

## A constrained cluster-based approach for tracking the S&P 500 index

Dexiang Wu, Roy H. Kwon\*, Giorgio Costa

Department of Mechanical and Industrial Engineering, University of Toronto, Toronto, Ontario, M5S 3G8, Canada



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### ABSTRACT

We consider the problem of tracking a benchmark target portfolio of financial securities in particular the S&P 500. Linear integer programming models are developed that seeks to track a target portfolio using a strict subset of securities from the benchmark portfolio. The models represent a clustering approach to select securities and also include additional constraints that aim to control risk and transactions costs. Lagrangian and semi-Lagrangian methods are developed to compute solutions to the tracking models. The computational results show the effectiveness of the linear tracking models and the computational methods in tracking the S&P 500. Overall, the models and methods presented can serve as the basis of the optimization module in an optimization-based decision support for creating tracking portfolios.

### 1. Introduction

The proliferation and demand of exchange-traded funds (ETFs) where the underlying asset is a market index such as the S&P 500 Index (SPDR) is a reflection of the demand in investment in broad markets as opposed to actively managed investments that try to beat the markets. There is good reason for this, for example, the average return of 769 all-equity actively managed funds was 2%–5% lower than the S&P 500 index during the period 1983–1989 (Zenios, 2007) and more recent studies have also found similar differences. ETFs allow a broader participation in investment in major market indices since it is the ETF company that is responsible for investing for replication of an index i.e. investing to mimic the risk and return profile of a market index. Clearly, a key strategic decision of an ETF company is the construction of a portfolio that mimics a given benchmark market index. However, this is not necessarily a straightforward process and is often referred to as index tracking. For example, it has been found in (Valle et al., 2014) that many commercial ETFs have higher variance than the underlying assets. For ETFs that aim to replicate both the risk and return properties of an index as closely as possible the task is not trivial. It should be noted that the models presented in this paper seek to track and not outperform the S&P 500 and so this motivates replicating both risk and return of the index as close as possible. This will also motivate the use of capitalization-style weightings for the assets selected by the models.

Index tracking is an important passive investing strategy where one seeks a portfolio of securities that emulates a given benchmark portfolio such as the S&P 500. Full replication of a benchmark portfolio is an

obvious strategy for tracking where all assets in the benchmark are held in the quantities as specified by the weightings of the benchmark portfolio, but full replication is not practical given the transaction costs this would entail. For example, fully replicating the S&P 500 index would require holding the 500 assets along with weightings for each asset. The weightings are based on market capitalization and so as soon as the prices of assets change the weights change as well. Constant rebalancing of the tracking portfolio would result in a prohibitive amount of transactions. An alternative strategy is to select a strict subset of assets from the benchmark, however, this results in tracking portfolios that do not match the benchmark as closely as in full replication. A well-known measure of this discrepancy is called tracking error and is defined as the difference between returns of the tracking portfolio and benchmark. In general, there will be a trade-off between tracking error and transactions costs. Models that seek to minimize tracking error have emerged as a popular approach for constructing tracking portfolios (Jorion, 2003). Such models exhibit non-linearity as it is the variance of tracking error that is often minimized or constrained. A further complication is that in enforcing only a strict subset of assets selected discrete variables must be introduced. This constraint is called the cardinality constraint and requires binary variables for its implementation. Incorporating this aspect along with tracking error minimization into a model will result in a non-linear integer optimization problem which can present substantial challenges in computing optimal or near-optimal solutions. Furthermore, most tracking models e.g. those minimizing tracking error require estimates of the expected returns of assets. It is challenging to estimate these quantities and the corresponding estimation errors can result in

\* Corresponding author.

E-mail address: [rkwon@mie.utoronto.ca](mailto:rkwon@mie.utoronto.ca) (R.H. Kwon).

substantial bias in optimized portfolios. However, it should be noted that there are models that minimize tracking error through linear objectives see (Guastaroba and Speranza, 2012) and the references therein.

Many companies that offer ETFs to the open public are large financial institutions that will invariably use portfolio management systems e.g. computer-based decision support to assist in construction of (tracking) portfolios (Xidonas et al., 2009). In particular, optimization-based decision support can be even more relevant for portfolio optimization where in addition to database and statistical modules, an optimization module is present that contains mathematical models and algorithms (Beraldi et al., 2011). But a central challenge for any optimization-based decision support is to have mathematical models that not only can track a given benchmark well, but that can be solved within a reasonable amount of time (Steuer et al., 2011). In this paper, we consider linear mixed integer optimization models for tracking broad market indices such as the S&P 500. The models we consider represent a cluster-based approach for tracking based on a model in (Cornuejols and Tutuncu, 2007). The cluster-based approach seeks to partition the assets in a benchmark portfolio into disjoint clusters such that a single (representative) asset is selected from each cluster. The set of representatives constitutes the tracking portfolio. The clusters are grouped to maximize similarity among assets in a cluster. The number of clusters to generate is a user controlled parameter and is implemented by a cardinality constraint that explicitly restricts the number of representatives to equal the user specified number of assets to hold. The presence of a cardinality constraint in a model makes the model more difficult to solve. However, it is a useful construct by which asset managers can explicitly control the size of portfolios. There are other approaches for limiting the size of portfolios that do not rely on a cardinality constraint (Bruni et al., 2015) but may require information related to returns of assets for which the proposed models in this paper do not rely on.

Any quantity can be used to represent similarity as long as the similarity of an asset with itself is 1 and the similarity of two different assets are less than or equal to one where the similarity of two different assets are larger (closer to 1) if they are more similar. Thus, a measure of similarity can be represented by correlations between returns of pairs of assets. There are many well established methods to estimate statistical correlation and makes it a practical choice as a measure of similarity. One of the advantages of the cluster-based models is that they only require information about similarity whereas most tracking models e.g. those that use tracking error require information about expected returns in addition to correlation estimations.

However, a tracking strategy based only on clustering may produce a tracking portfolio that tracks a benchmark portfolio well in terms of return, but could produce an insufficiently diverse portfolio when tracking a broad market index such as the S&P 500 thereby increasing the risk of the tracking portfolio. A market index such as the S&P 500 consists of approximately 500 large cap stocks from 10 different economic sectors such as energy, information technology, consumer discretionary, consumer staples, materials, financial, utilities, industrials, telecommunication and services, and health care. The sectors represent the broad and diverse economy of the United States. A pure clustering solution may result in concentration of assets into just a few sectors. As such, we consider constraints to ensure that a tracking portfolio for the S&P 500 contains reasonable representation from each sector. We also consider some additional important constraints that aim to control transaction costs such as buy-in thresholds and turnover constraints see (Zenios, 2007). Buy-in threshold constraints ensure that assets selected will have weights that are not unrealistically small and turnover constraints ensure that the tracking portfolio does not deviate excessively from a current tracking portfolio.

The main contribution of the paper is the development of a sector constrained linear clustering approach for tracking the S&P 500 with buy-in thresholds using the base model in (Cornuejols and Tutuncu, 2007). However, the addition of these elements is non-trivial as the resulting models are more difficult to solve than the base model. A

secondary contribution is the development of Lagrangian and Semi-Lagrangian relaxation approaches for computing near optimal solutions to these models. The models we propose are linear integer programming models. Recently, there have emerged several mixed-integer linear models related to tracking an index see (Guastaroba and Speranza, 2012), (Filippi et al., 2016), and (Mezali and Beasley, 2014) and related to constructions of ETFs (Valle et al., 2015). The main differences in the work in this paper are that (1) the use of the cluster-based approach will not require asset return information directly in the models and (2) the cluster-model has explicit additional (e.g. in addition to cardinality restrictions) diversification control through sector constraints. Also, the solution approach is based on Lagrangian relaxation. We believe the models proposed in this paper are the first constrained cluster-based approaches for tracking an index portfolio.

The rest of the paper is organized as follows: Section 2 briefly surveys the literature on index tracking. In Section 3 we formulate the index tracking models with sector limit and other practical constraints. In Section 4 we develop the Lagrangian relaxation-based methods for the models. In Section 5 computational results are given and we conclude the paper in Section 6.

## 2. Literature review for index tracking

A common approach to the index tracking problem is to formulate it as an integer optimization problem. One of the major challenges is to deal with the cardinality constraint and a diversity of algorithmic methods ranging from evolutionary heuristics to methods based on branch-and-bound have been considered to solve models with cardinality restrictions. Beasley et al. (2003) consider a general non-linear tracking model with transaction costs and cardinality constraint and solve it using evolutionary heuristics in testing five major markets in the world. Bertsimas et al. (1999) consider a mixed integer program to construct a portfolio to track a given benchmark portfolio with the aim of having fewer stocks with turnover and transaction costs. Coleman et al. (2006) minimize tracking error in the index tracking problem with cardinality constraints using a graduated non-convexity algorithm to satisfy the cardinality restriction. Jansen and Van Dijk, 2002 convert the cardinality constraint into a continuous non-convex power function which results in better portfolio weights with lower tracking errors through diversifying the allocation into small stocks. Oh et al. (2005) consider genetic algorithms to generate the optimal weights for selected stocks to track a benchmark (where the tracking portfolio has strictly fewer assets) where first stocks are distributed into the sectors with larger market capitalization. Ruiz-Torrubiano and Suárez (2009) apply a hybrid approach that uses a genetic algorithm to select the assets that track different market indices with fewer assets and use quadratic programming to determine the weights of the assets; other practical constraints such as transaction cost are not included in their model. Stoyan and Kwon (2010) consider a two-stage stochastic mixed integer programming model with several discrete choice constraints such as buy-in thresholds, cardinality constraints, as well as round lots to track the Toronto Stock Exchange (TSX). Lejeune and Samatłō-Paç, 2013 consider a chance constrained stochastic programming formulation used for the risk averse indexing problem with cardinality constraints and develop an associated outer approximation method. Cornuejols and Tutuncu (2007) consider an index tracking model which maximizes similarity between selected assets and the assets of the target index. The model represents a clustering-based approach for constructing a tracking portfolio. Chen and Kwon (2012) consider a robust version of the model in Cornuejols and Tutuncu (2007). Canakgoz and Beasley (2009) consider an enhanced index tracking problem via a mixed integer program where the objective is to allow outperformance of a benchmark, the model includes transaction cost and is tested on eight large market indices. Bruni et al. (2015) consider another approach for enhanced indexation based on bi-objective linear programming and a linear risk-return model where the number of securities is limited by requiring that a portfolio have a certain level of return. This approach

could easily be used for index tracking as well as enhanced indexation. [Chavez-Bedoya and Birge \(2014\)](#) consider a multi-objective non-linear programming approach applied for enhanced indexation that decomposes the variance of the tracking error of the portfolio so that a model is solved with fewer variables. [Gaivoronski et al. \(2005\)](#) consider different types of risk measurement for index tracking ranging from mean-variance and conditional value at risk (CVaR) models to track with fewer assets.

Most models described above require estimates of expected price or return of assets. In general, it is difficult to estimate expected returns accurately and portfolio optimization models can be sensitive to estimation errors of returns see ([Chopra and Ziemba, 1993](#)) and often they maximize the errors found in estimates see ([Michaud, 1989](#)). In the next section we develop models for index tracking which do not require expected return estimates, but only require information about similarity e.g. correlation between the returns of assets.

### 3. Model formulations

#### 3.1. Basic cluster-based index tracking model

We adopt and describe the basic index tracking model in ([Cornuejols and Tutuncu, 2007](#)). Suppose the target portfolio has  $n$  securities. The model seeks to partition the  $n$  securities of the target portfolio into  $q$  disjoint groups (clusters) of securities where securities in a group are the most “similar” to each other. Then, the model will select a “representative” from each group. The  $q$  representatives will constitute the tracking portfolio.

Let  $\rho_{ij}$  represent the correlation (similarity) between security  $i$  and security  $j$  and let  $q$  denote the size of tracking portfolio where  $q < n$ . For  $i, j = 1, \dots, n$  let  $x_{ij}$  represent whether stock  $j$  is a representative of stock  $i$  where  $x_{ij} = 1$  if  $j$  is chosen as a representative for  $i$  or 0 otherwise. For  $j = 1, \dots, n$  let  $y_j$  represent the selection of a security to be part of the tracking portfolio where  $y_j = 1$  if security  $j$  is selected or 0 otherwise. Then, the problem of creating a tracking portfolio can be formulated as follows:

$$\max \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} x_{ij} \quad (1.1)$$

$$s.t. \quad \sum_{j=1}^n y_j = q \quad (1.2)$$

$$\sum_{j=1}^n x_{ij} = 1, \forall i = 1, \dots, n \quad (1.3)$$

$$x_{ij} \leq y_j, \forall i = 1, \dots, n; j = 1, \dots, n \quad (1.4)$$

$$x_{ij}, y_j \in \{0, 1\} \quad (1.5)$$

The model above will be referred to as model (1). The objective is to select securities so that total similarity of all groups is maximized. The first constraint enforces that the tracking portfolio will have exactly  $q$  securities and is called a cardinality constraint. The second constraint ensures that each security has exactly one representative in the portfolio. The third constraint prohibits a security to be a representative of any security if it is not selected to be part of the tracking portfolio.

The model above only selects securities for the tracking portfolio, but once the model is solved the investment weight for each selected security expressed as proportion of total investment can be calculated. In particular, a weight  $w_j$  can be calculated for each selected asset  $j$  using total market value of all securities in the group that security  $j$  represents divided by the total market value of all securities in the target portfolio (index), i.e.,  $w_j = \sum_i V_i x_{ij} / \sum_i V_i$ . For example, if stock 1 represents stock 2 and 3 in the portfolio, we sum the market values of stock 1, 2 and 3, and

then divide the sum by the market value of the  $n$  securities in the target portfolio. The weight for security 1 in the tracking portfolio would be positive assuming that all securities have positive prices and the weights for securities 2 and 3 would be set to 0 as they would not be in the tracking portfolio. This follows the capitalization-based weighting that is found in the S&P 500 and other major indices. It should be noted that the models presented in this paper seek to track and not outperform the S&P 500 and so this motivates the use of capitalization-style weightings for the assets selected by the models.

