UChicago Honors Analysis Quarter 1 2019 Notes

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This is a collection of definitions and theorems from the first quarter of Honors Analysis. It is intended to help me, in rewriting them, remember the definitions, theorems, and proofs of the course and to be a reference in the future for what the course covered.

1 10/2 - Linear Algebra

Definition 1.1 (Vector Space)**.**
A ***vector space*** *is a set with addition and scalar multiplication defined such that for any* $\mathbf{v}, \mathbf{u}, \mathbf{w} \in V$ *and* $\alpha, \beta \in \mathbb{R}$

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \in V$ *(Commutivity)*
2. $(\mathbf{v} + \mathbf{u}) + \mathbf{w} = \mathbf{v} + (\mathbf{u} + \mathbf{w})$ *(Associativity of Addition)*
3. *There exists an element* $\mathbf{0} \in V$ *such that for all* $\mathbf{v} \in V$*,* $\mathbf{0} + \mathbf{v} = \mathbf{v}$ *(Identity)*
4. $\alpha\mathbf{v} \in V$ *(Closed under scalar multiplication)*
5. $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$ *(Associativity of multiplication)*
6. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$ *and* $\alpha(\mathbf{v} + \mathbf{u}) = \alpha\mathbf{v} + \alpha\mathbf{u}$ *(Distributivity)*

Definition 1.2 (Span).
*For a set $B \subset V$, the **span** of B is defined as*

$$\operatorname{span} B = \sum_{\mathbf{v} \in B} a_{\mathbf{v}} \mathbf{v}$$

For any $a_{\mathbf{v}} \in \mathbb{R}$.
For a finite set $B = \{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n\}$, span can also be written as

$$\operatorname{span} B = \sum_{i=1}^{n} a_i \mathbf{v}_i$$

For any $a_i \in \mathbb{R}$.
If $\operatorname{span} B = V$, then we say B spans (or generates) V.

Definition 1.3 (Linear Dependence and Independence).
*A set $B \subset V$ is **linearly independent** if*

$$\sum_{\mathbf{v} \in B} a_{\mathbf{v}} \mathbf{v} = \mathbf{0}$$

*implies that $a_{\mathbf{v}} = 0$ for all $\mathbf{v} \in B$. Otherwise, B is **linearly dependent**.*

Definition 1.4 (Basis and Dimensions).
*A set $B \subset V$ is a **basis** of V if it is linearly independent and is a basis of V. The **dimension** of V equals the cardinality of B. In other words,*

$$\dim V = |B|$$

If B is finite, we say V is finite dimensional, otherwise we say V is infinite dimensional.

Theorem 1.1 (Bases of the same vector space have the same length).
If V is finite dimensional and has bases A, B, then $|A| = |B|$.

Proof. Let V be a finite dimensional vector space with bases

$$A = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

and

$$B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$$

Suppose $k \neq n$, and without loss of generality, let $n > k$. Since $\mathbf{u}_1 \in \operatorname{span} A$,

$$\mathbf{u}_1 = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

Without loss of generality, let $\alpha_1 \neq 0$. Since A is a basis, so is $\{\alpha_1 \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and thus so is

$$A_1 = \{\alpha_1 \mathbf{v}_1 + \sum_{i=2}^{n} \alpha_i \mathbf{v}_i, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\mathbf{u}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

We can repeat this process with $\mathbf{u}_2$.

$$\mathbf{u}_2 = \beta \mathbf{u}_1 + \sum_{i=2}^{n} \alpha_i \mathbf{v}_i$$

Suppose $\alpha_i = 0$ for $i = 2, 3 \ldots, n$. Then $\mathbf{u}_2 - \beta \mathbf{u}_1 = 0$, which contradicts linear independence of $\mathbf{u}_1$ and $\mathbf{u}_2$. Thus, there exists some $\alpha_i \neq 0$. Without loss of generality, we have $\alpha_2 \neq 0$, so by the same process, let

$$A_2 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$$

Repeating this for all other elements of B gives a basis

$$A_k = \{\mathbf{u}_1 \ldots \mathbf{u}_k \mathbf{v}_{k+1} \ldots \mathbf{v}_n\}$$

Since $\mathbf{v}_{k+1} \in \operatorname{span} B$, $\mathbf{v}_{k+1}$, we have

$$\mathbf{v}_{k+1} = \sum_{i=1}^{k} \alpha_i \mathbf{u}_i$$

and thus

$$\mathbf{v}_{k+1} - \sum_{i=1}^{k} \alpha_i \mathbf{u}_i = 0$$

hence A_k is not linearly independent, which is a contradiction. Thus, $k = n$, so $|A| = |B|$. $\square$

Subsequently, vector spaces will be assumed to be finite dimensional unless otherwise stated.

