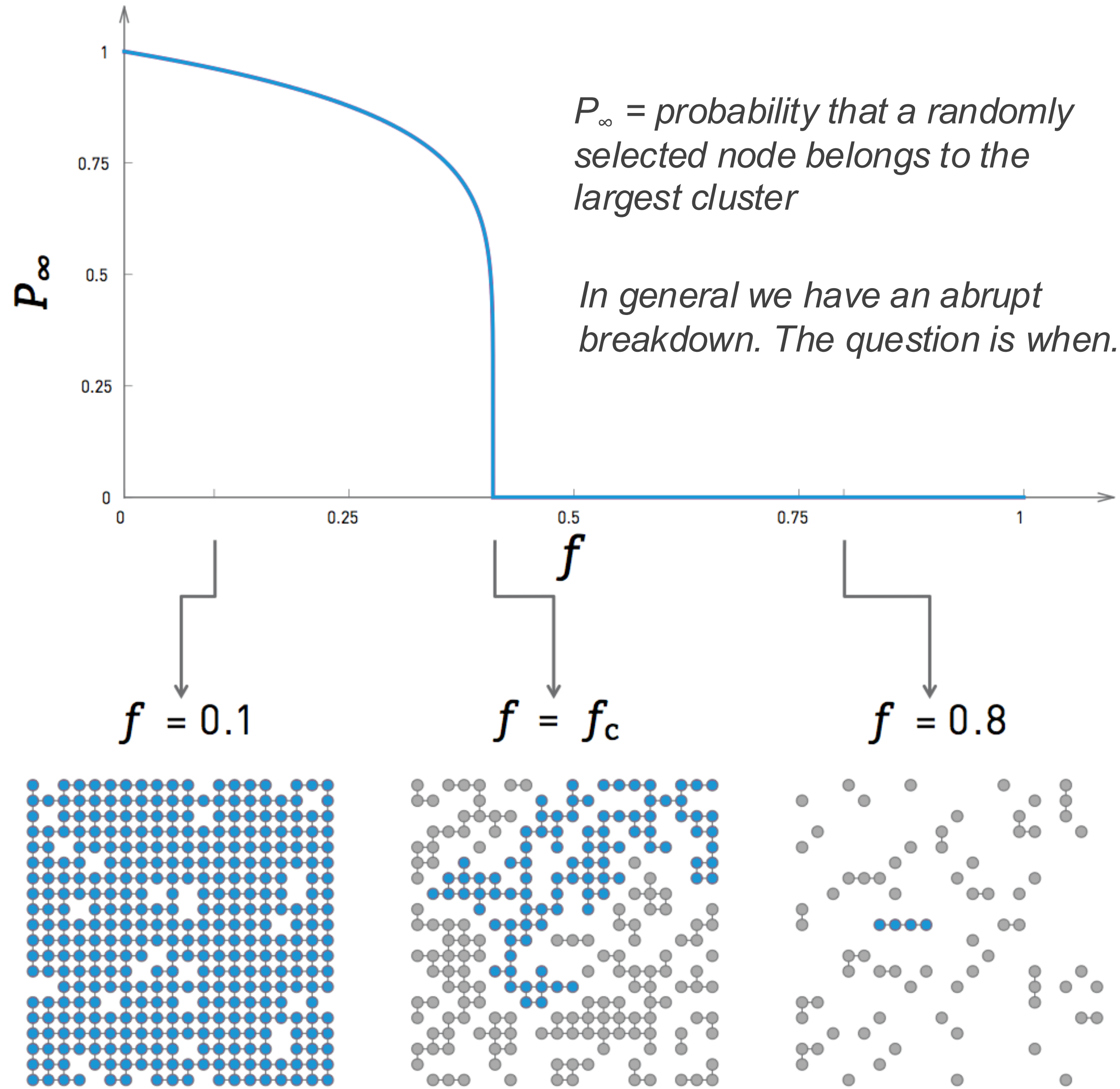
Percolation and simulation of forest fires







Robustness as an inverse percolation problem



Fraction of removed nodes:

$$f = 1-p$$

e.g., the fraction of nodes that fail

$0 < f < f_c$:

There is a giant component.

$$P_\infty \sim |f - f_c|^\beta$$

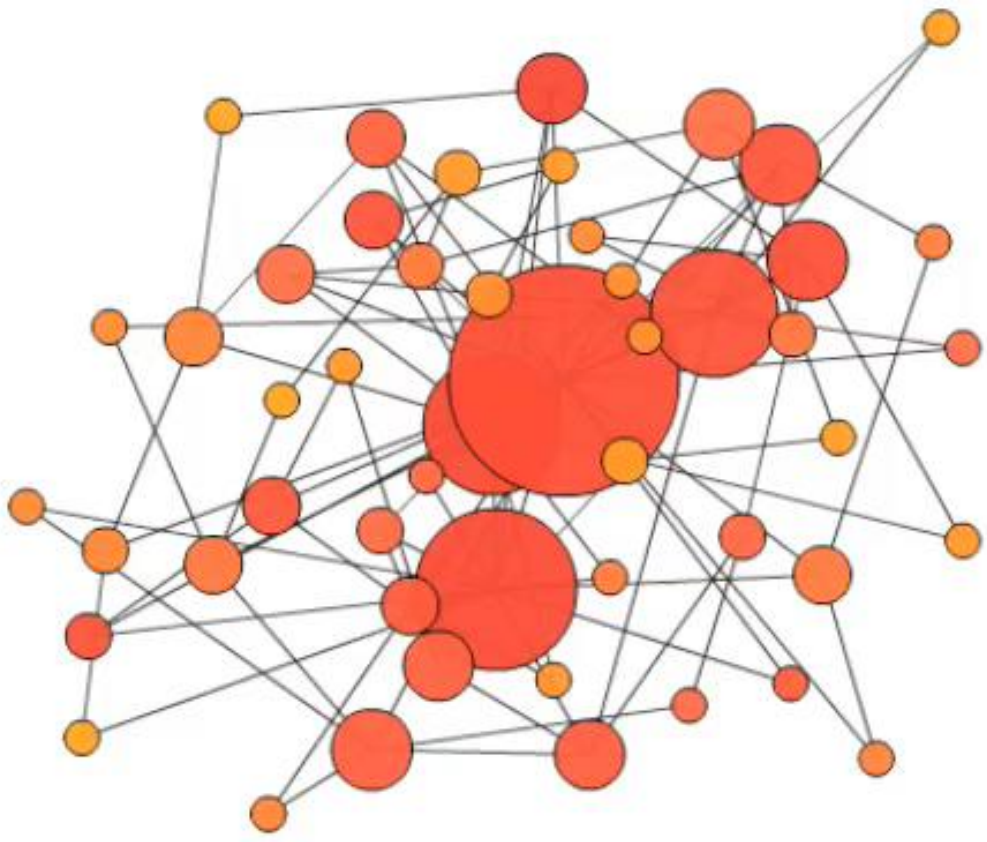
$f = f_c$:

The giant component vanishes.

$f > f_c$:

The lattice breaks into many tiny components.

Scale-free nets under random failures



Scale-free nets show an usual behavior: we must remove almost all of its nodes to destroy its giant component.

What’s the origin of this result?

Example of a scale-free network (BA model) under random node failure
Video by Dashun Wang

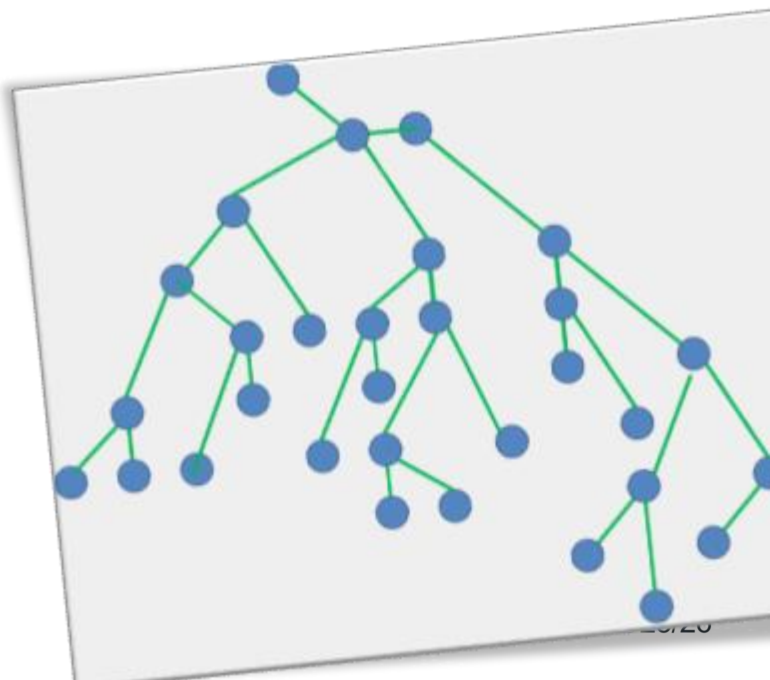
Analytical insights: Molloy-Reed criterion (1995)

For details see for instance, Barrat et al. Dynamical processes in complex networks, 2008, section 6.4.

- In general the breakdown of any complex network is not gradual: it occurs abruptly in a phase transition at some critical fraction of nodes removed.
- A (infinite) randomly wired complex network (without loops!) w/ arbitrary degree distribution
  shows a giant component if

$$\langle k^2\rangle - \langle k\rangle > \langle k\rangle \quad \text{or} \quad \frac{\langle k^2\rangle}{\langle k\rangle} > 2$$

nearest neighbors average degree



Analytical insights: Molloy-Reed criterion (1995)

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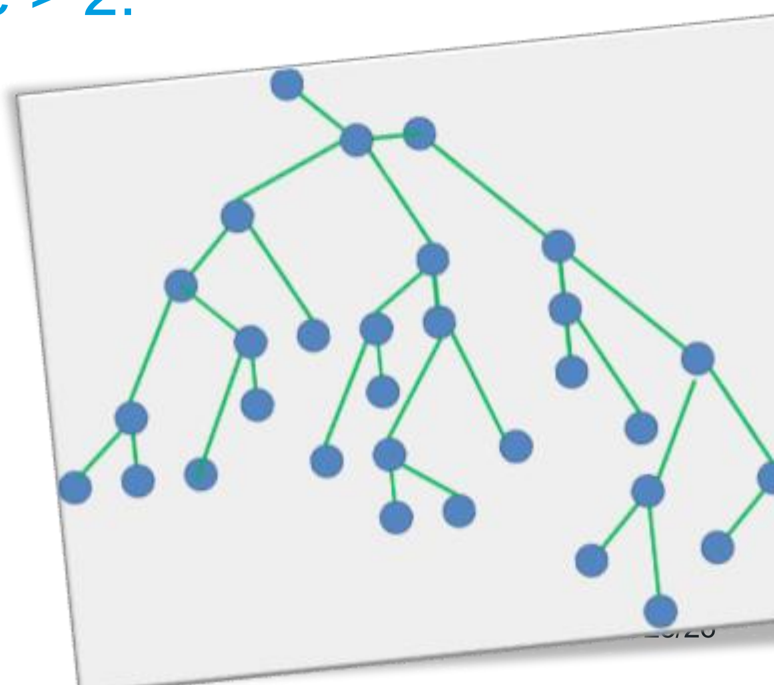
- In general the breakdown of any complex network is not gradual: it occurs abruptly in a phase transition at some critical fraction of nodes removed.
- Intuition: To form a chain each individual must hold the hand of two other individuals. This means that average degree of our neighbors should be > 2.

$$\langle k^2\rangle - \langle k\rangle > \langle k\rangle \quad \text{or} \quad \frac{\langle k^2\rangle}{\langle k\rangle} > 2$$

nearest neighbors
average degree



Molloy-Reed criterion (1995)

For details see for instance, Barrat et al. Dynamical processes in complex networks, 2008, section 6.4.

- In general the breakdown of any complex network is not gradual: it occurs abruptly in a phase transition at some critical fraction of nodes removed.
- …or when the mean number of second neighbors surpasses the mean number of neighbors.

$$\langle k^2 \rangle - \langle k \rangle > \langle k \rangle$$

or

$$\frac{\langle k^2 \rangle}{\langle k \rangle} > 2$$

nearest neighbors average degree

Molloy-Reed criterion (1995)

For details see for instance, Barrat et al. Dynamical processes in complex networks, 2008, section 6.4.

- In uncorrelated networks, the probability that an edge connects a node of degree k is given by the

$$\frac{\text{\# edges departing from nodes of degree k}}{\text{\# edges departing from nodes of any degree.}}$$

$$\frac{kN_k}{\sum_i iN_{k=i}} = \frac{kNP(k)}{N\sum_i iP(i)} = \frac{kP(k)}{\langle k \rangle}$$

the probability that the randomly picked edge leads to a vertex of degree *k*

Barrat et al. Dynamical processes in complex networks, 2008, Chapter 1

Nearest neighbors average degree

Analytics

- The average nearest neighbors degree is given by

$$k_{nn}(k) = \sum_{k'} k' P(k' \mid k)$$

- Thus

$$k_{nn}(k) = \frac{1}{\langle k \rangle} \sum_{k'} k' k' P(k') = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

Barrat et al. Dynamical processes in complex networks, 2008, Chapter 1

Molloy-Reed criterion (1995)

For details see for instance, Barrat et al. Dynamical processes in complex networks, 2008, section 6.4.

- Let us consider *q* as the probability that a randomly chosen edge does NOT lead to a vertex connected to a giant cluster.

$$q = \sum_{k} \frac{kP(k)}{\langle k \rangle} q^{k-1}$$

Average over all degrees

the probability $kP(k)/\langle k \rangle$ that the randomly picked edge leads to a vertex of degree k

The probability q^{k-1} that none of the remaining $k-1$ edges lead to a vertex connected to a giant cluster.

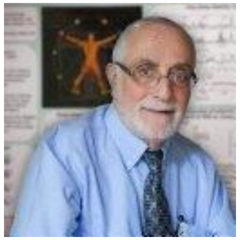
- The Molloy-Reed Criterion emerges from solving this implicit function for q<1.

In general…

For details see for instance, Barrat et al. Dynamical processes in complex networks, 2008, section 6.4.

R Cohen et al. Resilience of the Internet to random breakdowns, Phys. Rev. Lett. (2000)

R Cohen



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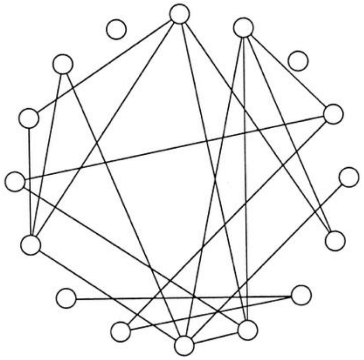
- From this, it can be shown that the giant component is present if the fraction f of removed nodes is

$$f < f_C = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1}$$

For instance for a random network we have $\langle k^2 \rangle = \langle k \rangle \left(1 + \langle k \rangle\right)$ getting

$$f_c^{ER} = 1 - \frac{1}{\langle k \rangle}$$



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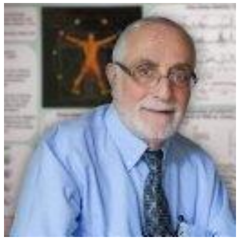
S Havlin

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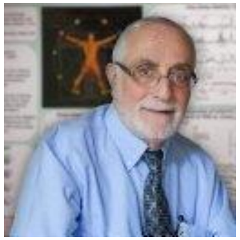
$$f < f_C = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1}$$

- As you may remember, in ***scale-free nets*** with $2 \le \gamma < 3$ the second moment $\langle k^2 \rangle$ □ □
- Thus, when $2 \le \gamma < 3$, we get a giant component whenever ***p=1-f>0*** (or ***f<1***), i.e., **always**!!

Molloy-Reed criterion

R Cohen et al.
Resilience of the Internet to random breakdowns, Phys. Rev. Lett. (2000)

R Cohen



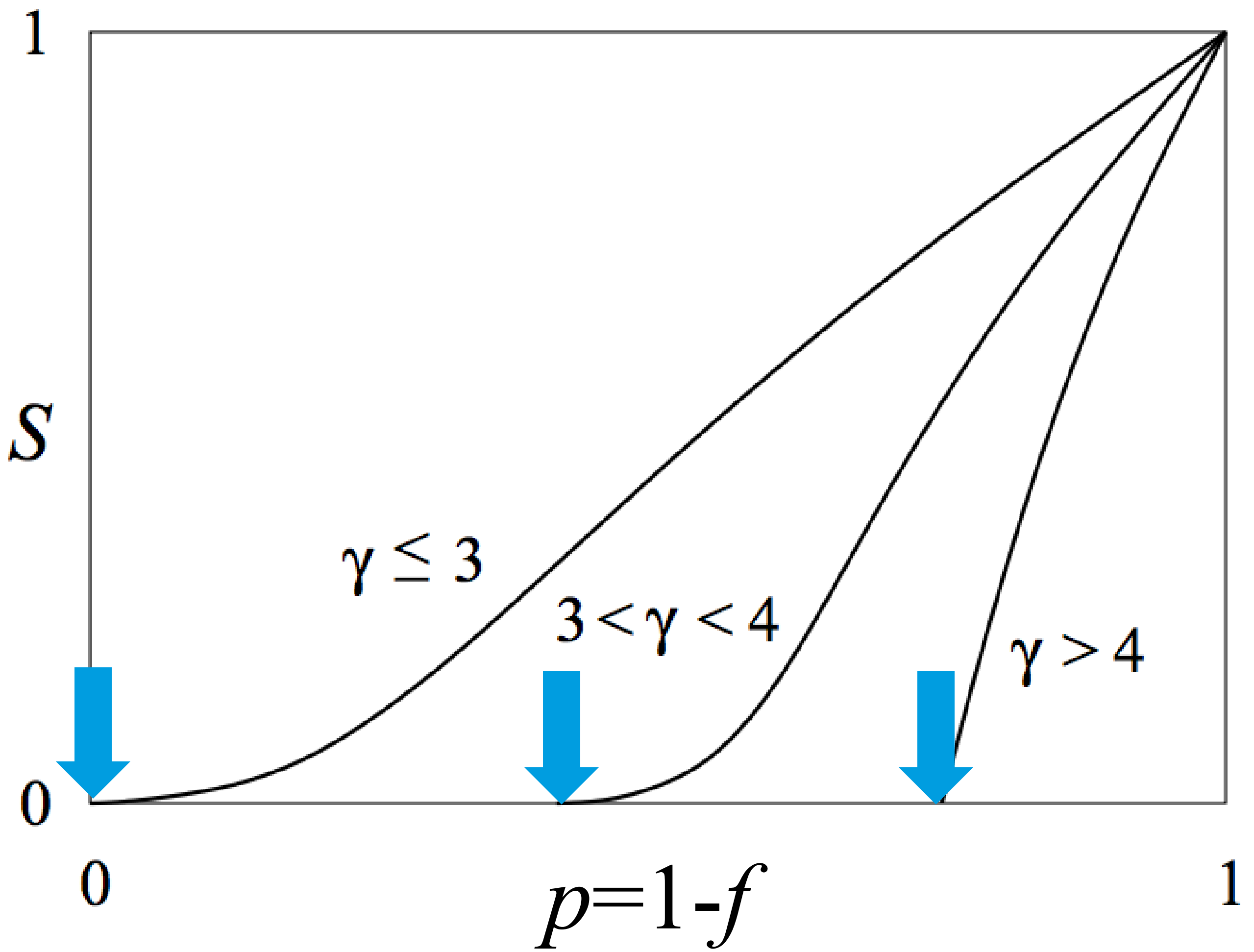
S Havlin

- From this, it can be shown that the giant component is present if the fraction f of removed nodes is

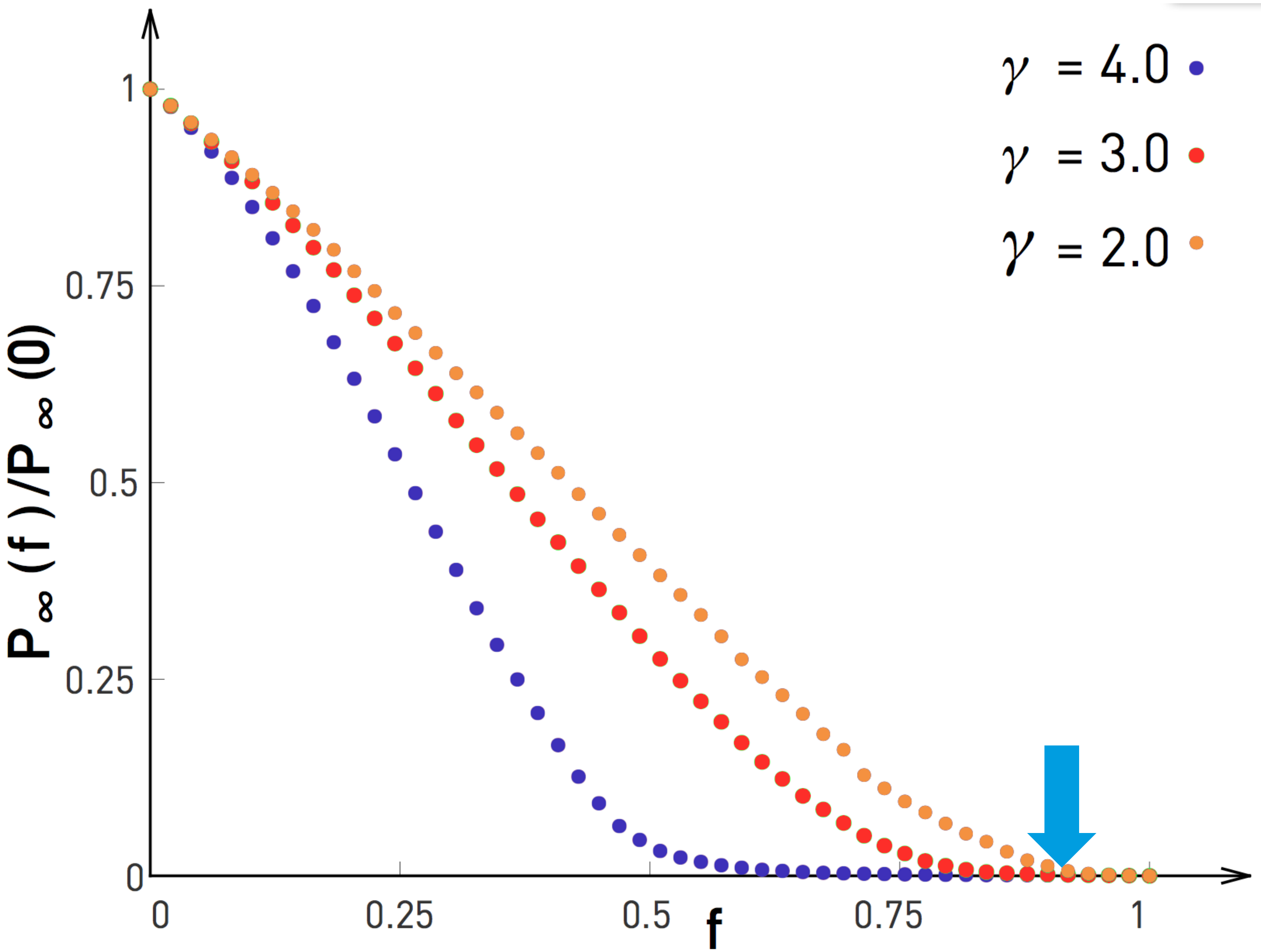
This means that
to fragment a scale-free network we must remove all its nodes!!

- As you may remember, in ***scale-free nets*** with $2 \leq \gamma < 3$ the second moment $\left\langle k^2 \right\rangle$ □ □
- Thus, when $2 \leq \gamma < 3$, we get a giant component whenever ***p=1-f>0*** (or ***f<1***), i.e., **always**!!