The clustering based model utilizes only linear constraints and therefore is a pure 0–1 linear integer program. The quality of the tracking portfolio generated by the model is measured ex-post i.e. tracking error and metrics to measure closeness to the benchmark index portfolio are computed after the tracking portfolio is generated. An alternative would be to explicitly have tracking error minimized as the objective in a tracking model. This has been a popular approach in the practice and literature see ([Jorion, 2003](#)). However, this would create a non-linearity in the objective as the variance of the difference of the returns of the tracking and benchmark portfolios would need to be minimized and in conjunction with cardinality constraint requirements would result in a quadratic non-linear integer program which is known to be very challenging to solve see ([Bertsimas and Shioda, 2009](#); [Pardalos and Vavasis, 1991](#)).

Model (1) has been shown in ([Chen and Kwon \(2012\)](#)) to track a benchmark portfolio (S&P 100) well where the number of securities in the benchmark portfolio is  $n = 100$ . Instances of model (1) were able to be solved adequately with exact methods. However, there are several important practical elements that have not been considered. First, model (1) above lacks transaction costs. It will be most likely in practice that some tracking portfolio is already existing. It will be important to make sure that a new tracking portfolio is not too different from the currently existing one as substantial differences will result in higher turnover and thus higher transaction costs. Model (1) will be extended to have turnover constraints that limit transaction costs. Further, tracking portfolios with small positions are also limited by incorporating buy-in thresholds in model (1).

Second, the tracking portfolio generated from model (1) may track a benchmark well in terms of return, but the portfolio itself may be insufficiently diversified as there is no constraints that limits portfolio risk. This is an important issue when tracking market indices such as the S&P 500 as any tracking portfolio should include securities across the 10 different sectors (Consumer Discretionary, Consumer Staples, Energy, Financials, Health Care, Industrials, Information Technology, Materials, Telecommunications Services, and Utilities) that comprise a market index. Model (1) can be shown to produce tracking portfolios with securities from only a few e.g. 2 or 3 sectors. This would be problematic for most portfolio managers concerned about risk and diversification. To this end, constraints that ensure sector diversification will be incorporated in model (1).

#### 3.2. Model with buy-in threshold and turnover constraints

We now consider the addition of buy-in threshold and turnover constraints in model (1). The resulting model is given in the following formulation:

$$\max \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} x_{ij} \quad (2-1.1)$$

$$s.t. \quad \sum_{j=1}^n x_{ij} = 1, \forall i = 1, \dots, n \quad Assignment \quad (2-1.2)$$

$$x_{ij} \leq y_j, \forall i, j = 1, \dots, n \quad (2-1.3)$$

$$\sum_{j=1}^n y_j = q \quad (2-1.4)$$

$$l_j y_j \leq w_j \leq u_j y_j, \forall j = 1, \dots, n \quad Buy-in \quad (2-1.5)$$

$$w_j = \sum_{i=1}^n V_i x_{ij} / \sum_i V_i \quad \forall j = 1, \dots, n \quad (2-1.6)$$

$$\sum_{j=1}^n |w_j^0 - w_j| \alpha \leq \gamma \quad Transaction-Cost \quad (2-1.7)$$

$$x_{ij}, y_j \in \{0, 1\}, \forall i, j = 1, \dots, n \quad (2-1.8)$$

The model above will be referred to as model (2-1) and includes decision variables and parameters of model (1), but now has the following additional parameters:  $\alpha$  is a proportional transaction cost,  $\gamma$  is the limit on transaction cost,  $V_i$  denotes the market capitalization of stock  $i$  at current time,  $w_j^0$  denotes the proportion of stock  $j$  in current portfolio. In addition, model (2-1) has the variable  $w_j$  denoting the proportion of wealth invested in stock  $j$  for  $j = 1, \dots, n$ .

The buy-in threshold constraints set the weight of a stock to be  $\sum_i V_i x_{ij} / \sum_i V_i$  which is the standard market capitalization based weight of assets in indices such as the S&P 500 and is set to 0 if asset  $j$  is not selected. If asset  $j$  is selected, then the weight for asset  $j$  must be between the prescribed lower and upper bounds  $l_j$  and  $u_j$ , respectively. The transaction cost constraint,  $|w_j^0 - w_j|$  denotes the turnover of stock  $j$  from buying or selling and the cost of turnover of an asset  $j$  is proportional to the amount of turnover given by  $|w_j^0 - w_j| \alpha$ . The transaction constraint limits the total proportional turnover (transaction) cost to  $\gamma$ .

The absolute value terms in transaction cost constraint can be removed by introducing auxiliary variables  $z_j$ , after which the model (2-1) becomes equivalent to the following model which we refer to as model (2):

$$\max \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} x_{ij} \quad (2-2.1)$$

$$s.t. \quad (2-1.2) - (2-1.6), (2-1.8) \quad (2-2.2)$$

$$\left. \begin{array}{l} \sum_{j=1}^n z_j \leq \frac{\gamma}{\alpha} \\ z_j \geq w_j^0 - w_j \quad \forall j = 1, \dots, n \\ z_j \geq -(w_j^0 - w_j) \quad \forall j = 1, \dots, n \\ z_j \geq 0 \quad \forall j = 1, \dots, n \end{array} \right\} Transaction-Cost \quad (2-2.3)$$

However, computational experiments in section 5 show that optimal tracking portfolios from models (1) and (2) are often concentrated in a few sectors which may result in high portfolio variance or lack of diversification. Therefore, constraints that impose diversification in a natural way are considered in section 3.3.

### 3.3. Basic model with sector limits

For simplicity of exposition, we first consider diversification (sector limit) constraints for model (1) and then consider the addition of these constraints to model (2). The idea is to classify assets in a tracking model according to what sector an asset belongs to. For example, in the S&P 500 index the constituent assets are classified as belonging to one of 10 sectors collectively representing the broad economy of the United States.

In general, we assume that the benchmark index consists of  $K$  sectors. Let  $x_{ijk}$  equal 1 if stock  $j$  represents stock  $i$  in sector  $k$ , 0 otherwise.  $y_{jk}$  is equal to 1 if stock  $j$  from sector  $k$  is selected to the tracking portfolio from, 0 otherwise.  $|K|$  is the number of sectors, and  $n_k$  denotes the number of assets (stocks) in sector  $k$ .

The idea of the sector constrained model is to ensure that there is sufficient investment across all sectors by creating sub-portfolios for each sector where each sub-portfolio is sought that maximizes similarity of the sub-portfolio with respect to its sector. Let  $\rho_{ijk}$  denote the similarity between assets  $i$  and  $j$  in sector  $k$ .  $\Delta_k$  and  $\nabla_k$  denote the lower and upper bounds on the cardinality of the sub-portfolio from sector  $k$ . In our computation we set  $\Delta_k$  and  $\nabla_k$  to 0 and the cardinality of the sector the asset belongs to, respectively.  $q_k$  denotes the sub-portfolio size of sector  $k$  and  $q$  denotes total portfolio size. Then, model (1) modified for sector constraints is as follows which we refer to as model (3):

$$\max \sum_{k=1}^{|K|} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} \rho_{ijk} x_{ijk} \quad (3-1.1)$$

$$s.t. \quad \sum_{j=1}^{n_k} y_{jk} = q_k \quad \forall k = 1, \dots, |K| \quad (3-1.2)$$

$$\Delta_k \leq q_k \leq \nabla_k \quad \forall k = 1, \dots, |K| \quad (3-1.3)$$

$$\sum_{k=1}^{|K|} q_k = q \quad (3-1.4)$$

$$\sum_{j=1}^{n_k} x_{ijk} = 1 \quad \forall i = 1, \dots, n_k; \forall k = 1, \dots, |K| \quad (3-1.5)$$

$$x_{ijk} \leq y_{jk} \quad \forall i = 1, \dots, n_k; \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \quad (3-1.6)$$

$$x_{ijk}, y_{jk} \in \{0, 1\} \quad \forall i = 1, \dots, n_k; \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \quad (3-1.7)$$

### 3.4. The model with trading and sector diversification constraints

We now consider a comprehensive version of a cluster-based model for tracking, called model (4), that includes the buy-in thresholds, trading constraints, and the sector diversification constraints as seen in models (2) and (3).

$$\max \sum_{k=1}^{|K|} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} \rho_{ijk} x_{ijk} \quad (4-1.1)$$

$$s.t. \quad \sum_{j=1}^{n_k} x_{ijk} = 1, \forall i = 1, \dots, n_k, \forall k = 1, \dots, |K| \quad Assignment \quad (4-1.2)$$

$$\left. \begin{array}{l} \sum_{j=1}^{n_k} y_{jk} = q_k, \quad \forall k = 1, \dots, |K| \\ \Delta_k \leq q_k \leq \nabla_k \quad \forall k = 1, \dots, |K| \\ \sum_{k=1}^{|K|} q_k = q \end{array} \right\} Cardinality \quad (4-1.3)$$

$$x_{ijk} \leq y_{jk} \quad \forall i = 1, \dots, n_k; \quad j = 1, \dots, n_k, \quad k = 1, \dots, |K| \quad (4-1.4)$$

$$l_{jk} y_{jk} \leq w_{jk} \leq u_{jk} y_{jk}, \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \quad Buy-in \quad (4-1.5)$$

$$w_{jk} = \sum_{i=1}^{n_k} V_{ik} x_{ijk} / \sum_{i=1}^n V_i, \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \quad (4-1.6)$$

**Table 1**  
Applicability of different models.

Models	Suitable problem
Model (1)	Construct the tracking portfolio via maximizing the co-movement between the target and its subset. Does not require an existing tracking portfolio.
Model (2)	Control the transaction cost from rebalancing a current portfolio to the new tracking portfolio from model (1). It can be used to restrict and reduce the operational cost for rebalancing the existing portfolio.
Model (3)	Force the allocation of tracking portfolio from model (1) to distribute into different sectors so that portfolio risk can be reduced. It is an alternative tool for a risk-averse investor.
Model (4)	The portfolio aims to control both risk and transaction cost, therefore it is suitable for rebalancing an existing tracking portfolio with a large number of sectors.

**Table 2**  
Model test by Gurobi.

n	$q = 10$			
	Zmax	model (1)	model (2)	model (3)
100	52.0822	46.9684	44.3890	40.0673
300	125.9684	118.1528	119.8971	104.0347
500	215.8263	209.2602	Out of memory	Out of memory

$$\sum_{k=1}^{|K|} \sum_{j=1}^{n_k} \left| w_{jk}^0 - w_{jk} \right| \alpha \leq \gamma \quad \text{Transaction - Cost} \quad (4-1.7)$$

$$x_{ijk}, y_{jk} \in \{0, 1\} \quad \forall i = 1, \dots, n_k; \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \quad (4-1.8)$$

The parameter  $w_{jk}^0$  denotes the initial proportion of wealth invested in stock  $j$  (from sector  $k$ ) which is needed when considering transaction costs (turnover) in the presence of sector constraints and the decision variable  $w_{jk}$  denotes the proportion of wealth invested in stock  $j$  (from sector  $k$ ). The absolute values that appear in the turnover constraints can be removed by introducing auxiliary continuous variables  $z_{jk}$  that represents the turnover amount for asset  $j$  (from sector  $k$ ) and  $p_k$  which represents the aggregate turnover of assets in sector  $k$  to get the following constraints for turnover:

$$\begin{aligned} \sum_{j=1}^{n_k} z_{jk} &= p_k \quad \forall k = 1, \dots, |K| \\ \sum_{k=1}^{|K|} p_k &\leq \frac{\gamma}{\alpha} \\ z_{jk} &\geq w_{jk}^0 - w_{jk}, \quad \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \\ z_{jk} &\geq -\left( w_{jk}^0 - w_{jk} \right), \quad \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \\ z_{jk} &\geq 0, \quad \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \end{aligned} \quad \left. \begin{array}{l} \\ \\ \\ \text{Cost} \end{array} \right\} \text{Transaction}$$

The applicability of the proposed models (1), (2), (3), and (4) can be seen in Table 1.

### 3.5. Tractability of the cluster-based models

The size of model (2) is larger than models (1), (3), and (4) in terms of the number of variables and constraints. However, computational results indicate that it is the presence of the cardinality constraint that makes computation of solutions most challenging.

We solve instances of each of these models using the commercial solver Gurobi on a 1.58 GHz PC with 2GB of RAM. Random instances of the tracking problems were generated where for each instance  $q$  assets

will be selected from a benchmark portfolio of  $n$  assets where  $n$  is chosen as 100, 200, and 500. We randomly generated multivariate normal distributions for different  $n$  through the mvnrnd function in MATLAB, and calculated the associated correlations  $\rho_{ij}$ . Computational results are presented in Table 2. Each row in Table 2 is for an instance of  $n$  assets. Moving across each row from left to right we see that as more constraints are incorporated into model (1), the objective values decreases. Moving down each column we see that instances with larger  $n$  have better objective values for each type of model. Gurobi cannot solve models (3) and (4) when  $n = 500$ . This motivates the development of algorithms for model (4) so that quality solutions for instances with  $n = 500$  are possible. Important and popular market indices such as the S&P 500 have 500 assets and so it will be critical to have methods to deal with indices of this size.