Definition 1.5. *Subspace of Vector Space*
*A **subspace** of a vector space V is a subset $W \subset V$ such that for all $\mathbf{v}, \mathbf{u} \in W$ and $\alpha \in \mathbb{R}$, $\alpha\mathbf{v} + \mathbf{u} \in W$*

Theorem 1.2. *Span of vectors is subspace Let V be a vector space and $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$. Then* $\operatorname{span} B$ *is a subspace of V.*

Proof. Let $\mathbf{u}, \mathbf{w} \in \operatorname{span} B$ and $\lambda \in \mathbb{R}$. Then

$$\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

and

$$\mathbf{w} = \sum_{i=1}^{n} \beta_i \mathbf{v}_i$$

so

$$\lambda\mathbf{u} + \mathbf{w} = \sum_{i=1}^{n} (\lambda\alpha_i + \beta_i)\mathbf{v}_i$$

Since $(\lambda\alpha_i + \beta_i) \in \mathbb{R}$ for $i = 1, \ldots, n$, $\lambda\mathbf{u} + \mathbf{w} \in \operatorname{span} B$, and thus B is a subspace of V. □

Theorem 1.3. *Basis of subspace is subset of basis of space*
Let W be a subspace of V. For any basis A of W with

$$A = \{\mathbf{u}_1, \ldots \mathbf{u}_k\}$$

there exists a basis B of V such that

$$B = \{\mathbf{u}_1 \ldots \mathbf{u}_k, \mathbf{u}_{k+1} \ldots \mathbf{u}_n\}$$

Proof. Let A be as defined in the theorem. Either $V - \operatorname{span} A = \emptyset$ or otherwise (where $-$ is defined as set difference). If $V - \operatorname{span} A = 0$, then $V \subset \operatorname{span} A$. Since $\operatorname{span} A \subset V$, it follows that $V = \operatorname{span} A$ and thus A is a basis of V. Otherwise, there exists a $\mathbf{u}_{k+1} \in V - \operatorname{span} A$. Define $A_1 = A \cup \mathbf{u}_{k+1}$. Since $\mathbf{u}_{k+1} \notin \operatorname{span} A$, A_1 is linearly independent. Repeat this until $V - \operatorname{span} A_j = 0$. The $A_j = B$ is linearly independent and spanning and thus is a basis of V with

$$B = \{\mathbf{u}_1 \ldots \mathbf{u}_k, \mathbf{u}_{k+1} \ldots \mathbf{u}_n\}$$

□

2 10/4 - Linear Algebra

Definition 2.1. *Linear Map*
A ***linear map*** $T : V \to W$ *is a function such that for all* $\mathbf{v}, \mathbf{u} \in V$ *and* $\alpha \in \mathbb{R}$,

$$T(\alpha\mathbf{v} + \mathbf{u}) = \alpha T\mathbf{v} + T\mathbf{u}$$

The set of all linear maps from V *to* W *is denoted* $\mathcal{L}(V, W)$ *and the set of all linear maps from* V *to* V *is denoted* $\mathcal{L}(V)$*. Linear maps from* V *to* V *are also known as linear operators.*

Theorem 2.1. *Properties of Linear Maps*
Let $T : V \to W$ *and* $S : W \to Z$ *be linear maps. The following all hold:*

1. $S \circ T : V \to Z$ *is a linear map.*
2. *If* T *is bijective, it has a unique inverse* $T^{-1} : W \to V$ *such that* $TT^{-1} = T^{-1}T = I$.
3. T *is uniquely determined by its values over its bases.*
4. $\mathcal{L}(V, W)$ *is a vector space.*

1. *Proof.* Let $\alpha \in \mathbb{R}$ and $\mathbf{v}, \mathbf{u} \in V$. From the properties of linear maps we have:

$$\begin{aligned} (S \circ T)(\alpha\mathbf{v} + \mathbf{u}) &= S(T(\alpha\mathbf{v} + \mathbf{u})) && (1) \\ &= S(\alpha T\mathbf{v} + T\mathbf{u}) && (2) \\ &= \alpha(S \circ T)\mathbf{v} + (S \circ T)\mathbf{u} && (3) \end{aligned}$$

   Thus $S \circ T$ is a linear map. □

2. *Proof.* First we will prove a linear inverse exists. Bijective functions have inverses, so if T is bijective, then it has an inverse T^{-1}. Let $\alpha \in \mathbb{R}$ and $\mathbf{v}, \mathbf{u} \in V$.