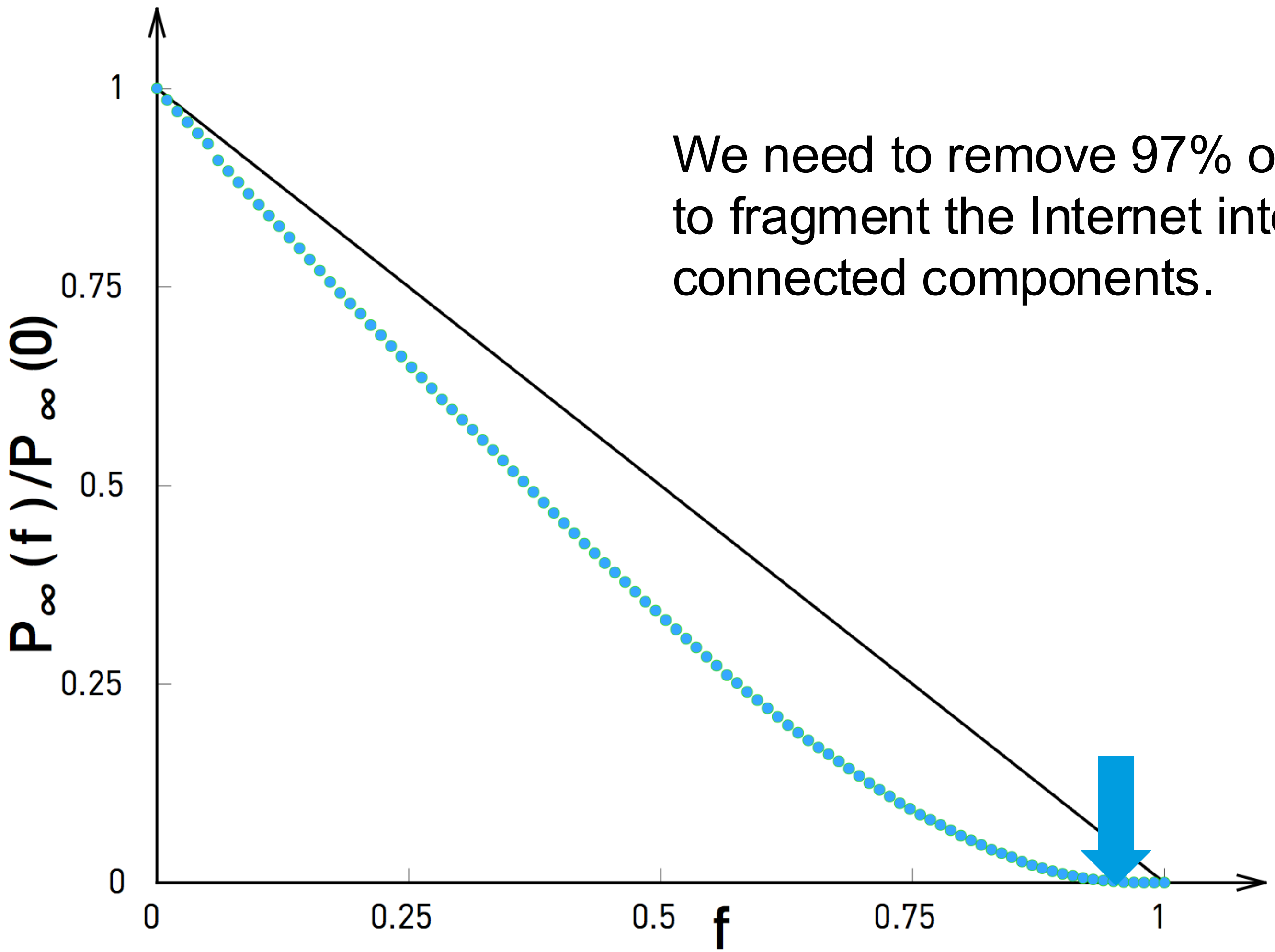
Scale-free networks (analytics)



Scale-free networks (simulations)



Scale-free networks (Internet / Simulations)



We need to remove 97% of the routers to fragment the Internet into dis-connected components.

Finite size effects

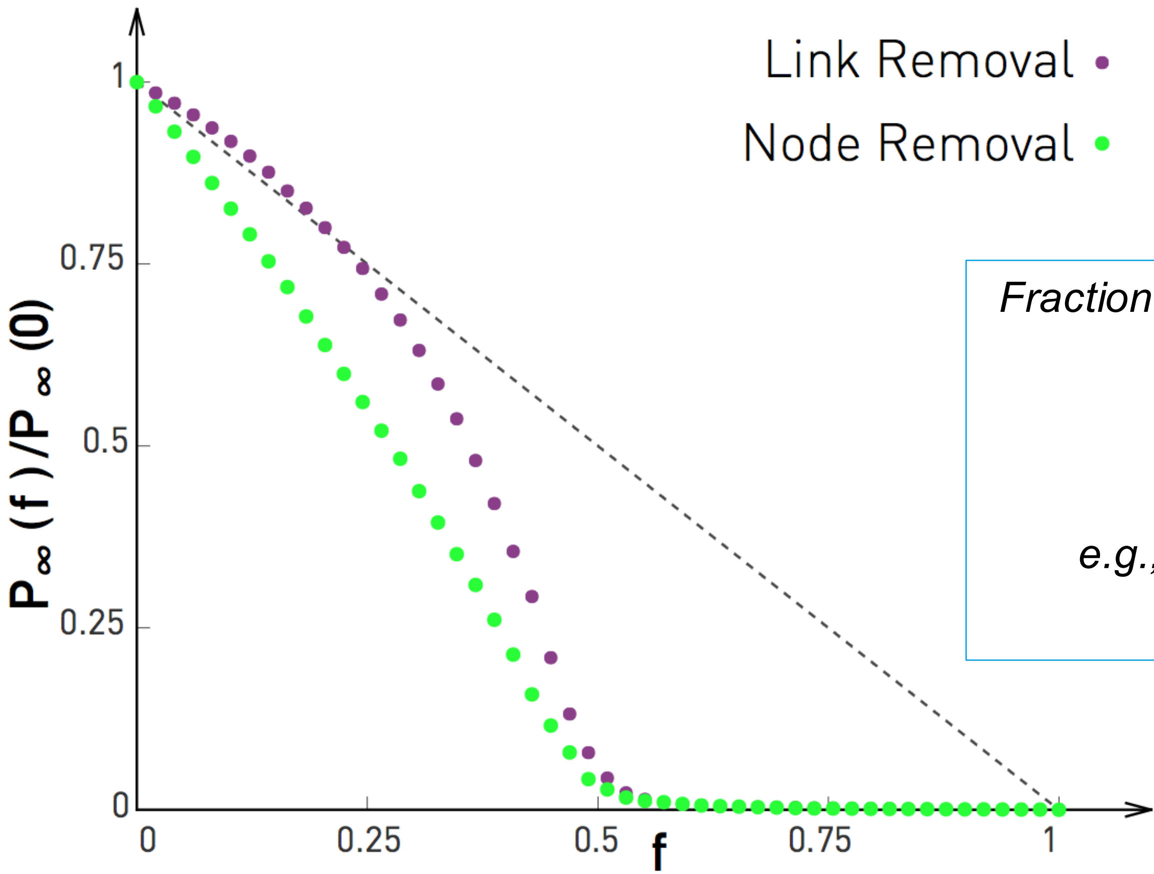
- For finite networks, naturally, $\langle k^2 \rangle$ does not diverge. Also the “abrupt” transition becomes smoother. If you consider a finite N and a power-law with $2 \leq \gamma < 3$ one gets

$$f_c \approx 1 - \frac{3-\gamma}{\gamma-2} k_{\min}^{2-\gamma} k_{\max}^{\gamma-3}$$

- Example: for N=10^3, minimum degree (k_{min}) = 1, γ=2.5, we get a maximum degree (k_{max}) ~ $N^{1/(1-\gamma)}$ ~100 and a critical f_c = 0.9<1.

What happens if we randomly remove links instead of nodes?

ex: Random network with <k>=2



Fraction of removed nodes / links:

$$f = 1\text{-}p$$

e.g., the fraction of nodes or links that fail

On average each node removes <k> links. Hence the removal of a fraction f of nodes is equivalent to a removal of a fraction f<k> links

Conclusion

- We discussed a fundamental property of real world networks: robustness to random failures
- The breakdown threshold of a network depends of $\langle k \rangle$ and on $\langle k^2 \rangle$, which are uniquely defined by the degree distribution.
- For $\gamma<3$ the breakdown threshold rapidely converges to one, which means that we have to remove almost all nodes such that the network falls apart.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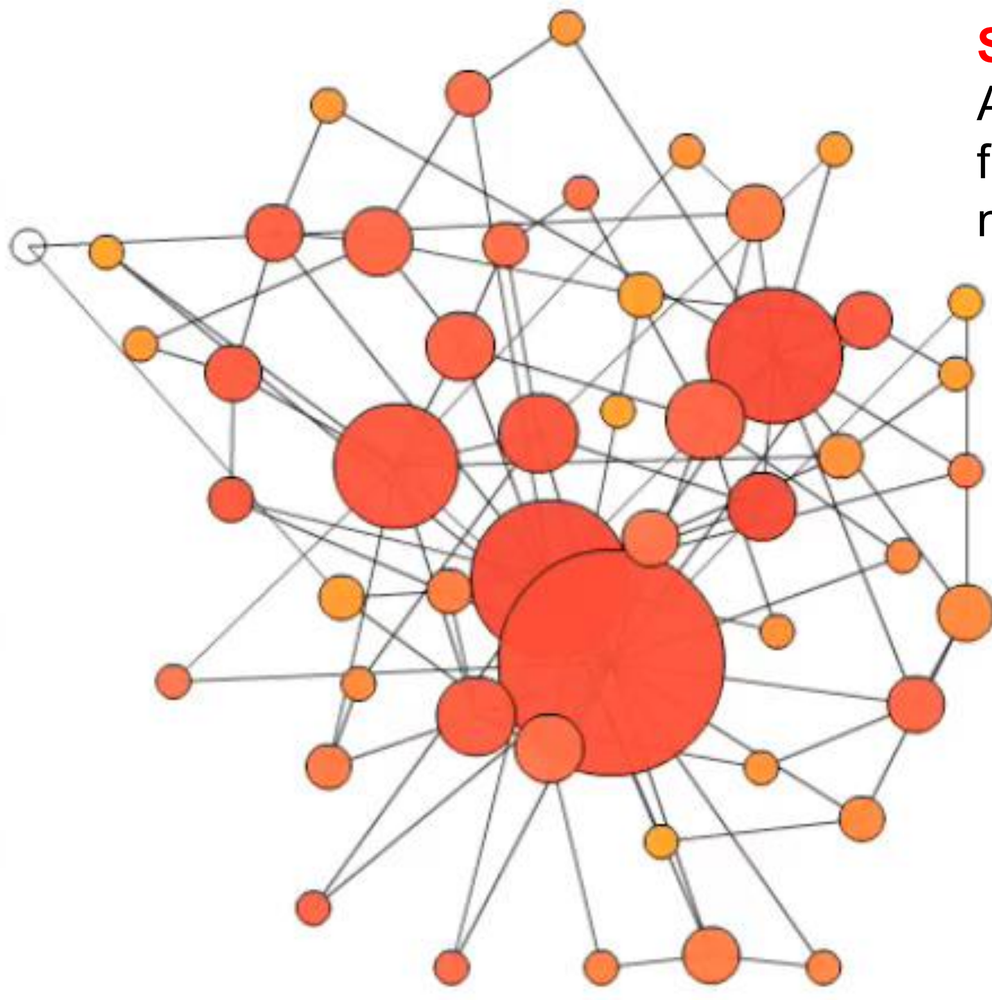
Achilles' heel of the Internet
Obesity Mice that eat more but weigh less
Ocean anoxic events Not all at sea
Cell signalling Fringe sweetens Notch
new on the market

Achilles' heel of scale-free networks

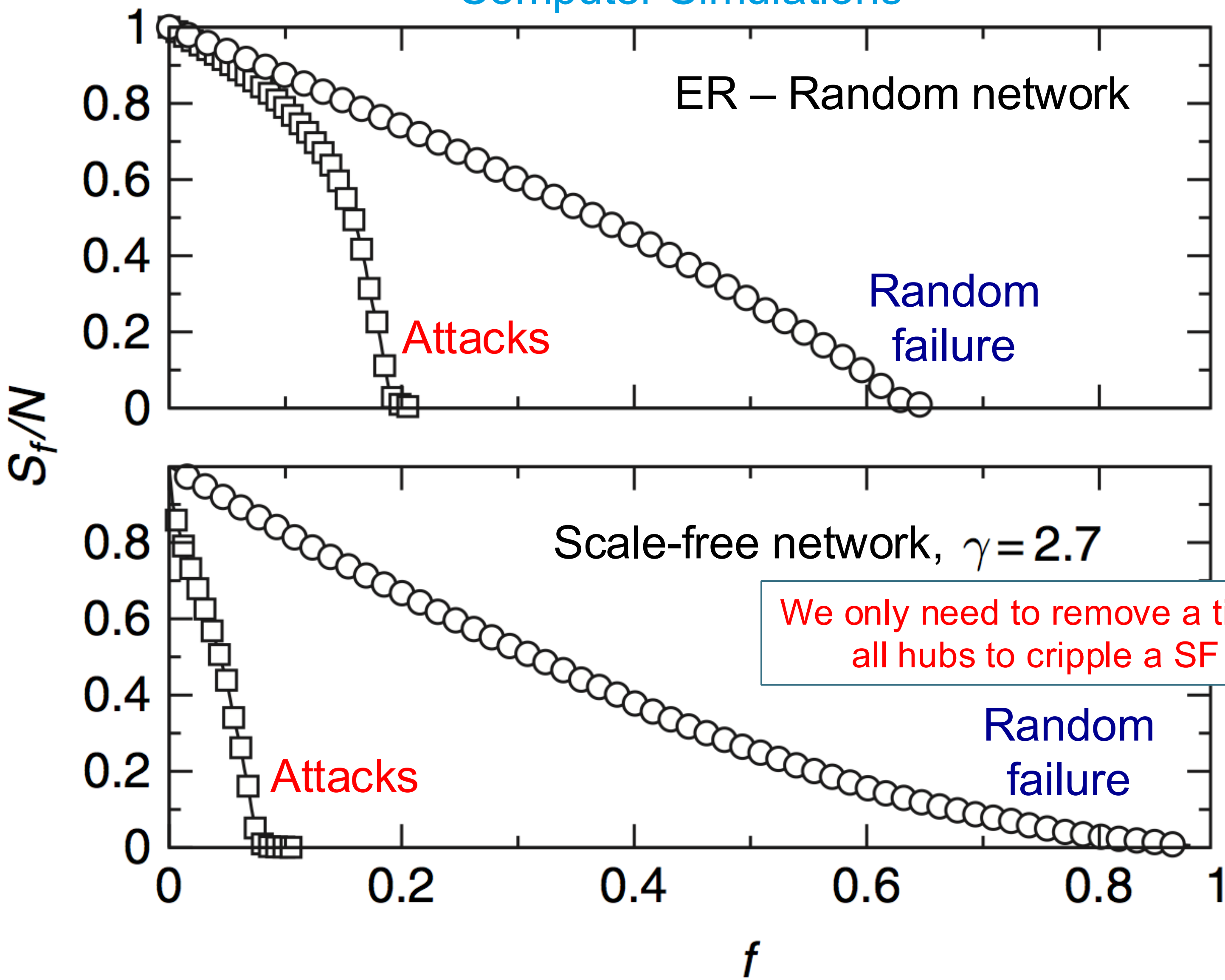


Scale-free networks under attack
Attack first the highest-degree node, followed by the next highest degree node, and so on.

Video by Dashun Wang

Robustness against targeted attacks

Computer Simulations



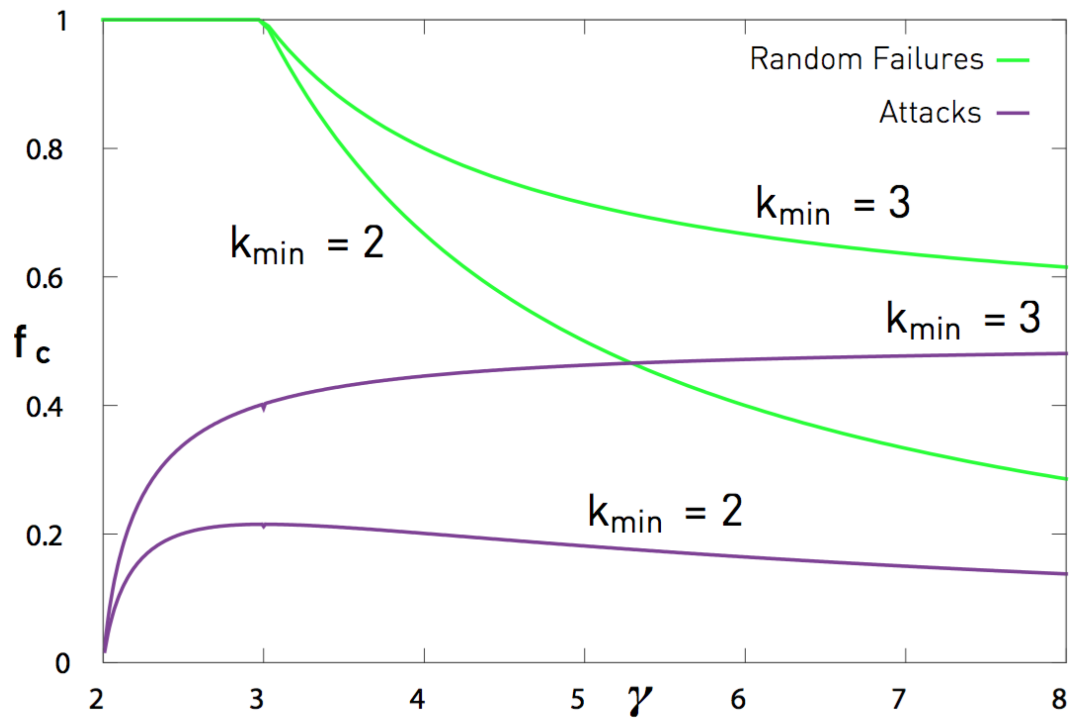
We only need to remove a tiny fraction of all hubs to cripple a SF network.

Attack tolerance

Cohen et al., (2001)
Dorogovtsev and Mendes (2001)

- Hub removal changes the maximum degree of the network ($k_{\max}$ □ $k'_{\max}$).
- Hub removal reduces the heterogeneity of the degree distribution (P_k □ P'_k).

We can analyze this situation in the same way, yet with a different max. degree and degree dist.



Alternative strategies?

Petter Holme et al. Phys. Rev. E 65, 056109 (2002)

Besides the degree we may also try:

- Targeting nodes with large betweenness centrality?

- Should we recalculate the degree and betweenness after each attack?

- Does it change much if I remove edges instead of nodes?

Robustness beyond the degree distribution

- What's the impact of clustering on the robustness of a network? Holme et al. (2002)

- What's the role of degree-degree correlations?

- Resilience and robustness of weighted networks?
  See Dall'Asta et al., 2005, 2006

- Beyond topology: Failure of a single node leads to a redistribution of traffic on the network which may trigger subsequent overloads and failure of the next most-loaded node…

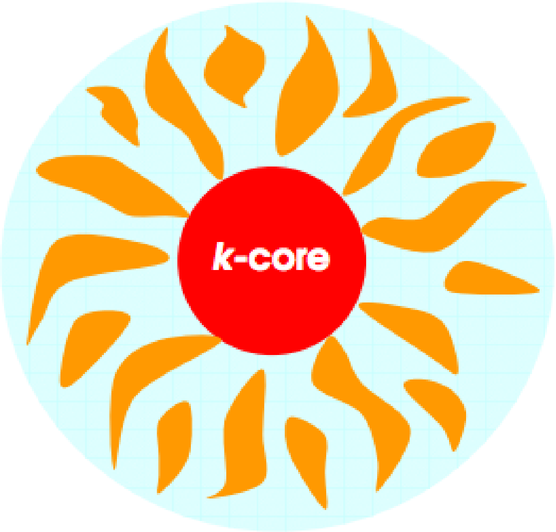
Robustness and assortativity?

NETWORK	RANDOM FAILURES (REAL NETWORK)	RANDOM FAILURES (RANDOMIZED NETWORK)	ATTACK (REAL NETWORK)
Internet	0.92	0.84	0.16
WWW	0.88	0.85	0.12
Power Grid	0.61	0.63	0.20
Mobile-Phone Call	0.78	0.68	0.20
Email	0.92	0.69	0.04
Science Collaboration	0.92	0.88	0.27
Actor Network	0.98	0.99	0.55
Citation Network	0.96	0.95	0.76
E. Coli Metabolism	0.96	0.90	0.49
Yeast Protein Interactions	0.88	0.66	0.06

Extracting the most interconnected sub-graphs

K-core: largest subgraph whose nodes have at least ***K*** interconnections.

Indicates the best interconnected parts of the network.



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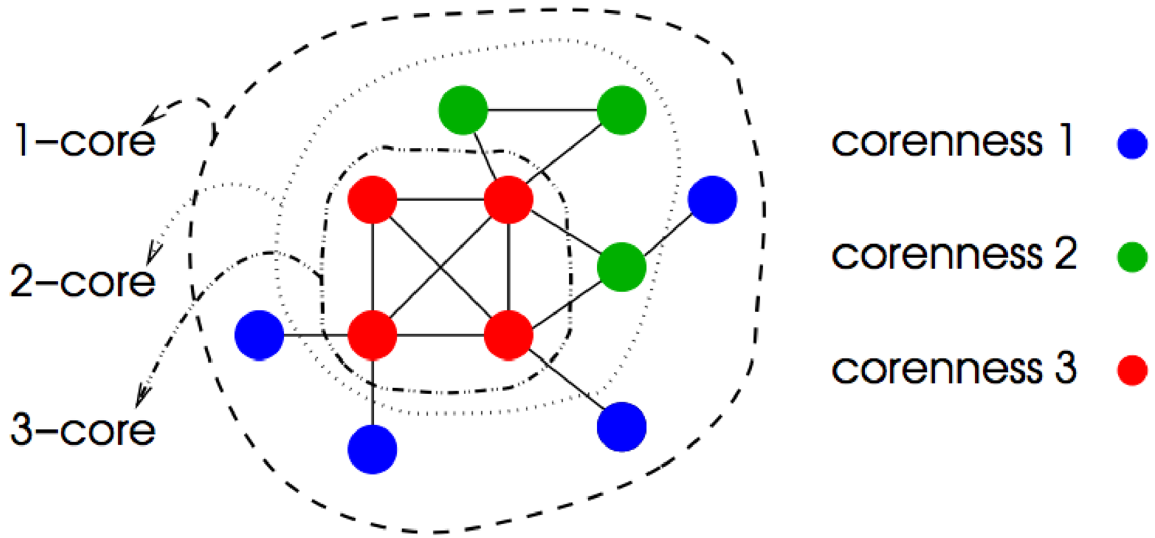
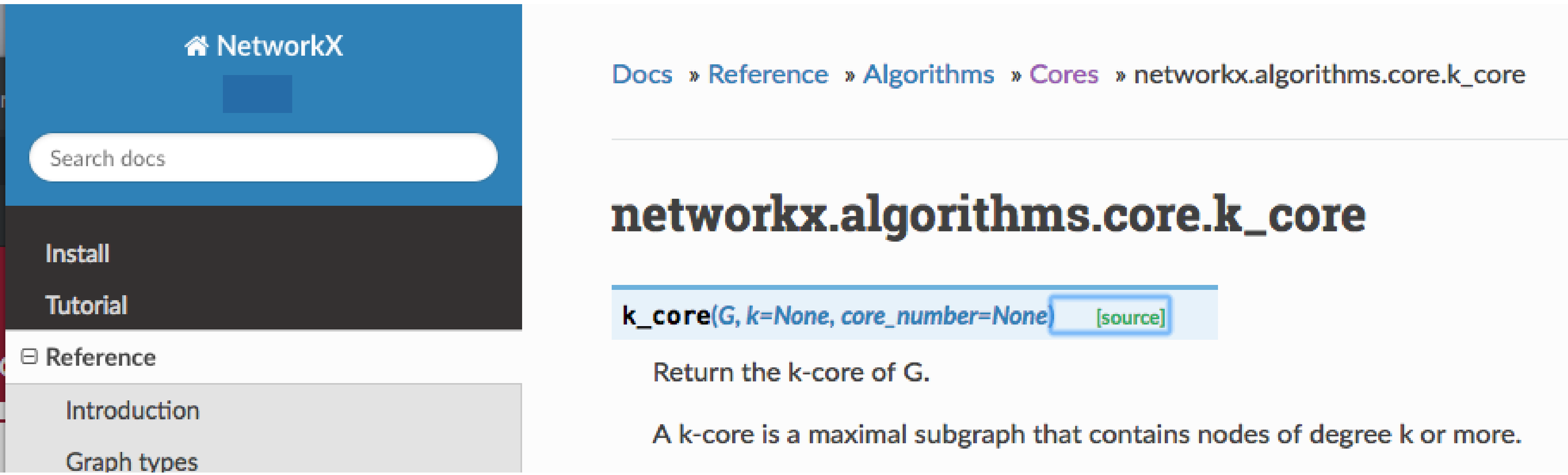


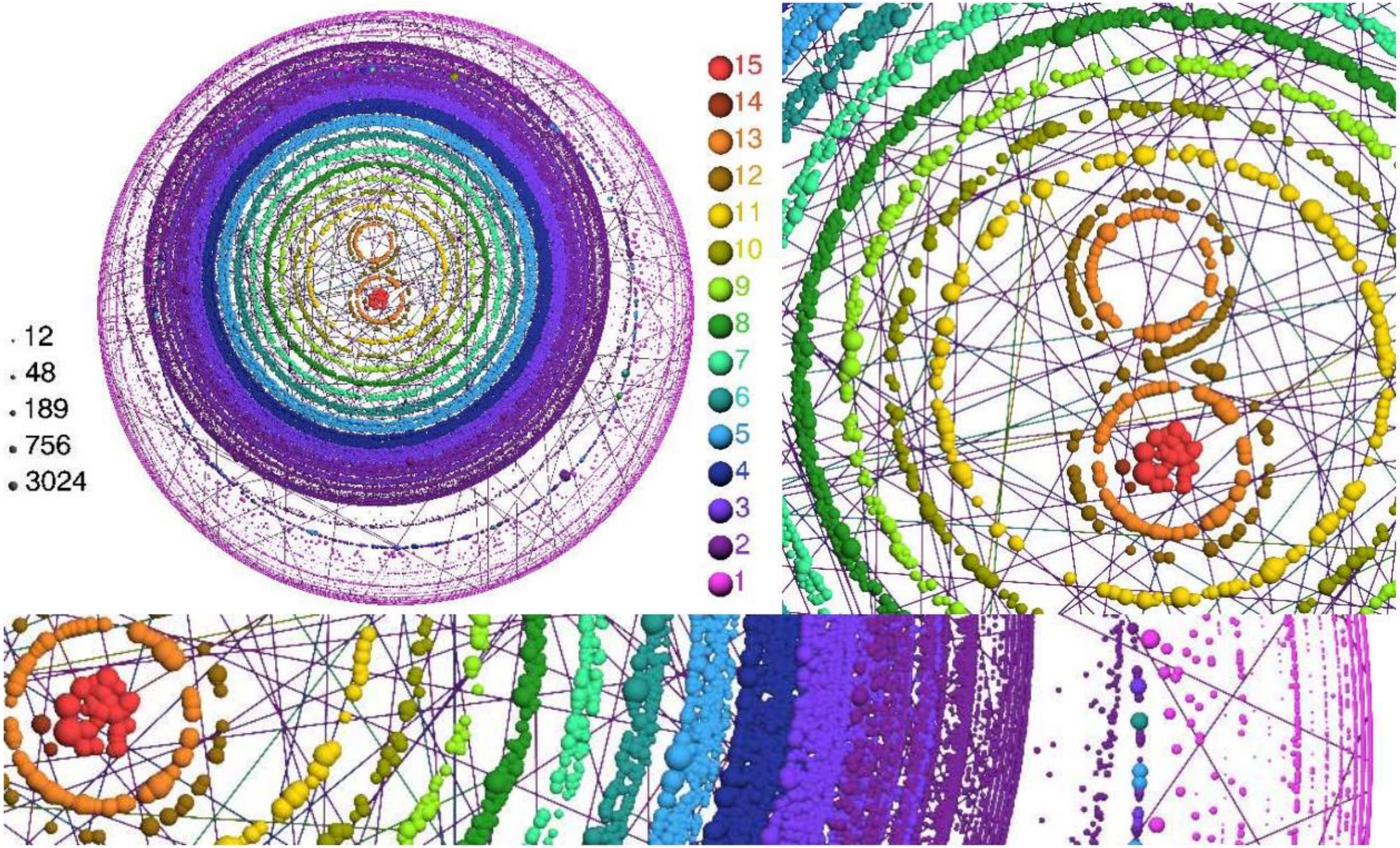
For scale-free nets with γ<3 all k-core are "ultra-resilient" against random damage. In finite networks, first the highest k-core disappears, then the 2nd, and so on.