### 4. Lagrangian Relaxation algorithms

We present both Lagrangian relaxation and Semi-Lagrangian relaxation methods for (4). Hard constraints of (4) are identified and relaxed by putting these constraints in the objective. The resulting problem is called the Lagrangian relaxation and represents an upper bound on the optimal solution of (4) see (Cornuejols et al., 1977; Fisher, 1981; Geoffrion & Bride, 1978; Geoffrion, 2010). In particular, two constraints in model (4)  $\sum_{k=1}^{|K|} q_k = q$  and  $\sum_{k=1}^{|K|} p_k \leq \frac{\gamma}{\alpha}$  can be put into the objective function by using the Lagrange multipliers  $\lambda$  and  $\mu$ , respectively. Then a Lagrangian objective function can be derived as follows:

$$\begin{aligned} L(x, y, z, \lambda, \mu) &= \max_{(x,y,z)} \sum_{k=1}^{|K|} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} \rho_{ijk} x_{ijk} - \lambda \left( \sum_{k=1}^{|K|} q_k - q \right) - \mu \left( \sum_{k=1}^{|K|} p_k \right. \\ &\quad \left. - \frac{\gamma}{\alpha} \right) \\ &= \max_{(x,y,z)} \sum_{k=1}^{|K|} \left[ \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} \rho_{ijk} x_{ijk} - \lambda q_k - \mu p_k \right] + \lambda q + \mu \frac{\gamma}{\alpha} \\ &= \sum_{k=1}^{|K|} \max_{(x,y,z)} \left[ \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} \rho_{ijk} x_{ijk} - \lambda q_k - \mu p_k \right] + \lambda q + \mu \frac{\gamma}{\alpha} \end{aligned}$$

Then for any sector  $k$ , the associated  $k$ th Lagrangian sub-problem ( $L_k$ ) is:

$$\max_{(x,y,q_k,p_k)} O_k = \sum_i^{n_k} \sum_j^{n_k} \rho_{ijk} x_{ijk} - \lambda q_k - \mu p_k \quad (L.1)$$

$$\text{s.t. } (4-1.2), (4-1.4), (4-1.5) \quad (L.2)$$

$$\left. \begin{array}{l} \sum_{j=1}^{n_k} y_{jk} = q_k \quad \forall k = 1, \dots, |K| \\ \Delta_k \leq q_k \leq \nabla_k \quad \forall k = 1, \dots, |K| \end{array} \right\} \text{Cardinality} \quad (L.3)$$

$$\left. \begin{array}{l} w_{jk} = \sum_{i=1}^{n_k} V_{ik} x_{ijk} / \sum_{i=1}^n V_i, \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \\ \sum_{j=1}^{n_k} z_{jk} = p_k \quad \forall k = 1, \dots, |K| \\ z_{jk} \geq w_{jk}^0 - w_{jk}, \quad \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \\ z_{jk} \geq -\left( w_{jk}^0 - w_{jk} \right), \quad \forall j = 1, \dots, n_k; \forall k = 1, \dots, |K| \end{array} \right\} \text{Transaction cost} \quad (L.4)$$

$$x_{ijk}, y_{jk} \in \{0, 1\}; \quad z_{jk} \geq 0; \quad \forall i, j = 1, \dots, n_k; \forall k = 1, \dots, |K| \quad (L.5)$$

Solving ( $L$ ) is easier than solving problem (4) since the size of the subproblems is much smaller than the original problem and each of them can

be solved to optimality by using a commercial solver. After obtaining all optimal sub-objectives, we then simply calculate the Lagrangian function by adding the fixed term  $\lambda q + \mu\gamma/\alpha$  for the given  $\lambda$  and  $\mu$ .

Then, the Lagrangian dual problem is  $\min_{(\lambda \in R, \mu \geq 0)} L(x, y, z, \lambda, \mu)$ , whose optimal solution will provide the lowest upper bound for problem (4). The Lagrangian dual will be solved with a sub-gradient method with heuristics for feasibility. This forms the basis of the Lagrangian relaxation algorithm for solving problem (4). We summarize the Lagrangian relaxation algorithm as follows:

In Step 1, Heuristic I is applied to obtain a initial solution. If the solution is infeasible, a more sophisticated heuristic (Heuristic II) is applied

to satisfy the global constraints,  $\sum_{k=1}^{|K|} q_k = q$  and  $\sum_{k=1}^{|K|} p_k \leq \frac{\gamma}{\alpha}$ , and the associated lower bound can be updated.

Let  $n(k)$  denote the size of sector  $k$ . Let  $Q = \{q_k, k = 1, \dots, |K|\}$  be a vector that satisfies the cardinality constraint in model (4) and let  $Q' = \{q'_k, k = 1, \dots, |K|\}$  be another vector that also satisfies the cardinality constraint in model (4), but different than  $Q$ .  $Q^{LR} = \{q_k^{LR}, k = 1, \dots, |K|\}$  be a vector satisfying the cardinality constraint in model (L).  $I$ ,  $I'$  and  $I^{LR}$  are the associated index sets of  $Q$ ,  $Q'$  and  $Q^{LR}$ , respectively. We first describe Heuristic I as follows:

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### Lagrangian Relaxation Algorithm (LR)

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#### Step 0: (Initialization)

$$v \leftarrow 0, \lambda^{(v)} \leftarrow 1, \mu^{(v)} \leftarrow 0$$

#### Step 1: (Dual Decomposition)

For any  $k$ , solve the corresponding sector sub-problem (L.1) - (L.5), and

$$UBD \leftarrow \sum_{k=1}^{|K|} O_k^* + \lambda^{(v)} q + \mu^{(v)} \gamma / \alpha.$$

If  $(x^{(v)}, y^{(v)}, z^{(v)})^T$  is **feasible for (4)**,  $LBD \leftarrow UBD$ , STOP.

Else find a feasible solution (and associated  $LBD$ )  
by Heuristic I ( $v = 0$ ) or Heuristic II ( $v > 0$ ).

$$gap^{(v)} = (UBD - LBD) / |LBD|$$

#### Step 2: (Lagrangian Multiplier Update)

Build Lagrangian dual problem  $\min_{\mu \geq 0} L(x^{(v)}, y^{(v)}, z^{(v)}, \lambda, \mu)$

Update step size  $t^{(v)}$  by Bi-section method.

$$\lambda^{(v+1)} = \lambda^{(v)} + t^{(v)} \left( \sum_{k=1}^{|K|} q_k - q \right)$$

$$\mu^{(v+1)} = \max \left( 0, \mu^{(v)} + t^{(v)} \left( \sum_{k=1}^{|K|} p_k - \gamma / \alpha \right) \right)$$

Calculate (L) with new multiplier  $(\lambda^{(v+1)}, \mu^{(v+1)})^T$

#### Step 3: (Move to next Iteration)

If  $gap^{(v)} \geq \eta, v \leq V$

$v \leftarrow v + 1$ , Go to Step 1

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Note:  $\mu \geq 0$  for any iteration.

**Heuristic I:****Step (0): (Sort Market Capitalization)**

Sort market capitalization of assets in descending order and put in vector  $V$

Choose the first  $q$  assets in  $V$  that satisfy sector cardinality bounds and weight

bounds; Obtain a  $Q$  vector.

**Step (1): (Divide the index of  $Q$  into 3 groups)**

$$I_1 = \{h | Q_h = 0\} \text{ sort } I_1 \text{ in descending order according to } \{n(h) | h \in I_1\}$$

$$I_2 = \{i | Q_i \neq 0, i \in \{\text{index set of first } l \text{ largest } Q_i\}\}$$

$$I_3 = \{j | Q_j \neq 0, j \in I / I_1 \cup I_2\}, \text{ sort } I_3 \text{ according to } \{(Q_j, n(j)) | j \in I_3\}$$

Switch portion of indices between  $I_1$ ,  $I_2$  and  $I_3$ .

Generate  $N$  neighborhood points  $Q'$  around  $Q$ .

**Step (2): (Solve Relaxed Lagrangian sub-problem)**

Solve  $(L)$  without constraint  $\sum_{k=1}^{|K|} P_k \leq \frac{\gamma}{\alpha}$  under  $Q'$ 's.

**Step (3): (Test transaction cost constraint (TC))**

Choose solution  $Q'$  better than  $Q$  and satisfy (TC) if it exists, STOP; else

GO TO Step (1).

Step (0) in Heuristic I guarantees that a starting solution will satisfy the transaction cost constraint by emphasizing the selection of assets with larger market capitalization. For example, suppose  $V = (10000, 100, 10)^T$  and associated  $w_j^0 = (0.9891, 0.0099, 0.0010)^T$ , if the first asset is not selected to the tracking portfolio, the turnover weight is 98.91% and is much larger than the maximal turnover weights of the second and third assets, so the turnover constraint will be easily violated. We then generate the neighborhood of points around  $Q$  in Step (1) by choosing pairs of sectors as indexed by the subsets  $I_1$ ,  $I_2$  and  $I_3$  and swapping pair-wise.

The philosophy behind the swap rules is to generate only a small size  $N$  of neighborhood points such that swaps attempt to distribute the assets to more sectors so that the objective value becomes better. For our computations,  $N$  was set to 3. In Step (1), we sort  $I_3$  in increasing order according to  $\{Q_j | j \in I_3\}$ . If elements in  $\{Q_j | j \in I_3\}$  are equal, we then sort  $I_3$  in descending order according to  $\{n(j) | j \in I_3\}$ . We always select sectors

at the front position of the index sets  $I_1$ ,  $I_2$ ,  $I_3$ , and switch 2 assets between pairs of these three groups in Step (1). If no improvement occurs at the current iteration, the sectors with different positions in the index sets are selected in the next iteration. For example, parallel swapping steps include:

- Pick 2 assets from  $a$ th sector in  $I_2$  and move into  $b$ th sector in  $I_3$ , this generates a  $Q'$ ;
- Or pick 2 assets from  $a$ th sector in  $I_2$  and move into  $b$ th sector in  $I_1$ , this generates a  $Q'$ ;
- Or pick 2 assets from  $a$ th sector in  $I_2$ , add 1 asset to  $b$ th sector in  $I_3$  and 1 asset to  $c$ th sector in  $I_1$ , this generates a  $Q'$ ;
- Or pick 1 asset from  $a$ th sector in  $I_2$  and 1 asset from  $b$ th sector in  $I_3$ , add them to  $c$ th sector in  $I_1$ , this generates a  $Q'$ .

Here the indices  $a$ ,  $b$ , and  $c$  are generally set as small values since switching other indices may be inefficient in improving the objective, e.g.  $a$ ,  $b$ ,  $c$  are set no more than 2 in our computation.

**Heuristic II:****Step (1): (Adjust  $Q^{LR}$  vector)**

Pick  $\{k \mid \min(Q_k^{LR})\}$ , if  $Q_k^{LR} \leq n(k)$ ,  $q_k' = q_k^{LR}$ , else  $q_k' = n(k)$

Repeat above steps unless  $q - \sum_{k=1}^{|K|} q_k' = 0$ ; if  $q - \sum_{k=1}^{|K|} q_k' > 0$ , add the difference

**into the sector with maximal number of assets;** solve  $(L)$  with  $Q'$  vector;

**Step (2): (Test transaction cost constraint (TC))**

If solution satisfies TC, STOP, else, GO TO Step (3);

**Step (3): (Swapping assets)**

Within each sector  $k$ , do:

$$I_1 = \{w_{j,k}^0 \mid j \in \{n(k)\} \setminus \{Q_k'\}\}, \text{ sort } I_1 \text{ in increasing order.}$$

$$I_2 = \{w_{j,k}^0 \mid j \in \{n(k)\} \setminus \{Q_k'\}\}, \text{ sort } I_2 \text{ in decreasing order.}$$

Swap first  $\Delta$  assets between  $I_1$  and  $I_2$ ; Solve  $(L)$  with new  $Q'$  vectors;

Test TC, if TC satisfied, STOP, else GO TO Step (4)

**Step (4): (Additional Swapping of Assets)**

Pick two sectors that have large  $(k_1)$  and small  $(k_2)$  asset numbers in  $Q'$ , do:

$$I_1 = \{w_{j,k_1}^0 \mid j \in \{n(k_1)\}\}, \text{ sort } I_1 \text{ in increasing order.}$$

$$I_2 = \{w_{j,k_2}^0 \mid j \in \{n(k_2)\}\}, \text{ sort } I_2 \text{ in decreasing order.}$$

Swap first  $\Delta$  assets between  $I_1$  and  $I_2$ . Obtain new  $Q^{LR}$  vectors, GO TO (1);

If the TC cannot be satisfied in Step (3) in Heuristic II, we adjust the portfolio by capital weights in the same sector, and then adjust the portfolio between the sectors in Step (4) if necessary. We selected the sectors with large and small stocks because so as to not lose too much objective value. Like in Heuristic I, we always go back to the assets that have larger capital weights to adjust the constructed portfolio. It is a trade-off between the Cardinality and Transaction Cost constraints. To exchange the assets between sectors in Step (4) we adopt the Variable Neighborhood Search (VNS) approach (Hansen and Mladenović, 2001). We describe the steps that we implemented: Step (1) Shaking - randomly perturb some assets between  $\theta_{ik}^{(v)} \leftarrow 0, \forall k, \forall i$  and  $\min(Q^{LR})$  from current

solution; Step (2) Local search - search the selected neighborhood region, i.e.  $Q'$  vectors. Step (3) Move or not – if an improved solution obtained. Our computational observation is that in most instances Step (3) and (4) are needed to achieve a feasible solution for transaction cost constraints, which indicates that the cardinality and TC constraints are a computational challenge to satisfy as they run in opposing directions.

In Step 2 in the LR algorithm, the step size is updated by the bi-section method. The bi-section method for one dimensional searches in the sub-gradient method has been widely used in LR algorithm see (Fisher, 1981; A. M. Geoffrion, 2010). The details of the bi-section method that we used for calculating the step size in Step 2 of LR algorithm are presented below.

**Bi-section search**

Set initial  $\sigma$ , do:

$$\lambda^{(v+1)} = \lambda^{(v)} + \sigma t^{(v)} \left( \sum_{k=1}^{|K|} q_k - q \right) \quad (1)$$

$$\mu^{(v+1)} = \max \left( 0, \mu^{(v)} + \sigma t^{(v)} \left( \sum_{k=1}^{|K|} p_k - \frac{\gamma}{\alpha} \right) \right) \quad (2)$$

$$\text{where } t^{(v)} = (UBD - LBD) / \left\| \left( \sum_{k=1}^{|K|} q_k - q; \sum_{k=1}^{|K|} p_k - \frac{\gamma}{\alpha} \right) \right\|$$

$$\text{Solve } (L) \text{ under new multiplier } (\lambda^{(v+1)}, \mu^{(v+1)}) \quad (3)$$

$$\text{While } L(x, y, z, \lambda^{(v+1)}, \mu^{(v+1)}) > UBD \text{ and } \sigma > \varepsilon$$

$\sigma = .5\sigma$ , repeat (1) - (3).

