Robotics: Composition of poses and landmarks

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November 2018

1 Introduction

The purpose of this exercise is to get familiar with the process of observing landmarks from robot poses. The main tools for that are:

- the **composition of two poses** and the **composition of a pose and a landmark**.
- the **propagation of uncertainty** through the Jacobians of these compositions.

We will address several problems in an incremental complexity way. The following figures will help you to follow the exercise.

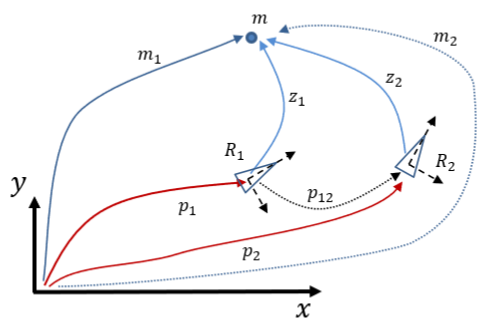

Figure 1: Composition of poses and landmarks

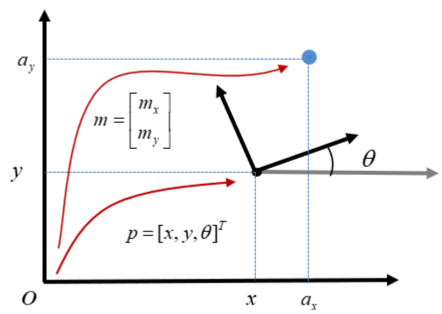

Figure 2: Composition of a pose and a landmark point

- Robot/Sensor pose: $p = [x, y, \theta]^T$
- Landmark observation: $z_c = \begin{bmatrix} z_x \\ z_y \end{bmatrix} = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$
- $m = p \oplus z_c = f(p, z_c) = \begin{bmatrix} x + z_x\cos\theta - z_y\sin\theta \\ y + z_x\sin\theta - z_y\cos\theta \end{bmatrix}$

2 First exercise

Let's consider a robot $R1$ at a perfectly known pose $p1 = [1, 2, 0.5]^T$ which observes a landmark m with a range-bearing (polar) sensor affected by a zero-mean Gaussian error with covariance `W1p = diag([0.25, 0.04])`. The sensor provides the measurement $z_{1p} = [4m., 0.7rad.]^T$. Compute the Gaussian probability distribution (mean and covariance) of the landmark in the world frame (the same as the robot) and plot its corresponding ellipse (in magenta, $\sigma = 1$).

Hint: Prior to propagate the measurement uncertainty, we need to compute the covariance of the observation in the cartesian robot $R1$ frame:

$$z_c = \begin{bmatrix} z_x \\ z_y \end{bmatrix} = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} = f(r, \alpha)$$

Solution

To follow the code we will use the following suffixes to name variables:

```
% _w: world reference frame
% _r: robot reference frame
% Other codes:
% p: in polar coordinates
% c: in cartesian coordinates
% e.g. z1p_r represents an observation z1 in polar (robot frame)
```

First of all we need to convert the given polar coordinates z_{1p} into cartesian (in the robot frame). To do that we use the hint supplied in the problem definition.

```
% Landmark
z1p_r = [4,0.7]'; % Measurement/observation (polar)

% 1. Convert polar coordinates to cartesian (in the robot frame)
z1xc_r = z1p_r(1) * cos(z1p_r(2)); % r * cos alfa
z1yc_r = z1p_r(1) * sin(z1p_r(2)); % r * sen alfa
zc_r = [z1xc_r, z1yc_r]'; % Landmark position in cartesian (robot frame)
```

Secondly, we obtain the sensor covariance in cartesian coordinates in the robot frame. For that we need the Jacobian built from the expression that converts from polar to cartesian coordinates.

```
r = z1p_r(1); % Useful variables
alpha = z1p_r(2);
c = cos(alpha);
s = sin(alpha);

J_pc = [c -r*s; s r*c]; % Build the Jacobian

Wzc_r = J_pc*W1p_r*J_pc'; % Sensor covariance in cartesian
```