Extracting the most interconnected sub-graphs



Visualizing networks using k-cores

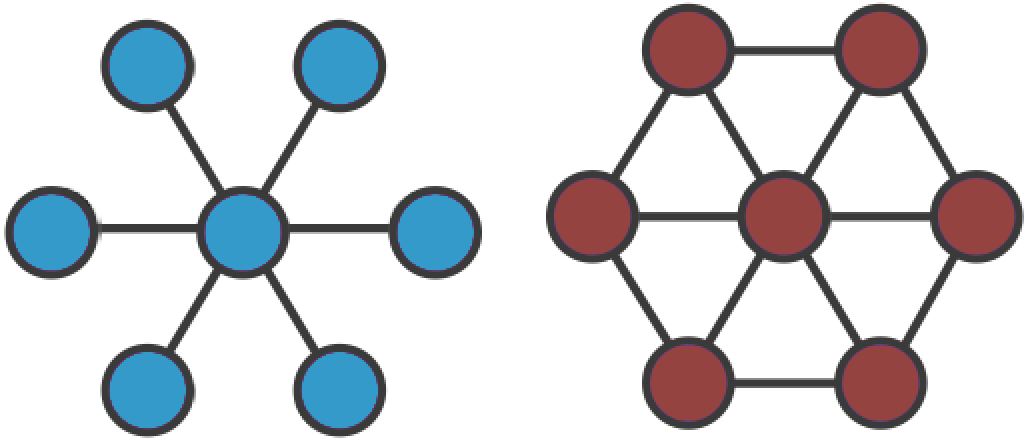
WWW (.fr domain)



Designing robust networks

against random failures and attacks

R.V. Sole, et al. Phys. Rev. E, 77: 026102, 2008.



Increase robustness through additional links… ☺

OK, fine… but if one wants to keep the average degree constant or even minimize <k>…
What should I do?

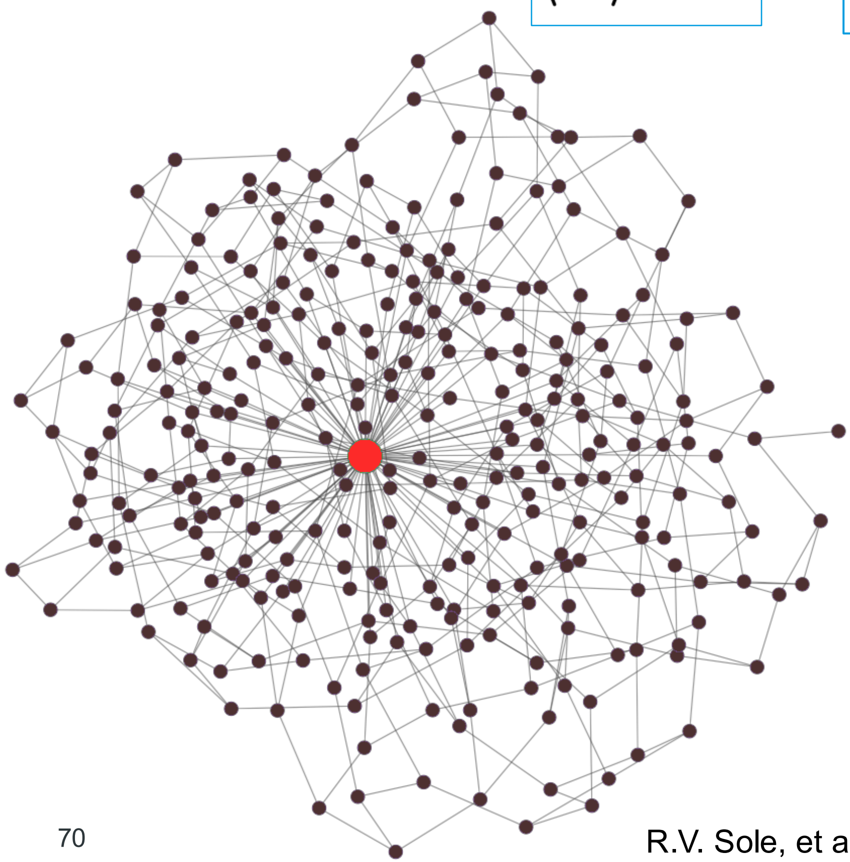
Maximizing the sum: $$F = f_c^{RANDOM} + f_c^{ATTACK}$$

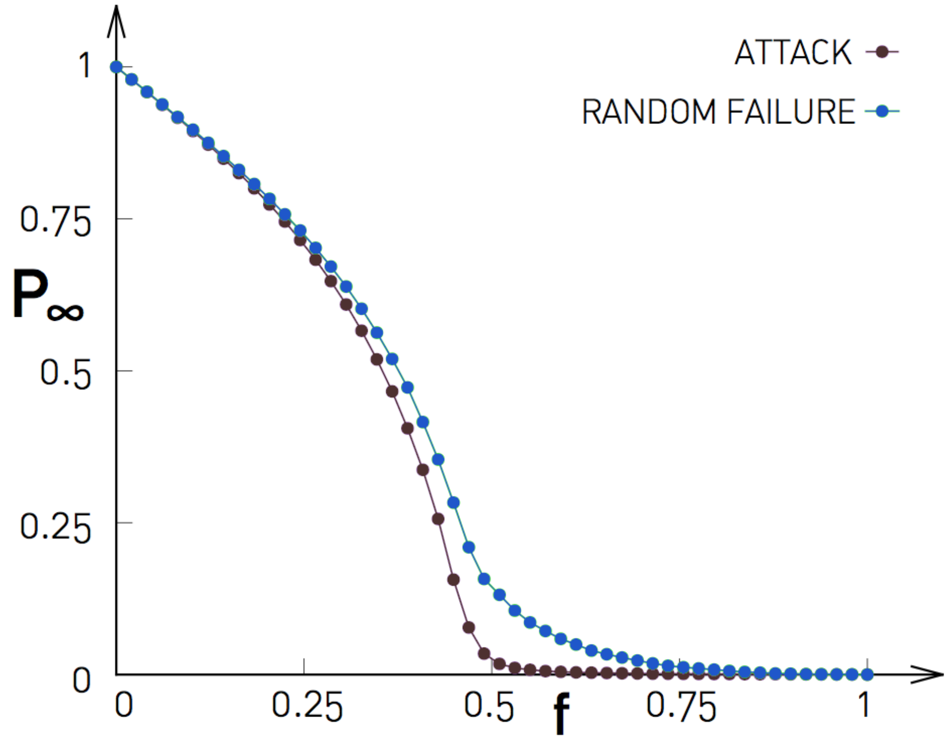
Optimizing against random failures & attacks

- Bimodal distributions (maximizing $<k^2>$)

$$\langle k \rangle = 3$$

$$p_k = (1-r)\delta(k-k_{\min}) + r\delta(k-k_{\max})$$





R.V. Sole, et al. Robustness of the European power grids under intentional attack. Phys. Rev. E, 77: 026102, 2008.

Other measures of robustness

- Other possible quantitative measures of topological damage can be defined in networks.
- For example, the increase in the distances (l_{ij}) between pairs of nodes indicates how communication becomes less efficient. Example: Latora and Marchiori (2001) “efficiency”:

$$\frac{1}{N(N-1)} \sum_{i \neq j} \frac{1}{\ell_{ij}}$$

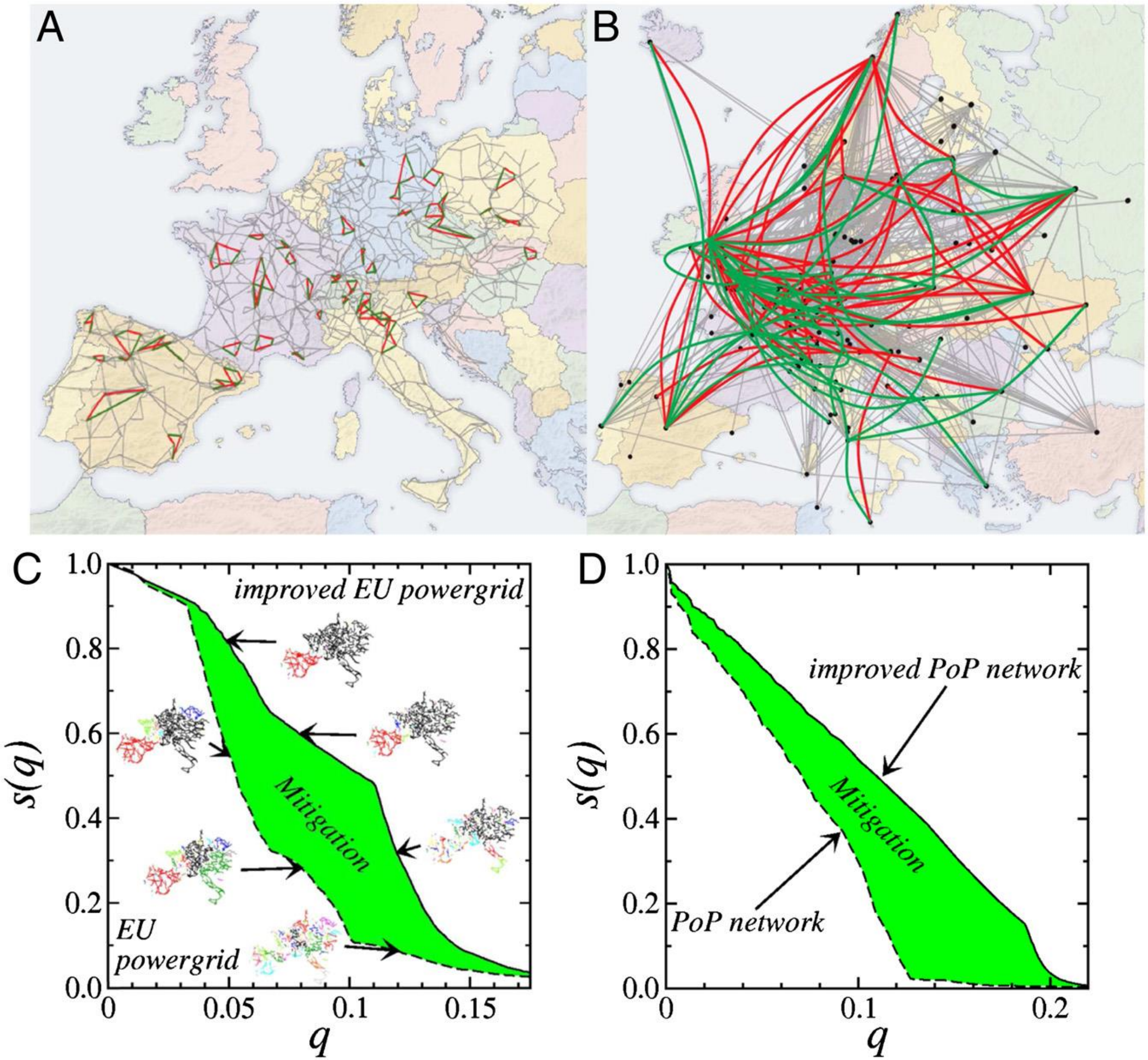
- In weighted networks, the integrity of a network may be quantified by the largest sum of the weights in the largest connected component

Challenge: Check if the critical fraction of removed nodes previously obtained describe the dependence on these quantities.

Mitigation of malicious attacks on networks

Mitigation of malicious attacks on

- the power supply system in Europe and
- the global Internet at the level of service providers (a.k.a., point of presence (PoP)).

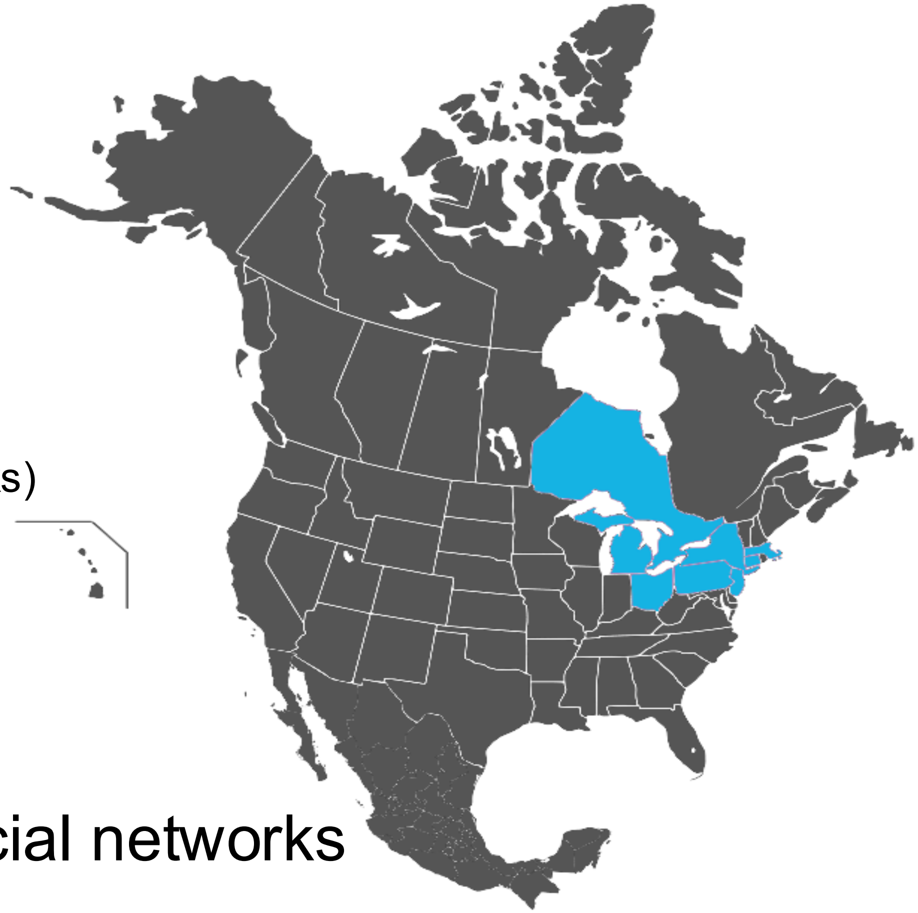


Schneider, C. M., Moreira, A. A., Andrade, J. S., Havlin, S., & Herrmann, H. J. Mitigation of malicious attacks on networks. PNAS 2011

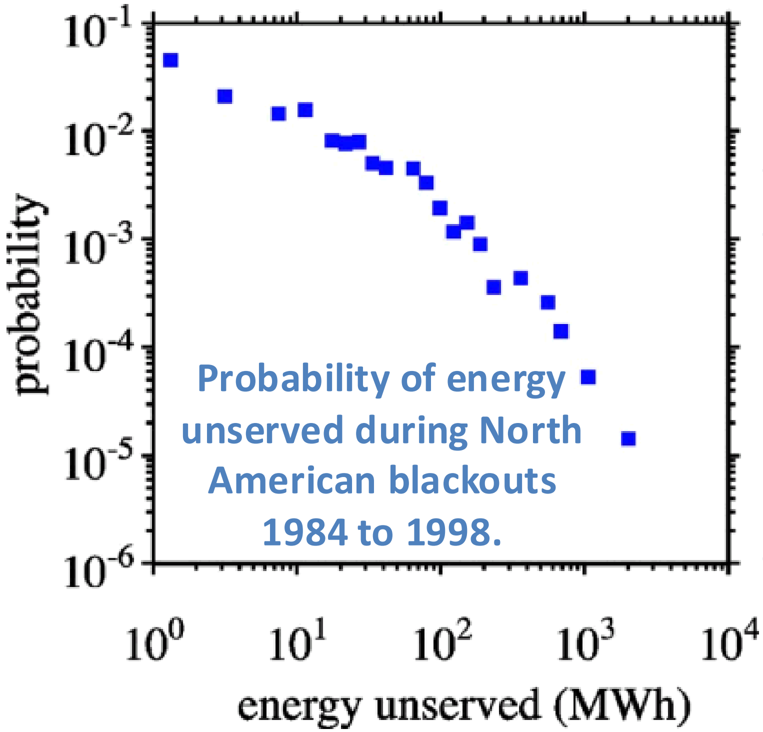
Cascading events & avalanches

Nodes of a network fail independently of each other?

- Airports
- Power grids
- Internet (e.g. Denial of Service Attacks)
  C. Labovitz, A. Ahuja and F. Jahasian. Proc. of IEEE FTCS, Madison, WI, 1999
- Financial networks
  Haldane & May. Nature, 469: 351-355, 2011.
  Roukny, Bersini, et al. Sci. Rep., 3: 2759, 2013.
  Tedechi et al., PLoS One 7: e52749, 2012
- Information cascades in social networks



Power-stations networks, blackouts and power-laws: *Avalanche exponents*

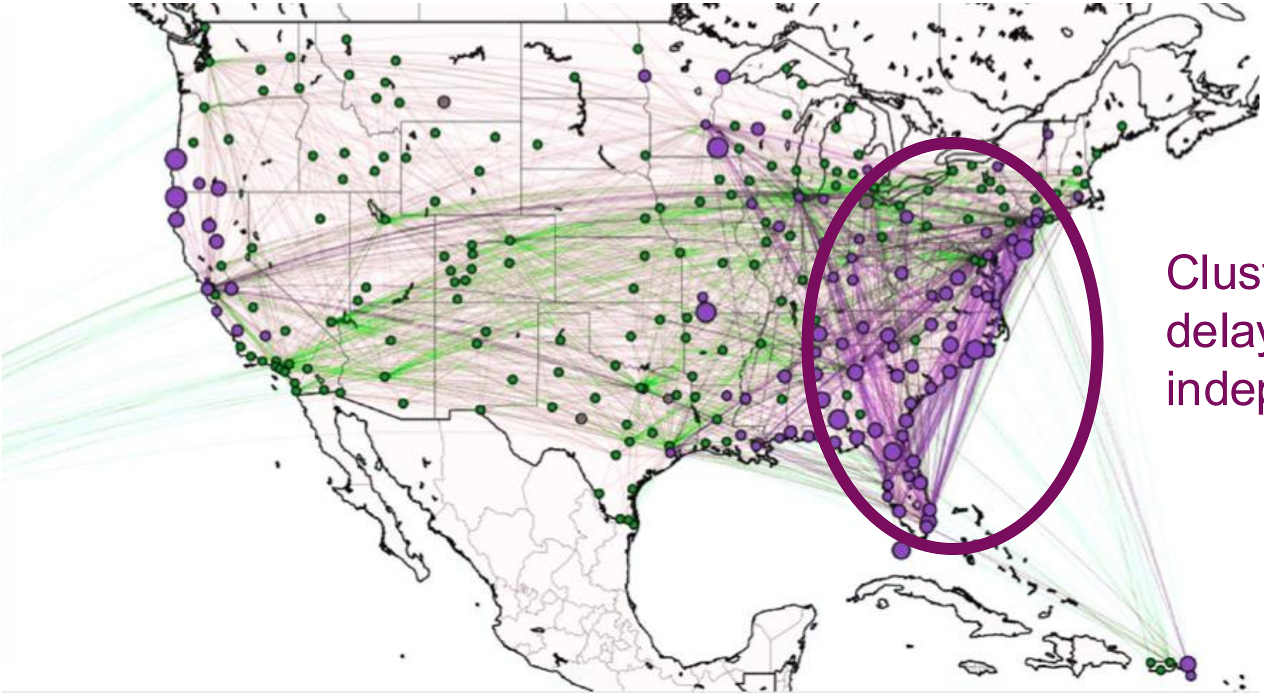


$$P(S) \sim S^{-\alpha},\ 1 < \alpha < 2$$

Source	Exponent	Quantity
North America	2.0	Power
Sweden	1.6	Energy
Norway	1.7	Power
New Zealand	1.6	Energy
China	1.8	Energy

Cascading airport congestions

Flight delays in the U.S. have an economic impact of over $40 billion per year

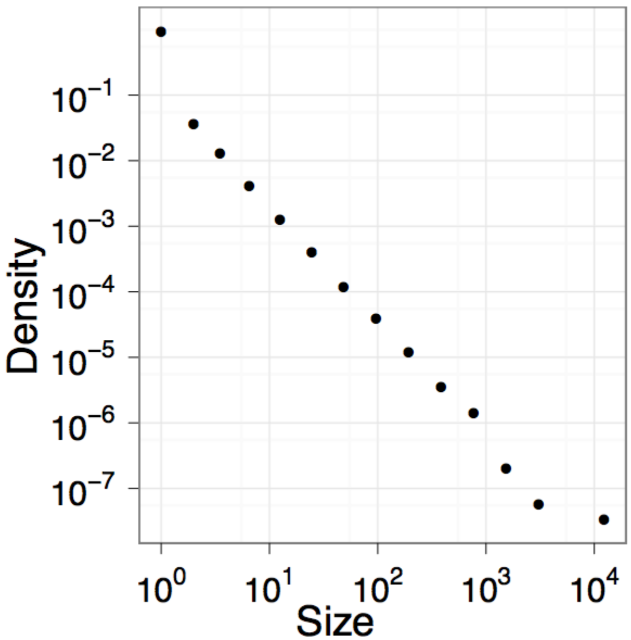


Clusters indicate that delays are not independent of each other

The delay distribution follows a power law, implying that while most flights are delayed by just a few minutes, a few were hours behind schedule.

Information cascades in Twitter

The distribution of cascade sizes on Twitter. While most tweets go unnoticed, a tiny fraction of tweets are shared thousands of times.



