Games on Graphs

Giuseppe Perelli

giuseppeperelli.github.io
perelli@diag.uniroma1.it

Sapienza University of Rome

Reasoning Agents Seminar

Where do games come from?

Games on graphs are a useful mathematical model of many phenomena in computer science, economics, biology, that involve dynamic interaction among two or more agents.

Main idea: Capture strategic interaction

Model: Two players moving a token along the edges of a graph, resulting in a infinite (sometime finite) path called a play. Player 0 is trying to ensure the play satisfies a given property, and player 1 is trying to fail such property.

No playing around: game theory is serious business!

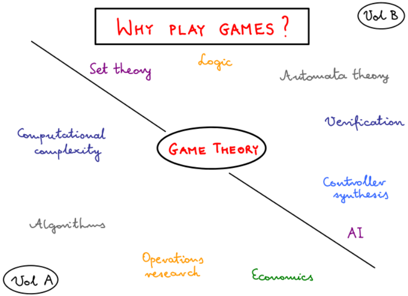
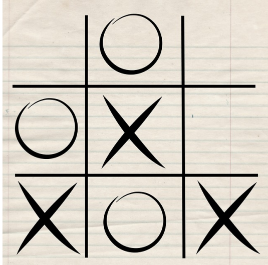
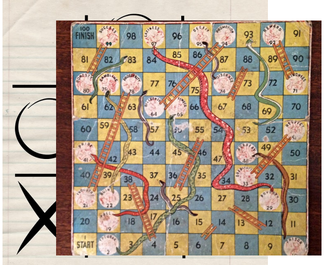


Image credits: Martin Zimmerman

- ▷ It's fun!
- ▷ Model reactive systems
- ▷ Solve synthesis problems
- ▷ Evaluate logic formulas

Examples of games




100
FINISH
START

Examples of games

Examples of games

Synthesis

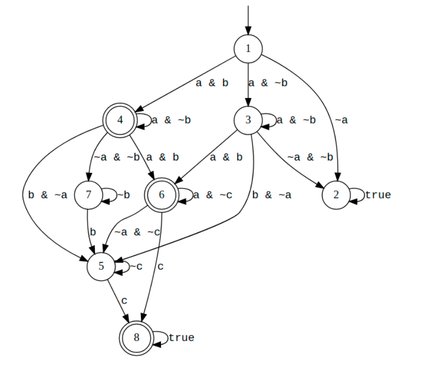


Image credits: ltlf2dfa.diag.uniroma1.it

Classification of games

Players

- 1 player;
- 2 players;
- multi-players.

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Interaction
- Turn-based;
- Concurrent.

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- Reachability;
- Safety;
- Buchi;
- co-Buchi;
- ...

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Today

2-player turn-based perfect information games.

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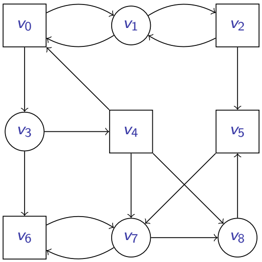
Today

2-player turn-based perfect information games.

Possible projects

See the blue and red.

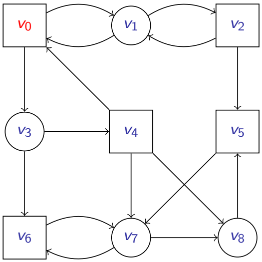


A Game is played over a (finite) graph (V, E), whose vertexes are under the control of the two players $\mathrm{V} = \mathrm{V}_0 \cup \mathrm{V}_1$.

A token moves along the vertexes and sent to a successor by the controlling player.

The outcome or play is an infinite sequence of vertexes in the graph.

A winning condition/objective is a subset $\mathsf{Obj} \subseteq \mathrm{V}^\omega$ of plays that Player 0 wants to occur.



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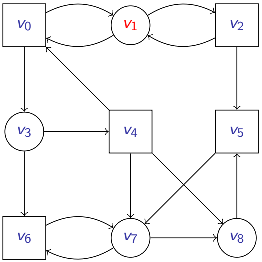
Sample play

$\pi = v_0$



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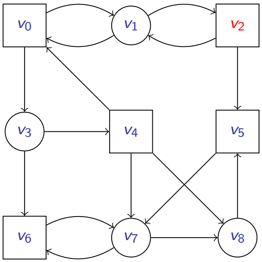
Sample play

$\pi = v_0 \cdot v_1$



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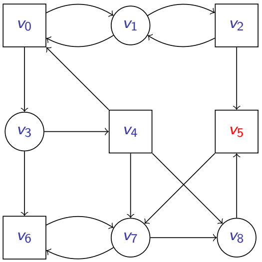
$\pi = v_0 \cdot v_1 \cdot v_2$



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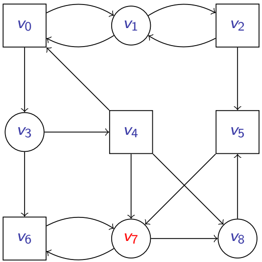

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Sample play

$\pi = v_0 \cdot v_1 \cdot v_2 \cdot v_5 \cdot v_7 \cdot \ldots \in \mathrm{V}^\omega$

Type of objectives

For a subset of the vertexes $T \subseteq \mathrm{V}$:

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Question: what if we have more alternations of existential and universal quantifiers?