Now we can compute the sensor measurement in the world's coordinate system (mean and covariance). For do that, we use the supplied function `tcomp.m` which calculates de composition of transformations given by poses and `J1, J2` to calculate `J_ap` and `J_aa`.

```
z1_w = tcomp(p1_w, [zc_r; 1]) % Compute coordinates of the landmark in the world. The '1' has no meaning

% Now build the Jacobians for obtain the covariance matrix
J_ap = J1(p1_w, zc_r);
J_aa = J2(p1_w, z1_w);

Wzc_w = J_ap*Qp1_w*J_ap' + J_aa*Wzc_r*J_aa' % Finally, propagate the covariance!
```

Results

```
z1_w = [2.4494, 5.7282, 1.5000]'
```

$$\texttt{Wzc\_w} = \begin{bmatrix} \texttt{0.5888} & \texttt{-0.1317} \\ \texttt{-0.1317} & \texttt{0.3012} \end{bmatrix}$$

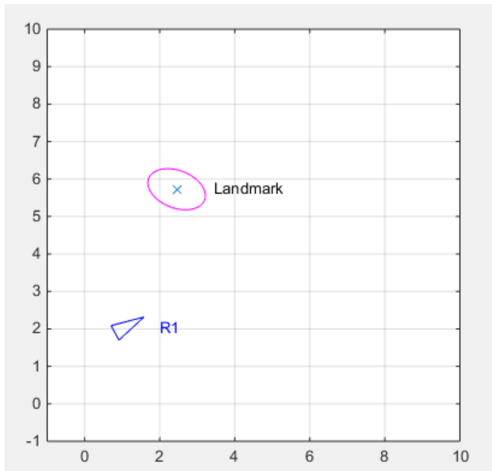

Figure 3: Gaussian probability distribution of the landmark. Ellipse covariance centered at the mean

Conclusions

Note that in the use of the supplied functions `J1` and `J2`, we only make use of the 2×3 and 2×2 submatrices respectively.

We have used the pose composition (`tcomp`):

$$a = p \oplus a' = f(p, a') = z_c = \begin{bmatrix} x + a'_x \cos\theta - a'_y \sin\theta \\ y + a'_x \sin\theta - a'_y \cos\theta \end{bmatrix}$$

to obtain the position of a landmark in cartesian coordinates in the world frame and using the jacobians (`J1, J2`):

$$\frac{\partial a}{\partial p} = \frac{\partial f(p,a')}{\partial p} = \frac{\partial\{a_x, a_y\}}{\partial\{x, y, \theta\}} = \begin{bmatrix} 1 & 0 & -a'_x \sin\theta - a'_y \cos\theta \\ 0 & 1 & a'_x \cos\theta - a'_y \sin\theta \end{bmatrix}$$

$$\frac{\partial a}{\partial a'} = \frac{\partial f(p,a')}{\partial a'} = \frac{\partial\{a_x, a_y\}}{\partial\{a'_x, a'_y\}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

to propagate the covariance in the expression.

```
Wzc_w = J_ap*Qp1_w*J_ap' + J_aa*Wzc_r*J_aa'
```

In summary, mean and covariance has been calculated using the next mathematical background:

- Mean: $\bar{m}^i = \bar{p}^i \oplus \bar{z}^i$

- Covariance: $\underbrace{\Sigma_{m^i}}_{2\times 2} = \overbrace{\underbrace{\frac{\partial m^i}{\partial p_i}}_{2\times 3} \Sigma_i (\frac{\partial m^i}{\partial p_i})^T}^{\text{uncertainty in } m^i \text{ due to } \Sigma_i} + \overbrace{\underbrace{\frac{\partial m^i}{\partial z_i}}_{2\times 2} Q_i (\frac{\partial m^i}{\partial z_i})^T}^{\text{uncertainty in } m^i \text{ due to } Q_i}$

Where $m_i = p_i \oplus z^i = f(p_i, z^i)$, $m^i \sim N(\bar{m}^i, \Sigma_{m^i})$ and Q_i is the covariance of the landmark in the robot frame.