Avalanche exponent: $\alpha \approx 1.75$

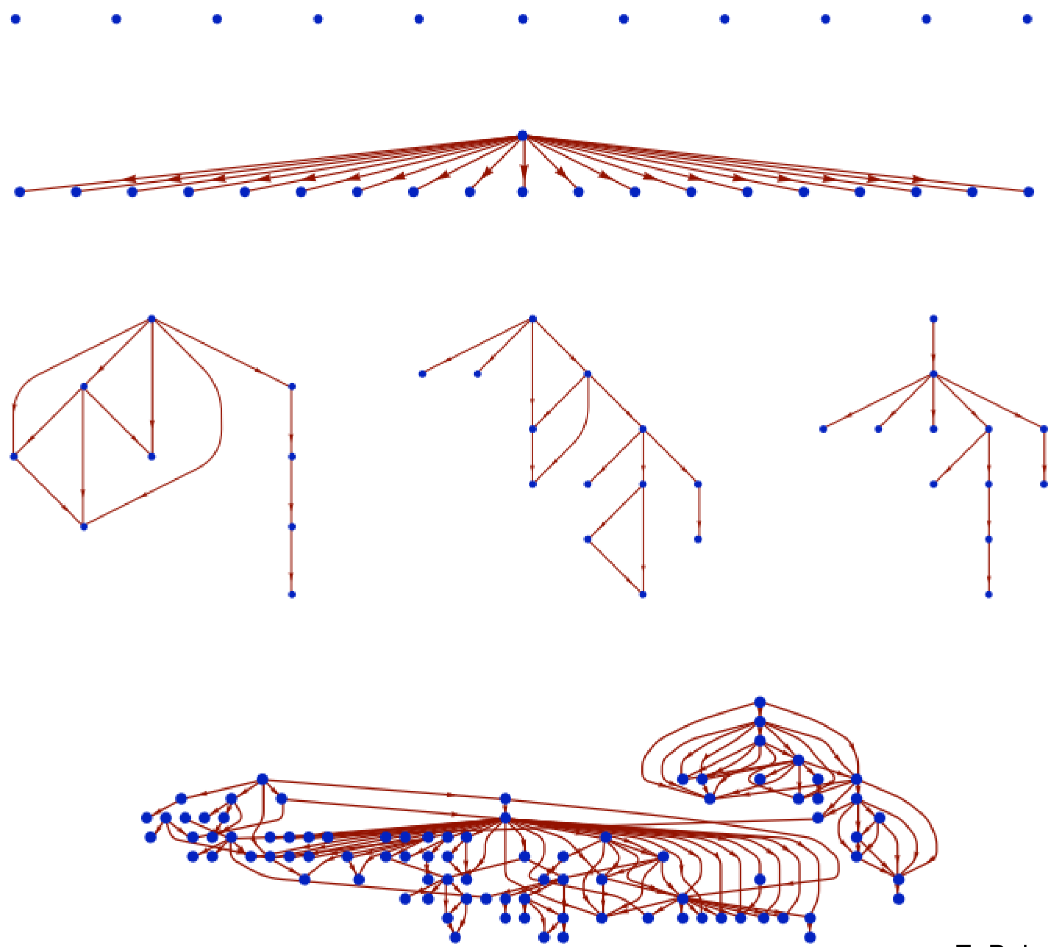
The power law indicates that the vast majority of posted URLs do not spread at all!!

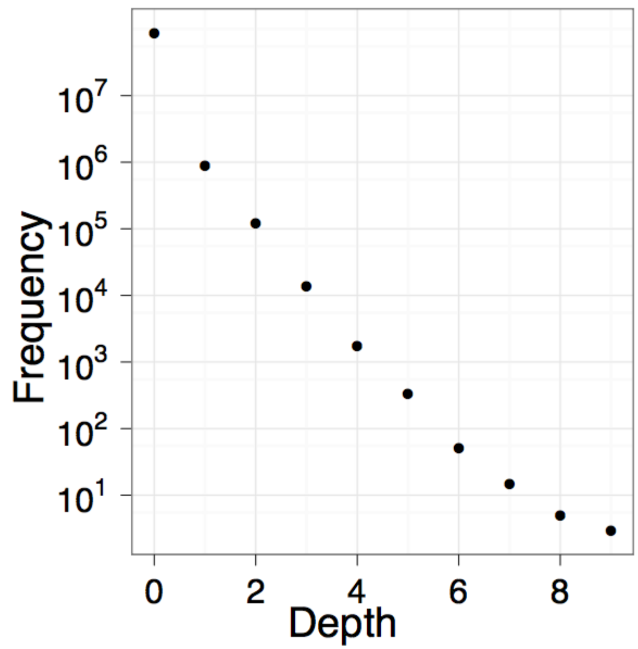
Indeed, the average cascade size is only 1.14.

Yet, a small fraction of URLs are reposted thousands of times.

E. Bakshy, et al. ***Everyone's an influencer: quantifying influence on twitter.*** WSDM '11, 65-74, 2011.

Information cascades in Twitter

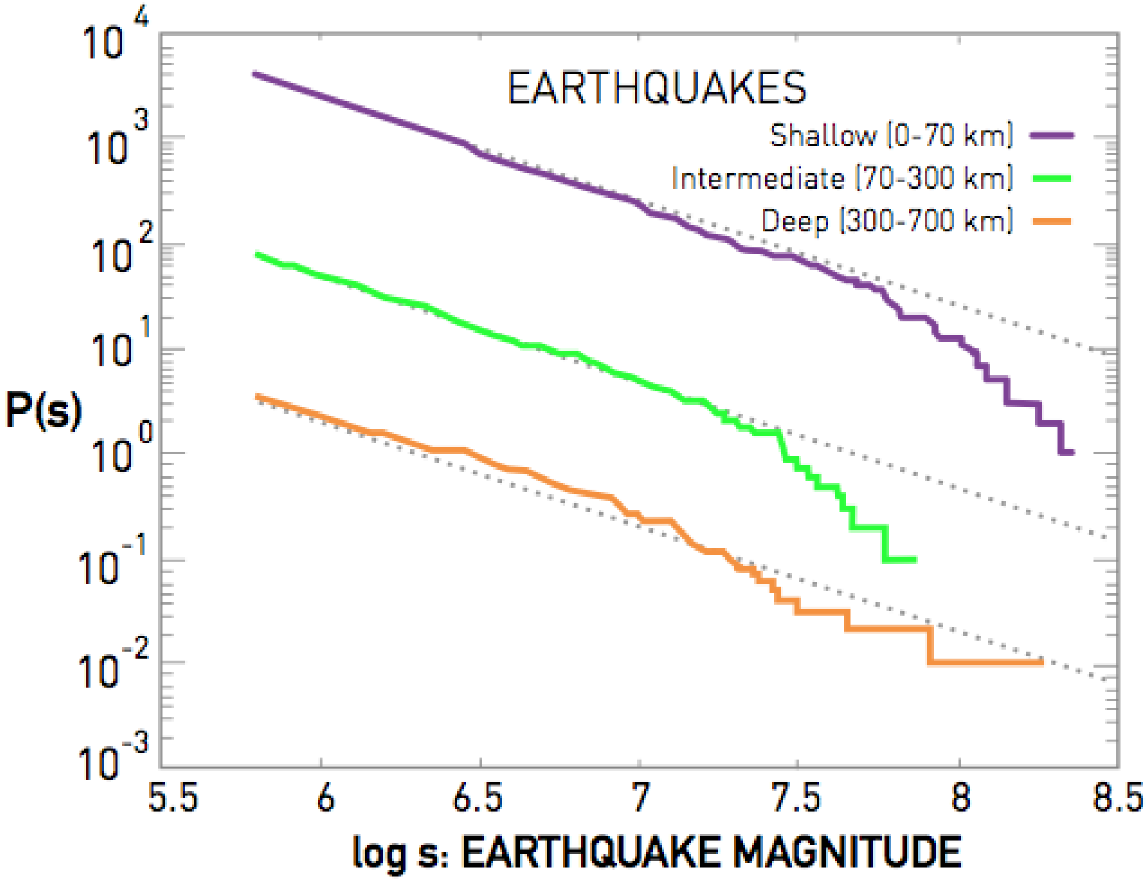




(b) *Cascade Depths*

E. Bakshy, et al. ***Everyone's an influencer: quantifying influence on twitter.*** WSDM '11, 65-74, 2011.

Earthquakes magnitudes



Cascading events and power-laws

Avalanche exponents, from earthquakes and power failures, to twitter and flights delay, are surprisingly close (1.6-2.0)

SOURCE	EXPONENT	CASCADE
Power grid (North America)	2.0	Power
Power grid (Sweden)	1.6	Energy
Power grid (Norway)	1.7	Power
Power grid (New Zealand)	1.6	Energy
Power grid (China)	1.8	Energy
Twitter Cascades	1.75	Retweets
Earthquakes	1.67	Seismic Wave

Cascading events and power-laws

Avalanche exponents, from earthquakes and power failures, to twitter and flights delay, are surprisingly close (1.6-2.0)

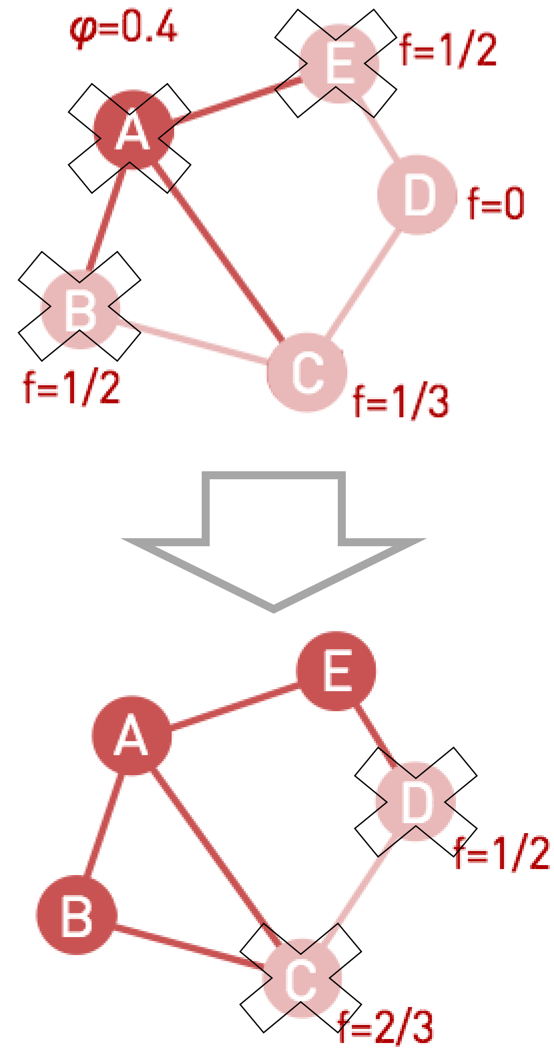
SOURCE	EXPONENT	CASCADE

Can we create a model that helps us to explain the emergence of these properties?

Power grid (Norway)	1.7	Power
Power grid (New Zealand)	1.6	Energy
Power grid (China)	1.8	Energy
Twitter Cascades	1.75	Retweets
Earthquakes	1.67	Seismic Wave

A simple model for cascading events

- Consider a network of size N
- All nodes have a state initialized to “0” (healthy)
- At each time step, each node will turn “1” if at least a fraction φ of its neighbors is also “1” (i.e., have also failed).



Kong & Yeh, Resilience of degree-dependent and cascading node failures in random geometric networks, IEEE Transactions in Information Theory, 2010

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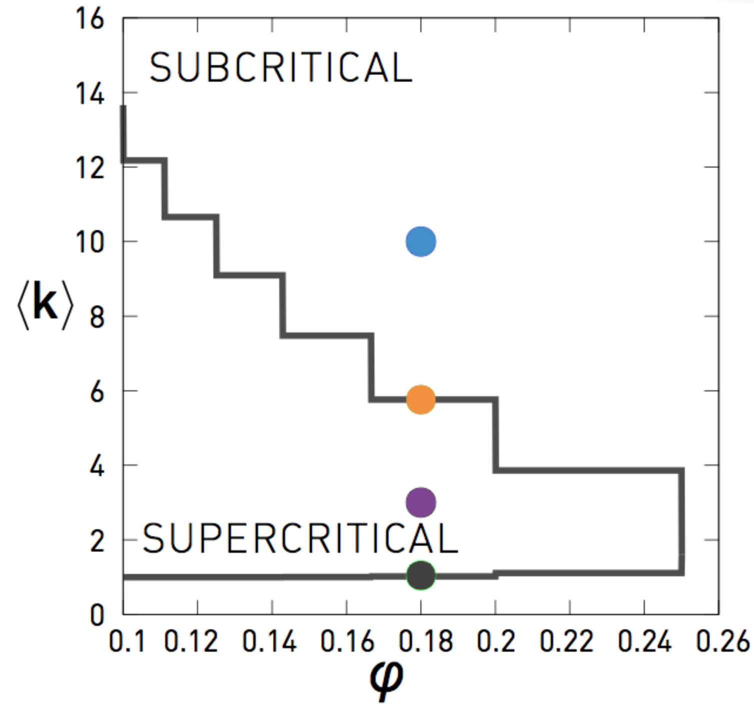
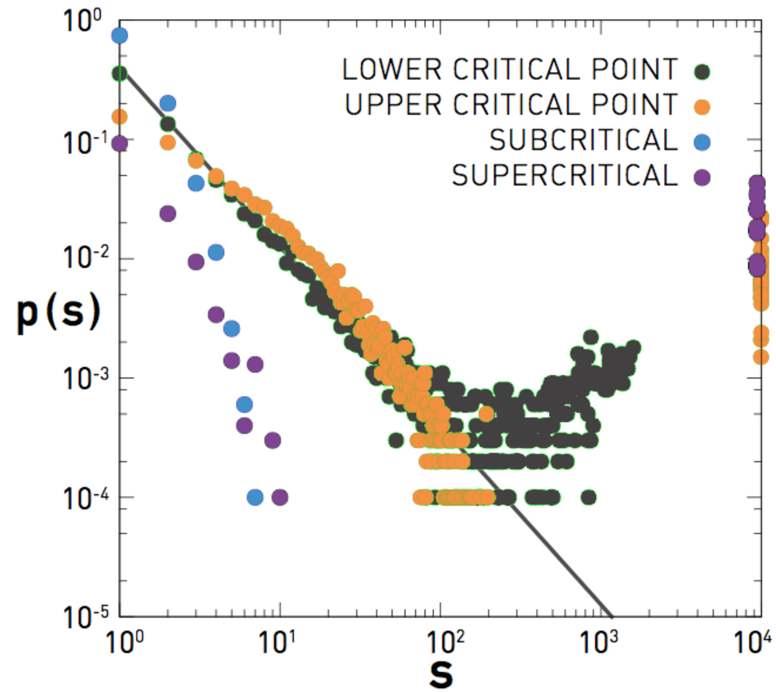




Kong & Yeh, Resilience of degree-dependent and cascading node failures in random geometric networks, IEEE Transactions in Information Theory, 2010

Motter, Lai, PRE 66 (6), 065102 (2002)
Motter, PRL 93(9), 098701 (2004)

Other example
Motter-Lai congestion model (2002)

- All nodes, at equal rate, send data packages to each other along shortest paths.
- Therefore, the permanent load of a node *i* is proportional to its betweeness centrality, $B_{0,i}$
- This model assumes that each node i has a limiting capacity given by

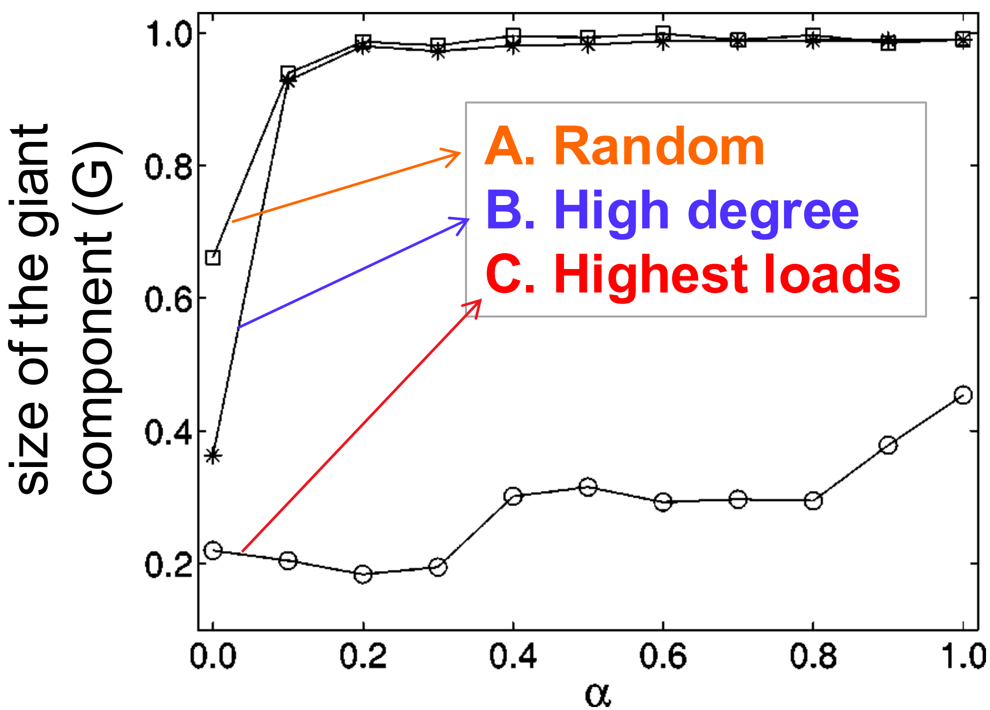
$$c_i = (1+\alpha) B_{0,i}$$

above this, the node colapses, leading to further collapses elsewhere.

Motter, Lai, PRE 66 (6), 065102 (2002)
Motter, PRL 93(9), 098701 (2004)

Other example
Motter-Lai congestion model (2002)

1. Compute all Betweeness centralities, *B*
2. Remove a given node using a given criterion: **A**, **B** or **C**.
3. Compute all B's and delete nodes with $B_i > (1+\alpha) B_{0,i}$
4. Repeat (3) until no overloaded nodes remain.



Degree-based attacks are surprisingly ineffective

Halting avalanches

Can we avoid cascading failures?

Reinforce the network after the first failure by adding new links? Does not work... The time needed to establish a new link is much larger than the timescale of a cascading failure (e.g.. financial and legal barriers, new transmission line on the power grid, etc.)

Can we reduce cascading failures through selective node and link removal (fighting fire with fire)?

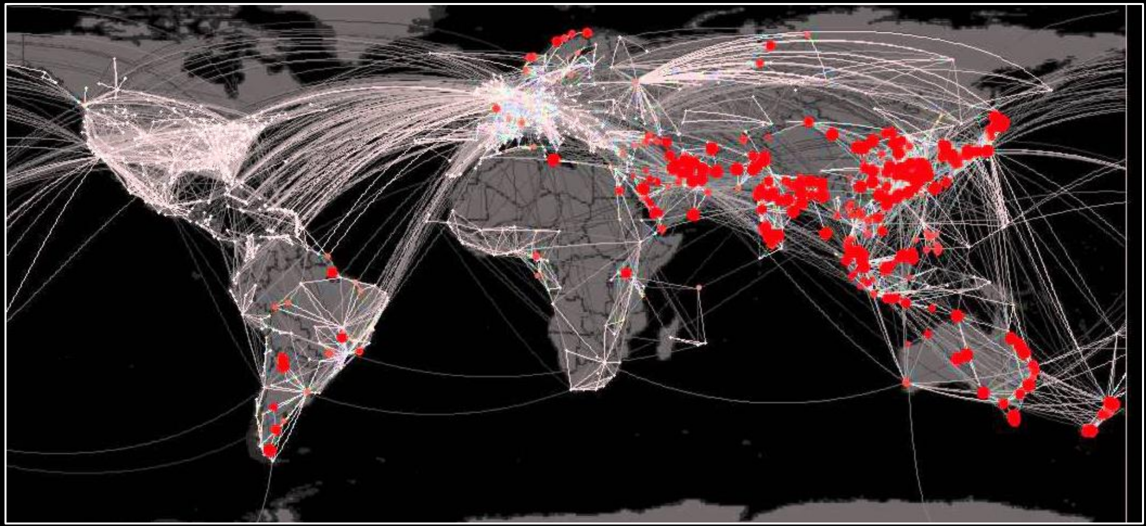
Challenge: What is the most efficient strategy?

- Remove nodes with low or high loads?
- Remove links with low or large loads?

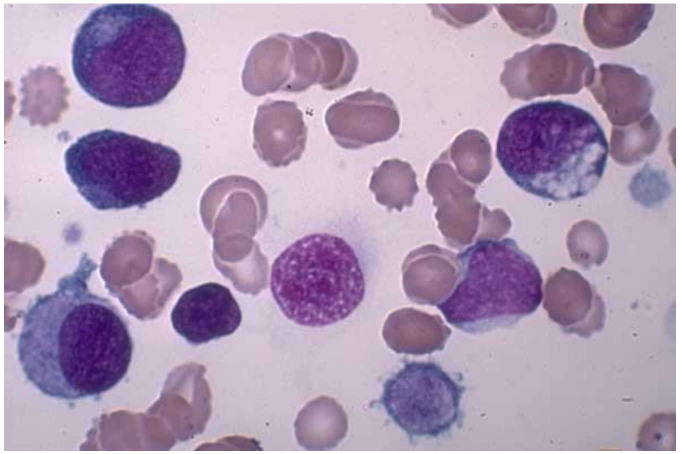
AE Motter, Cascade control and defense in complex networks, Phys Rev Lett 93(9), 098701 (2004)

How does the shape of our contact networks influence the spreading of information, diseases, human decisions, etc.?

What's next? Modelling and forecasting of epidemics, opinions, fads, etc.

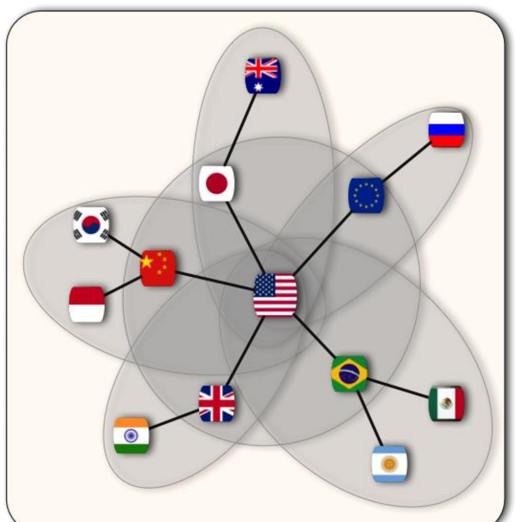


What's next? Evolutionary dynamics

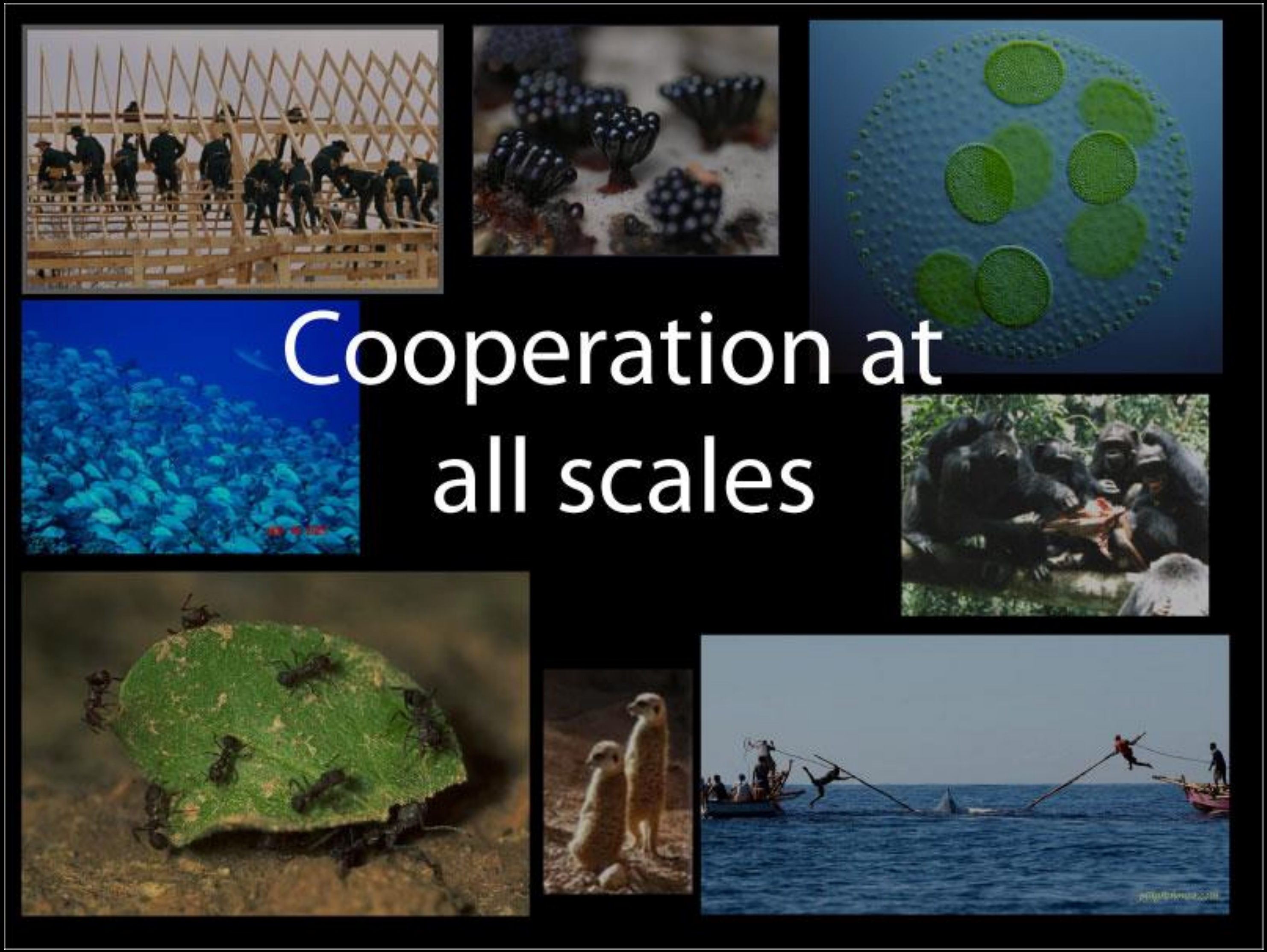


Genetic evolution

Cultural evolution

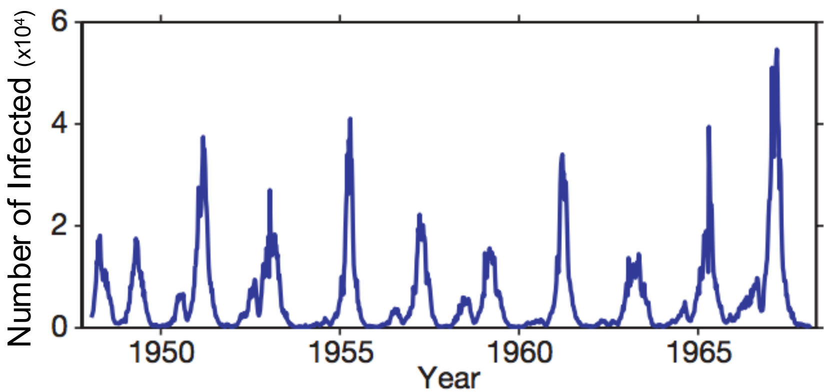


Why we cooperate?



