Math 207 HW2

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Exercise 5.1.5a and c. Explain why the following are not inner products on the given vector space.

a) $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 - x_2 y_2$ over $\mathbb{R}^2$.

b) Skip

c) $f \cdot g = \int_0^1 f'(t)g(t)dt$ over the space of polynomials

Solution. a) Let $\mathbf{x} = (1, 1)$. Then $\mathbf{x} \cdot \mathbf{x} \cdot \mathbf{x} = 1 - 1 = 0$ but $\mathbf{x} \neq 0$. Therefore $\cdot$ is not an inner product.

b) Skip

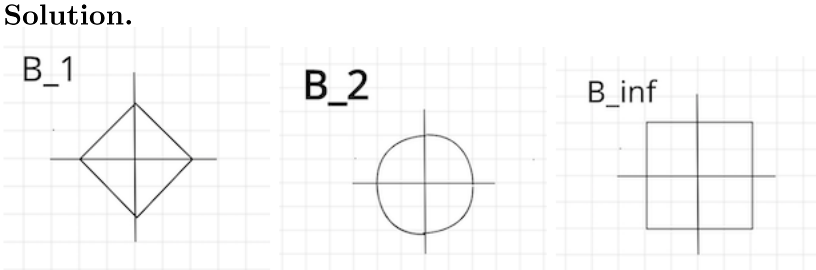
c) Let $f = x - \frac{1}{2}$. Then $f'(x) = 1$. $f \cdot f = \int_0^1 f'(t)f(t)dt = \int_0^1 f(t)dt = 0$, but $f \neq 0$. Therefore $\cdot$ is not an inner product. $\cdot$ also isn't commutative.

Exercise 5.1.9. Consider the space $\mathbb{R}^2$ with the norm $|\cdot|_p$, introduced in Section 1.5. For $p = 1, 2, \inf$ draw the "unit ball" B_p in the norm $|\cdot|_p$

$$B_p = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}|_p < 1\}$$

Can you guess what the balls B_p for other p look like?

Solution.



As p increases, the ball becomes less circular, with a higher slope near $|x| = 1$ and $|y| = 1$ and turning more sharply around the line $y = \frac{x}{2}$ and $y = -\frac{x}{2}$.

Exercise 5.3.4. Find the distance from a vector $(2, 3, 1)^T$ to the subspace spanned by the vectors $(1, 2, 3)^T, (1, 3, 1)^T$.

Solution. We will let $W = \text{span}(1,2,3),(1,3,1)$. Then, by the cross product we find $(-7,2,1) \perp (1,2,3)$ and $(-7,2,1) \perp (1,3,1)$, so $W^\perp = \text{span}(-7,2,1)$. Then by exercise 5.2, $(2,3,1) = \alpha(1,2,3) + \beta(1,3,1) + \gamma(-7,2,1)$. Solving the system of equations gives $\alpha = \frac{1}{54}, \beta = \frac{29}{27}, \gamma = \frac{7}{54}$. By 5.3, the closest point in W to $(2,3,1)$ is $\alpha(1,2,3) + \beta(1,3,1) = (\frac{59}{54}, \frac{88}{27}, \frac{61}{54})$. The distance from $(2,3,1)$ to $(\frac{59}{54}, \frac{88}{27}, \frac{61}{54})$ is $\frac{7}{3\sqrt{6}}$.

Exercise 5.3.13. Suppose P is the orthogonal projection onto a subspace E, and Q is the orthogonal projection onto the orthogonal complement $E^\perp$.

a) What are $P+Q$ and PQ?

b) Show that $P-Q$ is its own inverse.

Solution.

1. Let $\mathbf{v} \in V$. Then $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ for $\mathbf{w} \in E$, $\mathbf{w}^\perp \in E^\perp$. Then $P(\mathbf{v}) = \mathbf{w}$ and $Q(\mathbf{v}) = \mathbf{w}^\perp$. Thus, $P+Q(\mathbf{v}) = \mathbf{w}+\mathbf{w}^\perp = \mathbf{v}$, so $P+Q = I$. For any $\mathbf{v} \in V$, $Q(\mathbf{v}) = \mathbf{w}^\perp \in E^\perp$. Since $\mathbf{w}^\perp \in E^\perp$, its orthogonal projection in E is $\mathbf{0}$, so $PQ(\mathbf{v}) = \mathbf{0}$ and thus $PQ = \mathbf{0}$.