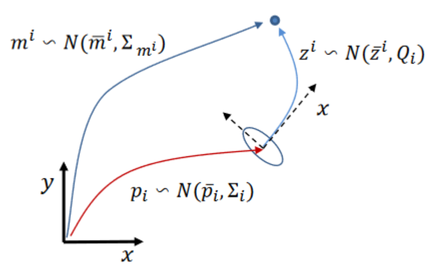

Figure 4: Covariance of an observed landmark

3 Second exercise

Now, let's assume that the robot pose is not known, but a RV that follows a Gaussian probability distribution: $p1 \sim N([1, 2, 0.5]T, \Sigma_1)$ with Σ_1 =`diag([0.08,0.6, 0.02])`.

- Compute the covariance matrix Σ_{m1} of the landmark in the world frame and plot it as an ellipse centered at the mean $m1$ (in blue, sigma = 1). Plot also the covariance of the robot pose (in blue, sigma = 1).

- Compare the covariance with that obtained in the previous case. Is it bigger? Is it bigger than that of the robot? Why?

Solution

Now, we have uncertainty in the robot pose given by the covariane matrix `Qp1_w`. And again, we propagate the covariances but using this new covariance matrix:

```
% Now, we have uncertainty in the robot pose!
Qp1_w = diag([0.08, 0.6, 0.02]); % New R1 covariance diag
    (x, y, theta)

% Propagate the covariances again, taken into account
    this new info.
Wzc_w = J_ap*Qp1_w*J_ap' + J_aa*Wzc_r*J_aa'
```

Results

$$\texttt{Wzc\_w} = \begin{bmatrix} 0.9468 & -0.2398 \\ -0.2398 & 0.9432 \end{bmatrix}$$

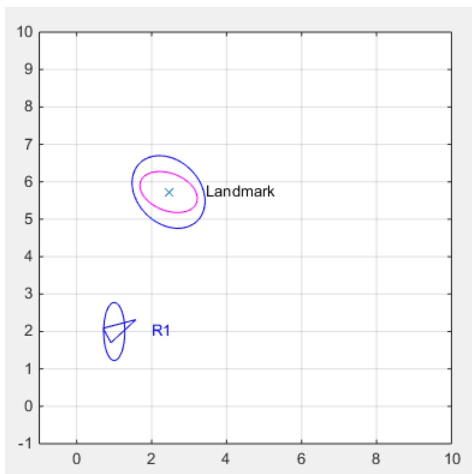

Figure 5: Covariance of an observed landmark and robot pose. New uncertainty represented by the blue ellipse around the landmark

Conclusions

In this case, we see how the ellipse that represents the covariance matrix of the landmark in the world frame got bigger as we add uncertainty in the robot pose.

Note that this new covariance is also bigger than that of the robot because:

$$\text{Covariance: } \underbrace{\Sigma_{m^i}}_{2\times 2} = \overbrace{\underbrace{\frac{\partial m^i}{\partial p_i}}_{2\times 3} \Sigma_i (\frac{\partial m^i}{\partial p_i})^T}^{\text{uncertainty in } m^i \text{ due to } \Sigma_i} + \overbrace{\underbrace{\frac{\partial m^i}{\partial z_i}}_{2\times 2} Q_i (\frac{\partial m^i}{\partial z_i})^T}^{\text{uncertainty in } m^i \text{ due to } Q_i}$$

That is, Σ_{m^i} is the sum of two covariance matrices and the uncertainties are added resulting in a bigger ellipse.

In summary, if we don't know exactly where the robot is, we don't know from where exactly it is taken the measurement. In consequence, the landmark has more possibles positions in which to be.

4 Thrid exercise

Another robot $R2$ is at pose $p2 \sim ([6m., 4m., 2.1rad.]^T, \Sigma_2)$ with $\Sigma_2 =$ `diag([0.20,0.09, 0.03])`. Plot $p2$ and its ellipse (covariance) in green (sigma=1). Compute the relative pose $p_{1,2}$ between $R1$ and $R2$. For that, take a look at the file "Clarifying the relative pose between to poses" and implement the two possible ways to obtain such pose.