The step size is reduced by half in each iteration since the dual variables can be quickly updated, so we embed bi-section search into the LR algorithms. The initial  $\sigma$  we set as 20 in our calculation. The norm is step (2) is a Euclidean norm of the vector consisting of the two sub-gradients.

The LR method cannot always produce small gaps, but with the heuristics above always generate a feasible solution. One possible extension of the LR algorithm to generate tighter bounds is Semi-Lagrangian Relaxation (SLR), a LR approach that considers stricter feasible regions and therefore gives the opportunity to obtain tighter bounds. Due to the decomposition requirement in the main LR algorithm structure, the global constraints cannot be returned into the constraint set. However, other types of constraint can be relaxed and then returned to the constraint set to partially satisfy the SLR framework. This procedure is called partial SLR see (Beltran et al., 2006) and suitable for our problem. In particular, in addition to relaxing the assignment constraint, we put the assignment constraint back and formulate the (partial SLR) Lagrangian function as follows:

where  $P_{ijk} = \rho_{ijk} - \theta_{ik}$ .

Then the associated  $k$ th Lagrangian sub-problem can be formulated by

$$\max_{(x,y,z,q_k,p_k)} \bar{O}_k = \sum_i^{n_k} \sum_j^{n_k} P_{ijk} x_{ijk} - \lambda q_k - \mu p_k + \sum_{i=1}^{n_k} \theta_{ik} \quad (\text{SL.1})$$

$$\text{s.t. } (4 - 1.4), (4 - 1.5), (L.3), (L.4), (L.5) \quad (\text{SL.2})$$

$$\sum_{j=1}^{n_k} x_{ijk} \leq 1, \forall i = 1, \dots, n_k; \forall k = 1, \dots, |K| \quad \text{Relaxed Assignment} \quad (\text{SL.3})$$

And the dual problem becomes  $\min_{(\lambda \in R, \mu \geq 0, \theta \geq 0)} \bar{L}(x, y, z, \lambda, \mu, \theta)$ . Then, the LR framework can be applied to the partial SLR construct. We present the Semi-Lagrangian-based Algorithm as follows:

$$\begin{aligned} \bar{L}(x, y, z, \lambda, \mu, \theta) &= \max_{(x,y,z,q_k,p_k)} \sum_{k=1}^{|K|} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} \rho_{ijk} x_{ijk} - \lambda \left( \sum_{k=1}^{|K|} q_k - q \right) - \mu \left( \sum_{k=1}^{|K|} p_k - \frac{\gamma}{\alpha} \right) - \sum_{k=1}^{|K|} \sum_{i=1}^{n_k} \theta_{ik} \left( \sum_{j=1}^{n_k} x_{ijk} - 1 \right) \\ &= \sum_{k=1}^{|K|} \max_{(x,y,z,q_k,p_k)} \left[ \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} (\rho_{ijk} - \theta_{ik}) x_{ijk} - \lambda q_k - \mu p_k \right] + \lambda q + \mu \frac{\gamma}{\alpha} + \sum_{k=1}^{|K|} \sum_{i=1}^{n_k} \theta_{ik} \\ &= \sum_{k=1}^{|K|} \max_{(x,y,z,q_k,p_k)} \left[ \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} P_{ijk} x_{ijk} - \lambda q_k - \mu p_k + \sum_{i=1}^{n_k} \theta_{ik} \right] + \lambda q + \mu \frac{\gamma}{\alpha} \end{aligned}$$

**Semi-Lagrangian Relaxation Algorithm (SLR)**

## Step 0: (Initialization)

$$v \leftarrow 0, \lambda^{(v)} \leftarrow 1, \mu^{(v)} \leftarrow 0$$

$$\theta_{ik}^{(v)} \leftarrow 0, \forall k, \forall i$$

## Step 1: (Dual Decomposition)

$$\text{For } k \in \{1, \dots, |K|\}, \text{ do } P_{ijk}^{(v)} \leftarrow \rho_{ijk}^{(v)} - \theta_{ik}^{(v)}, \forall i, \forall j$$

Solve the corresponding sector sub-problem (SL.1) – (SL.3), and

$$UBD \leftarrow \sum_{k=1}^{|K|} \bar{O}_k + \lambda^{(v)} q + \mu^{(v)} \gamma / \alpha$$

If  $(x^{(v)}, y^{(v)}, z^{(v)})^T$  is feasible to the (4),  $LBD \leftarrow UBD$ , STOP

Else find a feasible solution (and associated  $LBD$ )  
by Heuristic I ( $v = 0$ ) or Heuristic II ( $v > 0$ ).

$$gap^{(v)} = (UBD - LBD) / |LBD|$$

## Step 2: (Lagrangian Multiplier Update)

$$\text{Build Lagrangian dual problem } \min_{\mu \geq 0, \theta \geq 0} \bar{L}(x^{(v)}, y^{(v)}, z^{(v)}, \lambda, \mu, \theta)$$

Update step size  $t^{(v)}$  by Bi-section methods.

$$\lambda^{(v+1)} = \lambda^{(v)} + t^{(v)} \left( \sum_{k=1}^{|K|} q_k - q \right)$$

$$\mu^{(v+1)} = \max \left( 0, \mu^{(v)} + t^{(v)} \left( \sum_{k=1}^{|K|} p_k - \gamma / \alpha \right) \right)$$

$$\theta_{ik}^{(v+1)} = \max \left( 0, \theta_{ik}^{(v)} + t^{(v)} \left( \sum_{j=1}^{n_k} x_{ijk} - 1 \right) \right), \forall k, \forall i$$

Calculate  $(\bar{L})$  with new multiplier  $(\lambda^{(v+1)}, \mu^{(v+1)}, \theta^{(v+1)})^T$

## Step 3: (Move to next Iteration)

If  $gap^{(v)} \geq \eta, v \leq V$

$v \leftarrow v + 1$ , Go to Step 1

The feasible lower bound is generated by the same heuristics as in the LR algorithm. In Step 2, the sub-gradient method with bi-section search was applied to calculate the dual variables  $(\lambda^{(v+1)}, \mu^{(v+1)}, \theta^{(v+1)})^T$  for the SLR algorithm. The computation is terminated if an optimal solution is obtained in Step 1 or gap tolerance or maximum iteration number reached in Step 3. As observed in (Beltran et al., 2006), partial SLR cannot guarantee a tighter bound or small duality gaps. However, with the embedded heuristics SLR generates a feasible solution and returns a better bound than LR in some instances in our computation. We compare the solutions resulting from LR and SLR methods in next section.

## 5. Computational results: tracking the S&P500

In this section we give the computational results from using the LR and SLR methods to solve model (4). The S&P 500 index is used as the target benchmark.

### 5.1. Parameter estimation

To generate the correlation matrix  $\rho_{ij}$ , we collected the historical price information of all components of S&P500, and calculated the daily returns by  $r_{it} = (P_{it} - P_{i,t-1}) / P_{i,t-1}$ , where  $P_{it}, P_{i,t-1}$  are the adjusted

closing prices at time  $t$  and  $t - 1$ . Then daily returns were used to calculate the mean returns of assets and covariance matrix between different assets:

$$\bar{r}_i = \frac{1}{T} \sum_{t=1}^T r_{it}, \quad \text{cov}_{ij} = \frac{1}{T} \sum_{t=1}^T (r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)$$

Here we use one year's daily return ( $T = 252$ ) to generate the correlation matrix, i.e.  $\rho_{ij} = \text{cov}_{ij} / \sqrt{\text{cov}_{ii} * \text{cov}_{jj}}$ , for all models, and we calculate the correlation matrices by using data from 4 time intervals which were [2006 2007], [2007 2008], [2008 2009] and [2010 2011] respectively. Some stocks in the S&P500 index may be replaced by some other stocks outside of the index since they do not satisfy the selection criteria of S&P500 in the designed time period, we retrieved the stocks that were moved out into the designed intervals and the associated price information. For example, ABK was replaced by LO in June 10, 2008, and then we used the price of ABK rather than that of LO to calculate  $r_{it}$  before 2008. Usually this replacement is rare and the components of S&P500 are stable. We note that we use non-log returns in our computations whereas most finance studies use log returns. It is reasonable in our setting to use non-log returns as we are dealing with returns of the S&P which are often negatively skewed whereas log returns are positively skewed.

According to Global Industry Classification Standard (GICS) Sector

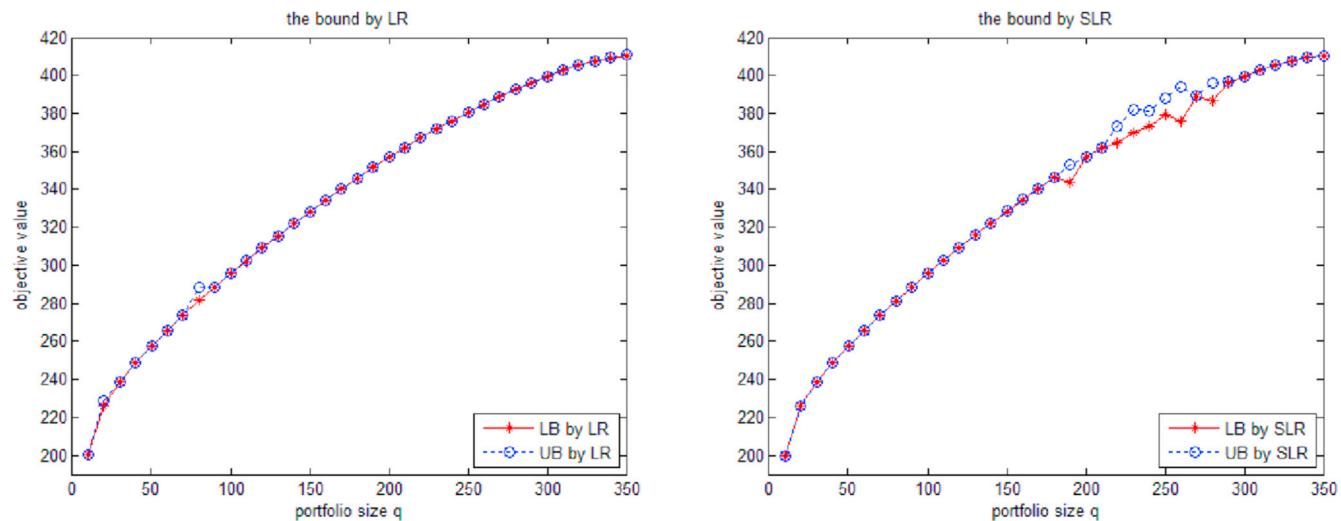


Fig. 1. Gap comparison between LR and SLR.

criterion, the components of S&P500 index are selected from 10 main sectors in US market and we indicate sector 1–10 represent Consumer Discretionary, Consumer Staples, Energy, Financials, Health Care, Industrials, Information Technology, Materials, Telecommunications Services, and Utilities in this research. The vector of sector sizes  $n(k) = [82\ 41\ 41\ 81\ 51\ 62\ 70\ 29\ 8\ 35]^T$  at the time of this research. We adjusted the number of stocks  $I$  in each sector over the intervals if necessary and computed the associated correlation matrix  $\rho_{ijk}$  for the models.

We normalized the market value of each component to calculate the component weight, and used these weights as previous proportion, i.e.  $w_j^0$ , for transaction cost constraint in model (2) and (4). All necessary data were obtained from the Financial Research and Trading Lab at University of Toronto. All tracking models were solved by Gurobi 4.5.1 with a MATLAB interface. We initialize the Lagrangian multipliers  $(\lambda^0, \mu^0)' = (1, 0)'$  for LR and  $(\lambda^0, \mu^0, \theta)' = (1, 0, 0)'$  for SLR, and set  $\alpha = 0.001$ ,  $\gamma = 0.05$ ,  $\Delta_k = 0$ ,  $V_k$  equals the maximum stock number of sector  $k$ ,  $l_{jk} = 0.001$ ,  $u_{jk} = 1$  and  $w_{jk}^0$  set as normalized market capitalization of component of S&P500 at sector  $k$ .

## 5.2. LR versus SLR

We computed solutions for model (4) over portfolio sizes ranging from 10 to 350 in increments of 10 assets, the upper bound (UB) decreased and the lower bound (LB) increased iteratively in the LR algorithm. Fig. 1 depicts the computational comparison of the LR and partial SLR methods, where the maximal gaps between the lower and upper bound were 2.37% and 4.59%, respectively. Most of the gaps were under 0.5%, especially in the interval, [50 200]. In some cases, SLR returned a better bound and a smaller gap than LR (see  $q = 20, 80$ ), and in other cases SLR was worse than LR (see  $q = 250$ ). Therefore, we use a partial SLR algorithm to approximate the optimal solution in the next section since it returned better solutions compared with those by LR method in the region  $q \in [10, 200]$ .

## 5.3. Comparison of different tracking models

We now solve the models (1), (2), (3), and (4) over various portfolio sizes  $q$  and compare the associated portfolios by different models. The experiments in this section are static in that experiments are done in-sample and out-of-sample but at relatively few out-of-sample points and with no re-balancing and no periodic rolling updates of parameters in the out-of-sample horizon. This type of out-of-sample experiment is

relevant for construction of portfolios that are infrequently re-balanced due to say avoidance of excessive transaction costs. Interesting results include: (1) The computational results for all period intervals demonstrate that without the sector limit constraint, the portfolio allocation are concentrated in fewer sectors (see  $q = 10, 30$ ). This sector diversification can explain the reason why the portfolio with sector limit has a lower variance in the next section. (2) The model with sector limit generates solutions that have constant sector weights regardless of size  $q$  of the tracking portfolio. Although the investment in different sectors is limited (Bertsimas and Shiota, 2009), the authors have not explored how to decide the best sector investments. In this paper, our numerical results shown that the optimal sector weights were consistent with the sector weights of the target index.