$$\begin{aligned} T(\alpha T^{-1}\mathbf{u} + T^{-1}\mathbf{v}) &= \alpha TT^{-1}\mathbf{u} + TT^{-1}\mathbf{v} && (1) \\ &= \alpha\mathbf{u} + \mathbf{v} && (2) \\ &= TT^{-1}(\alpha\mathbf{u} + \mathbf{v}) && (3) \end{aligned}$$

Thus, we have $T(\alpha T^{-1}\mathbf{u} + T^{-1}\mathbf{v}) = TT^{-1}(\alpha\mathbf{u} + \mathbf{v})$, which implies $T^{-1}T(\alpha T^{-1}\mathbf{u} + T^{-1}\mathbf{v}) = T^{-1}TT^{-1}(\alpha\mathbf{u} + \mathbf{v})$, which in turn implies $\alpha T^{-1}\mathbf{u} + T^{-1}\mathbf{v} = T^{-1}(\alpha\mathbf{u} + \mathbf{v})$. Thus, T^{-1} is a linear map.
To prove uniqueness, suppose L, G are inverses of T. Then $TL = I = TG$, so $LTL = LTG$, and since $LT = I$, we have $L = G$. Thus, inverses are unique. □

3. *Proof.* Let $\{vv_1, \dots, \mathbf{v}_n\}$ be a basis of V. Suppose $S, T : V \to W$ are such that $S\mathbf{v}_1 = T\mathbf{v}_1, \dots, S\mathbf{v}_n = T\mathbf{v}_n$. Let $\mathbf{v} \in V$, then

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

so

$$\begin{aligned} S\mathbf{v} &= S\sum_{i=1}^{n} \alpha_i \mathbf{v}_i \\ &= \sum_{i=1}^{n} \alpha_i S\mathbf{v}_i \\ &= \sum_{i=1}^{n} \alpha_i T\mathbf{v}_i \\ &= T\sum_{i=1}^{n} \alpha_i \mathbf{v}_i \\ &= T\mathbf{v} \end{aligned}$$

Thus for all $\mathbf{v} \in V$, $S\mathbf{v} = T\mathbf{v}$, so $S = T$. Hence, linear maps are uniquely determined by the image of a basis of the domain. □

4. *Proof.* By the definition of function addition, all the axioms of vector spaces hold except closure over addition and closure of scalar multiplication. It suffices to show that for any $T, S \in \mathcal{L}(V, W)$ and $\alpha \in \mathbb{R}$, $\alpha T + S \in \mathcal{L}(V, W)$ Suppose $\beta \in \mathbb{R}$, and $\mathbf{v}, \mathbf{u} \in V$. Then

$$\begin{aligned} (\alpha T + S)(\beta\mathbf{v} + \mathbf{u}) &= \alpha T(\beta\mathbf{v} + \mathbf{u}) + S(\beta\mathbf{v} + \mathbf{u}) && (1) \\ &= \alpha\beta T(\mathbf{v}) + \alpha T(\mathbf{u}) + \beta S(\mathbf{v}) + S(\mathbf{u}) && (2) \\ &= \beta(\alpha T + S)(\mathbf{v}) + (\alpha T + S)(\mathbf{u}) && (3) \end{aligned}$$

Thus $\alpha T + S$ is a linear map, so $\mathcal{L}(V, W)$ is a vector space. □

Theorem 2.2. $\dim \mathcal{L}(V, W) = \dim V \dim W$
For finite dimensional vector spaces V, W, $\dim \mathcal{L}(V, W) = \dim V \dim W$.

Proof. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V and $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a basis of W. Let $T \in \mathcal{L}(V, W)$. Define

$$T_{ij}(\mathbf{v}_k) = \begin{cases} \mathbf{w}_j & k = i \\ 0 & \text{otherwise} \end{cases}$$

For any linear map $S \in \mathcal{L}(V, W)$, S is uniquely determined by its values over $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

$$S(\mathbf{v}_i) = \sum_{j=1}^{k} \alpha_j \mathbf{w}_j = \sum_{j=1}^{k} \alpha_{ij} T_{ij}(\mathbf{v}_i)$$

Thus

$$S(\mathbf{v}) = \sum_{j=1}^{k} \sum_{i=1}^{n} \alpha_{ij} T_{ij}(\mathbf{v})$$

Thus $\{T_{ij} : i \leq n, j \leq k\}$ is a basis of $\mathcal{L}(V, W)$ which clearly has dimension kn, so $\dim \mathcal{L}(V, W) = \dim V \dim W$. □

Definition 2.2. *Kernel and Image of a Linear Map*
The ***kernel*** *of a linear map $T : V \to W$ is the set $\ker T = \{\mathbf{v} \in V : T\mathbf{v} = 0\}$.*
The ***image*** *of a linear map is the set $\operatorname{Im} T = \{T\mathbf{v} \in W : \mathbf{v} \in V\}$.*

Theorem 2.3. *Kernel and Image are subspaces*
Let $T : V \to W$ be a linear map. Then:

1. $\ker T$ *is a subspace of* V.
2. $\operatorname{Im} T$ *is a subspace of* W.