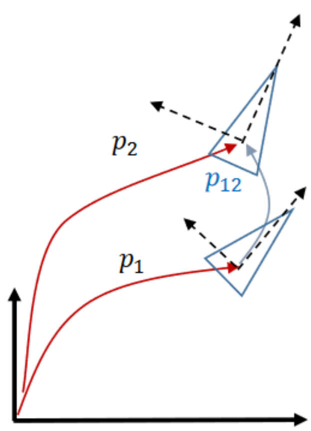

Figure 6: Relative pose $p_{1,2}$ between the poses p_1 and p_2

Solution

There are two ways for computing the relative pose $p_{1,2}$ between $R1$ and $R2$.

- Through the composition of poses, but using $\ominus p_1$ instead of p_1
- Using the inverse composition of poses: $p_{1,2} = \ominus p1 \oplus p2 = p2 \ominus p1$

Using the composition of poses with the inverse pose, we calculate the inverse pose `p1inv_w` making use of the supplied function `tinv`, that calculates the inverse of one or more transformations. Then we use the supplied function `tcomp` to compose the inverse pose just computed with the pose of the second robot p_2 and we obtain the mean.

To obtain the covariance, we compute Qinvp1_w using the supplied funtion Jinv, that returns the Jacobian of the inverse of a pose. Once we have Qinvp1_w, we can sum the covariances and obtain Qp12_w.

```
% First way: composition of poses with inverse pose
p1inv_w = tinv(p1_w);
p12_w = tcomp(p1inv_w, p2_w)

Qinvp1_w = Jinv(p1_w)*Qp1_w*Jinv(p1_w)';
Qp12_w = J1(p1inv_w,p2_w) * Qinvp1_w * J1(p1inv_w,p2_w)'
    + J2(p1inv_w,p2_w) * Qp2_w * J2(p1inv_w,p2_w)'
```

On the other hand, using the inverse composition, we calculate the mean as $p_2 \ominus p_1$ given by the matrix p12_w2. Next we need to build two jacobians $\frac{\partial p_{1,2}}{\partial p_1}$ and $\frac{\partial p_{1,2}}{\partial p_2}$, implemented with variable names J_p12p1 and Jp12p2. Finally, we can add them and obtain the convariance.

```
% Second way: Inverse Composition
c = cos(p1_w(3)); % Useful variables
s = sin(p1_w(3));
xp1 = p1_w(1); yp1 = p1_w(2);
xp2 = p2_w(1); yp2 = p2_w(2);

p12_w2 = [(xp2-xp1)*c+(yp2-yp1)*s;
          (yp2-yp1)*c-(xp2-xp1)*s;
           p2_w(3)-p1_w(3)]

J_p12p1 = [-c, -s, -(xp2-xp1)*s + (yp2 - yp1)*c;
           s, -c, -(xp2-xp1)*c - (yp2 - yp1)*s;
           0, 0, -1];

Jp12p2 = J2(p1_w, p2_w)';

Qp12_w2 = J_p12p1*Qp1_w*J_p12p1' + Jp12p2*Qp2_w*Jp12p2'
```

Results

p12_w = [5.3468, -0.6420, 1.6000]'

$$\texttt{Qp12\_w = =} \begin{bmatrix} 0.3825 & 0.2411 & 0.2411 \\ 0.2411 & 1.1675 & 0.1069 \\ 0.0128 & 0.1069 & 0.0500 \end{bmatrix}$$

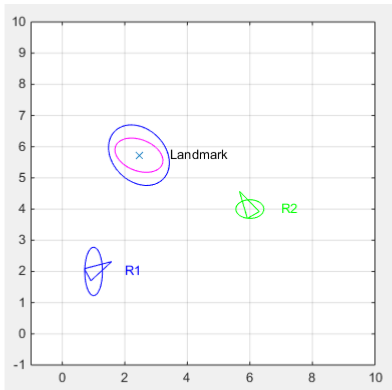

Figure 7: Introduction of a robot $R2$ with uncertainty in the pose

Conclusions

As for the first form of calculate the relative pose $p_{1,2}$ (using the inverse of a pose), we calculate the probability distribution as:

