Solving for orbital elements given observations in 6-d phase space

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1. Introduction

This document describes the method and equations I used for adopting the methodology of OFTI to make use of 3-d velocities for fitting for orbital elements.

If the two components of a binary system are resolved in Gaia, we can make use of Gaia's astrometric observations of RA/Dec and proper motion in RA/Dec as constraints for fitting orbital elements. If Gaia also obtained radial velocities for both bodies (or if they're been attained independently), then that gives a 5th measurement, leaving only one unconstrained phase space measurement of Z position. If accelerations have been measured for any coordinate, then this provides even more constraints, and allows Z to be removed as a free parameter.

The orbital elements are: a (semi-major axis) [and thus T (period - derived from Kepler's 3rd law)], t_0 (time of periastron passage), e (eccentricity), i (inclination), ω (argument of periastron - angle from ascending node to periapse), and Ω (longitude of periastron - angle of location of ascending node from reference direction). (Note - other sybmols are often used from these variables [for example, P rather than T for period]. I will be using the symbols as they are defined in Seager (2010) for consistency).

These formulae are taken from the book *Exoplanets* (Seager (2010)), Part 1: Keplerian Orbits and Dynamics of Exoplanets, by C.D. Murray and A.C.M. Correia (Murray & Correia 2010).

2. Derivation of equations for positions, velocities, and accelerations given orbital elements

For fitting orbits to stellar binaries in *Gaia*, we reduce the two-body system to the relative motion of one mass-less point particle around a central object of mass equal to the total system mass. Taking the central body of a 2-body Keplerian orbit to be at the origin of the 3-d cartesian coordinate system, the position of the orbiting body is given by the coordinates (X,Y,Z), where +X is the reference direction, equal to +Declination in the on-sky coordinates. +Y is the +RA direction, and +Z is the line of sight direction towards the observer. This is the coordinate system presented in Murray & Correia (2010), from which we base our derivation.

2.1. Positions

Murray & Correia (2010) eqns 53, 54, and 55 derives the following formulae for projecting orbital elements onto the plane of the sky:

$$X = r[\cos\Omega\cos(\omega + f) - \sin\Omega\sin(\omega + f)\cos i] \tag{1}$$

$$Y = r[\sin\Omega\cos(\omega + f) - \cos\Omega\sin(\omega + f)\cos i] \tag{2}$$

$$Z = r\sin(\omega + f)\sin i \tag{3}$$

+X and +Y correspond to the observed +ΔDec ($\Delta\delta$) and +ΔRA ($\Delta\alpha$) respectively between the orbiting body and central object. In this system +Z is defined **towards the observer** (contrary to radial velocity convention).

r is the radius of the orbiting body in the orbital plane, and is given as:

$$r = \frac{a(1-e^2)}{1 + e\cos f} \tag{4}$$

f is the true anomaly and is given by solving Kepler's equation at the observation date (for *Gaia* DR2 this is 2015.5):

$$M = \frac{2\pi}{T}\big(Date - t_o\big) \tag{5}$$

$$g(E) = E - e\sin E - M \tag{6}$$

which is a transcendental equation which must be solved numerically (such as Newton-Raphson method). $\big[T$ is derived from Kepler's 3rd law as $T = \sqrt{\frac{4\pi^2 a^3}{\mu}}$, where $\mu = G(m_1 + m_2).\big]$ The true anomaly then is given by:

$$f = 2\ \arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right) \tag{7}$$

2.2. Velocities

Murray & Correia (2010) derives in eqn 63 the formula for velocity in the Z direction as:

$$\dot{Z} = \dot{r}\sin(\omega + f)\sin i + r\dot{f}\cos(\omega + f)\sin i \tag{8}$$

which is the time derivative of Z. In the equations above, only r and f vary with time. This corresponds to the observed radial velocity.

Taking the time derivative of X and Y we obtain the velocity in the X and Y direction, which corresponds to proper motion in the Dec and RA directions respectively (μ_δ and μ_α).

$$\begin{aligned}\dot{X} = \dot{r}\,\big[\,\cos\Omega\cos(\omega + f) - \sin\Omega\sin(\omega + f)\cos i\,\big] + \\ r\dot{f}\,\big[\,-\cos\Omega\sin(\omega + f) - \sin\Omega\cos(\omega + f)\cos i\,\big]\end{aligned} \tag{9}$$

$$\begin{aligned}\dot{Y} = \dot{r}\,\big[\sin\Omega\cos(\omega + f) + \cos\Omega\sin(\omega + f)\cos i\,\big] + \\ r\dot{f}\,\big[\,-\sin\Omega\sin(\omega + f) + \cos\Omega\cos(\omega + f)\cos i\,\big]\end{aligned} \tag{10}$$

Thus equations (1)-(3) and (8)-(10) form the 6-dimensional phase space of position and velocity in three dimensions.

$\dot{r}$ and $r\dot{f}$ are the time rate of change of separation and angular distance from the focus of the ellipse (the central body). Eqns (31) and (32) in Murray & Correia (2010) define $\dot{r}$ and $r\dot{f}$ in terms of a , e , and f:

$$\dot{r} = \frac{na}{\sqrt{1-e^2}} e \sin f \tag{11}$$

$$r\dot{f} = \frac{na}{\sqrt{1-e^2}} \big(1 + e \cos f\big) \tag{12}$$

Where $n = \frac{2\pi}{T}$.