1. *Proof.* Let $\mathbf{v}, \mathbf{w} \in \ker T$ and $\alpha \in \mathbb{R}$. Then

$$T(\alpha\mathbf{v} + \mathbf{w}) = \alpha T(\mathbf{v}) + T(\mathbf{w}) = 0$$

Thus, $\alpha\mathbf{v} + \mathbf{w} \in \ker T$, so $\ker T$ is a subspace of V. □

2. *Proof.* Let $\mathbf{v}, \mathbf{w} \in \operatorname{Im} T$ and $\alpha \in \mathbb{R}$. Define $\mathbf{v}', \mathbf{w}' \in V$ such that $T\mathbf{v}' = \mathbf{v}$ and $T\mathbf{w}' = \mathbf{w}$. Then

$$T\alpha\mathbf{v}' + \mathbf{w}' = \alpha T\mathbf{v}' = t\mathbf{w}' = \alpha\mathbf{v} + \mathbf{w}$$

Thus $\alpha\mathbf{v} + \mathbf{w} \in \operatorname{Im} T$, so $\operatorname{Im} T$ is a subspace □

Definition 2.3. *Nullity and Rank of a Linear Map*
*The **nullity** of a linear map is the dimension of the kernal:* $\operatorname{null} T = \dim \ker T$*.*
*The **rank** of a linear map is the dimension of the image:* $\operatorname{rank} T = \dim \operatorname{Im} T$*.*

Definition 2.4. *Surjectivity, Injectivity, and Bijectivity*
*A linear transformation $T : V \to W$ is **surjective** if for every $\mathbf{w} \in W$ there exists a $\mathbf{v} \in V$ such that $T\mathbf{v} = \mathbf{w}$.*

*A linear transformation $T : V \to W$ is **injective** if for every $\mathbf{w}, \mathbf{v} \in V$, if $T\mathbf{v} = T\mathbf{w}$ then $\mathbf{v} = \mathbf{w}$.*

*A linear transformation $T : V \to W$ is **bijective** if it is surjective and injective. In otherwords, for an $\mathbf{w} \in W$ there exists a unique $\mathbf{v} \in V$ such that $T\mathbf{v} = \mathbf{w}$.*

Theorem 2.4. *Surjective, Injective, and Bijective Linear Maps*
Let $T : V \to W$ be a linear map and $B \subset V$.

1. *If B spans V and T is surjective, then $TB = \{Tb : b \in B\}$ spans W.*

2. *If B is linearly independent and T is injective, then TB is linear independent in W.*

3. *If B is a basis of V and T is bijective, then TB is a basis of W.*

1. *Proof.* Suppose $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ spans V and T is surjective. Fix $\mathbf{w} \in W$. Then there exists $\mathbf{v} \in V$ such that $T\mathbf{v} = \mathbf{w}$. Since B spans V,

$$\mathbf{v} = \sum_{i=1} \alpha_i \mathbf{u}_i$$

it follows that

$$\mathbf{w} = T\mathbf{v} = \sum_{i=1} \alpha_i T\mathbf{u}_i$$

Thus any element of W is a linear combination of TB, so TB spans W. □

2. *Proof.* Suppose $B = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is linearly independent in V, T is injective, and TB is linearly dependent. Then there exists $\alpha_1, \dots, \alpha_n \neq 0 \in \mathbb{R}$ such that
$$\sum_{i=1}^{n} \alpha_i T\mathbf{u}_i = \mathbf{0}$$
Then
$$T\sum_{i=1}^{n} \alpha_i \mathbf{u}_i = 0$$
but since B is linearly independent,
$$\sum_{i=1}^{n} \alpha_i \mathbf{u}_i = \mathbf{u} \neq 0$$
Thus, $T\mathbf{u} = T\mathbf{0} = \mathbf{0}$ but $\mathbf{u} \neq \mathbf{0}$, so T is not injective. A contradiction is reached, so if B is linearly independent and T is injective, TB is linearly independent. □

3. *Proof.* Suppose B is a basis and T is bijective. Then since B is spanning and T is surjective, TB is spanning. Since B is linearly independent and T is injective, TB is linearly independent. Then TB is a basis. □

Theorem 2.5 (Injective maps map $\mathbf{0}$ to $\mathbf{0}$)**.** *A linear map $T : V \to W$ is injective if and only if* $\ker T = \{\mathbf{0}\}$.