$$\text{Mean: } p_{1,2} = \ominus p_1 \oplus p_2 = f(\ominus p_1, p_2) = \begin{bmatrix} x_{\ominus p_1} + x_{p_2}\cos\theta_{\ominus p_1} - y_{p_2}\sin\theta_{\ominus p_1} \\ y_{\ominus p_1} + x_{p_2}\sin\theta_{\ominus p_1} + y_{p_2}\cos\theta_{\ominus p_1} \\ \theta_{\ominus p_1} + \theta_{p_2} \end{bmatrix}$$

$$\text{Covariance: } \Sigma_{p_{1,2}} = \underbrace{\frac{\partial p_{1,2}}{\partial \ominus p_1}\frac{\partial \ominus p_1}{\partial p_1}\Sigma_{p_1}\frac{\partial \ominus p_1}{\partial p_1}^T\frac{\partial p_{1,2}}{\partial \ominus p_1}^T}_{\text{chain rule}} + \frac{\partial p_{1,2}}{\partial p_2}\Sigma_{p_2}\frac{\partial p_{1,2}}{\partial p_2}^T$$

$$\Sigma_{p_{1,2}} = \frac{\partial p_{1,2}}{\partial \ominus p_1}\Sigma_{\ominus p_1}\frac{\partial p_{1,2}}{\partial \ominus p_1}^T + \frac{\partial p_{1,2}}{\partial p_2}\Sigma_{p_2}\frac{\partial p_{1,2}}{\partial p_2}^T$$

Being:

$$\frac{\partial p_{1,2}}{\partial \ominus p_1} = \begin{bmatrix} 1 & 0 & -x_{p_2}\sin\theta_{\ominus p_1} - y_{p_2}\cos\theta_{\ominus p_1} \\ 0 & 1 & x_{p_2}\cos\theta_{\ominus p_1} - y_{p_2}\sin\theta_{\ominus p_1} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial \ominus p_1}{\partial p_1} = \begin{bmatrix} -\cos\theta_{p_1} & -\sin\theta_{p_1} & x_{p_1}\sin\theta_{p_1} - y_{p_1}\cos\theta_{p_1} \\ \sin\theta_{p_1} & -\cos\theta_{p_1} & x_{p_1}\cos\theta_{p_1} + y_{p_1}\sin\theta_{p_1} \\ 0 & 0 & -1 \end{bmatrix}$$

$$\frac{\partial p_{1,2}}{\partial p_2} = \begin{bmatrix} \cos\theta_{\ominus p_1} & -\sin\theta_{\ominus p_1} & 0 \\ \sin\theta_{\ominus p_1} & \cos\theta_{\ominus p_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Sigma_{\ominus p_1} = \frac{\partial \ominus p_1}{\partial p_1}\Sigma_{p_1}\frac{\partial \ominus p_1}{\partial p_1}^T$$

For the other approach (inverse composition of poses), we calculate the probability distribution as:

Mean: $p_{1,2} = \ominus p_1 \oplus p_2 = p2 \ominus p1 f(p_1, p_2) =$

$$= \begin{bmatrix} (x_{p_2} - x_{p_1})\cos\theta_{p_1} + (y_{p_2} - y_{p_1})\sin\theta_{p_1} \\ (x_{p_2} - x_{p_1})\sin\theta_{p_1} - (y_{p_2} - y_{p_1})\cos\theta_{p_1} \\ \theta_{p_2} - \theta_{p_1} \end{bmatrix}$$

$$\text{Covariance: } \Sigma_{p_{1,2}} = \underbrace{\frac{\partial p_{1,2}}{\partial p_1}\frac{\partial p_1}{\partial \ominus p_1}\Sigma_{\ominus} p_1 \frac{\partial p_1}{\partial \ominus p_1}^T \frac{\partial p_{1,2}}{\partial p_1}^T}_{\text{chain rule}} + \frac{\partial p_{1,2}}{\partial p_2}\Sigma_{p_2}\frac{\partial p_{1,2}}{\partial p_2}^T$$

$$\Sigma_{p_{1,2}} = \frac{\partial p_{1,2}}{\partial p_1}\Sigma_{p_1}\frac{\partial p_{1,2}}{\partial p_1}^T + \frac{\partial p_{1,2}}{\partial p_2}\Sigma_{p_2}\frac{\partial p_{1,2}}{\partial p_2}^T$$