Fig. 2 shows the norm value of the difference in the sector weights between the tracking portfolios and target S&P500. “TC” in all figures represents transaction cost and turnover constraints in model (2), and “sector” in all figures refers to the sector limit constraints in model (3). It is clear that when the sector limit constraint is considered, i.e. models (3) and (4), the sector weight of the constructed portfolios was closer to the S&P500 than the model without the sector limit constraint, i.e. model (1) and (2). Fig. 2 was generated based on all computational results under different  $q$  values from 10 to 100.

Fig. 3 illustrates the sector diversification process. Sizes  $q$  equal 10, 30, and 100 represent the low, medium and high density portfolios. For a small portfolio size ( $q = 10$ ), the stocks only distribute in 5 sectors when the sector limit constraint was not incorporated (models (1) and (2)) while the stocks are distributed in 10 sectors if we considered sector limits (model (3) and (4)). One major advantage of the sector limit constraint is that the diversification in sectors can reduce the portfolio risk. The sector limit constraint can induce the investment allocation across 10 sectors, i.e. the maximal sector fraction without sector limit is constantly larger than the fractions with sector limit. For instance, people will invest 59.27% of budget using model (1) and 47.23% by using model (2), the largest weight of their budget, to the sector of financials if only transaction costs and buy-in threshold constraints were incorporated. In contrast, the maximum sector weight is only 17.47% for the sector of Information Technology by model (3-1) and 18.39% by model (4-1).

### 5.3.1. Comparison of Performance across portfolio metrics

In this section we compare the performance of the portfolios constructed by the different tracking models. The performance metrics include optimal objective values, portfolio return, portfolio variance,

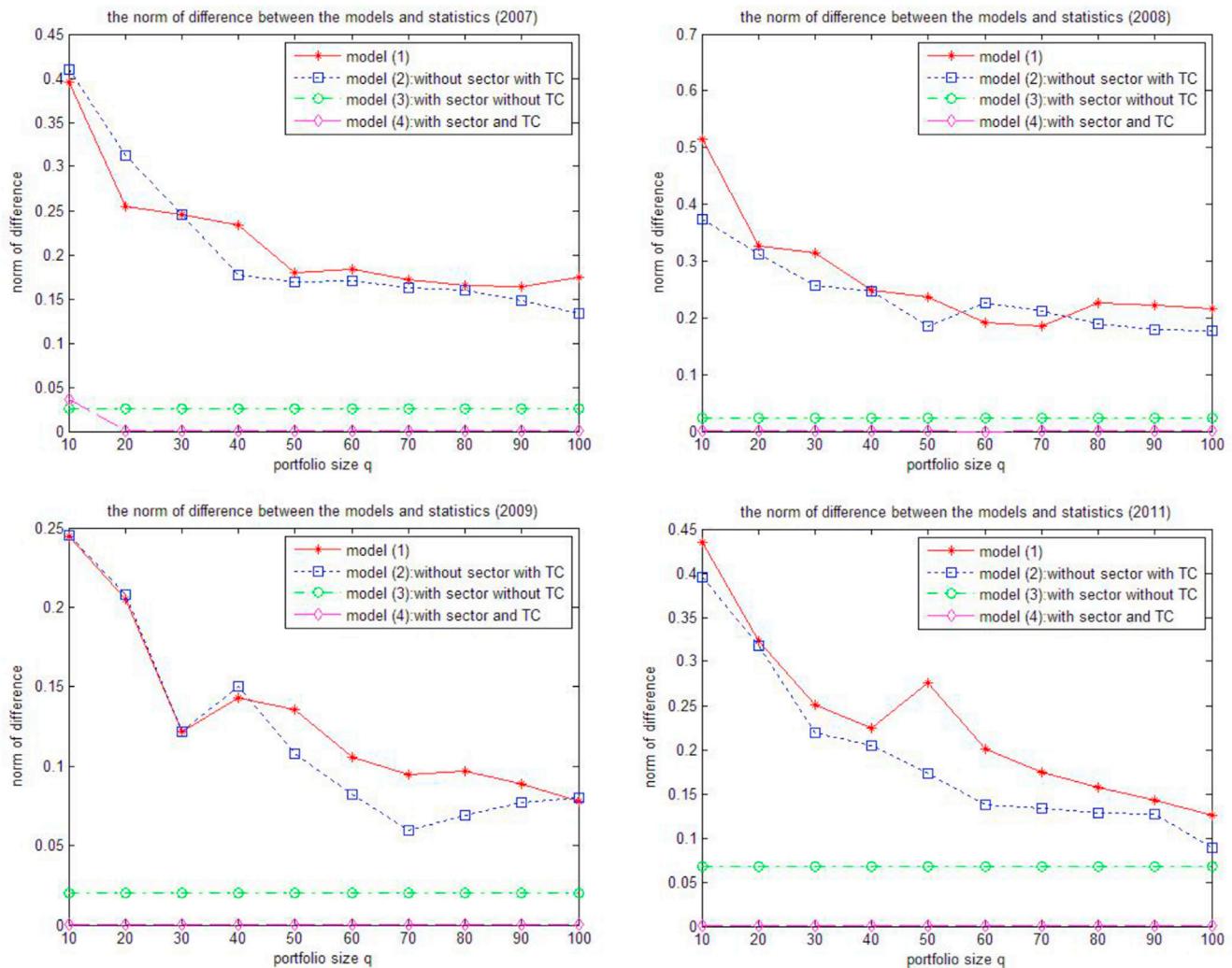


Fig. 2. Norm of sector differences between constructed portfolio and S&amp;P500.

portfolio Sharpe ratio and tracking ratio. Intuitively, the objective value is the first consideration of the comparison between different models since it denotes the similarity of the constructed portfolio with the original index that is tracked. The portfolio return is an important aspect of the performance of the generated tracking portfolios, and the portfolio variance is a prevalent risk measurement of the constructed portfolios. The Sharpe ratio (Sharpe, 1994), or the information ratio, which measures the risk/return efficiency of excess return is the third comparison because it can describe the trade-off between the excess return to the market and the associated portfolio risk. Finally, the tracking ratio was used to compare the tracking quality of the portfolio during different out-of-sample period under different restrictions. Figs. 4–8 show the numerical results with respect to different portfolio sizes  $q$  from 10 to 100 per 10 units for different time periods.

### 5.3.2. Comparison of tracking portfolio similarity

As shown in Fig. 4, the optimal objective value increased with respect to portfolio size  $q$ . Model (1) gives the greatest objective value while model (4) presents the smallest value, which is reasonable since model (4) is the most constrained. The objective value of models with sector limits are less than that without sector limits. For example, for any specific  $q$ , the value of model (1) is larger than that of the model (3) and the value of model (2) is larger than that of model (4). This is obvious as more strict constraints are added into the underlying model.

Compared with models (2) and (3), we can see that the sector limit constraints affected the objective value more significantly than the transaction costs and buy-in threshold constraints, i.e. the value of model (3) decreased faster than value of model (2). One explanation is that the sector limit constraint is a global restriction which dominates the local constraints such as transaction costs. When the local constraints were incorporated, the objective value changed progressively see Fig. 4. In contrast, the objective value changed dramatically with the impact of the global constraint.

### 5.3.3. Comparison of tracking portfolio return

Fig. 5 illustrates the portfolio returns achieved by the different models over different portfolio sizes. The straight line in each plot in Fig. 5 indicates the yearly return of the S&P500. The main goal of the tracking portfolio is to match the return and risk of the market index, as can be seen from Fig. 5, the portfolio returns under different models tend to move closer to the return of the target when the portfolio size became larger. For example, the returns with  $q = 10$  deviated further from the straight line than the returns when  $q = 100$  in 2008. The reason is that when more stocks were allowed in the tracking portfolio full replication could be closer to be achieved. In general, it is seen from Fig. 5 that portfolios generated from model (3) with sector limits and transaction costs generally performed very well e.g. dominated or were close to returns from other portfolios for cardinality

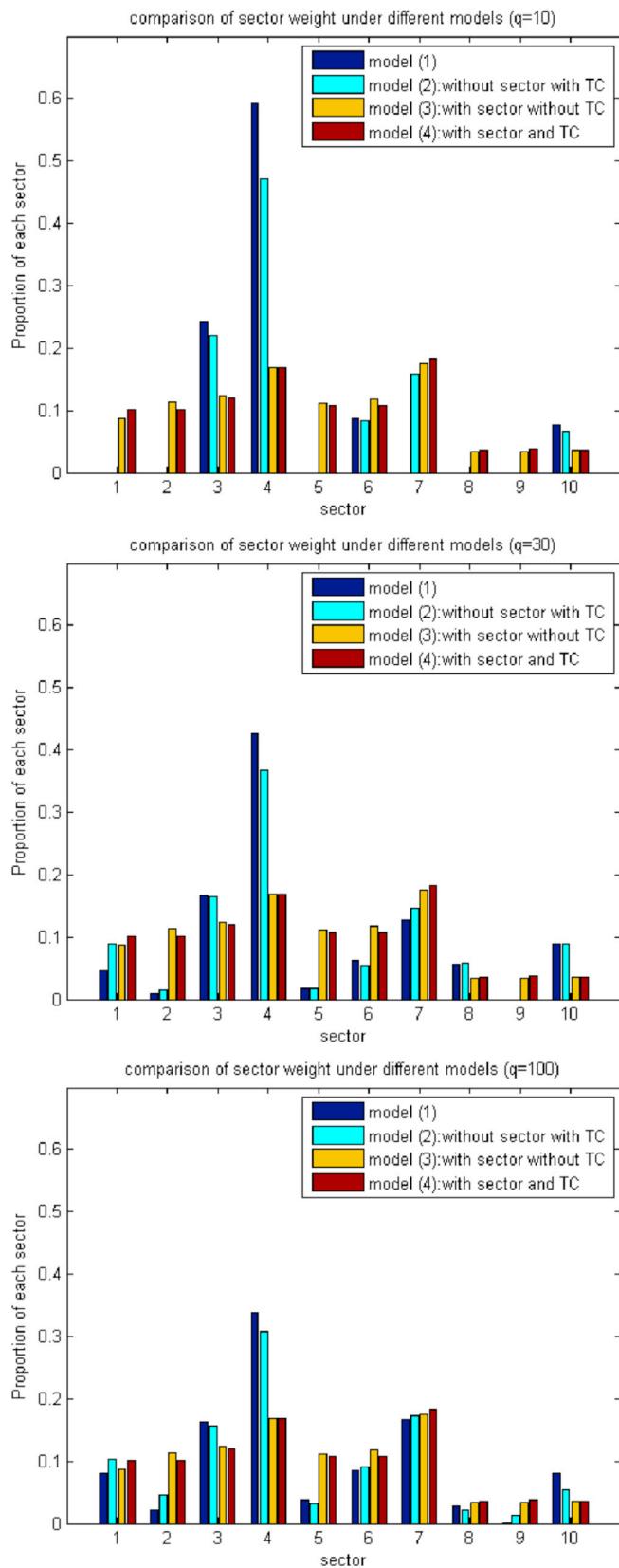


Fig. 3. Sector diversification.

values 30 to 70. The exception was for the 2007 results where model (3) was competitive with other models e.g. not too far below the return

from other portfolios for cardinality sizes 30 to 50, but worse from 50 to 70. For cardinality values larger than 70 portfolios from model (2) with transaction costs but no sector limit constraints generally dominated other portfolios. An interesting observation is that the path of model (2) matches the path of model (1) in every sub-figure, while the lines of model (3) did not follow that of model (1). As mentioned, the local constraints such as transaction costs may gradually affect the solution structure with the changes in the portfolio size, so the path of model (2) was close to path of the underlying model (1). On the other hand, the global constraint such as sector limit may lead to a totally different solution, which created different portfolio returns compared with the returns from model (1).

#### 5.3.4. Comparison of tracking portfolio variance

The portfolio variance under different models with respect to the portfolio size is plotted in Fig. 6. The straight line indicates the yearly variance of the S&P500. The smaller the value of the variance, the better. Model (1) and (3) produced the upper bound and lower bound of the variance plots. It can be seen from Fig. 6 that the variance of portfolios from model (1) was 3–7 times higher than the variance values by the model (3). It is also apparent from Fig. 6 that the portfolio variance from portfolios generated by models with the sector limit constraint was less than the variance from models without the sector limit constraint. In summary, the sector constraints impose diversification by distributing the limited number of the stocks into different independent sectors and thereby reduces portfolio risk.

The portfolios generated by model (2) tended to perform worse than the portfolios by model (3) in terms of the portfolio risk. It is not hard to see that the portfolio variance can decrease when the sector limit constraint is incorporated into model (2), i.e. the line of model (2) moved down to the line of model (4) in 2008, 2009, and 2011.

Interestingly, the portfolio variance increased if the transaction costs and buy-in threshold constraints were added into model (3). The reason is that the solution structure of each sector sub-problem became worse due to costly sector transactions that prevent more investments that would reduce risk as the local constraints were incorporated, which in the end results in the higher portfolio variance, i.e. the line of model (3) moved up to the line of model (4) for every sub-figure in Fig. 6.