Proof. Suppose $T : V \to W$ is injective. $T\mathbf{0} = \mathbf{0}$, so if $T\mathbf{v} = \mathbf{0}$, then since T is injective $\mathbf{v} = \mathbf{0}$. Hence $\ker T = \{\mathbf{0}\}$.

Suppose $\ker T = \{\mathbf{0}\}$, $\mathbf{v}, \mathbf{w} \in V$, and $T\mathbf{v} = T\mathbf{w}$. Then $T\mathbf{v} - \mathbf{w} = 0$, so $\mathbf{v} - \mathbf{w} = 0$ and thus $\mathbf{v} = \mathbf{w}$. Hence T is injective. □

Definition 2.5. *Isomorphism, Isomorphic Vector Space*
*An **isomorphism** is a bijective linear map $T : V \to W$. Two vector spaces V, W are **isomorphic** if there exists an isomorphism between them.*

Theorem 2.6. *Isomorphic Vector Spaces have the same dimension*
V, W are isomorphic if and only if they have the same dimension.

Proof. Suppose V, W are isomorphic. Then there exists a bijective linear map $T : V \to W$. For a basis B of V with length n, TB is a basis of W with length n. Thus, V and W have the same dimension.

Suppose V, W have the same dimension. Let $A = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V and $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be a basis of W. Define $T\mathbf{v}_i = \mathbf{w}_i$. Fix $\mathbf{w} \in W$. Then

$$\mathbf{w} = \sum_{i=1}^{n} \alpha_i \mathbf{w}_i = \sum_{i=1}^{n} \alpha_i T\mathbf{v}_i = T \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

The last sum is an element $\mathbf{v} \in V$, so T is surjective. Let $\mathbf{v} \in \ker T$ and suppose $\mathbf{v} \neq \mathbf{0}$. Then

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

so

$$T\mathbf{v} = T \sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \sum_{i=1}^{n} \alpha_i T\mathbf{v}_i = \sum_{i=1}^{n} \alpha_i \mathbf{w}_i$$

Since $T\mathbf{v} = 0$, we have

$$\sum_{i=1}^{n} \alpha_i \mathbf{w}_i$$

This contradicts the linear independence of B. Thus $\ker T = \{\mathbf{0}\}$, so T is injective. Since T is injective and surjective, it is bijective. $\square$

Theorem 2.7. *Rank + Nullity Theorem*
For a linear transform $T : V \to W$, $\operatorname{null} T + \operatorname{rank} T = \dim V$.

Proof. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a basis of $\ker T$. Extend that with $\{\mathbf{v}_{p+1}, \ldots, \mathbf{v}_n\}$ to a basis of V and define $\ker T^* = \operatorname{span}\{\mathbf{v}_{p+1}, \ldots, \mathbf{v}_n\}$. We will prove the restriction of T, $S : \ker T^* \to \operatorname{Im} T$ is bijective. First, suppose $\mathbf{v} \in \ker T^*$. Then $\mathbf{v} \in V$, so $S\mathbf{v} \in \operatorname{Im} T$, so S is well-defined. Suppose for some $\mathbf{v} \in \ker T^*$ $T\mathbf{v} = 0$. Then $\mathbf{v} \in \ker T^* \cap \ker T$. Then

$$\mathbf{v} = \sum_{i=1}^{p} \alpha_i \mathbf{v}_i = \sum_{i=p+1}^{n} \alpha_i \mathbf{v}_i$$

so

$$\sum_{i=1}^{p} \alpha_i \mathbf{v}_i - \sum_{i=p+1}^{n} \alpha_i \mathbf{v}_i = 0$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, it is linearly independent, so $\alpha_i = 0$ for $i = 1, \ldots, n$. Thus $\mathbf{v} = 0$, so S is injective. Suppose $\mathbf{w} \in \operatorname{Im} T$. Then there exists

$$\mathbf{v} = \sum_{i=1}^{p} \alpha_i \mathbf{v}_i + \sum_{i=p+1}^{n} \alpha_i \mathbf{v}_i$$

such that $T\mathbf{v} = \mathbf{w}$. Let

$$\mathbf{u} = \sum_{i=p+1}^{n} \alpha_i \mathbf{v}_i \in \ker T^*$$

Then

$$T\mathbf{v} = T\sum_{i=1}^{p} \alpha_i \mathbf{v}_i + T\mathbf{u} = T\mathbf{u} = \mathbf{w}$$

so there exists a $\mathbf{u} \in \ker T^*$ such that $T\mathbf{u} = \mathbf{w}$. Hence T is surjective and thus is bijective.