Strategies

A strategy maps partial outcomes (i.e., finite sequences of vertexes) into successors and it is of the form

Player 0 strategy $\sigma_0 : \mathrm{V}^* \cdot \mathrm{V}_0 \to \mathrm{V}$

Player 1 strategy $\sigma_1 : \mathrm{V}^* \cdot \mathrm{V}_1 \to \mathrm{V}$

Consistent plays

Strategies "restricts" the game only to those play π that are consistent with σ_0, that is such that $\pi_{i+1} = \sigma_0(\pi_0 \cdot \pi_1 \cdot \ldots \cdot \pi_i)$, if $\pi_i \in \mathrm{V}_0$.
For given strategies σ_0, σ_1, there is only one consistent play starting from v.

Winning strategies

A strategy σ_0 is winning for Player 0 in v if every consistent path π starting from v belongs to Obj. (Winning set $\mathrm{Win}_0 \subseteq V$)

A strategy σ_1 is winning for Player 1 in v if every consistent path π starting from v does not belong to Obj. (Losing set $\mathrm{Win}_1 \subseteq V$)

Solving a game

The solution of a game $\mathbf{G}$ is the set Win_0 of vertexes that are winning for Player 0.

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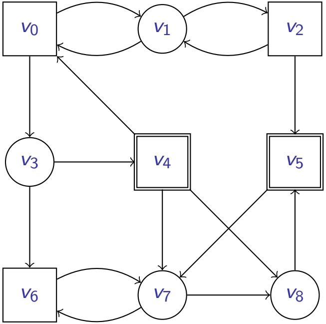
Solving a game

The solution of a game **G** is the set Win_0 of vertexes that are winning for Player 0.

Warning! While $\mathrm{Win}_0 \cap \mathrm{Win}_1 = \emptyset$, it is not always the case that $\mathrm{V} = \mathrm{Win}_0 \cup \mathrm{Win}_1$.

Consider again the arena below and let $T = \{v_4, v_5\}$ (the double bordered nodes).



What is the winning set of $\mathbf{G}$?

Consider the function force_0 defined as follows:

$$\text{force}_0(X) = \{v \in V_0 : E(v) \cap X \neq \emptyset\} \cup \{v \in V_1 : E(v) \subseteq X\}$$

Player 0 has a move to enter the region X;

Player 1 cannot avoid to enter the region X.

The function computes the vertexes from which Player 0 can enforce the token to move in the subset X of vertexes.

Constrained problem

$\mathsf{Reach}^{n}(\mathbf{G}) :=$ "Player 0 can reach T in at most n moves".

$n = 0$: T I have to be in T already.

$$\mathsf{Reach}^{0}(\mathbf{G}) = T$$

$n > 0$: either I am or can force to a vertex winning in at most $n - 1$ moves.

$$\mathsf{Reach}^{n}(\mathbf{G}) = \mathsf{Reach}^{n-1}(\mathbf{G}) \cup \mathsf{force}_{0}(\mathsf{Reach}^{n-1}(\mathbf{G}))$$

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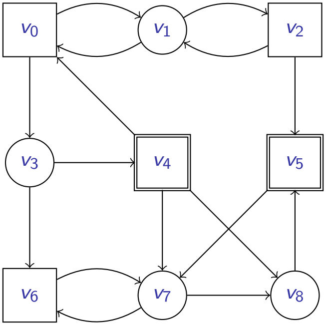
Fix-point calculation $\mu X.(T \cup \text{force}_0(X))$

Algorithm 1 Reachability game

1: $\mathsf{Win} := T$
2: **while** $\mathsf{Win} \neq \mathsf{Win} \cup \mathsf{force}_{\mathrm{o}}(\mathsf{Win})$ **do**
3: $\quad \mathsf{Win} := \mathsf{Win} \cup \mathsf{force}_{\mathrm{o}}(\mathsf{Win})$
4: **end while**
5: **return** Win

Question: What is the complexity this procedure?
Question: Can we do (theoretically) better?

Sample Execution



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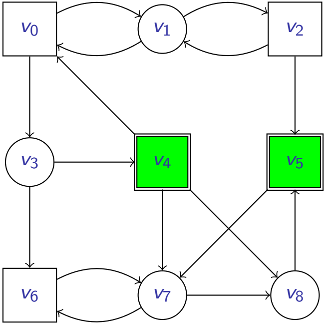

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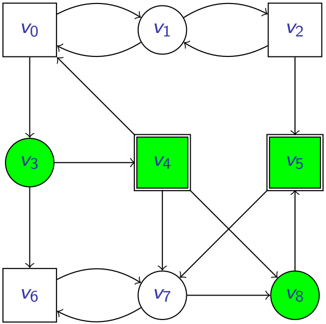