2. Let $\mathbf{v} \in V$. By part a), $P-Q(\mathbf{v}) = \mathbf{w}-\mathbf{w}^\perp$ with $\mathbf{w} \in E$ and $\mathbf{w}^\perp \in E^\perp$. Clearly $P(\mathbf{w}-\mathbf{w}^\perp) = \mathbf{w}$ and $Q(\mathbf{w}-\mathbf{w}^\perp) = -\mathbf{w}^\perp$, so $P-Q(\mathbf{w}-\mathbf{w}^\perp) = \mathbf{w}+\mathbf{w}^\perp = \mathbf{v}$. Thus $(P-Q)\circ(P-Q) = I$, so $(P-Q)^{-1} = P-Q$.

Exercise 0. Consider a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$.

a) Show that T is continuous.

b) Show that if there is a nonempty open set O so that if $T(O)$ is open, then $T(U)$ is open for every open set U.

c) Show that T is bijective if and only if there is a nonempty open set O whose image $T(O)$ is an open set.

Solution. a) Set

$$\alpha = \max_{i\le n, j\le n} (Te_i)\cdot e_j$$

Fix $\epsilon > 0$ and $\delta = \frac{\epsilon}{|\alpha|\sqrt{n}}$. Let $\mathbf{w} \in \mathbb{R}^n$ such that $|\mathbf{w}| < \delta$. Then $\sqrt{n}|\alpha\mathbf{w}| < \epsilon$.

$$\mathbf{w} = \sum_{i=1}^n \alpha_i e_i$$

so

$$T\mathbf{w} = \sum_{i=1}^n \alpha_i Te_i$$

which implies

$$T\mathbf{w}\cdot T\mathbf{w} = \sum_{i=1}^n (\alpha_i Te_i)^2 = \sum_{i=1}^n \alpha_i^2 \left(\sum_{j=1}^n (Te_i\cdot e_j)^2\right)$$

since $T(e_i)\cdot e_j \le \alpha$, we have

$$(T\mathbf{w})^2 = \sum_{i=1}^n \alpha_i^2 \left(\sum_{j=1}^n (Te_i\cdot e_j)^2\right) \le \sum_{i=1}^n \alpha_i^2 n\alpha^2 = n\alpha^2\mathbf{w}^2$$

Then we have

$$|T\mathbf{w}| \leq \sqrt{n}|\alpha\mathbf{w}| < \epsilon$$

so $|T\mathbf{w}| < \epsilon$, so $\lim_{\mathbf{w}\to\mathbf{0}} T\mathbf{w} = \mathbf{0}$.

Fix $\mathbf{v} \in \mathbb{R}^n$. Then

$$\lim_{\mathbf{w}\to\mathbf{0}} T(\mathbf{v}+\mathbf{w}) = \lim_{\mathbf{w}\to\mathbf{0}} T(\mathbf{v}) + \lim_{\mathbf{w}\to\mathbf{0}} T(\mathbf{w}) = T(\mathbf{v}) + \mathbf{0} = T(\mathbf{v})$$

Since $\lim_{\mathbf{w}\to\mathbf{0}} T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v})$, T is continuous.

b) Let U be an open subset of $\mathbb{R}^n$. Let $\mathbf{v} \in U$ and

$$B(\mathbf{v},\delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}-\mathbf{v}| < \delta\} \subset U$$

Since

$$S_n(\mathbf{v},\delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}-\mathbf{v}| < \delta\}$$

is compact and T is continuous, $T(S_n(\mathbf{v},\delta))$ is compact. Define $f : \mathbb{R}^n \to \mathbb{R}$ as $f(\mathbf{w}) = |\mathbf{w} - T\mathbf{v}|$, which is the compisition of continuous functions and is thus continuous. Then $f(T(S_n))$ is compact, so it has some minima $\alpha \geq 0$. Suppose $\alpha = 0$. Then there exists some $\mathbf{z} \in S_n(\mathbf{v},\delta)$ such that $|T\mathbf{z} - T\mathbf{v}| = 0$ and thus $T(\mathbf{z}-\mathbf{v}) = 0$. Since $\mathbf{z} \neq \mathbf{v}$, $\operatorname{null} T \geq 1$, so $\operatorname{Im} T \neq \mathbb{R}^n$, and thus there exists some $\mathbf{w} \in \mathbb{R}^n \setminus \operatorname{Im} T$. Let $\mathbf{v}' \in O$, there exists an open ball $B(T\mathbf{v}',\epsilon) \subset TO$. Then