Then $\dim kerT^* = \operatorname{rank} T$. By definition, $\dim\ker T + \dim\ker T^* = \dim V$ and thus $\operatorname{null} T + \ker T = \dim V$. $\square$

3 10/7 - Linear Algebra

Definition 3.1. *Inner Product on Vector Space*
*An **inner product** on a vector space V is a bilinear map $\cdot : V \times V \to \mathbb{R}$ such that for $\mathbf{v}, \mathbf{u}, \mathbf{w} \in V$ and $\alpha \in \mathbb{R}$.*

1. $(\alpha\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \alpha\mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
2. $\mathbf{u} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u}$
3. $\mathbf{u} \cdot \mathbf{u} > 0$ *if* $\mathbf{u} \neq \mathbf{0}$. $\mathbf{0} \cdot \mathbf{0} = 0$.

Definition 3.2. *Absolute Value, Orthonormal Bases*
*For $\mathbf{v} \in V$, the **(**absolute value) of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.*

*An **orthonormal basis** of V is a basis $B = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ such that*

$$v_i \cdot v_j = \begin{cases} 1 & i = j \\ 0 & \textit{otherwise} \end{cases}$$

Definition 3.3. *Orthogonal Complement*
*If $W \subset V$ is a subspace then the **orthogonal complement** $W^\perp$ is defined as*

$$W^\perp = \{v \in V : v \cdot w = 0 \textit{ for all } w \in W\}$$

Theorem 3.1. *$W^\perp$ is a Subspace Disjoint to W*

1. *$W^\perp$ is a subspace of V.*
2. *$W^\perp \cap W = \{0\}$.*

1. *Proof.* Suppose $\mathbf{u}, \mathbf{v} \in W^\perp$ and $\alpha \in \mathbb{R}$. Let $\mathbf{w}$ be an arbitrary element of W. Then
$$(\alpha\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \alpha\mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} = 0$$
Thus $\alpha\mathbf{u} + \mathbf{v} \in W^\perp$ so $W^\perp$ is a subspace. □

2. *Proof.* Suppose $\mathbf{v} \in W^\perp \cap W$. Then $\mathbf{v} \cdot \mathbf{v} = 0$, and thus $\mathbf{v} = 0$, so $W^\perp \cap W = \{0\}$. □

Theorem 3.2. *Inner Products and Orthonormal Bases Exist*

1. *V is an inner product.*
2. *Given a subspace $W \subset V$, there exists an orthonormal basis of W which can be extended to an orthonormal basis of V.*

1. *Proof.* Choose a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ of V. For $\mathbf{w}, \mathbf{u} \in V$ where
$$\mathbf{w} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i, \mathbf{u} = \sum_{i=1}^{n} \beta_i \mathbf{v}_i$$
define
$$\mathbf{u} \cdot \mathbf{w} = \sum_{i=1}^{n} \alpha_i \beta_i$$
Fix $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ such that
$$\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i, \mathbf{v} = \sum_{i=1}^{n} \beta_i \mathbf{v}_i, \mathbf{w} = \sum_{i=1}^{n} \gamma_i \mathbf{v}_i$$
and $\lambda \in \mathbb{R}$. Then
$$(\lambda\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \sum_{i=1}^{n} (\lambda\alpha_i + \beta_i)\gamma_i = \lambda_i \sum_{i=1}^{n} \alpha_i\gamma_i + \sum_{i=1}^{n} \beta_i\gamma_i = \lambda\mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$$
Since multiplication over $\mathbb{R}$ is commutative, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. $\mathbf{u} \cdot \mathbf{u} = \sum_{i=1} \alpha_i^2$. Thus $\mathbf{u} \cdot \mathbf{u} > 0$ unless $\alpha_1 = \cdots = \alpha_n = 0$, in which case $\mathbf{u} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{u} = 0$. Thus $\cdot$ is an inner product. □