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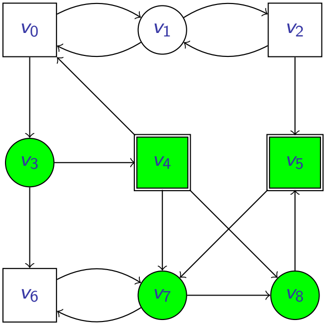

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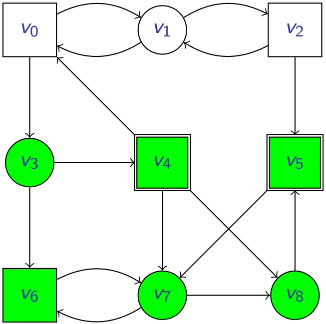

Memoryless strategy

A strategy σ_0 is memoryless if it is of the form

$$\sigma_0 : \cancel{\mathrm{V}^*\cdot}\mathrm{V}_0 \to \mathrm{V}$$

that is, at every vertex v, the next move does not depend on the past history (and thus it is always the same).

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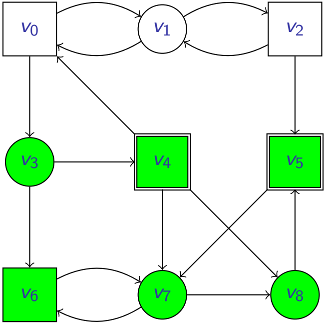
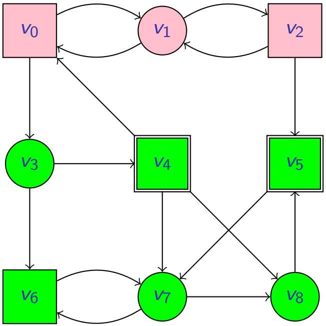
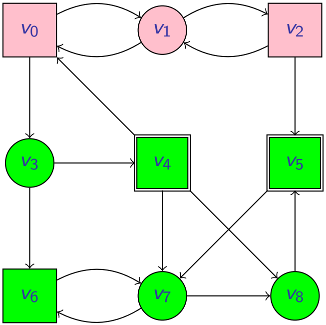
Theorem (Memoryless)

If $v \in \mathrm{Win}_o$, then there exists a memoryless strategy σ_0 that is winning from v.

Determinacy



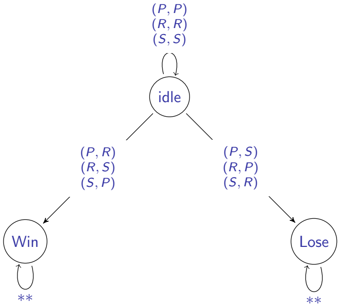

v_0
v_1
v_2
v_3
v_4
v_5
v_6
v_7
v_8



It holds that $\mathrm{Win}_0 \cup \mathrm{Win}_1 = \mathrm{V}$. When this is the case, we say that the game is determined.

Theorem (determinacy)

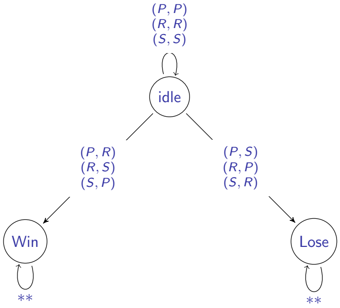
Every 2-player turn-based reachability game is determined.



A Concurrent Game is played over a structure $(\mathrm{V}, \mathrm{Ac}, \mathsf{tr})$ a set of actions Ac and a transition function $\mathsf{tr} : \mathrm{V} \times (\mathrm{Ac} \times \mathrm{Ac}) \to \mathrm{V}$

The token is sent to a successor by a coordinated action, following the transition function.

Plays and objectives are as for turn-based games.



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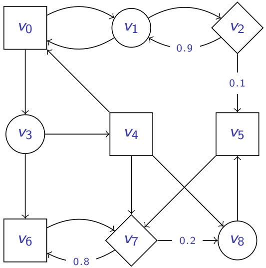
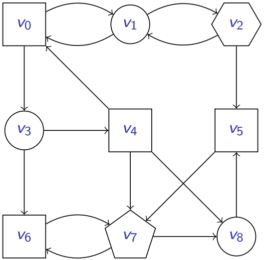
Plays and objectives are as for turn-based games.

Concurrent games are not determined!



A Stochastic Game is played by 2 players over (V, E) but $\mathrm{V} = \mathrm{V}_0 \cup \mathrm{V}_1 \cup \mathrm{V}_s$.

Player 0 has to maximize the probability of winning its objective.



A Multi-player Game is played by n players over (V, E) with $\mathrm{V} = \mathrm{V}_0 \cup \ldots \mathrm{V}_{n-1}$.

Player i is assigned an objective $\mathsf{Obj}_i \subseteq \mathrm{V}^\omega$.

Rather than winning strategies, we are interested in finding equilibria.

Project ideas

Implement, and validate algorithms for solving extensions of 2-player turn-based reachability games towards different directions

- Objective: Buchi, co-Buchi, ...
- Interaction: Concurrent games;
- Number of players: Multi-player games;
- Nature: Stochastic games.

Feel free to get in touch with me to discuss details of this perelli@diag.uniroma1.it