#### 5.3.5. Comparison of tracking portfolio Sharpe Ratio

The Sharpe ratio was calculated for portfolios generated from the four models. The higher Sharpe ratio value, the better performance of a portfolio. The straight line indicates the yearly Sharpe ratio of the S&P500. From Fig. 7, we can see that the difference of the Sharpe Ratios between model (3) and model (1) was larger than the difference of the Sharpe Ratios between model (2) and model (1), which indicated that the presence of the sector limit constraint improved the Sharpe Ratios more than the presence of the transaction costs and buy-in threshold constraints for the same underlying model. For example, for  $q = 20$  in 2007, the Sharpe Ratio difference between model (3) and model (1) was 0.8, but the Sharpe Ratio difference between model (2) and model (1) was -0.5, which means the sector limit constraint increased the Sharpe Ratio of model (1) but the transaction costs and buy-in threshold constraints decreased the Sharpe Ratio more than the increase. All the Sharpe Ratio values were negative in 2009 when the financial market dropped sharply. The Sharpe Ratio value of model (1) was close to the Sharpe Ratio value of the target, and the model (3) returned most negative value. Overall, from the Sharpe Ratio perspective the portfolios with the sector limit constraint had out-performed the portfolios without the sector limit constraint.

Next we calculate the Sharpe Ratio out-of-sample and compare the difference between Sharpe ratios from in-samples and out-of-samples. The periods of in-samples are the intervals of [2006 2007] and [2010 2011] and the associated out-of-samples are the daily returns in intervals of 2007.01–2008.01 and 2011.01–2011.06. The numerical results are given in the following Tables 3 and 4 with respect to the

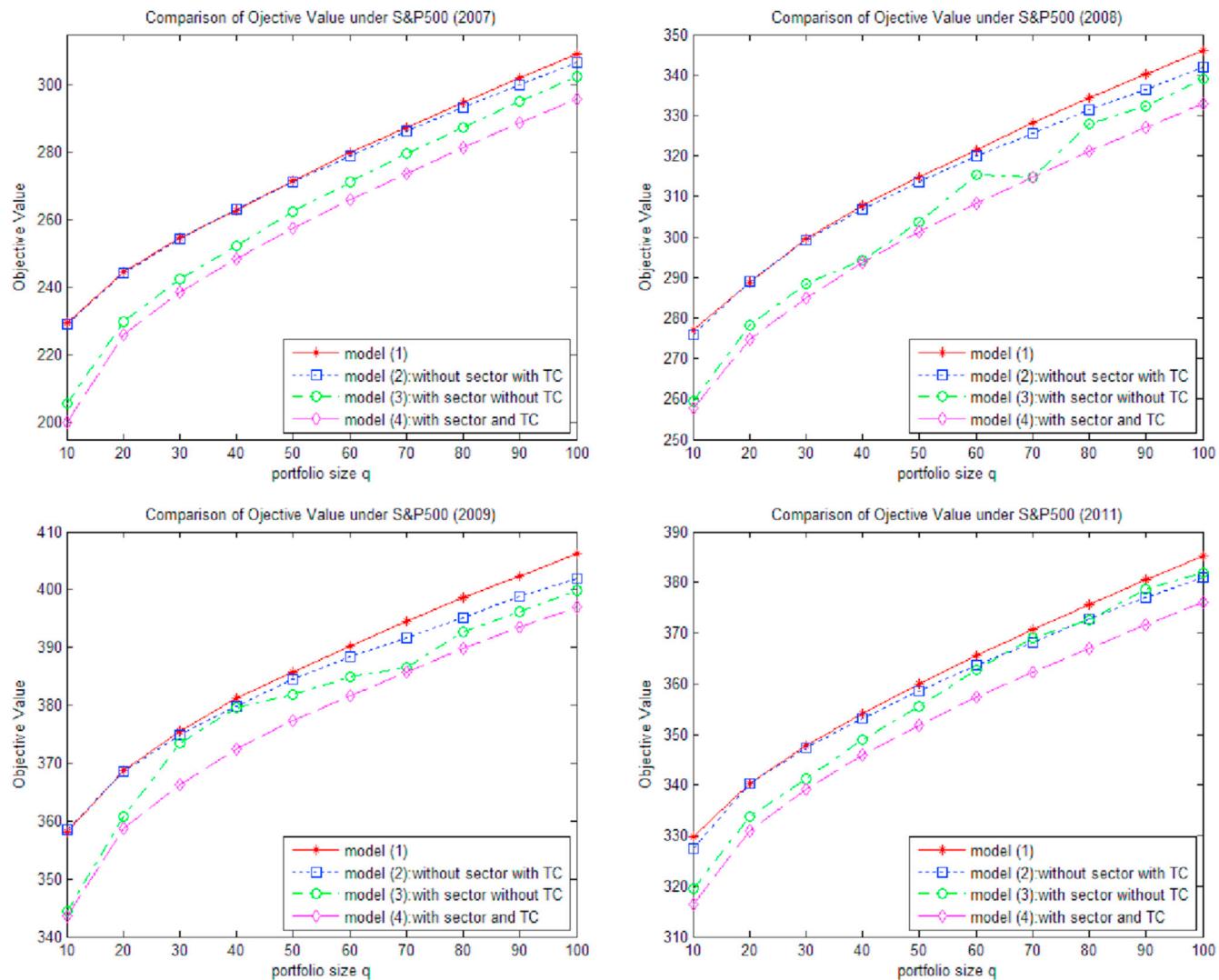


Fig. 4. Comparison of Performance – optimal objective value.

portfolio size. The 'diff' columns are the difference of Sharpe ratios between out-of-sample and in-sample values (Fig. 7), a positive number indicates a portfolio that maintained good performance during the out-of-sample periods, and a negative value means the portfolio constructed had relative underperformance in the associated period without any rebalancing.

From Table 3, we see that the Sharpe ratio values for out-of-sample by models (3) and (4) are generally larger than those from models (1) and (2). In terms of robustness of the Sharpe ratio testing, portfolios from model (3) decreased the most ( $-0.9435$  averagely) while those from model (4) increased  $14.64\%$  on average. Overall, portfolios from model (4) had the best performance for both in-sample and out-of-samples testing.

Table 4 lists the Sharpe ratios from out-of-sample testing during the 2011.01–2011.06. We see that the average Sharpe ratio of portfolios from model (4) and (2) are close to each other, but better than the average values from model (3). Meanwhile the difference between Sharpe ratio values of model (1) and (3) are negative, and model (3) has the largest average difference. Here we point out that for some particular data instances, transaction cost and turnover constraints can generate a portfolio with good average performance (see model (2) performance in Table 4) while for some other instances, models with sector limit constraints generate a better portfolio, e.g. the model (3) in Table 3. Model

(4) generally had good out-of-sample performance as seen in Table 3 to Table 4 since it incorporates both transaction cost and sector limit constraint sets.

#### 5.3.6. Comparison of tracking portfolio tracking ratio

Similar to the definition in (Cornuejols and Tutuncu, 2007), we calculated the tracking ratio by following formula:  $R_{0t} = \frac{\sum_{i=1}^n V_{it}}{\sum_{i=1}^n V_{i0}} / \frac{\sum_{j=1}^q w_j V_{jt}}{\sum_{j=1}^q w_j V_{j0}}$ ,  $t = 1, \dots, T$ , where  $\sum_{i=1}^n V_{it} / \sum_{i=1}^n V_{i0}$  indicates the target index's movement after investment,  $\sum_{j=1}^q w_j V_{jt} / \sum_{j=1}^q w_j V_{j0}$  denotes the portfolio's performance during the out-of-sample period. The ideal tracking ratio,  $R_{0t}$ , is 1, a higher value over than 1 means underperformance with respect to the target index, and a lower value less than 1 indicates excessive return. The straight line indicates the portfolio perfectly tracked the market index, S&P500. The out-of-sample periods were tested where the durations are 6 months and 12 months respectively, there was no rebalancing during the tracking period after investment.

Fig. 8 (a) and Fig. 8 (b) displayed the out-of-sample tracking ratios for four periods. As shown in Fig. 8 (a), the tracking portfolios might have better tracking performance in the near future (6 months) than the longer future (12 months). For example, all portfolios were superior to the

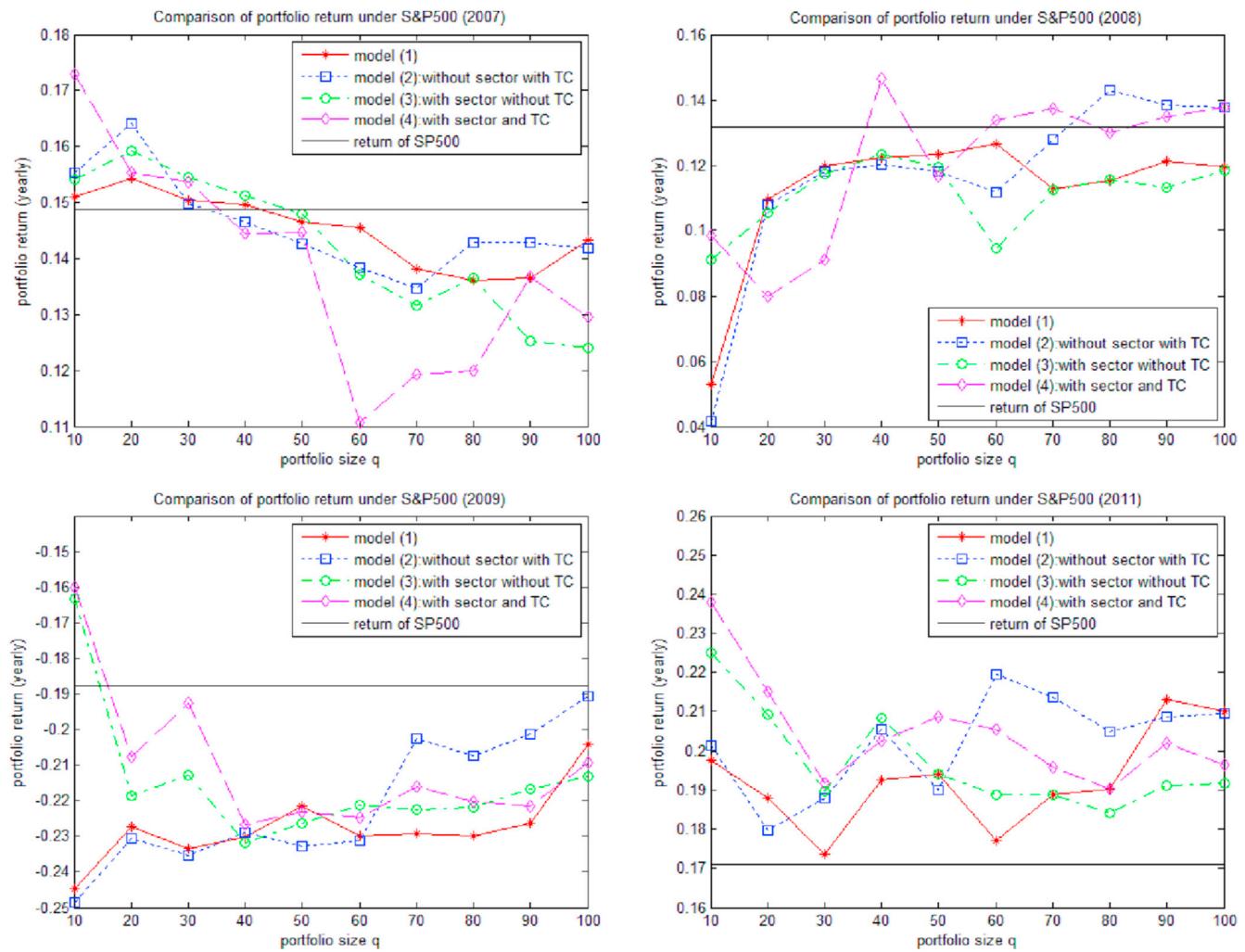


Fig. 5. Comparison of Performance –portfolio return.

market index during 2007.1–2007.6, i.e. all  $R_{0,6} < 1$ , while some portfolios had under-performed the market during 2007.1–2007.12, i.e. all  $R_{0,12} > 1$ . Another observation was that the models with the sector limit constraint had a more stable performance than the models without sector limit constraint.

### 5.3.7. Summary of static computational results

Taken together, the analysis from Figs. 4–8 provides important insights into the portfolio performance under different restrictions under static conditions i.e. in-sample and out-of-sample experiments with no rebalancing and no periodic rolling updates of parameters in the out of sample horizon. In particular, the sector limit constraint, as a global restriction, can induce generation of tracking portfolios that have better variance and Sharpe ratios than portfolios generated without these extra global constraints and can match the sector market capital weights of indices more closely.

As can be seen, model (4) has minimal objective value as it includes all local and global constraints. Model (1) achieved the best portfolio returns, but it also has the worst portfolio variance. Model (2) generally had poor portfolio returns and variances, while model (3) and (4) achieved good returns and variances, which indicates that the sector limit constraint can improve the portfolio risk but without sacrificing too much return. Model (3) had the best in-samples Sharpe ratios and model (4) had the best Sharpe ratio in (non-rolling horizon) out-of-

sample testing. Model (1) and (2) have poor Sharpe ratios for both in-sample and out of sample testing. With respect to the tracking ratio, we rank the absolute difference between the value in bracket and 1 with the smaller the absolute value the better. We cannot determine which model is the best since different sizes have different ranking under different periods. Finally, it is clear that model (4) and (3) can match the sector market capital weights better than the model (1) and (2).

### 5.4. Rolling window out-of-sample experiments

Next, we change the static framework of the experimental setup to study the relative performance of the different models over an extended timeframe. The investment horizon for this experiment ranges from 2007 to 2011 for a total of five years. The portfolios are rebalanced on a monthly basis, re-estimating the parameters at the start of each one-month investment period. As before, we use the daily returns to estimate the correlation matrix corresponding to the previous twelve months before the start of each investment period.