2. *Proof.* Pick $\mathbf{v}_1 \in W - \{0\}$, define $e_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}$, so $|e_1| = 1$. Set $W_1 = \text{span}\{e_1\}$ as a subspace of W. If $W_1^{\perp} \neq \{0\}$, choose $\mathbf{v}_2 \in W_1^{\perp} - \{0\}$. Set $e_2 = \frac{e_1}{|e_1|}$ and set $W_2 = \text{span}\{e_1, e_2\}$. Repeat this process until $W_k = W$. Since each vector in the basis is orthogonal to all the vectors before it, all vectors are orthogonal to all other vectors. All vectors are by definition length 1. Thus $e_1, \ldots, e_k$ is a orthonormal basis of W. By the same process it may be extended to a basis of V. □

Theorem 3.3. *Dimension of Orthogonal Complement*
Let W be a subspace of V. Then $\dim W + \dim W^{\perp} = \dim V$.

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Theorem 4.1 (Specific Rank + Nullity as Corollary of General Rank + Nullity)**.**
Let $T : V \to W$ be a linear map. Then

$$\dim V = \operatorname{null} T + \operatorname{rank} T$$

Proof.

$$\begin{aligned}
\operatorname{rank}(T) &= \operatorname{rank}(T^*) && (1)\\
&= \dim \operatorname{Im}(T^*) && (2)\\
&= \dim \ker(T)^{\perp} && (3)\\
&= \dim V - \dim \ker(T) && (4)\\
&= \dim V - \operatorname{null}(T) && (5)
\end{aligned}$$

Line 4 follows from $\dim W + \dim W^{\perp} = \dim V$, and the other from rank + nullity. Hence, we have $\operatorname{rank}(T) = \dim V - \operatorname{null}(t)$, so $\operatorname{rank}(t) + \operatorname{null}(t) = \dim V$. □

Theorem 4.2 (Index of operator corollary)**.**
For linear map $T : V \to W$:

$$\dim V - \dim W = \operatorname{null}(T) - \operatorname{null}(T^*)$$

$\operatorname{null}(T) - \operatorname{null}(T^*)$ *is known as the* ***index of the operator****.*

Proof. From rank + nullity, we have $\text{null}(T) + \text{rank}(T) = \dim V$, $\text{null}(T^*) + \text{rank}(T^*) = \dim W$, and $\text{rank}(T) = \text{rank}(T^*)$. Subtracting these gives

$$\dim V - \dim W = \text{null}(T) + \text{rank}(T) - \text{null}(T^*) - \text{rank}(T^*) = \text{null}(T) - \text{null}(T^*)$$

□

Definition 4.1. *Self-adjoint A linear map $T : V \to W$ is **self-adjoint** if $T = T^*$.*

Definition 4.2. *Eigenvalues, Eigenvectors, and Diagonalizable Maps Let $T : V \to V$ be a linear map. $\lambda \in \mathbb{R}$ is an **eigenvalue** if there exists an **eigenvector** $v \in V$ such that $Tv = \lambda v$.*

*T is **diagonalizable** if there exists an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that*

$$T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$$

Theorem 4.3. *Every self-adjoint map is diagonalizable.*

Proof. Let $P(n)$ be the proposition that for any vector space V such that $\dim V \leq n$, for any self-adjoint linear map $T : V \to V$, T is diagonalizable.

For the base case, since $\dim V = 1$, pick $\mathbf{v} \in V$ and let $\bar{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$. Then $V = \text{span}\, \bar{\mathbf{v}}$, so $T\bar{\mathbf{v}} = \lambda\bar{\mathbf{v}}$ for some $\lambda \in \mathbb{R}$. Since $\{\bar{\mathbf{v}}\}$ is an orthonormal basis of V and it has eigenvalue λ, T is diagonalizable.

Suppose for some $k \in \mathbb{N}$, $P(k)$ holds. Fix self-adjoint $T : V \to V$, where $\dim V = n + 1$. Since V is isomorphic to $\mathbb{R}^{n+1}$, we will let $V = \mathbb{R}^{n+1}$. Set $f : S^{n+1} \to \mathbb{R}$ such that $f(v) = T(v) \cdot v$. This f is continuous over a compact space, so it has a maximum of λ at some $\bar{\mathbf{v}} \in S^{n+1}$ and thus $f(\bar{\mathbf{v}}) = \lambda$.