NEXT CLASS

Microscopic interactions

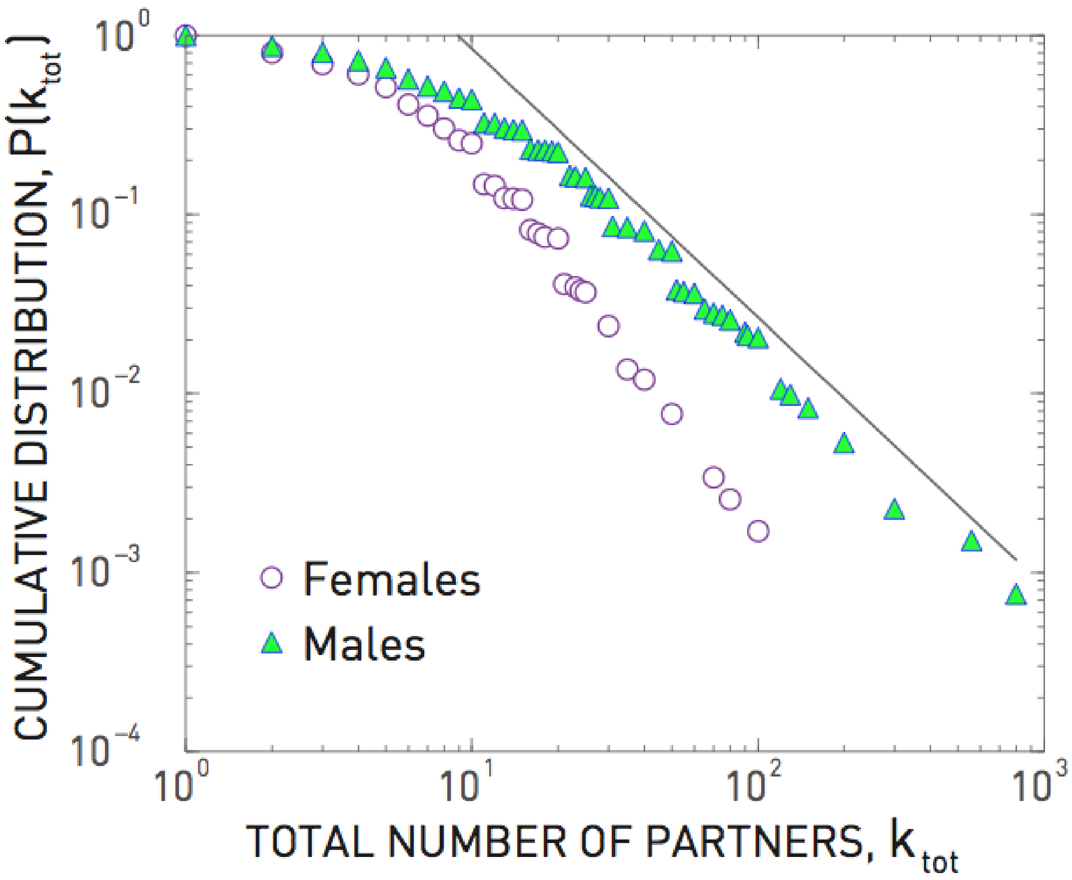


Emergent collective phenomena / properties

local

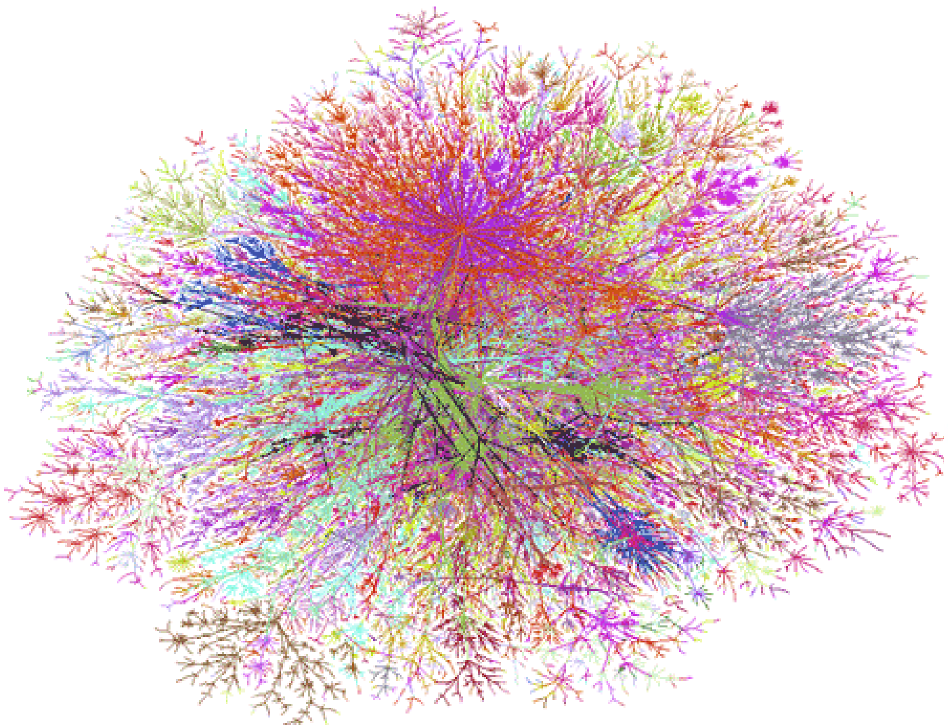
global

Sexually transmitted infections (STIs)

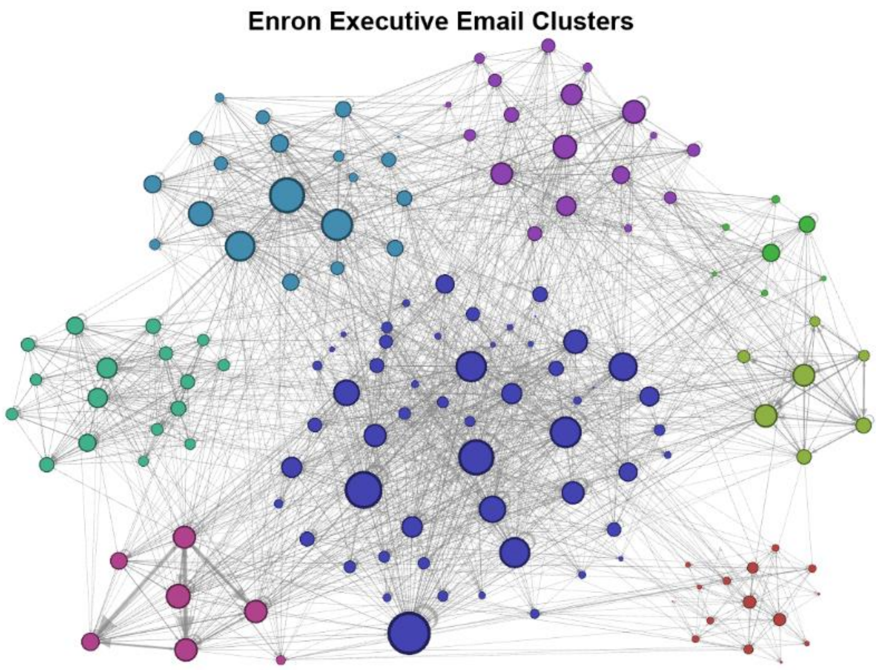


Computer viruses, mobile phone viruses, etc.

Internet

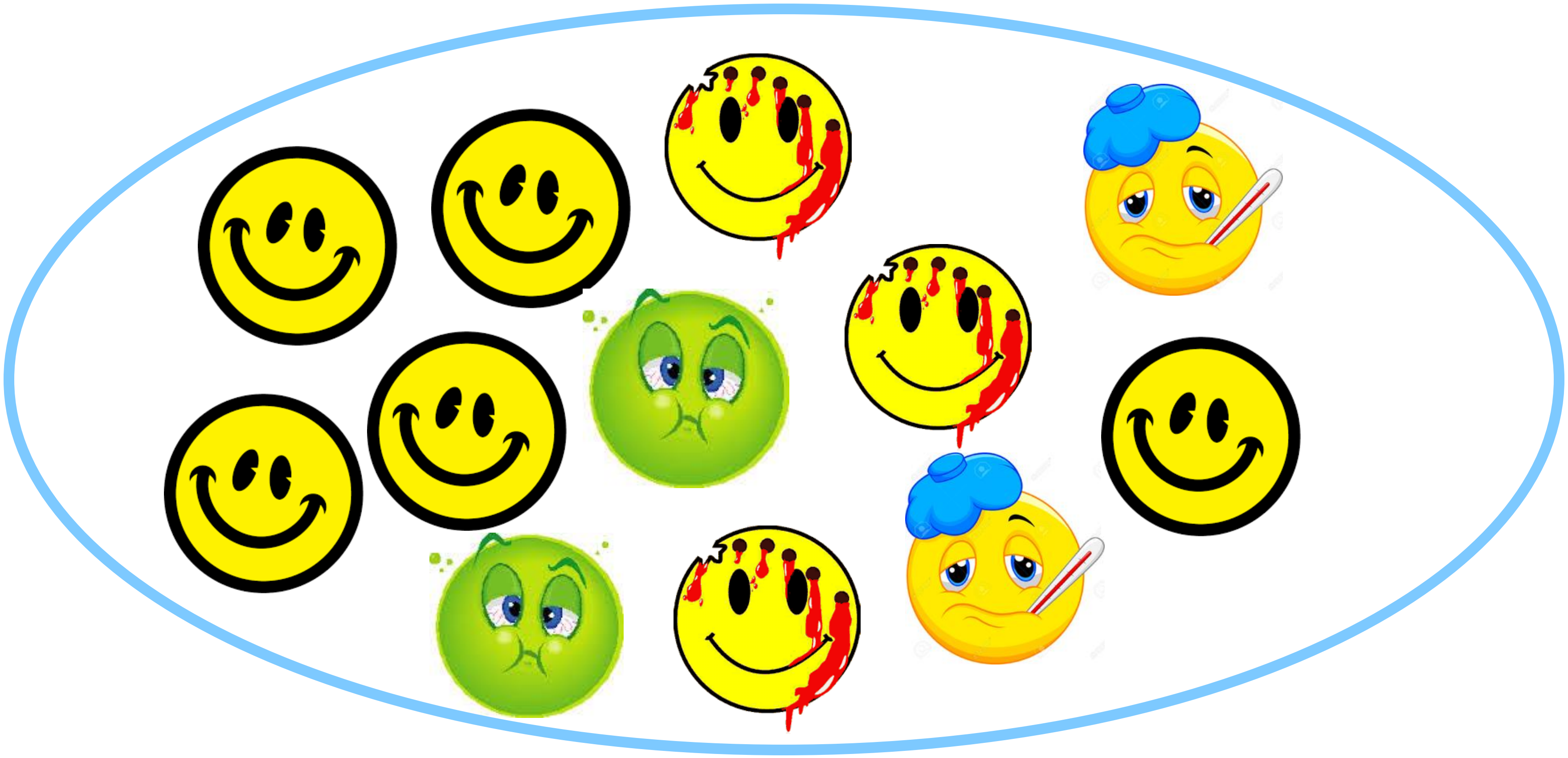


Email networks



Compartment models of disease spreading

Consider a population where each individual has a disease state



Compartment models of disease spreading

Each individual has a disease state. Examples:

S

Susceptible
(healthy)

x

I

Infected

y

E

Exposed
(latency period)

w

R

Vaccinated
or Recovered

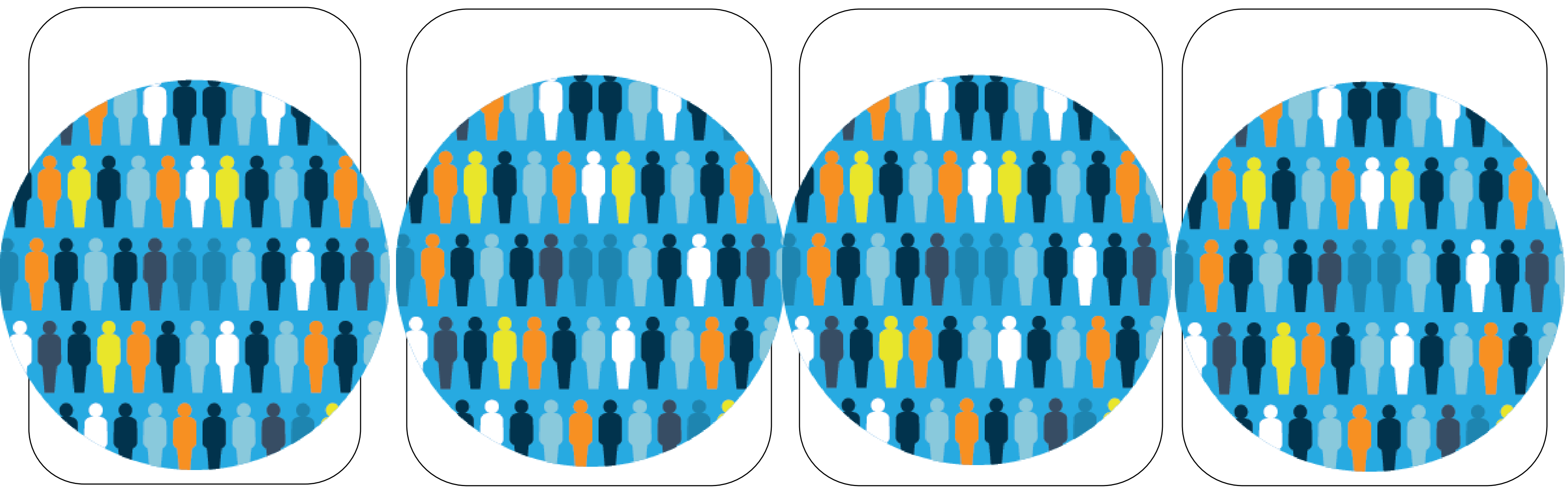
z

(…)

Compartment models of disease spreading

Populations are infinite and “well-mixed”

- anyone can potentially interact with anyone else
- all individuals in the same state are equivalent;
- the parameters defining the population dynamics are provided by the fraction of individuals that are in a given state.



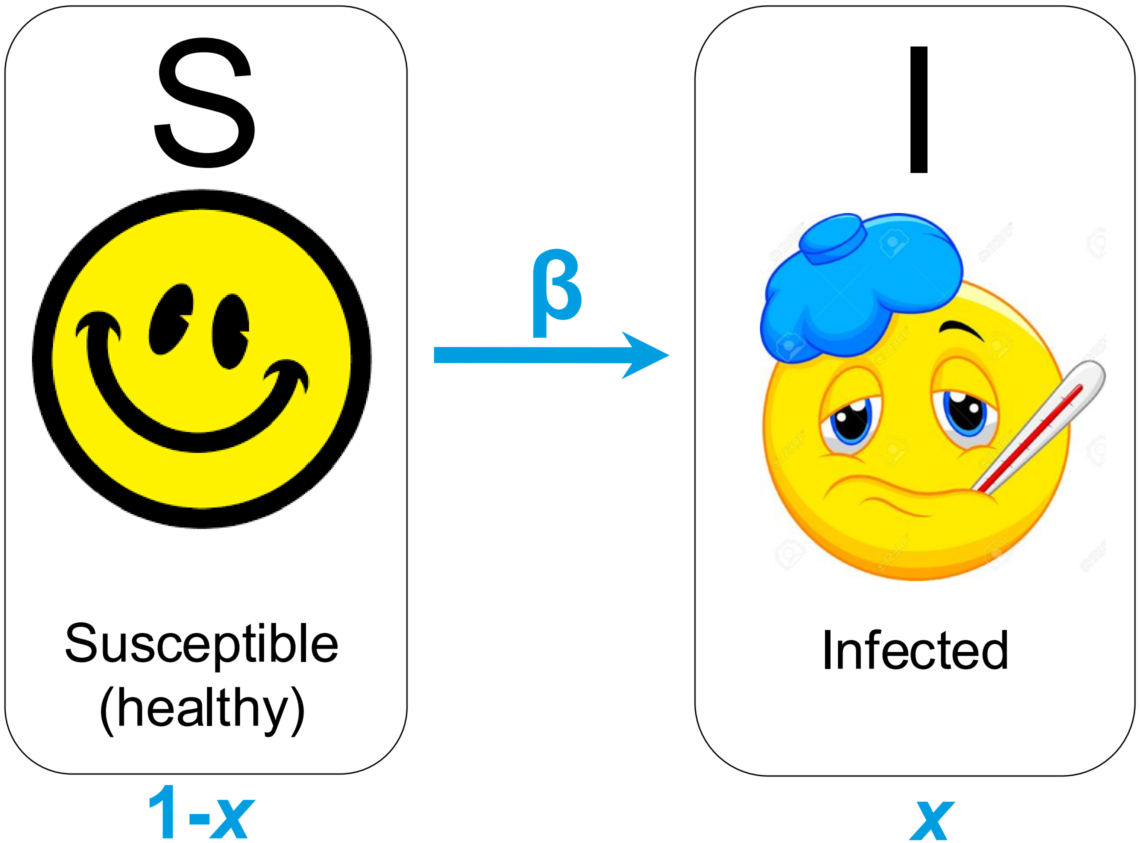
x y w z

(...)

SI model

β: contact infection rate

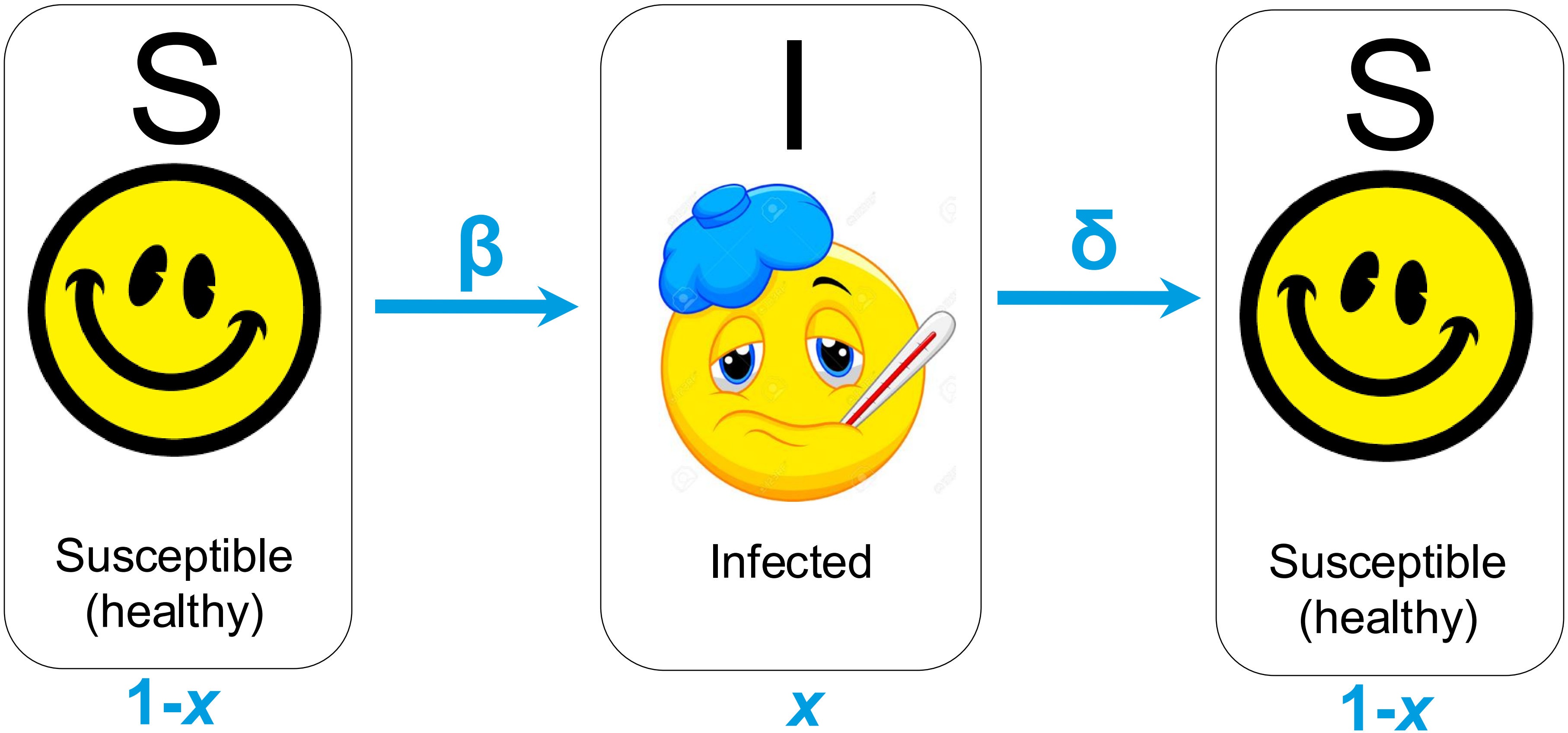
The very basic model…



SIS model

β: contact infection rate
δ: recovery rate

Let's add recovery!



SIR model

β: contact infection rate
δ: recovery rate

Often individuals develop immunity after recovery
(e.g., Influenza) or can be removed from the population.