Being:

$$\frac{\partial p_{1,2}}{\partial p_1} = \begin{bmatrix} -\cos\theta_{p_1} & -\sin\theta_{p_1} & -(x_{p_2} - x_{p_1})\sin\theta_{p_1} + (y_{p_2} - y_{p_1})\cos\theta_{p_1} \\ \sin\theta_{p_1} & -\cos\theta_{p_1} & -(x_{p_2} - x_{p_1})\cos\theta_{p_1} - (y_{p_2} + y_{p_1})\sin\theta_{p_1} \\ 0 & 0 & -1 \end{bmatrix}$$

$$\frac{\partial p_{1,2}}{\partial p_2} = \begin{bmatrix} \cos\theta_{p_1} & \sin\theta_{p_1} & 0 \\ -\sin\theta_{p_1} & \cos\theta_{p_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial p_1}{\partial \ominus p_1} = \left(\frac{\partial \ominus p_1}{\partial p_1}\right)^{-1}$$

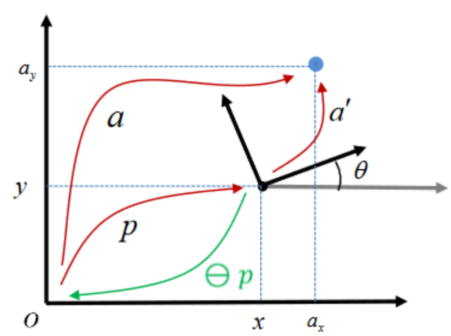

Figure 8: Inverse composition of a pose and a point $a' = \ominus p \oplus a$

5 Fourth exercise

According to the information that we have about the position of the landmark m in the world coordinates (provided by $R1$), compute the predicted observation distribution of $z2p = [r, \alpha] \sim N([z2p, W2p)$ by a range-bearing sensor from $R2$.

Hint: We need to compute the covariance of the predicted observation in polar coordinates ($W2p$). For that, use the following Jacobian:

$$\frac{\partial p}{\partial c} = \begin{bmatrix} \cos(\alpha+\theta) & \sin(\alpha+\theta) \\ -\sin(\alpha+\theta)/r & \cos(\alpha+\theta)/r \end{bmatrix}$$

Solution

We have to calculate r and α for taking the measurement using the range-bearing model. Next, we sum the θ of pose p_2 to α and use de jacobian supplied for obtain polar coordenates. Finally, propagate the uncertainty to polar coordinates in the robot frame.

```
% 1. Take a measurement using the range-bearing observation model!
r2 = sqrt( (z1_w(1)-xp2)^2 + (z1_w(2)-yp2)^2 );
alpha2 = atan2([z1_w(2), yp2], [z1_w(1), xp2]);

alpha2(:,1) = []

z2p_r = [r2,alpha2]'

% 2. Jacobian from cartesian to polar at z2p_r when the covariance is in
% global coordianes
alpha = alpha2 + p2_w(3);
Jcp_p2 = [cos(alpha) sin(alpha);
          -sin(alpha)/r2 cos(alpha)/r2];

% 3. Finally, propagate the uncertainty to polar coordinates in the
% robot frame
W2_p = Jcp_p2 * Wzc_w * Jcp_p2'  %dim: 2x2
```

Results

```
z2p_r = [3.9488, 0.5880]'
```

$$\texttt{W2\_p} = \begin{bmatrix} 1.1350 & -0.0371 \\ -0.0371 & 0.0484 \end{bmatrix}$$

Conclusions

To calculate `z2p_r = [r2,alpha2]'` it was used

$$z_i = \begin{bmatrix} d_i \\ \theta_i \end{bmatrix} = h(x, m_i) + w_i = \begin{bmatrix} \sqrt{(x_i - x)^2 + (y_i - y)^2} \\ atan2(\frac{y_i - y}{x_i - x}) - \theta \end{bmatrix} + w_i$$

being $h(x, m_i)$ the predicted value of z_i given x and m.