Choose $\mathbf{w} \perp \bar{\mathbf{v}}$. Set

$$v(t) = \frac{\bar{\mathbf{v}} + t\mathbf{w}}{|\bar{\mathbf{v}} + t\mathbf{w}|}$$

for t small. Then $v(0) = \bar{\mathbf{v}}$ and $v(t) \in S^{n+1}$. We want to show that $f(v(t))$ is maximized at $t = 0$, so $\frac{d}{dt} f(v(t))|_{t=0} = 0$. Computing $\frac{d}{dt} f(v(t))$ we see that

$$\tfrac{d}{dt} f(v(t)) = \tfrac{d}{dt} \frac{T(\bar{\mathbf{v}} + t\mathbf{w}) \cdot (\bar{\mathbf{v}} + t\mathbf{w})}{|\bar{\mathbf{v}} + t\mathbf{w}|^2}$$

Using the distributivity of the dot product and that $\mathbf{v} \cdot \mathbf{w} = 0$, we see

$$|(\bar{\mathbf{v}} + t\mathbf{w})|^2 = (\bar{\mathbf{v}} + t\mathbf{w}) \cdot (\bar{\mathbf{v}} + t\mathbf{w}) = |\bar{\mathbf{v}}|^2 + 2t\bar{\mathbf{v}}\mathbf{w} + t^2|\mathbf{w}|^2 = |\bar{\mathbf{v}}|^2 + t^2|\mathbf{w}|^2$$

Similarly

$$T(\bar{\mathbf{v}}+t\mathbf{w})\cdot(\bar{\mathbf{v}}+t\mathbf{w}) = T(\bar{\mathbf{v}})\bar{\mathbf{v}} + 2tT(\bar{\mathbf{v}})\cdot w + t^2T(\mathbf{w})\mathbf{w}$$

Finally, since $\bar{\mathbf{v}} \perp \mathbf{w}$, $\bar{\mathbf{v}}\cdot\mathbf{w} = 0$, so we have

$$\begin{aligned} &\tfrac{d}{dt}\frac{T(\bar{\mathbf{v}}+t\mathbf{w})\cdot(\bar{\mathbf{v}}+t\mathbf{w})}{|\bar{\mathbf{v}}+t\mathbf{w}|^2} \\ =\ &\frac{(|\bar{\mathbf{v}}|^2+t^2|\mathbf{w}|^2)(2T(\bar{\mathbf{v}})\cdot\mathbf{w})}{|\bar{\mathbf{v}}+t\mathbf{w}|^4} \end{aligned}$$

Since we know $f(\mathbf{v})$ is maximized when $\mathbf{v} = \bar{\mathbf{v}}$, we know $\frac{d}{dt}f(v(0)) = 0$. Thus at $t = 0$ we have

$$\begin{aligned} &\frac{|\bar{\mathbf{v}}|^2(2T(\bar{\mathbf{v}})\cdot\mathbf{w})}{|\bar{\mathbf{v}}|^4} \\ =\ &\frac{2T(\bar{\mathbf{v}})\cdot\mathbf{w}}{|\bar{\mathbf{v}}|^2} \end{aligned}$$

And therefore, $\frac{2T(\bar{\mathbf{v}})\cdot\mathbf{w}}{|\bar{\mathbf{v}}|^2}$, which in turn implies that $T(\bar{\mathbf{v}})\cdot\mathbf{w} = 0$. Let $\{\bar{\mathbf{v}}, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be an orthonormal basis of V. Then $T(\bar{\mathbf{v}}) = \lambda\bar{\mathbf{v}} + \sum_{i=2}^k \alpha_i\mathbf{w}_i$. Since $T(\bar{\mathbf{v}})$ is normal to all $\mathbf{w}_i$ in the basis, $\alpha_i = 0$ for all $i = 2...k$. Then since $T(\bar{\mathbf{v}}) \neq \mathbf{0}$, $T(\bar{\mathbf{v}}) = \lambda\bar{\mathbf{v}}$, so $\bar{\mathbf{v}}$ is an eigenvector of T with eigenvalue λ.

Now that we have an eigenvector $\bar{\mathbf{v}}$ of T, set $W = \text{span}\{\bar{\mathbf{v}}\}^\perp$. For all $\mathbf{w} \in W$, $T(\mathbf{w})\cdot\bar{\mathbf{v}} = T(\bar{\mathbf{v}})\cdot\mathbf{w} = \mathbf{w}\cdot\lambda\mathbf{v} = 0$, so $T(\mathbf{w}) \in W$, and thus it makes sense to restrict T to W as $T|_W : W \to W$. $T|_W$ remains self adjoint, and thus by $P(k)$ is diagonalizable over some orthonormal basis $\{\mathbf{v}_2, \ldots, \mathbf{v}_k\}$, so T is diagonalizable over $\{\bar{\mathbf{v}}, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. Thus $P(k)$ implies $P(k+1)$, so the proposition holds for all finite vector spaces. $\square$