S

Susceptible
(healthy)

$1-x-r$

β →

I

Infected

x

δ →

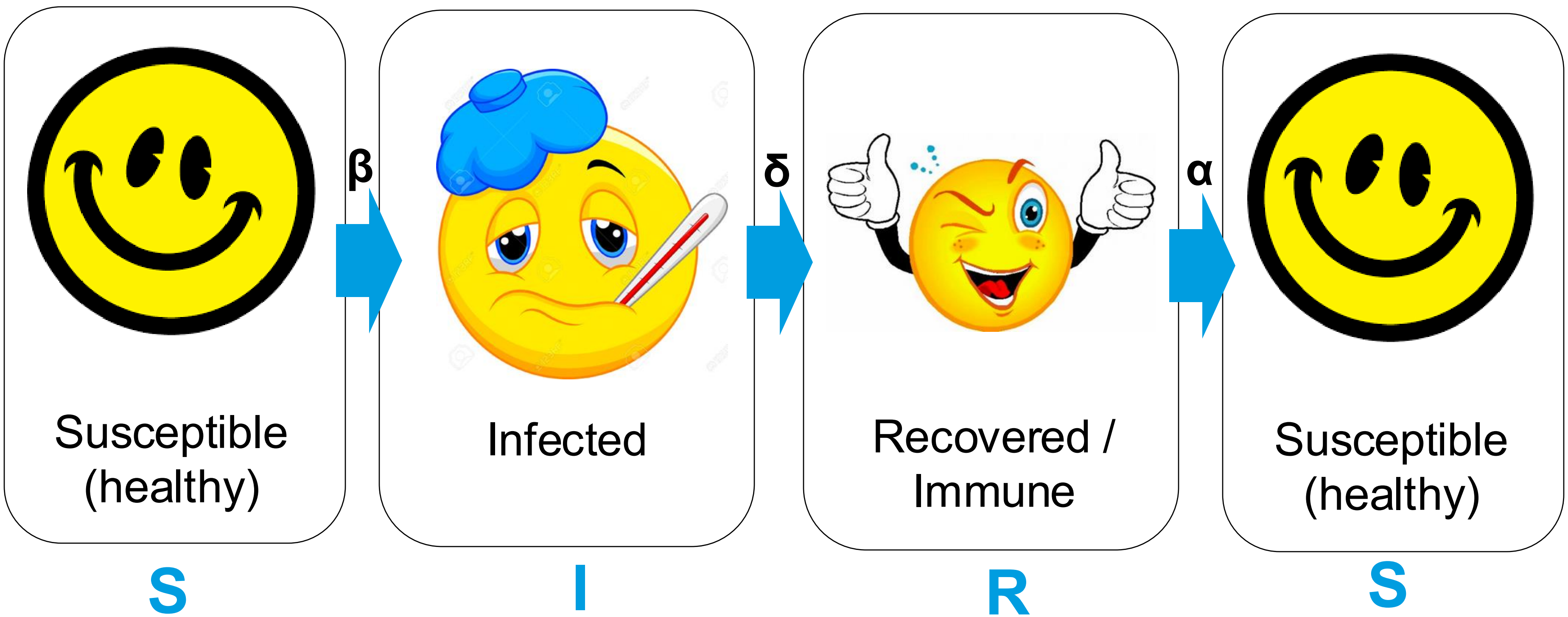
R

Recovered /
Immune

r

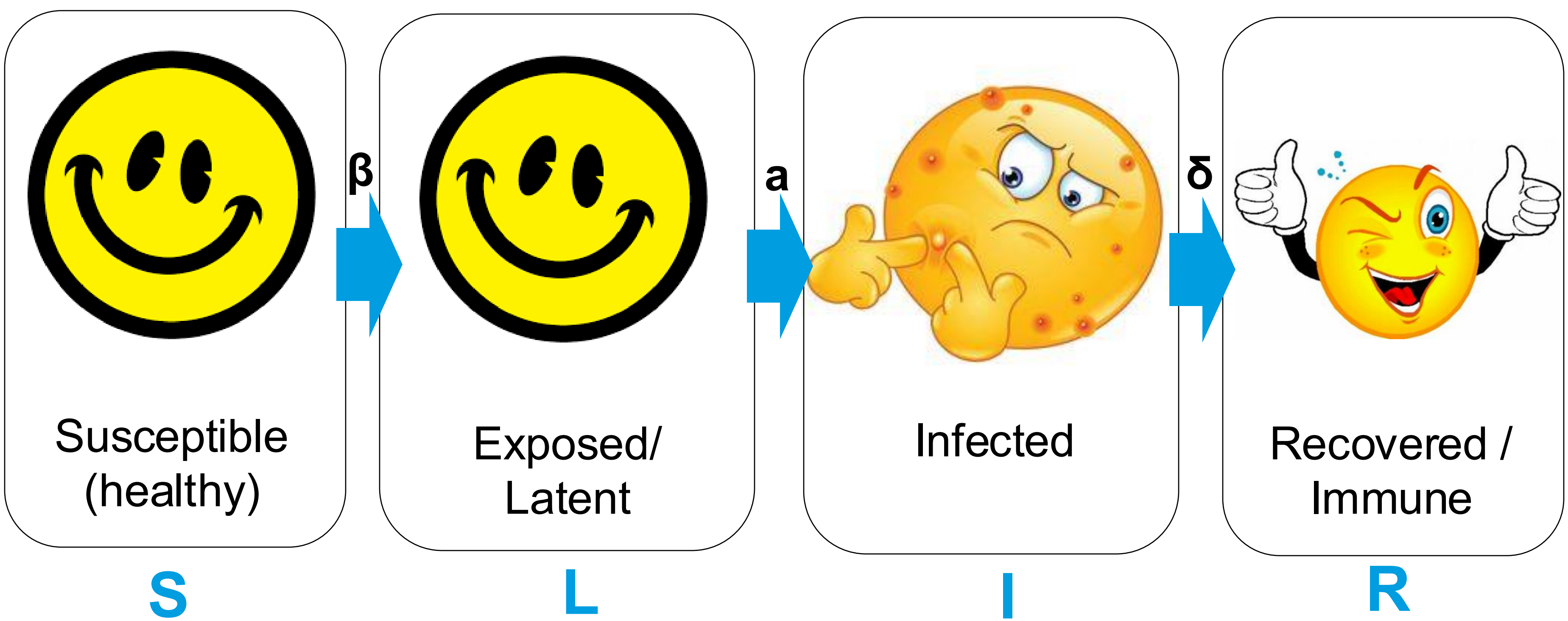
Other variants: SIRS model

This model is simply an extension of the SIR model. The only difference is that it allows members of the recovered class to be free of infection and rejoin the susceptible class.

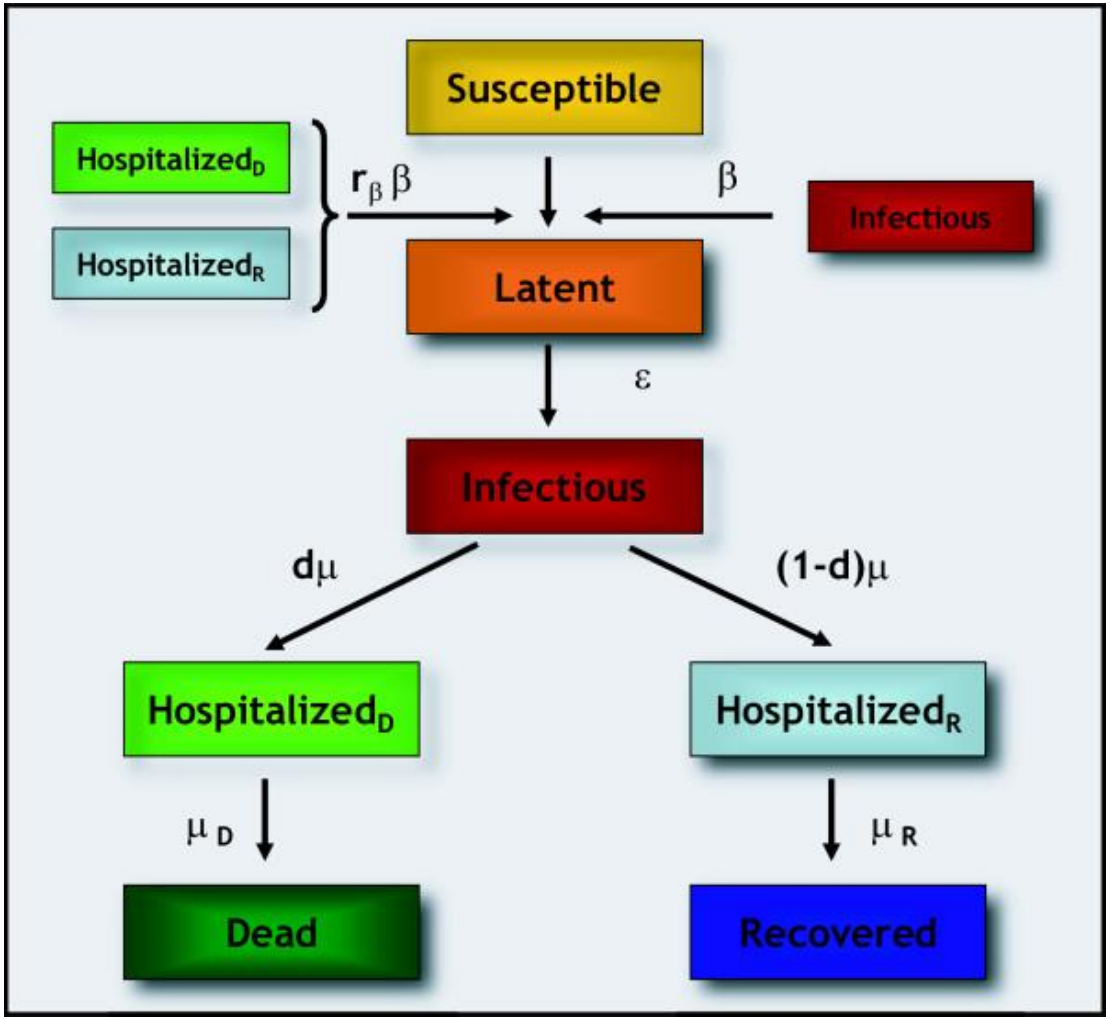


Other variants: SEIR and SEIR-S models

For many important infections there is a significant incubation period during which the individual has been infected but is not yet infectious themselves. During this period the individual is in compartment E (for exposed).



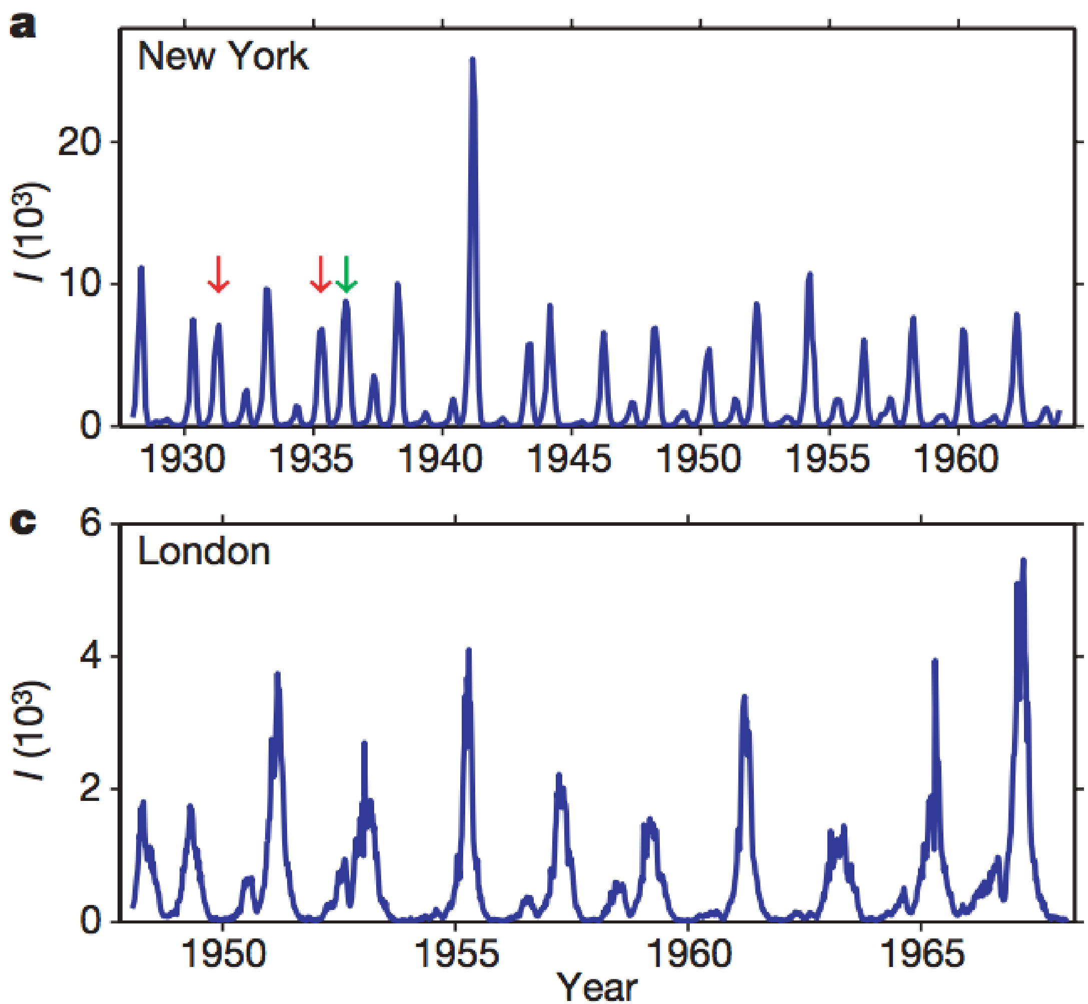
SARS (Severe acute respiratory syndrome)



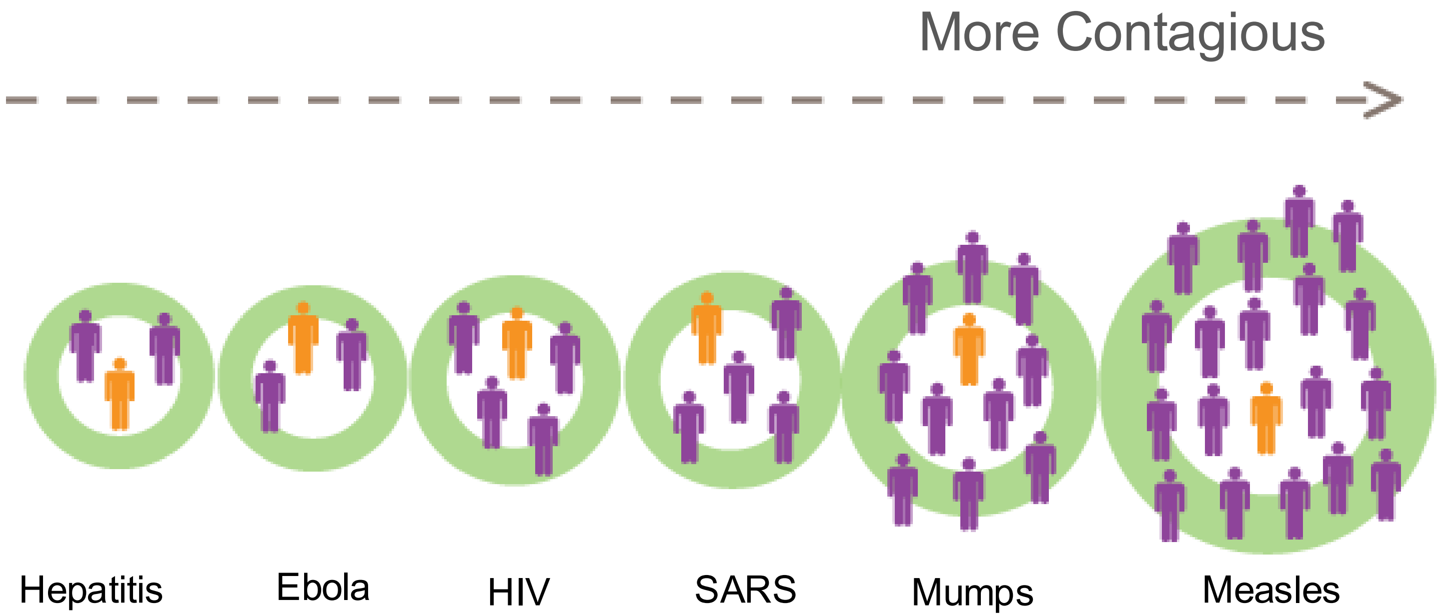
Colizza et al. BMC Medicine 2007 5:34

Epidemics of many infectious diseases occur periodically. Why?

Ex: Measles (“*Sarampo*”, “*Mässling*”, “*Mazelen*”, …)



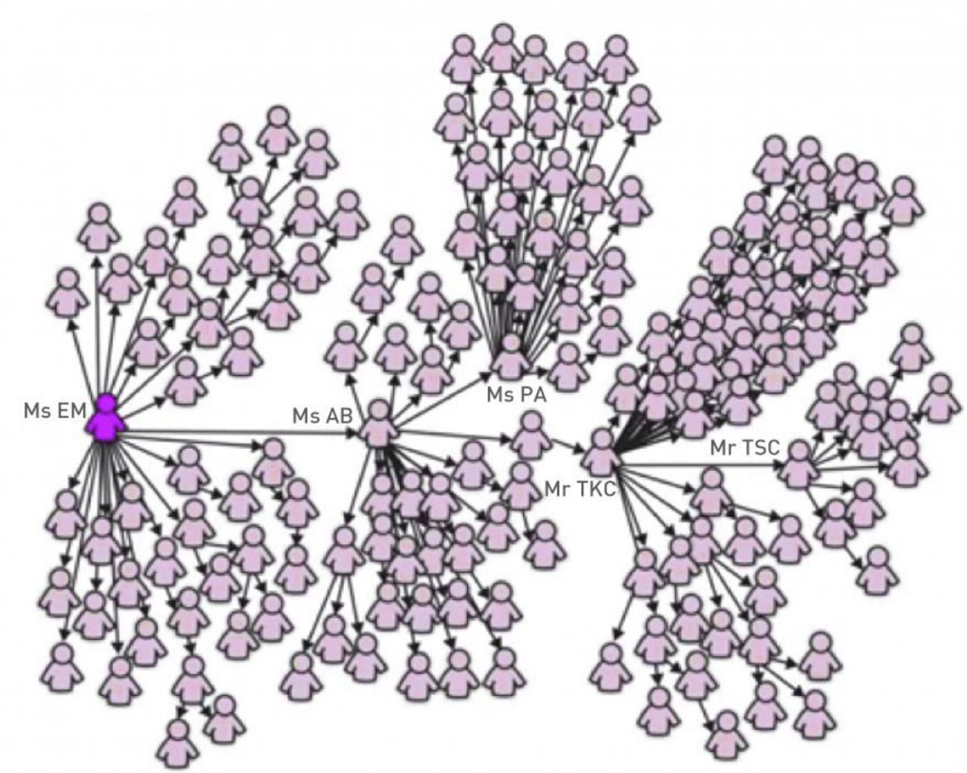
Stone, Ronen, Olinky, Huppert, Nature 2007

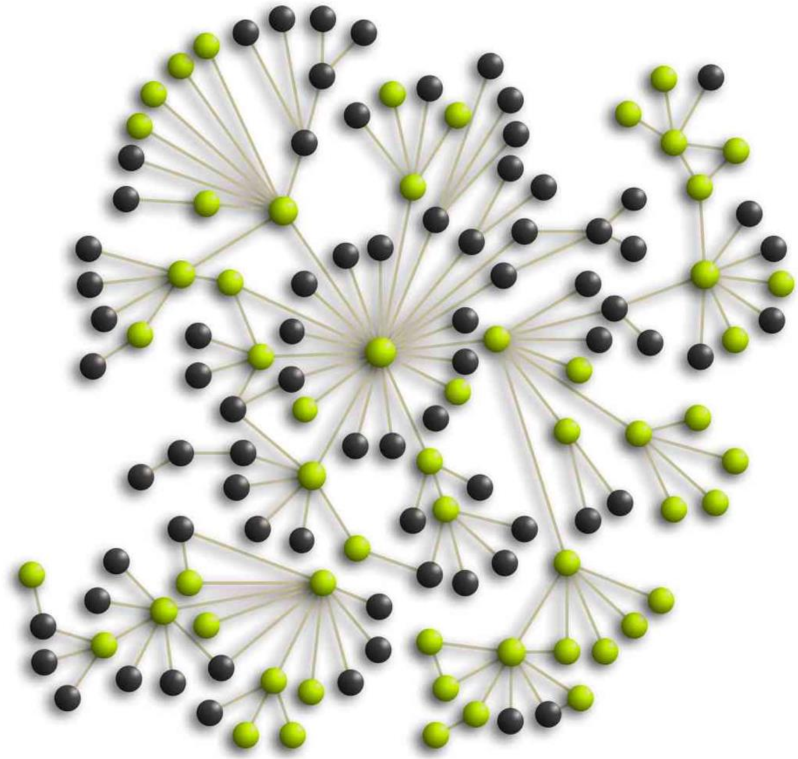


Endemic state
There's a disease outbreak

Disease free-state
The disease naturally dies out

The role of super-spreaders





In scale-free networks, the disease tends to spread through the hubs!

Endemic state
There's a disease outbreak

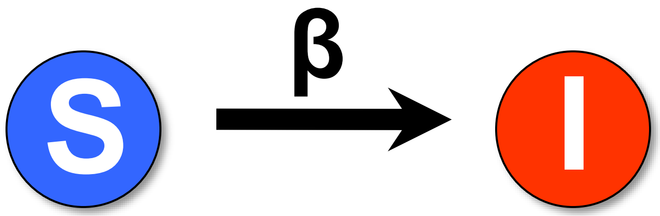
Disease free-state
The disease naturally dies out

Have a nice weekend
and
Thank you!!

Traditional models

β: contact infection rate

SI model



Ex: AIDS

Traditional models

β: contact infection rate
δ: recovery rate

SI model

$$S \xrightarrow{\beta} I$$

Ex: AIDS

SIS model

$$S \underset{\delta}{\overset{\beta}{\rightleftharpoons}} I$$

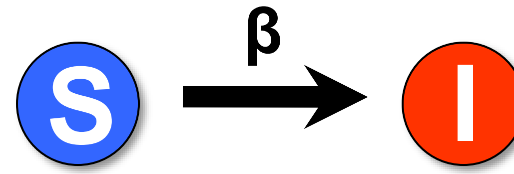
Ex: gonorrhea

SIR model

$$S \xrightarrow{\beta} I \xrightarrow{\delta} R$$

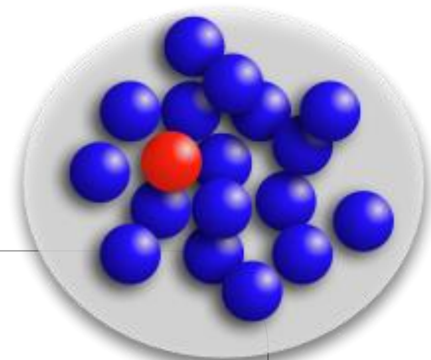
Ex: single season flu

Building a model (ex: SI model)



SI model

Assumptions:
N = population size (large number)
Everyone is equally likely to interact with everyone else.
S(t) susceptible and I(t) infected individuals at time t.

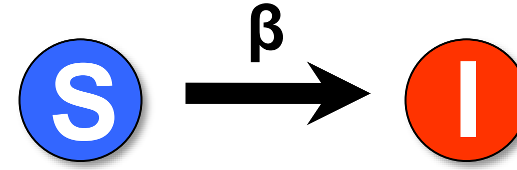


Goal:

Let us say that at time t=0 everyone is susceptible (S(0)=N) and no one is infected (I(0)=0).

If a single individual becomes infected at time *t=0* (i.e. *I*(0)=1), how many individuals will be infected at some later time *t*?