$$\left|\frac{\epsilon}{2|\mathbf{w}|}\mathbf{w} + T\mathbf{v} - T\mathbf{v}\right| = \frac{\epsilon}{2} < \epsilon$$

so $\frac{\epsilon}{2|\mathbf{w}|}\mathbf{w} + T\mathbf{v} \in B(T\mathbf{v}',\epsilon)$ and thus $\frac{\epsilon}{2|\mathbf{w}|}\mathbf{w} + T\mathbf{v} \in B(T\mathbf{v}',\epsilon) \in TO$. Then $\frac{\epsilon}{2|\mathbf{w}|}\mathbf{w} + T\mathbf{v} = T\mathbf{u}$ for some $\mathbf{u} \in O$, so $\mathbf{w} = T(\frac{2|\mathbf{w}|)}{\epsilon}(\mathbf{u}-\mathbf{v}))$, so $\mathbf{w} \in \operatorname{Im} T$, which is a contradiction. Hence, $\operatorname{null} T = 0$, so $\operatorname{rank} T = n$ and thus $\operatorname{Im} T = \mathbb{R}^n$.

From there we see that $\alpha > 0$. Let $\mathbf{w} \in B(T\mathbf{v},\alpha)$. As we proved in the previous paragraph, $\operatorname{Im} T = \mathbb{R}^n$, so there exists some $\mathbf{u} \in \mathbb{R}^n$ such that $T\mathbf{w} = \mathbf{u}$. Suppose $|\mathbf{u}-\mathbf{v}| \geq \delta$. Define $f : [0,1] \to \mathbb{R}$ such that $f(t) = |t\mathbf{u} + (1-t)\mathbf{v} - t\mathbf{v}|$. f is continuous, $f(0) = 0$, and $f(1) = |\mathbf{u}-\mathbf{v}| \geq \delta$, so by the intermediate value theorem, there exists some $t \in (0,1]$ such that $f(t) = \delta$. Then $|t\mathbf{u}+(1-t)\mathbf{v}-\mathbf{v}| = \delta$. Then $t\mathbf{u}+(1-t)\mathbf{v} \in S_n(\mathbf{v},\delta)$, so $T(t\mathbf{u}+(1-t)\mathbf{v}) \in TS_n(\mathbf{v},\delta)$, so $|T(t\mathbf{u}+(1-t)\mathbf{v})-T\mathbf{v}| \geq \alpha$. Since $|T(t\mathbf{u}+(1-t)\mathbf{v})-T\mathbf{v}| = t|T\mathbf{u}-T\mathbf{v}|$, we have $t|T\mathbf{u}-T\mathbf{v}| \geq \alpha$. $t \in (0,1]$ implies that $|T\mathbf{u}-T\mathbf{v}| \leq \frac{\alpha}{t} \leq \alpha$ which implies $T\mathbf{u} = \mathbf{w} \notin B(T\mathbf{v},\alpha)$, which is a contradiction. Therefore, if $\mathbf{w} \in B(T\mathbf{v},\alpha)$, then $\mathbf{w} = T\mathbf{u}$ for some $\mathbf{u} \in B(\mathbf{v},\delta)$. $B(\mathbf{v},\delta) \subset U$ implies $\mathbf{u} \in U$ which implies $\mathbf{w} \in TU$. Thus, $B(T\mathbf{v},\alpha) \subset TU$. Since there exists such a region around every $\mathbf{x} \in TU$, TU is open.

c) Suppose there exists a nonempty open set O such that TO is open. Following from that, in the first paragraph of part b, we proved that $\operatorname{null} T = 0$, so T is injective, and $\operatorname{rank} T = n$, so T is surjetive, and thus T is bijective.

Suppose T is bijective. Then T^{-1} is a bijective linear transformation. By a, it is continuous. Then T is the inverse of a continuous function, so for any open $U \in \mathbb{R}^n$, TU is open.

Exercise 1. Two finite dimensional vector spaces V, W with a corresponding inner product $\cdot$ and $\tilde{\cdot}$ are isomorphic if there is a bijective linear map $T : V \to W$ so that $T(x)\tilde{\cdot}T(y) = x \cdot y$ for all $x, y \in V$. The map T is called an isometry.
Show that any two finite dimensional vector spaces V, W with a corresponding inner product $\cdot$ and $\tilde{\cdot}$ are isomorphic if and only if $\dim V = \dim W$.