The five year investment horizon introduced the additional complexity of accounting for yearly changes to the constituents list of the S&P 500 index. This was resolved by only including assets which belonged to the S&P 500 during the entire investment horizon, reducing the list of constituents to 380 stocks. Although this introduces a

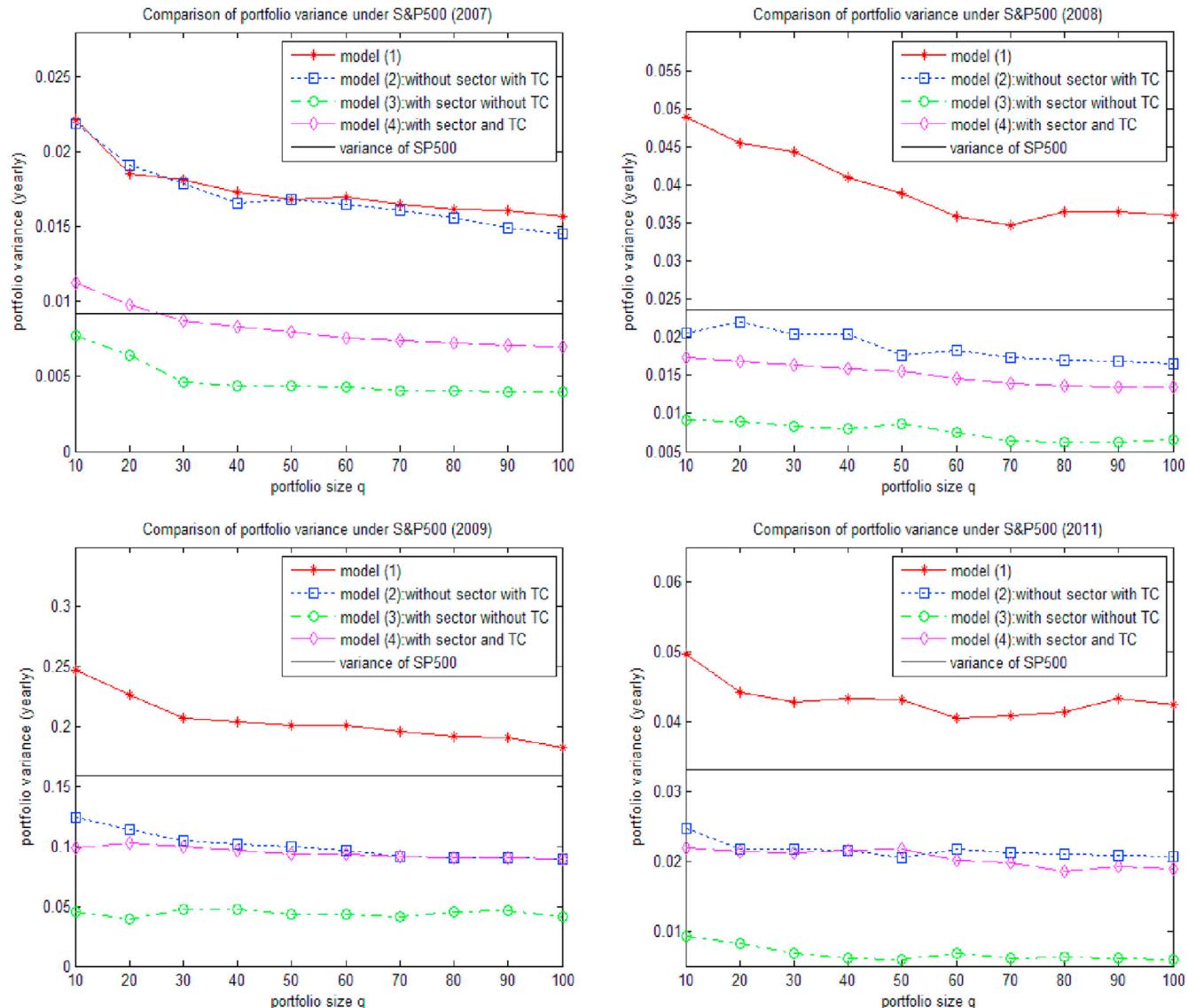


Fig. 6. Comparison of Performance –portfolio variance.

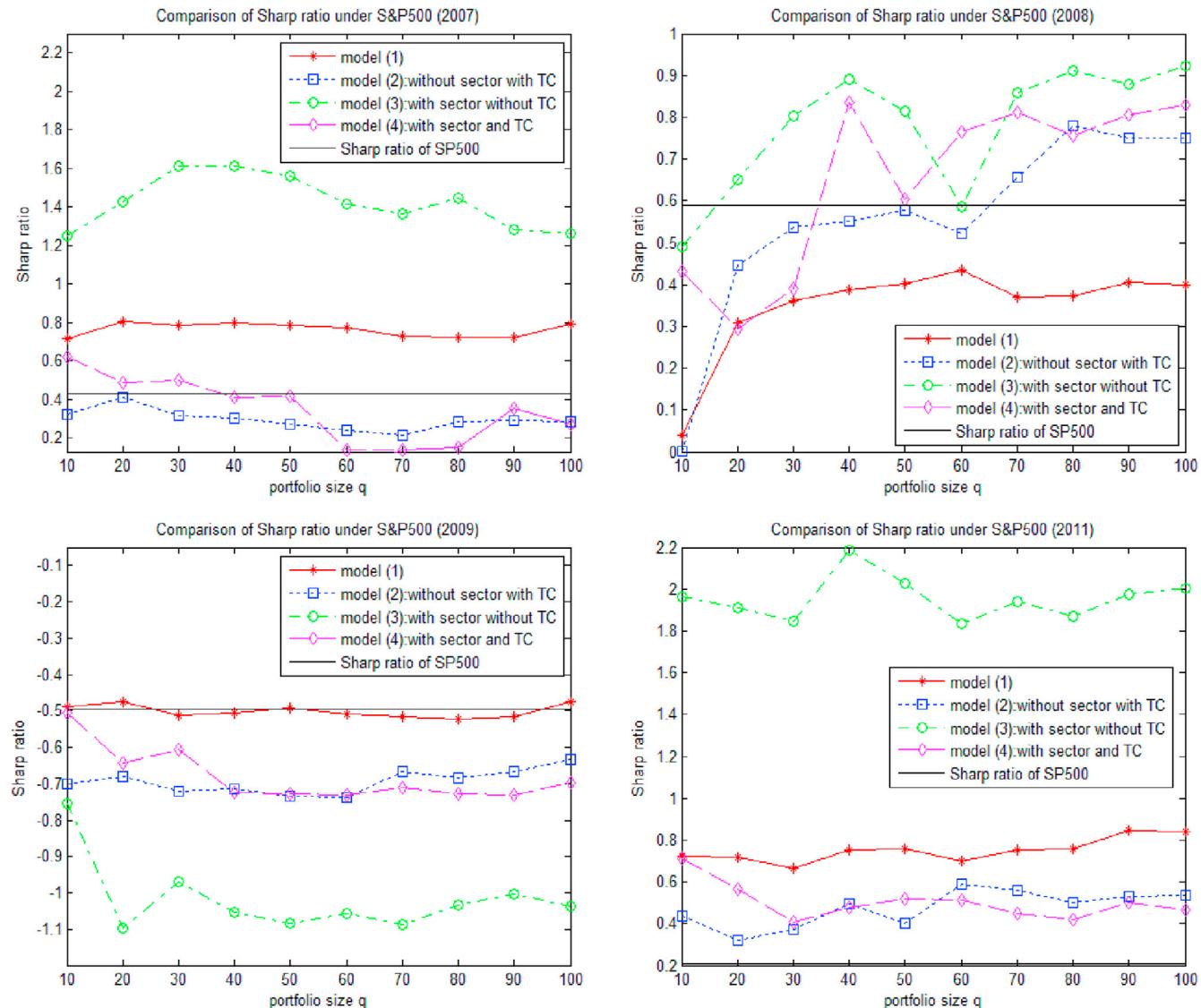


Fig. 7. Comparison of performance – Sharpe ratio.

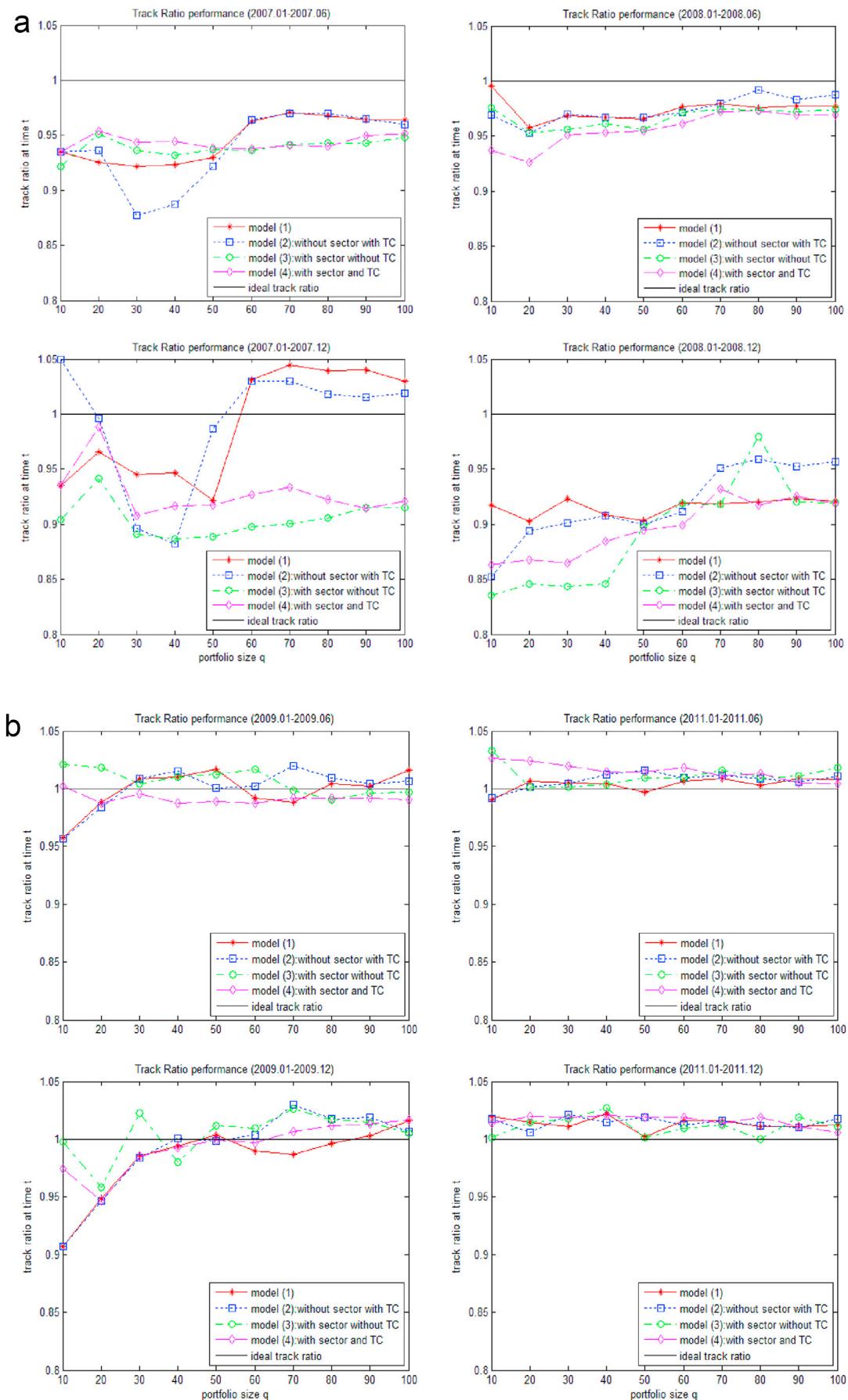


Fig. 8. (a): Comparison of Performance – Tracking Ratio of out-of-sample period (2007, 2008). (b): Comparison of Performance – Tracking Ratio of out-of-sample period (2009, 2011).

**Table 3**

Sharpe ratio for out-of-sample period 2007.01–2008.01.

q	Model (1)	diff	Model (2)	diff	Model (3)	diff	Model (4)	Diff
10	-0.0250	-0.7426	0.3918	0.0661	0.5313	-0.7192	0.6134	-0.0018
20	0.2774	-0.5298	0.0885	-0.3244	0.2845	-1.1435	0.2849	-0.2021
30	0.4229	-0.3665	0.3495	0.0298	0.5334	-1.0802	0.6320	0.1325
40	0.4448	-0.3566	0.2209	-0.0858	0.4309	-1.1776	0.5143	0.1034
50	0.4548	-0.3350	0.0952	-0.1789	0.5605	-1.0033	0.5156	0.0960
60	0.1035	-0.6733	0.1558	-0.0871	0.5134	-0.8991	0.6108	0.3383
70	0.0656	-0.6649	0.1160	-0.1012	0.5741	-0.7875	0.4562	0.2166
80	0.0504	-0.6702	0.1373	-0.1493	0.4573	-0.9863	0.5284	0.1433
90	0.0612	-0.6650	0.2083	-0.0841	0.4384	-0.8436	0.6103	0.2578
100	0.1295	-0.6611	0.1722	-0.1163	0.4708	-0.7944	0.6494	0.3797
Average	0.1985	-0.5665	0.1935	-0.1031	0.4795	-0.9435	0.5415	0.1464

**Table 4**

Sharpe ratio for out-of-samples 2011.01–2011.06.

q	Model (1)	diff	Model (2)	diff	Model (3)	diff	Model (4)	diff
10	0.7713	0.0471	1.1298	0.6921	0.4465	-1.5184	0.9845	0.2717
20	0.8116	0.0914	1.1298	0.8110	0.5405	-1.3674	0.9125	0.3467
30	0.7792	0.1156	0.6981	0.3241	0.6030	-1.2403	0.6983	0.2933
40	0.5514	-0.1996	0.8016	0.3052	0.4781	-1.7064	0.7322	0.2561
50	0.8077	0.0491	0.7605	0.3598	0.4650	-1.5620	0.7193	0.2033
60	0.6495	-0.0487	0.6087	0.0183	0.7241	-1.1084	0.6824	0.1689
70	0.6209	-0.1324	0.5740	0.0173	0.9563	-0.9827	0.9863	0.5382
80	0.6763	-0.0793	0.6740	0.1751	0.6803	-1.1866	0.6926	0.2706
90	0.6638	-0.1852	0.6674	0.1392	0.6685	-1.3072	0.6651	0.1645
100	0.6988	-0.1445	0.6934	0.1597	0.6958	-1.3081	0.7062	0.2426
Average	0.7030	-0.0487	0.7737	0.3002	0.6258	-1.3288	0.7779	0.2756

**Table 5**

Average yearly returns.

q	Yearly Returns				
	S&P 500	Model (1)	Model (2)	Model (3)	Model (4)
30	-0.032	-0.0367	-0.0332	-0.037	-0.0259
40	-0.032	-0.0388	-0.0273	-0.0269	-0.0176
50	-0.032	-0.012	-0.012	-0.034	-0.0116
60	-0.032	-0.0157	-0.0169	-0.0114	-0.0131
70	-0.032	-0.0119	-0.0198	-0.0189	-0.0145
80	-0.032	-0.0132	-0.0145	-0.0095	-0.0105

survivorship bias to the experiment, we expect the bias to have a similar effect on all the models under consideration. This, in turn, should still provide valuable relative results.