And our final position and velocity equations become:

$$X = \frac{a(1-e^2)}{1+e\cos f} (\cos\Omega \cos(\omega+f) - \sin\Omega \sin(\omega+f)\cos i) = \Delta\delta \tag{13}$$

$$Y = \frac{a(1-e^2)}{1+e\cos f} (\sin\Omega \cos(\omega+f) - \cos\Omega \sin(\omega+f)\cos i) = \Delta\alpha \tag{14}$$

$$Z = \frac{a(1-e^2)}{1+e\cos f} \sin(\omega+f) \sin i \tag{15}$$

$$\begin{aligned} \dot{X} = \frac{na}{\sqrt{1-e^2}} \, [\, & e \sin f (\cos\Omega \cos(\omega+f) - \sin\Omega \sin(\omega+f)\cos i) + \\ & \big(1+e\cos f\big)(-\cos\Omega \sin(\omega+f) - \sin\Omega \cos(\omega+f)\cos i)\,] = \mu_\delta \end{aligned} \tag{16}$$

$$\begin{aligned} \dot{Y} = \frac{na}{\sqrt{1-e^2}} \, [\, & e \sin f (\sin\Omega \cos(\omega+f) + \cos\Omega \sin(\omega+f)\cos i) + \\ & \big(1+e\cos f\big)(-\sin\Omega \sin(\omega+f) + \cos\Omega \cos(\omega+f)\cos i)\,] = \mu_\alpha \end{aligned} \tag{17}$$

$$\dot{Z} = \frac{na}{\sqrt{1-e^2}} \big[e \sin f \sin(\omega+f) \sin i + \big(1+e\cos f\big) \cos(\omega+f) \sin i\big] = RV \tag{18}$$

2.3. Accelerations

Beginning with eqns (8)-(10), we derive the second time derivative for X,Y, and Z position as

$$\begin{aligned} \ddot{X} = (\ddot{r} - r\dot{f}^2) \, \big[\, & \cos\Omega \cos(\omega+f) - \sin\Omega \sin(\omega+f \ \cos i \,\big] + \\ & (-2\dot{r}\dot{f} - r\ddot{f}) \, \big[\, \cos\Omega \sin(\omega+f) + \sin\Omega \cos(\omega+f)\cos i \,\big] \end{aligned} \tag{19}$$

$$\begin{aligned} \ddot{Y} = (\ddot{r} - r\dot{f}^2) \, \big[\, & \sin\Omega \cos(\omega+f) + \cos\Omega \sin(\omega+f)\cos i \,\big] + \\ & (2\dot{r}\dot{f} + r\ddot{f}) \, \big[\, \sin\Omega \sin(\omega+f) + \cos\Omega \cos(\omega+f)\cos i \,\big] \end{aligned} \tag{20}$$

$$\ddot{Z} = \sin i \, \big[\, (\ddot{r} - r\dot{f}^2\,) \sin(\omega+f) + (2\dot{r}\dot{f} + r\ddot{f}) \cos(\omega+f) \,\big] \tag{21}$$

Murray & Correia (2010) do not derive expressions for $\ddot{r}$ or $\ddot{f}$. As f depends on E, which varies with time, and is a transcendental, this gets messy quickly. Fortunately

Klioner (2016) has a lovely discussion of the two-body problem, and while they do not derive what we need directly, we can make use of their work to assemble needed expressions.

Klioner (2016) gives two helpful expressions for $\dot{E}$:

$$\dot{E} = \frac{n}{1 - e\cos E} \tag{22}$$

$$\dot{E} = \frac{an}{r} = \frac{n\,(1 + e\cos f)}{1 - e^2} \tag{23}$$

Thus we derive from (22):

$$\ddot{E} = \frac{-\,n\;e\;\sin E}{(1 - e\cos E)^2}\;\dot{E} = \frac{n^2\;e}{(1 - e\cos E)^2}\;\frac{\sin f}{\sqrt{1 - e^2}} \tag{24}$$

Or from (23):

$$\ddot{E} = \frac{-n\;e\;\sin f}{1 - e^2}\;\dot{f} \tag{25}$$

Reverse-engineering (12), we get that

$$\dot{f} = \frac{n\sqrt{1 - e^2}}{(1 - e\cos E)^2} = \dot{E}\frac{\sqrt{1 - e^2}}{1 - e\cos E} = \dot{E}\;\frac{\sin f}{\sin E} \tag{26}$$

where $\sin f = \frac{\sqrt{1-e^2}\sin E}{1-e\cos E}$. Which is kind of a nice result.

We can write r in a more derivative-friendly way as $r = a\,(1 - e\cos E)$, and thus:

$$\dot{r} = a\;e\;\sin E\;\dot{E} \tag{27}$$

$$\ddot{r} = a\,e\;\cos E\,\dot{E}^2\;+\;a\,e\;\sin E\;\ddot{E} \tag{28}$$

And from (26) we get:

$$\ddot{f} = \ddot{E}\;\frac{\sqrt{1 - e^2}}{1 - e\cos E} + \dot{E}^2\;\frac{e\sqrt{1 - e^2}\sin E}{(1 - e\cos E)^2} \tag{29}$$

which reduces to:

$$\ddot{f} = \ddot{E}\;\frac{\sin f}{\sin E} + \dot{E}^2\;\frac{e\sin f}{1 - e\cos E} \tag{30}$$

And now we have all the pieces for computing $\ddot{X}$, $\ddot{Y}$, and $\ddot{Z}$.

REFERENCES

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