Solution. Suppose V, W are isomorphic. Then there exists a bijection $T : V \to W$ between them. Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ be a basis of V. Since T is injective, $T\mathbf{v}_1, \cdots, T\mathbf{v}_k$ is linearly independent in W. Fix $\mathbf{w} \in W$. Since T is surjective, there exists a

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

such that

$$T\mathbf{v} = T\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \sum_{i=1}^{n} \alpha_i T\mathbf{v}_i$$

so $\mathbf{w} \in \operatorname{span}\{T\mathbf{v}_1, \cdots, T\mathbf{v}_k\}$. Since $T\mathbf{v}_1, \cdots, T\mathbf{v}_k$ is linearly independent and spanning, it is a basis. Therefore, $\dim V = \dim W$.

Suppose $\dim V = \dim W$. Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be an orthonormal basis of V and $\mathbf{w}_1, \cdots, \mathbf{w}_n$ be an orthonormal basis of W. Define $T : V \to W$ such that $T\mathbf{v}_i = \mathbf{w}_i$. Fix $\mathbf{v}, \mathbf{w} \in V$. Then

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

and

$$\mathbf{w} = \sum_{i=1}^{n} \beta_i \mathbf{w}_i$$

Then since $\mathbf{w}_1, \cdots, \mathbf{w}_n$ is an orthonormal basis we have

$$\begin{aligned}
T\mathbf{v} \cdot T\mathbf{w} &= T\left(\sum_{i=1}^{n} \alpha_i \mathbf{v}_i\right) \cdot T\left(\sum_{i=1}^{n} \beta_i \mathbf{v}_i\right) \\
&= \left(\sum_{i=1}^{n} \alpha_i T\mathbf{v}_i\right) \cdot T\left(\sum_{i=1}^{n} \beta_i T\mathbf{v}_i\right) \\
&= \left(\sum_{i=1}^{n} \alpha_i \mathbf{w}_i\right) \cdot \left(\sum_{i=1}^{n} \beta_i \mathbf{w}_i\right) \\
&= \sum_{i=1}^{n} \alpha_i \beta_i \mathbf{w}^2 \\
&= \sum_{i=1}^{n} \alpha_i \beta_i
\end{aligned}$$

Similarly, since $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is an orthonormal basis of V

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= \left(\sum_{i=1}^{n} \alpha_i \mathbf{v}_i\right) \cdot \left(\sum_{i=1}^{n} \beta_i \mathbf{v}_i\right) \\
&= \sum_{i=1}^{n} \alpha_i \beta_i \mathbf{w}^2 \\
&= \sum_{i=1}^{n} \alpha_i \beta_i
\end{aligned}$$

Thus $T\mathbf{v} \cdot T\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$, so an isometry exists between V and W.

Exercise 2. V and W are two finite dimensional vector spaces with corresponding inner products. (On the assignment, d was skipped. I did not skip it in my solution, so the following items are offset by one).

a) Show that the adjoint map T^* of $T : V \to W$ is unique.

b) Show that the adjoint of T^* is T.

c) Show that $\operatorname{rank}(T^*) = \operatorname{rank}(T)$.

d) Show that $\operatorname{Im}(T) = \ker(T^*)^\perp$

e) Show that $\operatorname{Im}(T^*) = \ker(T)^\perp$

f) Using c) and d) show that $\dim(V) = \operatorname{null}(T) - \operatorname{rank}(T)$

g) Show that $\dim(V) - \dim(W) = \operatorname{null}(T) - \operatorname{null}(T^*)$

Solution. a) Suppose there exists $S, S' : W \to V$ such that $\mathbf{w} \cdot T\mathbf{v} = \mathbf{v} \cdot S\mathbf{w} = \mathbf{v} \cdot S'\mathbf{w}$. Let $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ is a basis of W and $\mathbf{v} \neq \mathbf{0} \in V$ be arbitarary. Then for any $1 \leq i \leq n$, $\mathbf{w}_i \cdot T\mathbf{v} = \mathbf{v} \cdot S\mathbf{w}_i = \mathbf{v} \cdot S'\mathbf{w}_i$, so $\mathbf{v} \cdot (S\mathbf{w}_i - S'\mathbf{w}_i) = 0$. Since $\mathbf{v}$ is arbitrary, set $\mathbf{v} = (S\mathbf{w}_i - S'\mathbf{w}_i)$ then $(S\mathbf{w}_i - S'\mathbf{w}_i)^2 = 0$, so $S\mathbf{w}_i - S'\mathbf{w}_i = \mathbf{0}$ and thus $S\mathbf{w} = S'\mathbf{w}$. Since S equals S' over a basis of W, $S = S'$. Thus the adjoint map of T^* is unique.