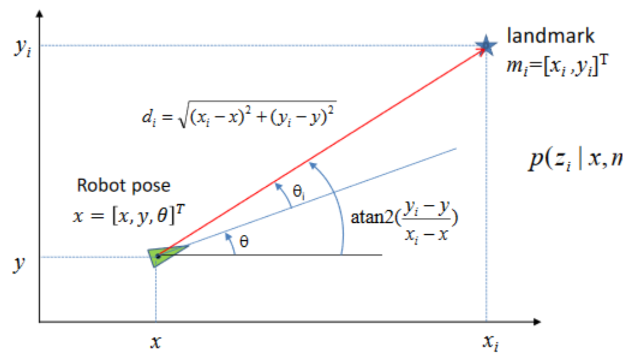

Figure 9: Distance and bearing from robot pose

6 Fifth exercise

Assume now that a measurement $z2 = [4m., 0.3rad.]^T$ of the landmark is taken from $R2$ with a sensor having the same precision as that of $R1$ ($W2p = W1p$).

- What is the pdf of the observed landmark according to this observation? Plot the corresponding ellipse (in green, sigma=1).
- Two different pdf's are now associated to the same landmark.

  Is that a contradiction?

  Can you work out a solution that combines these two "pieces of information"? Plot it (in red).

Solution

The pdf of the observed landmark according to this observation can be computed following the same procedure described in the first exercise, that is, converting polar coordinates to cartesian (in the robot frame), obtain the covariance matrix in cartesian coordinates in the frame of the robot using the jacobians and compute the sensor measurement in the world's coordinate system (mean and covariance).

Next, we combine the measurements of both sensors (product of gaussians):

```
% Combine the measurements from both sensors!
Wz_w = inv(inv(Wzc_w) + inv(Wz2c_w)) % sigma
```

```
z_w = Wz_w*inv(Wzc_w)*z1_w([1,2]) + Wz_w*inv(Wz2c_w)*z2_w
    ([1,2]) % mu
```

Results

$$\texttt{z2\_w = [3.0504, 6.7019, 3.1000]'}$$

$$\texttt{Wz2c\_w = =} \begin{bmatrix} \texttt{0.8469} & \texttt{0.4333} \\ \texttt{0.4333} & \texttt{0.8131} \end{bmatrix}$$

$$\texttt{z\_w = [2.5876, 6.1553]'}$$

$$\texttt{Wz\_w = =} \begin{bmatrix} \texttt{0.3797} & \texttt{0.0777} \\ \texttt{0.0777} & \texttt{0.3700} \end{bmatrix}$$

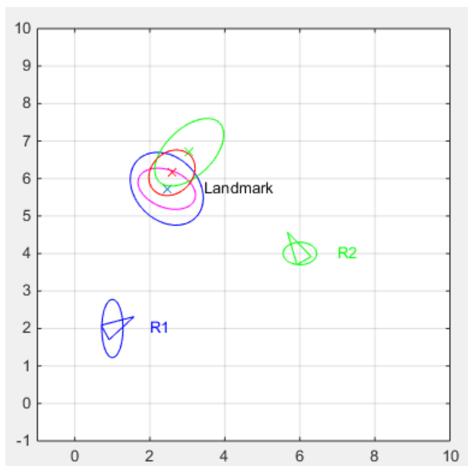

Figure 10: Combination (red ellipse) of two pdf's (blue and green ellipses) associated to the same landmark

Conclusions

It is not a contradiction that two different pdf's are associated to the same landmark. In fact, the combination of both measurements can give us new information concerning to the real position of the landmark.

For combine two measurements, given $G_1 = (\mu_1, \Sigma_1)$ and $G_2 = (\mu_2, \Sigma_2)$, we define de product $G_3 = (\mu_3, \Sigma_3)$ as:

$$\Sigma_3 = (\Sigma_1^{-1} + \Sigma_2^{-1})^{-1}$$

$$\mu_3 = \Sigma_3\Sigma_1^{-1}\mu_1 + \Sigma_3\Sigma_2^{-1}\mu_2)$$