The portfolio metrics were computed as before but with the additional dimension of time. Thus, the experiment was performed with a reduced set of cardinality values, ranging only from 30 to 80, to avoid overwhelming the reader with results. Similar to the previous experiment, the portfolio metrics consist of portfolio return, portfolio standard deviation, Sharpe ratio, tracking error and tracking ratio.

**Table 5** gives the average yearly return of the S&P 500 and all models. It can be seen that model (4) has the best returns for cardinality sizes 30 to 50 and model (4) is the only model that outperforms the S&P 500 across all cardinality sizes.

The yearly Sharpe ratios as seen in **Fig. 9** indicate that models (1) thru (4) generally had better yearly Shape ratios than the S&P 500 during the financial crisis of 2007–2008 and all models were very competitive with the S&P 500 during the aftermath (recovery) from 2009 to 2011, but there was no clear model or strategy including the S&P 500 that dominated all others on a yearly basis.

The results in **Table 6** show that model (4) has better Sharpe ratios over the entire 5 year period than most other models at the various cardinalities and dominated the S&P 500 over all specified cardinalities.

**Table 7** shows that the average daily standard deviations of all models were higher than the S&P 500 which is not surprising considering that all models had at most 80 assets, but there is no one model that dominates

other models.

All models give larger tracking error during the time of the financial crisis e.g. 2008–2009. As seen in **Fig. 10**, Model (4) exhibits less variability in tracking error while model (3) exhibits more variability for cardinalities 30 to 60. For larger cardinalities, model (1) exhibits more variability than other models. This highlights that it may be insufficient to just have larger cardinality in portfolios to reduce tracking error and that sector constraints are helpful in reducing tracking errors.

**Table 8** shows that the average daily tracking errors of the models are small and that as more assets are allowed tracking error improves across all models.

**Table 9** shows that all models have average tracking ratios near 1 indicating that all models are generating portfolios with value similar to that of the S&P 500.

#### 5.4.1. Summary of dynamic computational experiments

From **Tables 5–9** and **Figs. 9 and 10** it is seen that models with the sector constraint, especially model (4), performed well over the various performance metrics. These models had good Sharpe Ratio performance although they could not dominate the S&P at every instance but generally had higher returns than the S&P 500 but with higher volatility. All models had small tracking errors and tracking ratios near 1. In summary, the models with sector constraints exhibited similar advantages as in the static case in section 5.3.

## 6. Conclusions

In this work we have investigated portfolio tracking models that are linear mixed integer optimization problems that represent a constrained clustering approach for tracking a benchmark index, in particular the S&P 500. Motivated by real investment cases transaction costs and sector limits constraints were added to a base clustering model. We then developed both a Lagrangian Relaxation (LR) algorithm and the partial Semi-Lagrangian Relaxation (SLR) algorithm to solve the tracking problem with constraints. Numerical results have shown that both of the methods can achieve high quality solutions. Through the computational results we observe: (1) the sector limit constraint can diversify stocks into

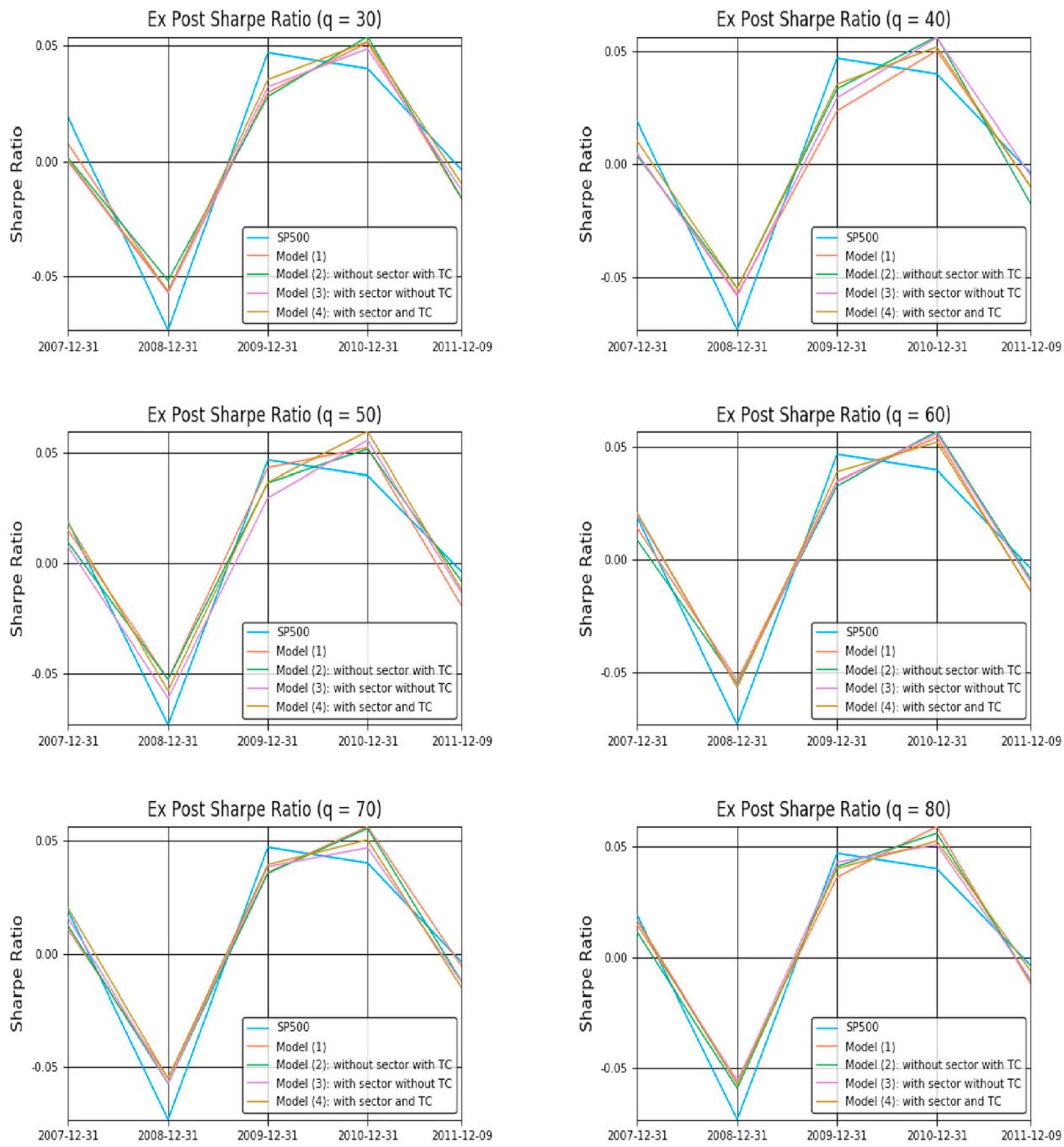


Fig. 9. Yearly Sharpe ratios.

**Table 6**  
Sharpe Ratios for single 5 year period from 2007–2011.

q	Sharpe Ratio				
	S&P 500	Model (1)	Model (2)	Model (3)	Model (4)
30	-0.0057	-0.00606	-0.00531	-0.00598	-0.00373
40	-0.0057	-0.00662	-0.00408	-0.00383	-0.00209
50	-0.0057	-0.00079	-0.00091	-0.00567	-0.0008
60	-0.0057	-0.00176	-0.00193	-0.00079	-0.00114
70	-0.0057	-0.00089	-0.00266	-0.00242	-0.00138
80	-0.0057	-0.00123	-0.00154	-0.00031	-0.00057

**Table 7**  
Average standard deviation of daily returns over 2007–2011 horizon.

q	Portfolio Standard Deviation				
	S&P 500	Model (1)	Model (2)	Model (3)	Model (4)
30	0.01673	0.01964	0.01965	0.01981	0.01957
40	0.01673	0.01923	0.01905	0.0194	0.01931
50	0.01673	0.01904	0.01887	0.01923	0.01898
60	0.01673	0.01898	0.01871	0.01919	0.01866
70	0.01673	0.01881	0.01855	0.01888	0.01858
80	0.01673	0.01878	0.01816	0.01868	0.0184

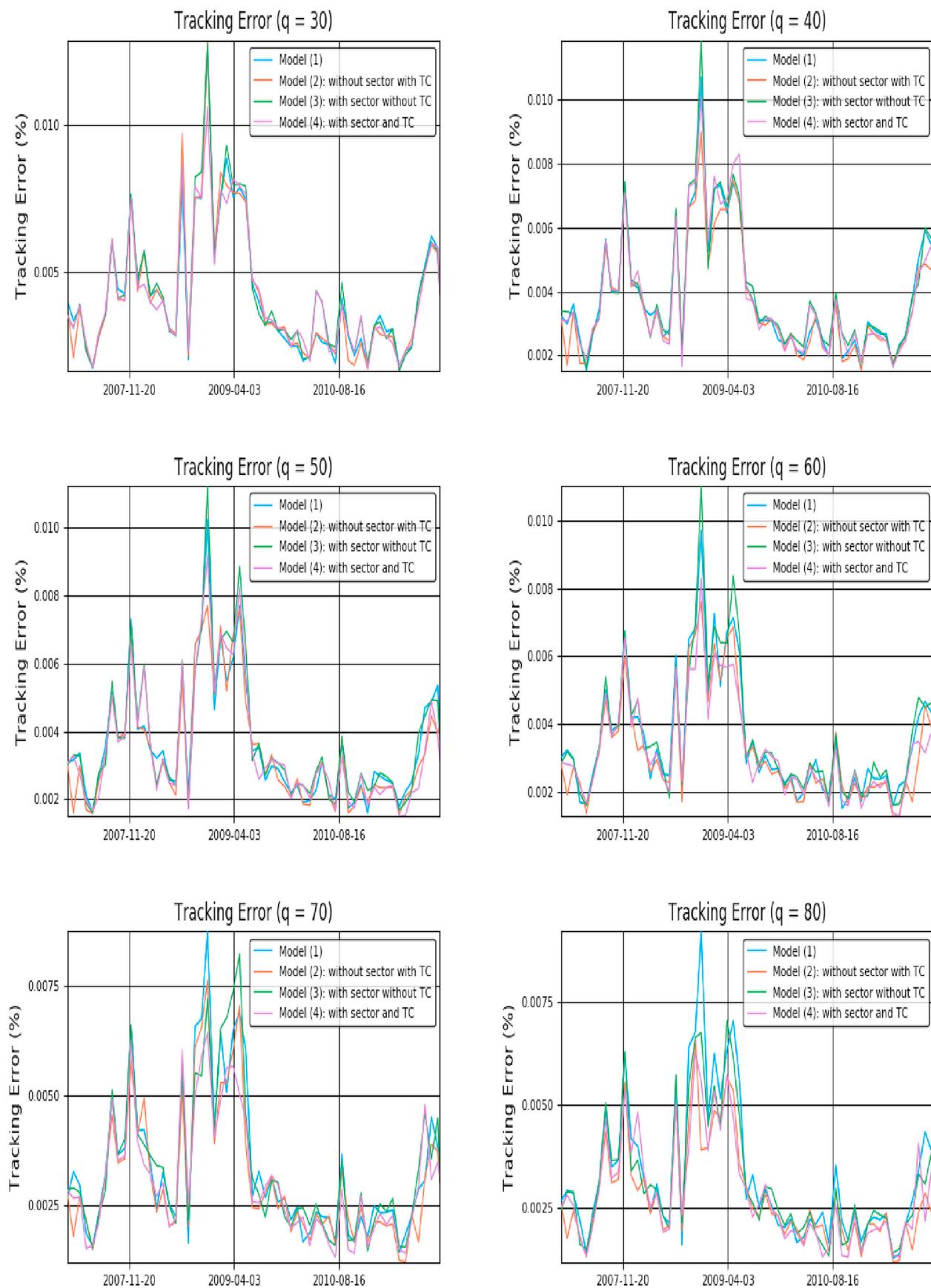


Fig. 10. Monthly tracking error.

**Table 8**

Average daily tracking error over 2007–2011 period.

Q	Average Tracking Error			
	Model (1)	Model (2)	Model (3)	Model (4)
30	0.00474	0.00473	0.00487	0.00467
40	0.00429	0.00406	0.00443	0.0043
50	0.00402	0.00379	0.00422	0.00394
60	0.00393	0.0036	0.00409	0.00356
70	0.00373	0.00344	0.00373	0.00333
80	0.00368	0.00303	0.00345	0.00312

**Table 9**

Average monthly tracking ratios.

q	Average Tracking Ratio			
	Model (1)	Model (2)	Model (3)	Model (4)
30	1.00071	1.00036	1.00065	1.0000
40	1.00072	0.99955	0.99976	0.99893
50	0.99866	0.99856	1.00035	0.99858
60	0.9988	0.9990	0.99848	0.99868
70	0.99854	0.99928	0.9990	0.99897
80	0.99858	0.99864	0.99862	0.99867

different sectors; (2) the optimal sector weights are consistent to the sector weights of the target index if the sector limit constraint is incorporated; and (3) models with sector constraints achieved good out-of-sample Sharpe Ratios and small tracking error.

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