Building a model (ex: SI model)



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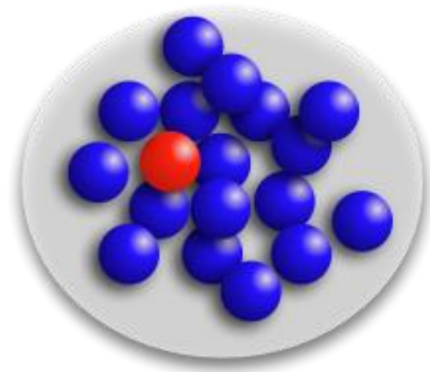
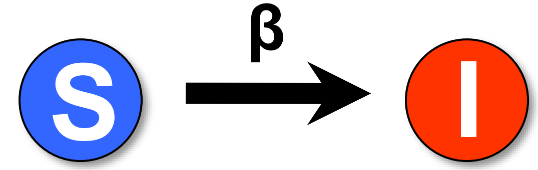


The probability that an infected person encounters a susceptible in a random interaction is

$$\frac{S(t)}{N}$$

Building a model (ex: SI model)



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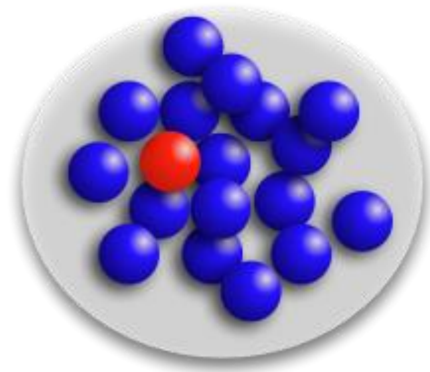
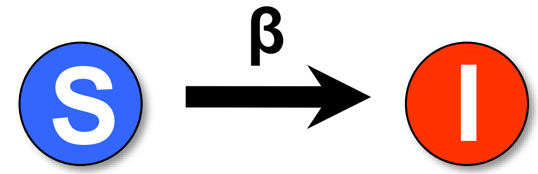


If everyone interacts <k> times at each time-step, we get the following rate of SI encounters (on average)

$$\langle k \rangle \frac{S(t)}{N}$$

Building a model (ex: SI model)



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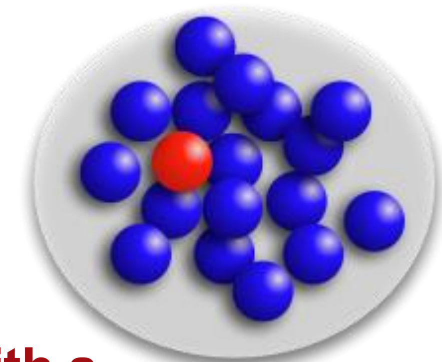
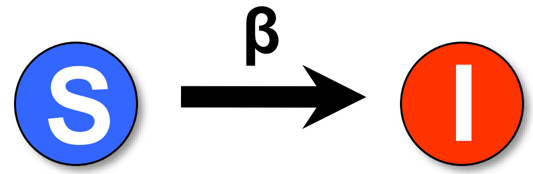


If each of the I(t) infected transmits the disease with a rate β the number of <u>new</u> infected individuals at each time-step is

$$I(t)\beta\langle k\rangle\frac{S(t)}{N}$$

Building a model (ex: SI model)



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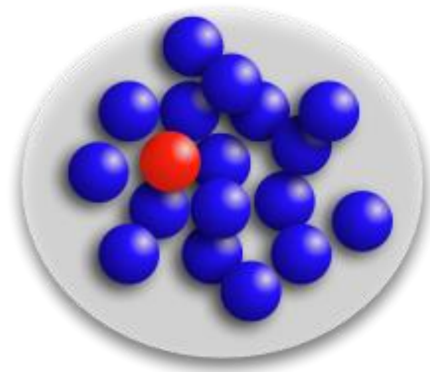
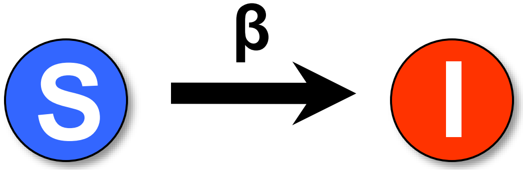


- **Therefore, the change in I(t) is given by**

$$I(t+1)-I(t)=I(t)\beta\langle k\rangle\frac{S(t)}{N}$$

Building a model (ex: SI model)



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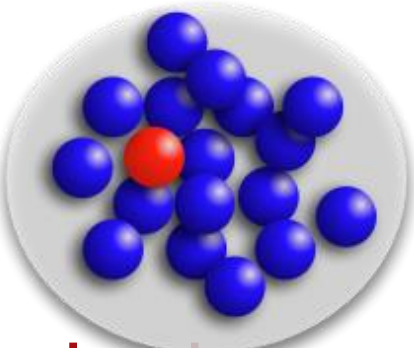
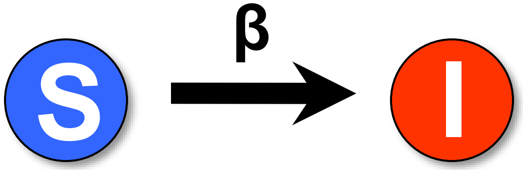


..or, in the other words, the rate of change dI(t)/dt is given by

$$\frac{dI(t)}{dt} = I(t)\beta\langle k\rangle\frac{S(t)}{N}$$

Building a model (ex: SI model)



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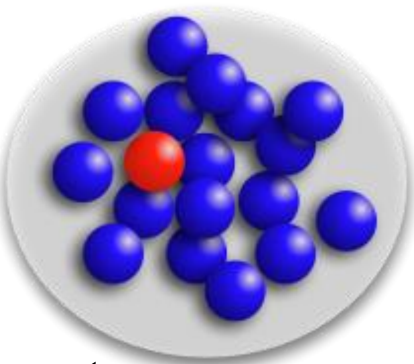

Using the frequencies $x=I(t)/N$ and $y=S(t)/N=1-x$ we get

$$\boxed{\frac{dx}{dt} = x(1-x)\beta\langle k\rangle}$$

SI model

β = infection rate
<k> = average number of contacts of a given individual
x = fraction of infected in the population
y = 1-x = fraction of susceptible

$$S \xrightarrow{\beta} I$$

$$\frac{dx}{dt} = x(1-x)\boxed{\beta\langle k\rangle}$$

Transmission rate / force of infection = β<k>

i.e., an infected individual is able to transmit the disease with β<k> others per unit time. Or, if you prefer, the characteristic timescale of the disease is

$$\tau = \left(\beta\langle k\rangle\right)^{-1}$$

SI model

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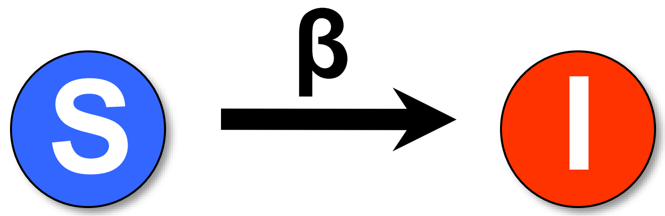
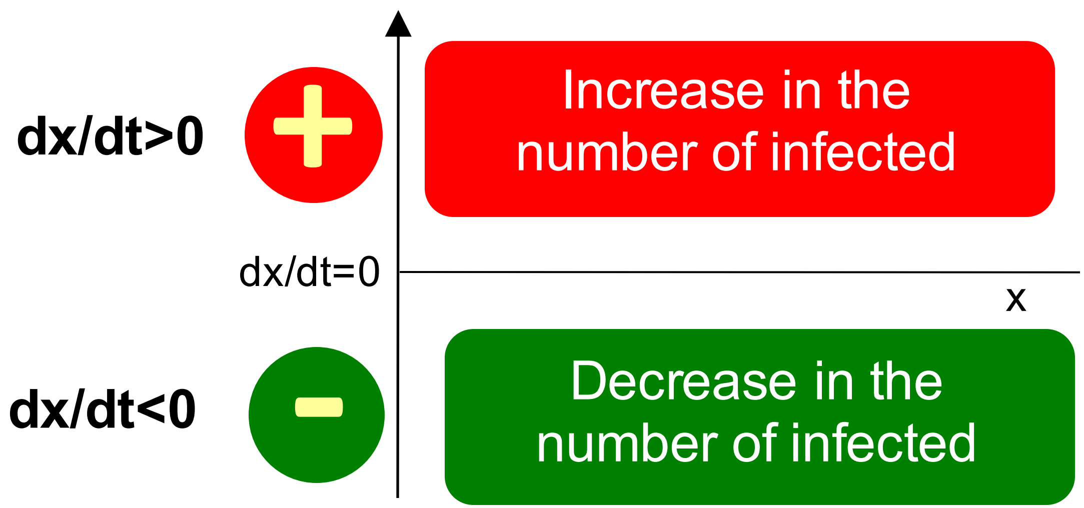


$$\frac{dx}{dt} = x(1-x)\beta\langle k\rangle$$

Gradient of infection:



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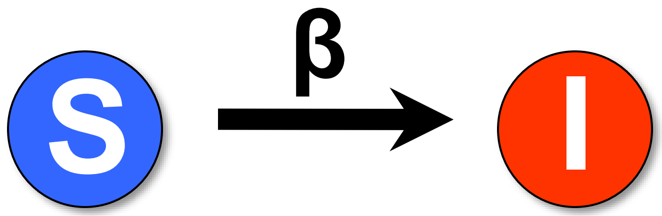

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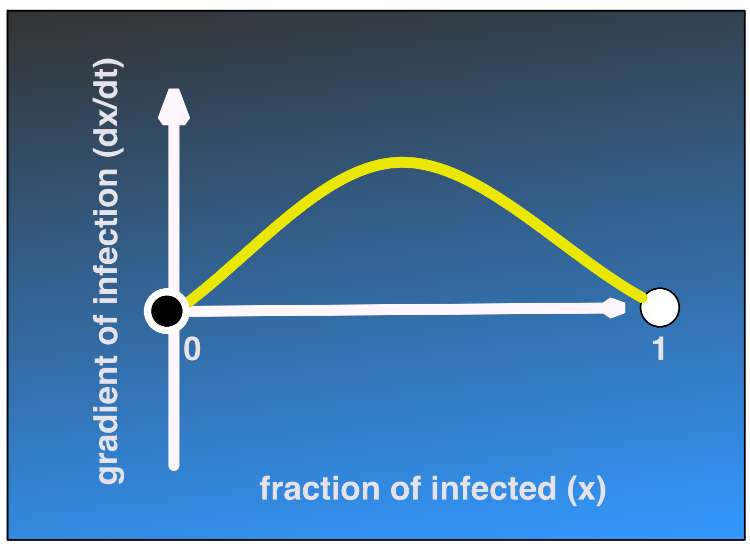

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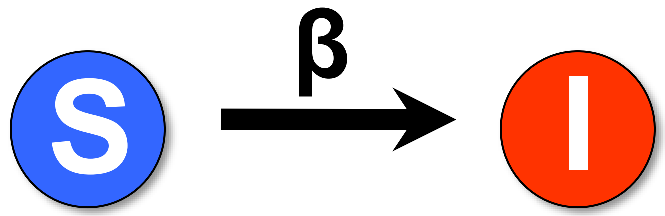

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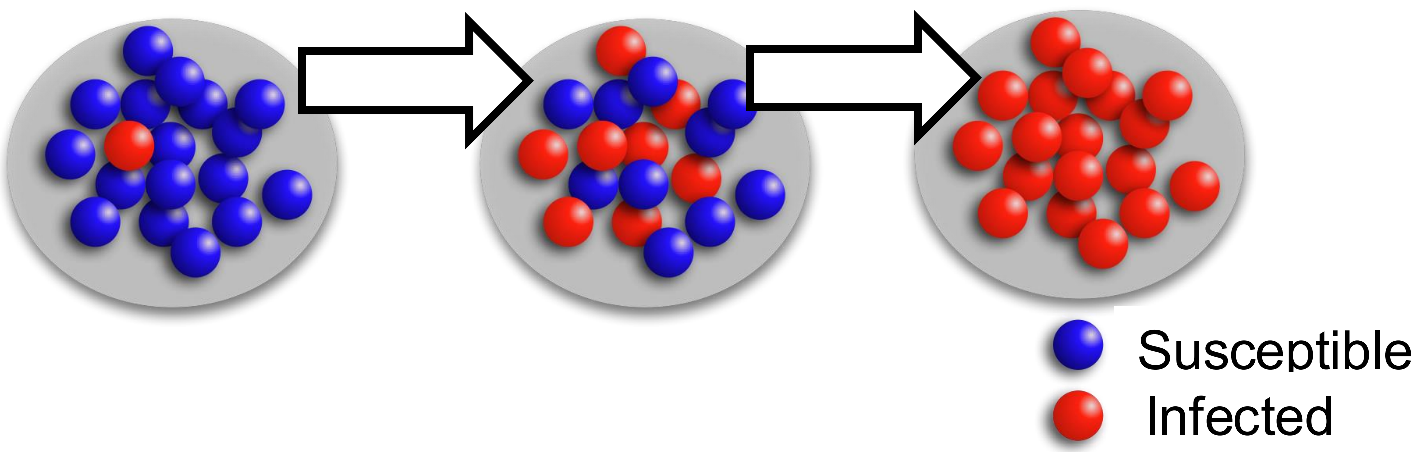

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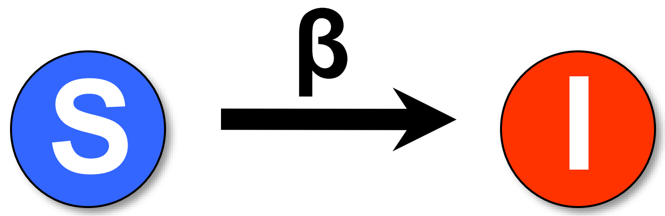


$$\frac{dx}{dt} = x(1-x)\beta\langle k\rangle$$

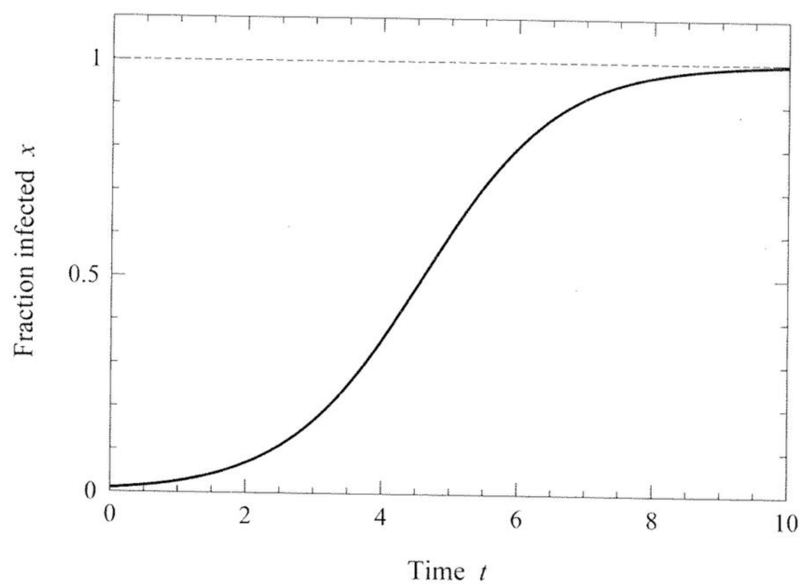
$$\tau = \left(\beta\langle k\rangle\right)^{-1}$$

$$x(t) = \frac{x_0 e^{t/\tau}}{1 - x_0 - x_0 e^{t/\tau}}$$

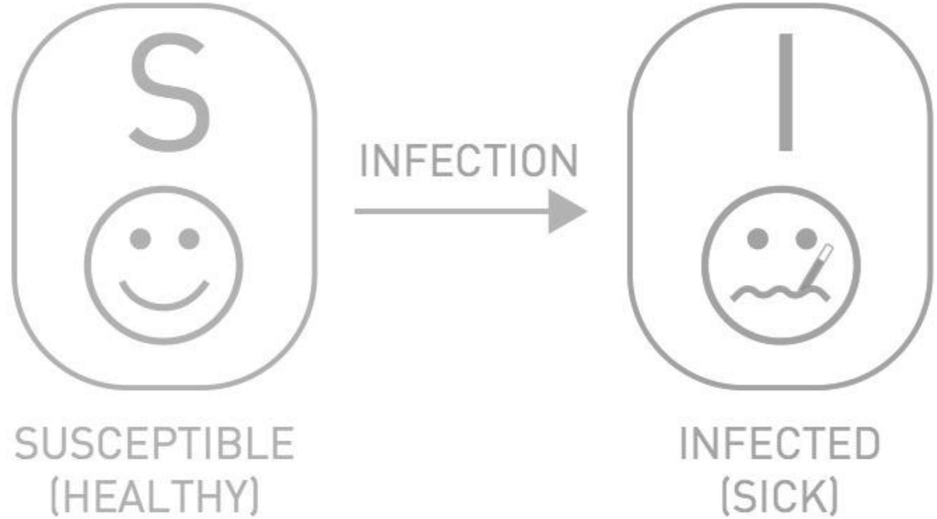
$$x(t) << 1$$

$$x(t) \approx x_0 e^{t/\tau}$$

Logistic curve:



SI model

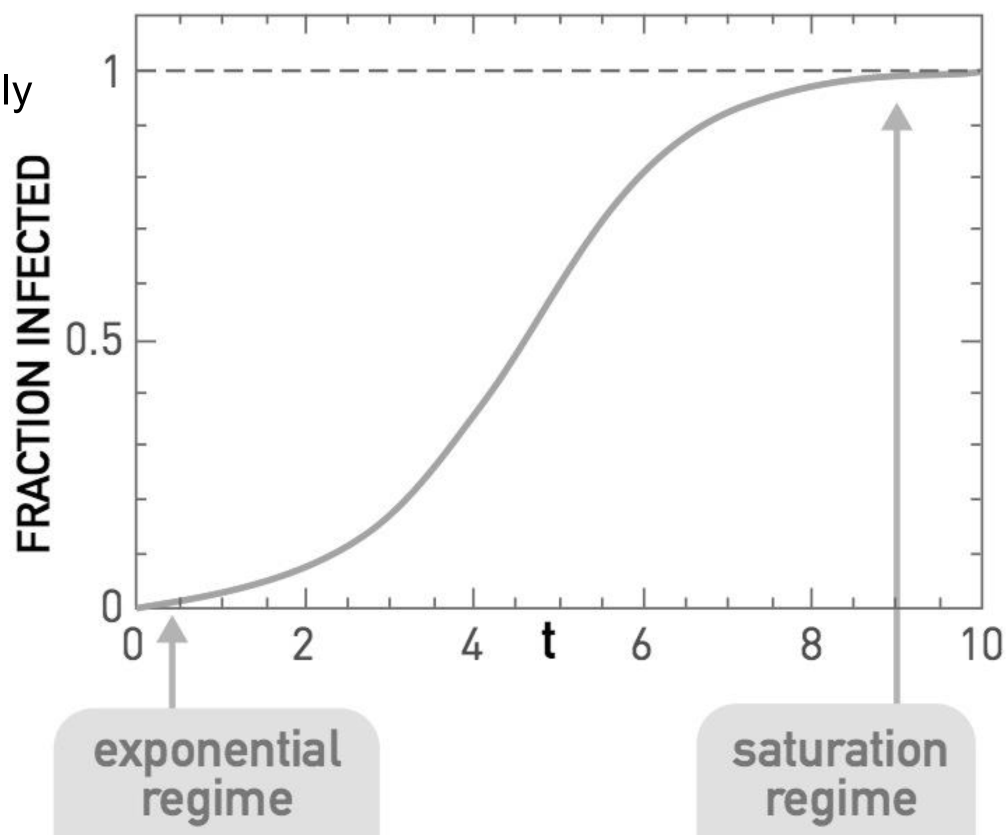


What are we missing here?

In some diseases pathogens are eventually defeated by the immune system.

Thus we need to consider the possibility that individuals recover, stopping to spread the disease.

Thus, we need an extra parameter:
The rate of recovery: **δ**



Traditional models

β: contact infection rate
δ: recovery rate

SIS model

$$S \underset{\delta}{\overset{\beta}{\rightleftarrows}} I$$

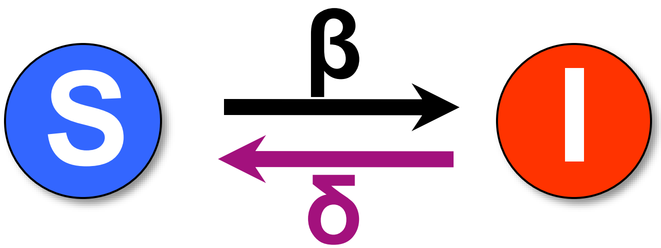
$$\dot{x} \equiv {}^{dx}\!/_{dt} = \underbrace{x(1-x)\beta\langle k\rangle}_{\text{infection}} \underbrace{-\delta x}_{\text{recovery}}$$

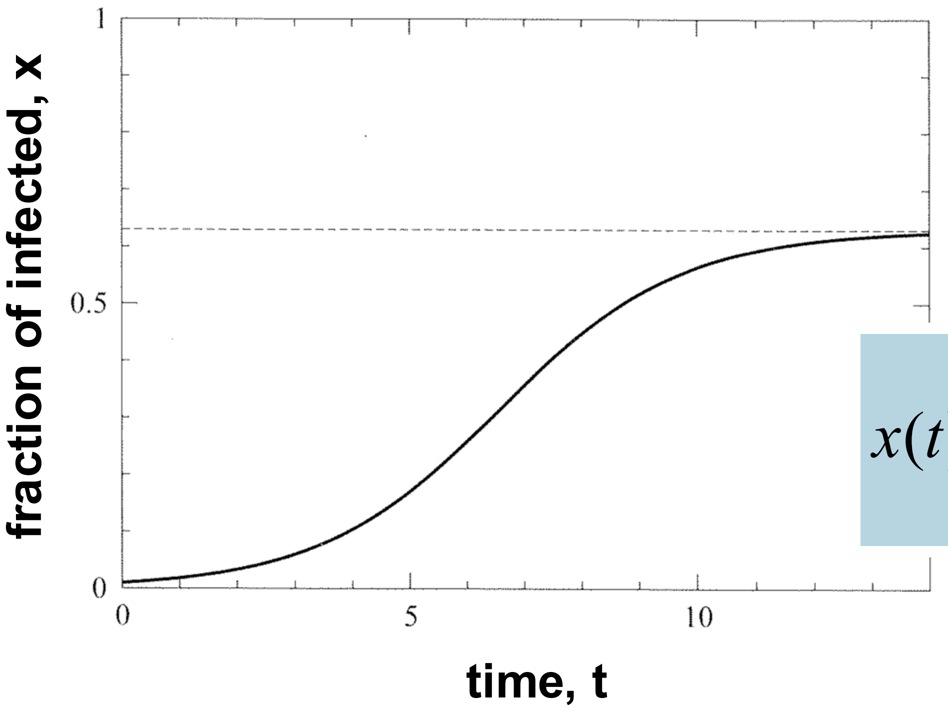
<k> – average number of contacts of a given individual
x – fraction of infected in the population
y = 1-x – fraction of susceptible

Traditional models

β: contact infection rate
δ: recovery rate

SIS model





→ **Endemic disease**

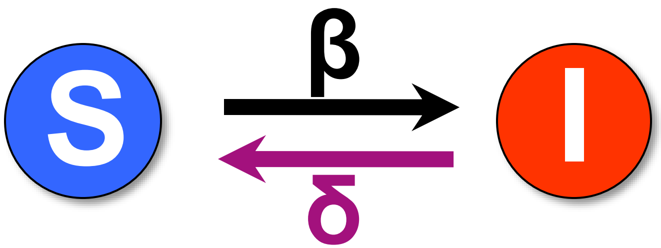
$$x(t) = \left(1 - \frac{\delta}{\beta\langle k\rangle}\right)\frac{Ce^{t/\tau}}{1+Ce^{t/\tau}}$$

$$\tau = \left(\beta\langle k\rangle - \delta\right)^{-1}$$

Traditional models

β: contact infection rate
δ: recovery rate

SIS model



- Endemic state:

  For low recovery rate, the disease will never disappear.

$$\delta < \beta \langle k \rangle$$

- Disease free-state:

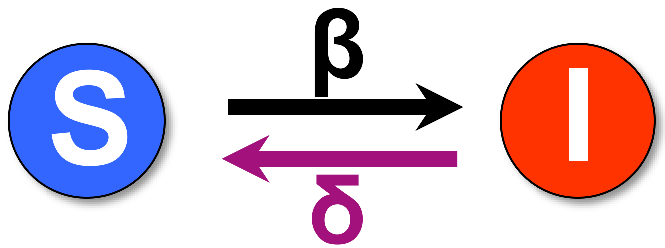
  For large recovery rates the number of new infections will be lower than the number of new recovered individuals, and the disease decreases exponentially in time.

$$\delta > \beta \langle k \rangle$$

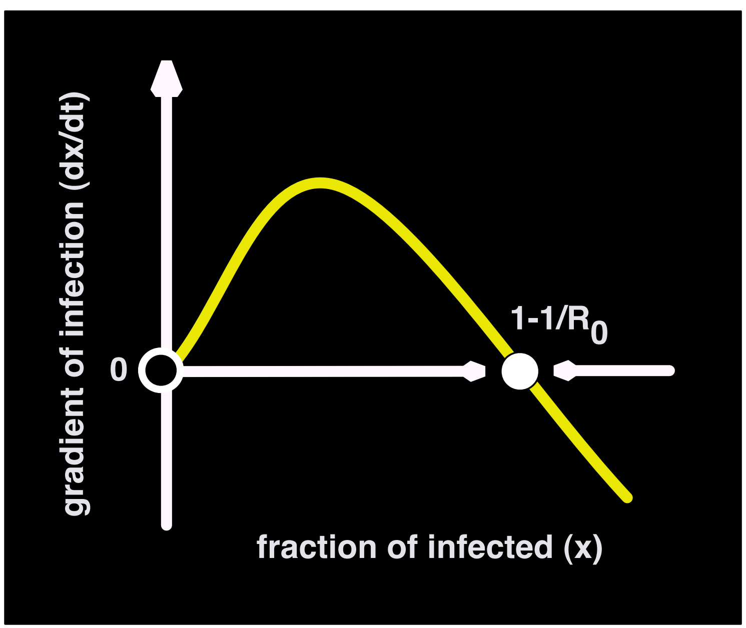
Endemic states

β: contact infection rate
δ: recovery rate

SIS model



$$\dot{x} = x(1-x)\beta\langle k\rangle - \delta x$$

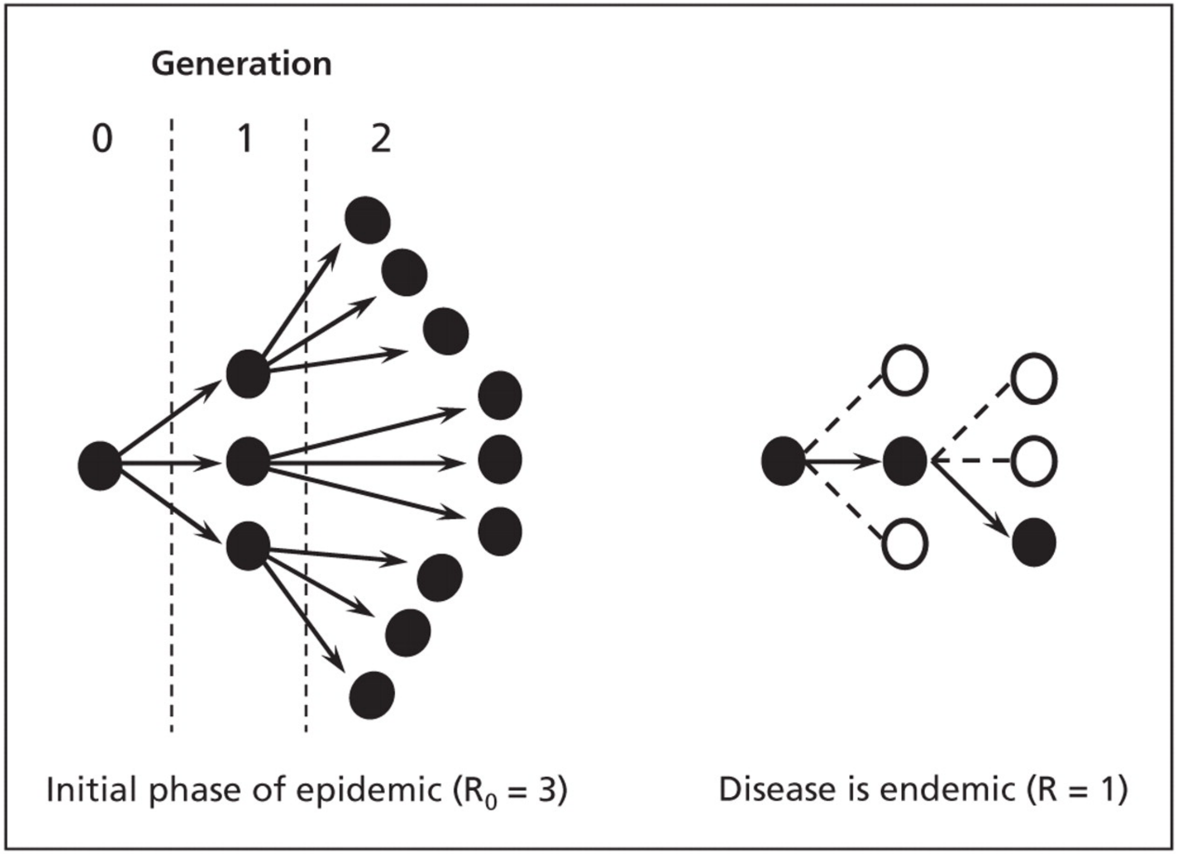


Basic Reproductive Ratio & Epidemic Threshold

Endemic for $R_0 \equiv \frac{\beta\langle k\rangle}{\delta} > 1$

The basic reproductive number (R-naught)

The reproductive number R_0 provides the number of individuals an infected infects if all its contacts are susceptible.



The basic reproductive number (R-naught)

The reproductive number R_0 provides the number of individuals an infected infects if all its contacts are susceptible.

For R_0<1 the disease dies out.

For R_0>1 the pathogen will spread and persist in the population.

Higher the value of R_0, faster the the spreading process.

The basic reproductive number (R-naught)

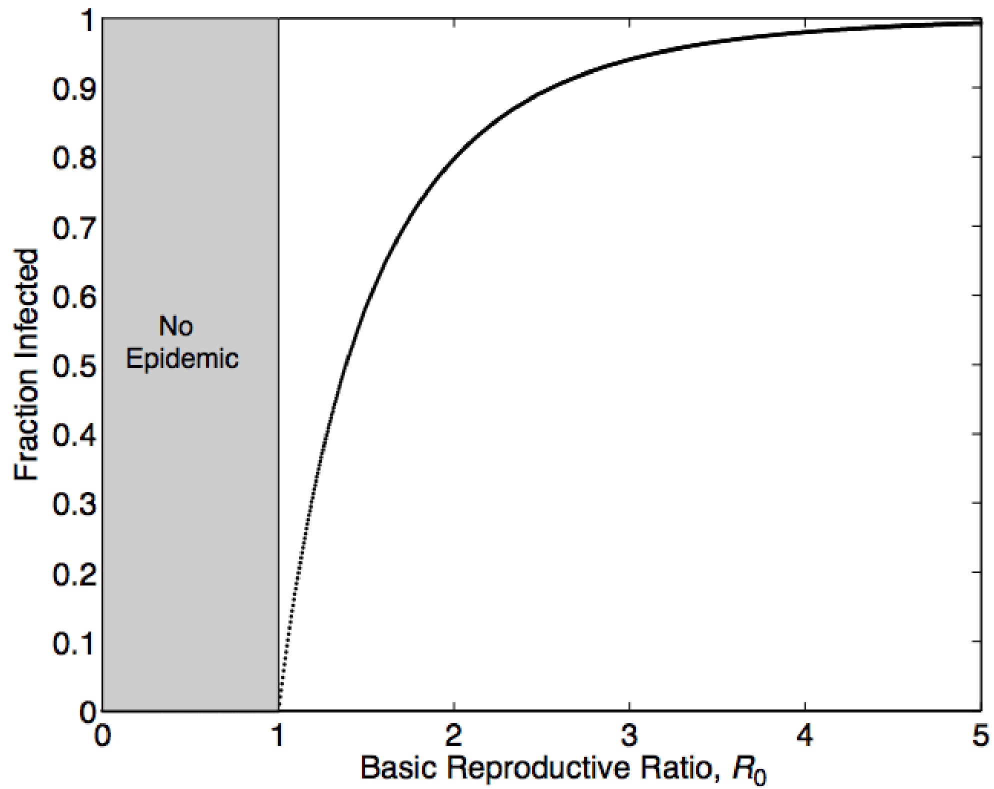
The reproductive number R_0 provides the number of individuals an infected infects if all its contacts are susceptible.

Disease	Transmission	R_0
Measles	Airborne	12-18
Pertussis	Airborne droplet	12-17
Diphtheria	Saliva	6-7
Smallpox	Social contact	5-7
Polio	Fecal-oral route	5-7
Rubella	Airborne droplet	5-7
Mumps	Airborne droplet	4-7
HIV/AIDS	Sexual contact	2-5
SARS	Airborne droplet	2-5
Influenza (1918 pandemic strain)	Airborne droplet	2-3

The basic reproductive number

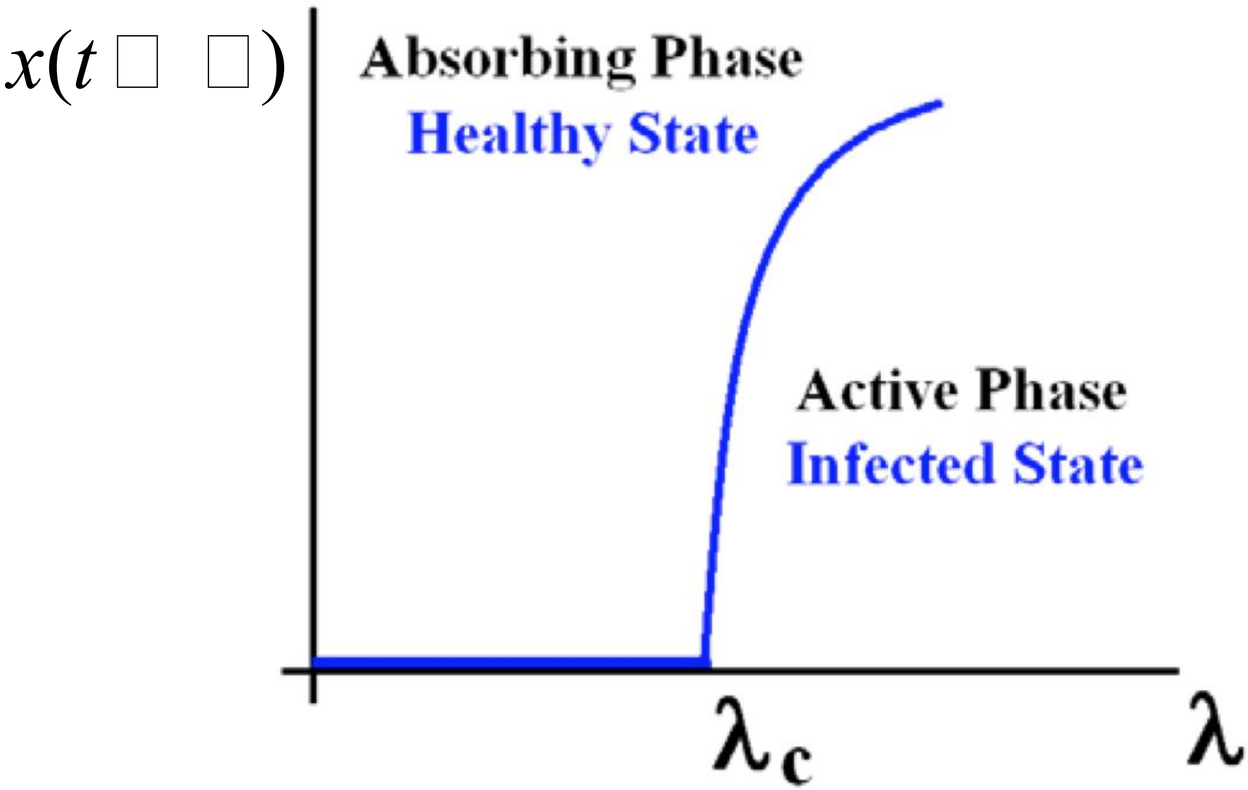
The reproductive number $\boldsymbol{R_0}$ provides the number of individuals an infected infects if all its contacts are susceptible.

$$R_0^{SIS,SIR} \approx \frac{\beta\langle k\rangle}{\delta}$$



Epidemic threshold

Equivalently, one may define an epidemic threshold $\lambda_c=\beta_c/\delta$ which also splits the phase space into the endemic state and the healthy state.

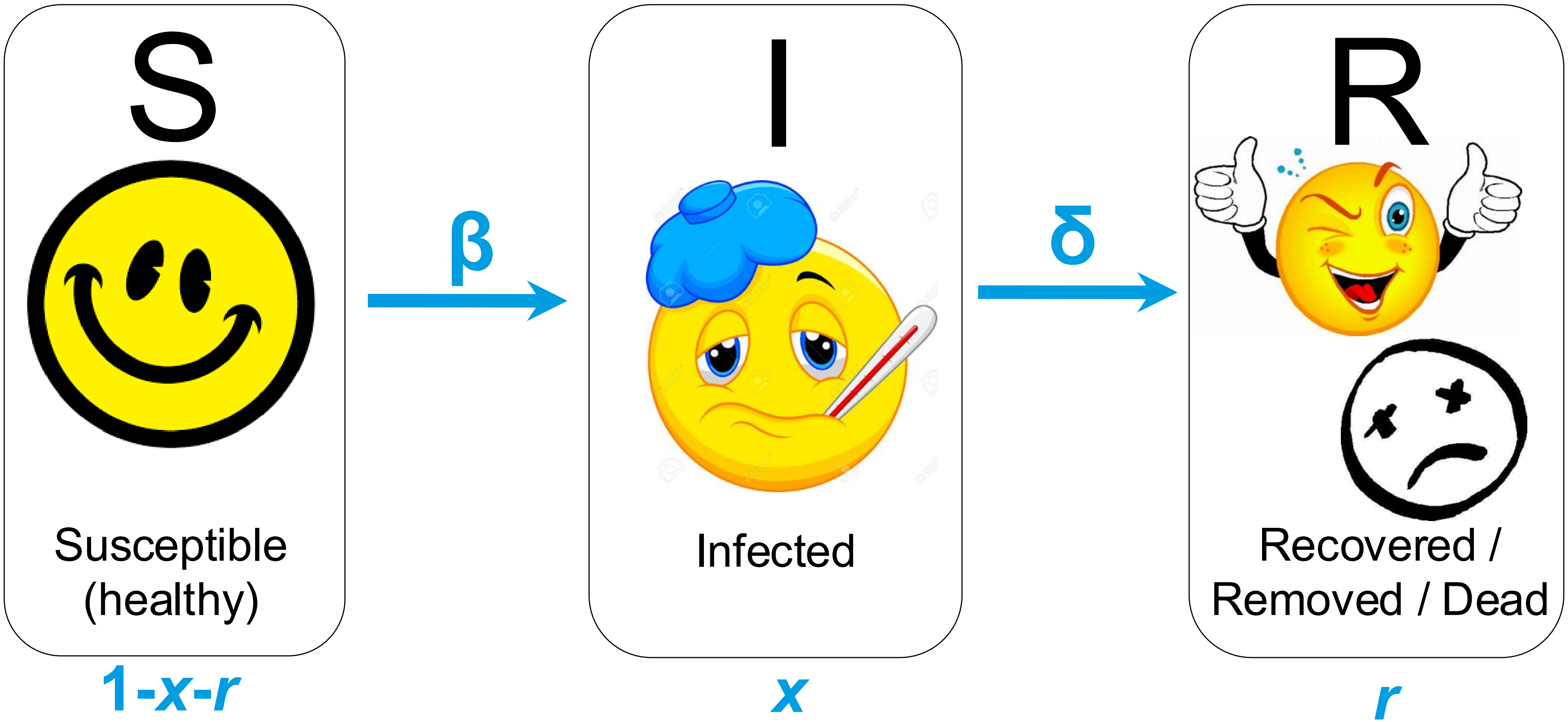


$$R_0^{SIS,SIR} \square \frac{\beta\langle k\rangle}{\delta}$$

SIR model

β: contact infection rate
δ: recovery rate

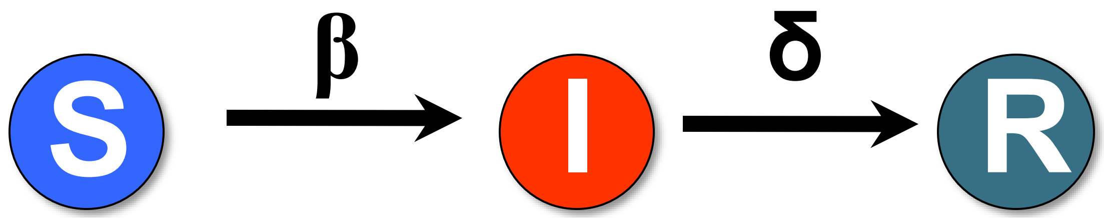
Often individuals develop immunity after recovery
(e.g., Influenza) or can be removed from the population.



Traditional models

β: contact infection rate
δ: recovery rate

SIR model



$$\dot{r} = \delta x$$

$$\dot{x} = \beta \langle k \rangle x(1 - x - r) - \delta x$$

<k> – average number of contacts of a given individual
x – fraction of infected in the population
r – fraction of recovered in the population
y = 1-x-r – fraction of susceptible in the population

Traditional models

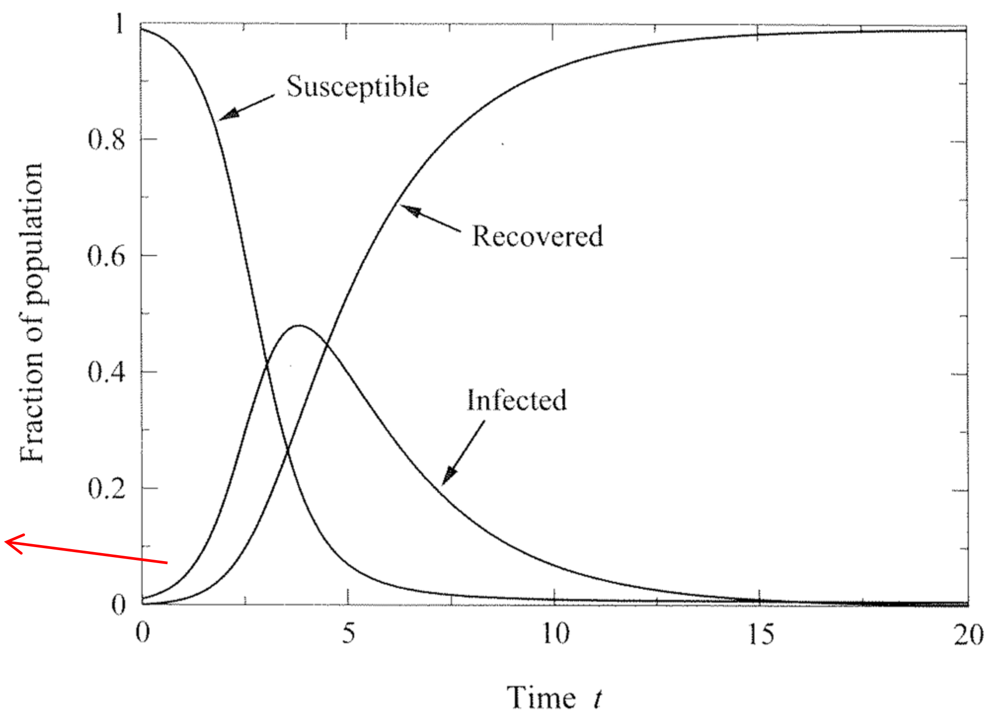
β: contact infection rate
δ: recovery rate

SIR model

$$S \xrightarrow{\beta} I \xrightarrow{\delta} R$$

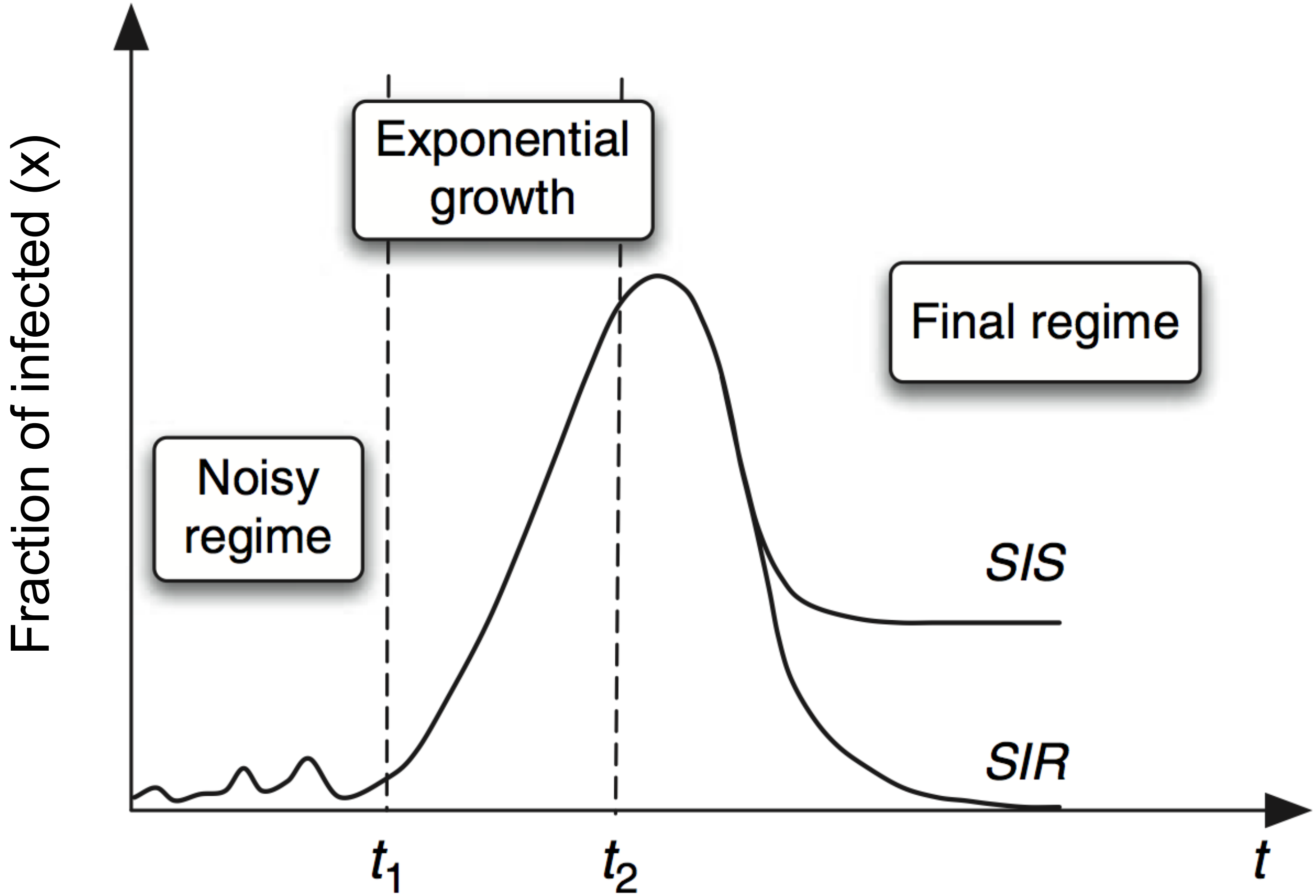
$$\dot{x} = \beta \langle k \rangle x(1 - x - r) - \delta x$$

$$\dot{r} = \delta x$$



$$R_0 = \frac{\beta \langle k \rangle}{\delta} > 1$$

Stochastic effects

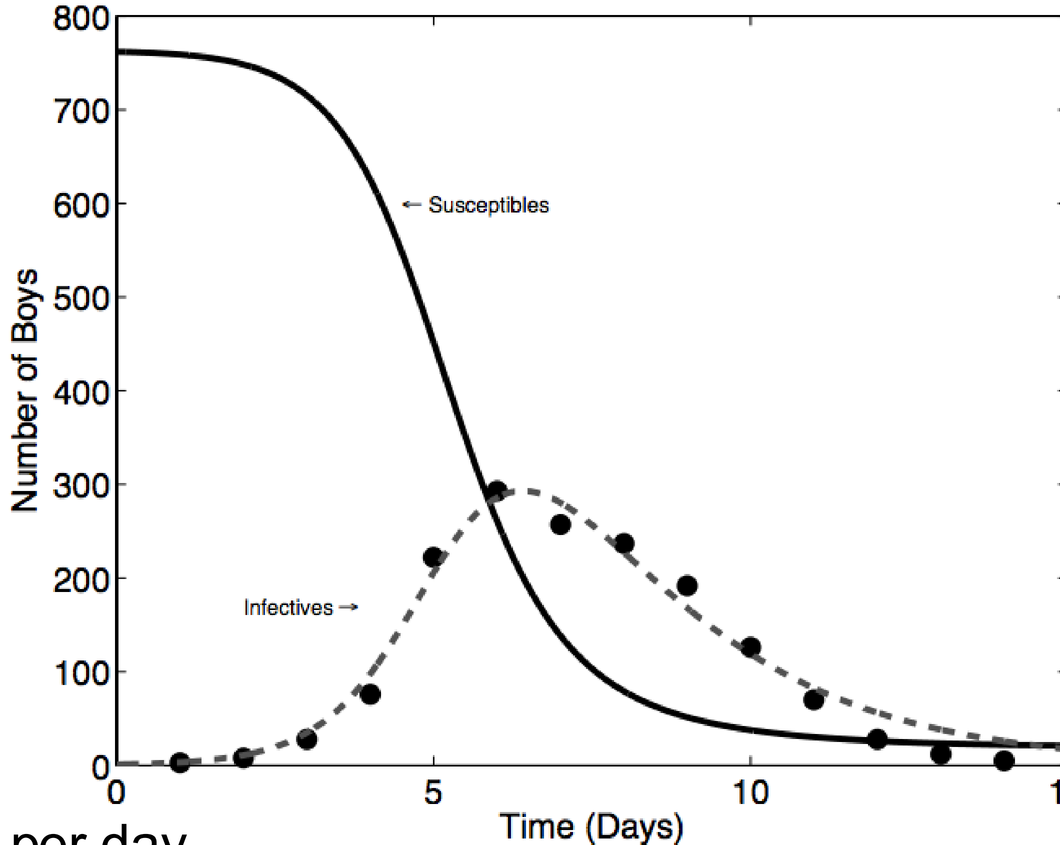


Working SIR example

- Outbreak of influenza in a British boarding school in 1978.
- Soon after the start of the Easter term, three boys were reported to the school infirmary with the typical symptoms of influenza.
- Over the next few days, a very large fraction of the 763 boys in the school had contracted the infection

Working SIR example

Within two weeks, the infection had become extinguished, as predicted by the simple SIR model



Estimated parameters:
transmission rate (β<k>) =1.66 per day
$1/\delta$ = 2.2 days,
giving an R_0 of 3.65.

(Murray 1989)
Keeling & Rohani, Chapter 2