b) By definition for any $\mathbf{v} \in V, \mathbf{w} \in W$,

$$\mathbf{v} \cdot T\mathbf{w} = \mathbf{w} \cdot T^*\mathbf{v} = \mathbf{v} \cdot T^{**}\mathbf{w}$$

Then by the previous proof, $T = T^**$.

c) Let $\mathbf{y} = T(\mathbf{x})$ for some $\mathbf{x} \in V$. Choose $\mathbf{w} \in W$ such that $T^*\mathbf{w} = \mathbf{0}$. Then

$$\mathbf{w} \cdot \mathbf{y} = \mathbf{w} \cdot T\mathbf{x} = \mathbf{x} \cdot T^*\mathbf{w} = \mathbf{x} \cdot \mathbf{0} = 0$$

Then for all $\mathbf{w} \in \ker T^*$, $\mathbf{w} \cdot \mathbf{y} = 0$ so $y \in (\ker T^*)^\perp$Therefore $\operatorname{Im}(T) \subset \ker(T^*)^\perp$. Similarly $\operatorname{Im}(T^*) \subset \ker(T)^\perp$. It follows that $\operatorname{rank} T \leq \operatorname{null}(T^*)^\perp$ and $\operatorname{rank} T^* \leq \operatorname{null}(T)^\perp$. Since $\operatorname{rank} T + \operatorname{null} T = \dim V$ and

$$\dim(\ker(T^*)^\perp) + \dim(\ker T^*) = \dim(\ker(T^*)^\perp) + \operatorname{null} T^* = \dim W = \operatorname{rank} T^* + \operatorname{null} T^*$$

we have

$$\operatorname{rank} T + \operatorname{null} T^* \leq \dim(\ker(T^*)^\perp) + \operatorname{null} T^* = \dim W = \operatorname{rank} T^* + \operatorname{null} T^*$$

and therfore we know $\operatorname{rank} T^* \leq \operatorname{rank} T$. Since $\operatorname{rank} T^* + \operatorname{null} T^* = \dim W$ and

$$\dim(\ker(T)^\perp) + \dim(\ker T) = \dim(\ker(T)^\perp) + \operatorname{null} T = \dim V = \operatorname{rank} T + \operatorname{null} T$$

we have

$$\operatorname{rank} T^* + \operatorname{null} T \leq \dim(\ker(T)^\perp) + \operatorname{null} T = \dim V = \operatorname{rank} T + \operatorname{null} T$$

and therfore we know $\operatorname{rank} T \leq \operatorname{rank} T^*$. Since both $\operatorname{rank} T \leq \operatorname{rank} T^*$ and $\operatorname{rank} T^* \leq \operatorname{rank} T$ and thus $\operatorname{rank} T = \operatorname{rank} T*$.

d) By the previous proof, $\operatorname{Im}(T) \subset \ker(T^*)^\perp$ and $\operatorname{rank}(T) = \operatorname{rank}(T^*)$.

$$\operatorname{rank}(T^*) + \operatorname{null}(T^*) = \dim W = \operatorname{null}(T^*) + \dim \ker(T^*)^\perp$$

so $\operatorname{rank}(T^*) = \dim \ker(T^*)^\perp$ and thus $\operatorname{rank}(T) = \dim \operatorname{Im} T = \dim \ker(T^*)^\perp$. It follows that $\operatorname{Im} T = \ker(T^*)^\perp$.

e) By part c, $\operatorname{Im}(T^*) \subset \ker(T)^\perp$ and $\operatorname{rank}(T) = \operatorname{rank}(T^*)$.

$$\operatorname{rank}(T) + \operatorname{null}(T) = \dim V = \operatorname{null}(T) + \dim \ker(T)^\perp$$

so $\operatorname{rank}(T) = \dim \ker(T)^\perp$ and thus $\operatorname{rank}(T^*) = \dim \operatorname{Im} T^* = \dim \ker(T)^\perp$. It follows that $\operatorname{Im} T^* = \ker(T)^\perp$.

f) By part c and d,

$$\begin{aligned}
\operatorname{rank}(T) &= \operatorname{rank}(T^*) && (1)\\
&= \dim \operatorname{Im}(T^*) && (2)\\
&= \dim \ker(T)^\perp && (3)\\
&= \dim V - \dim \ker(T) && (4)\\
&= \dim V - \operatorname{null}(T) && (5)
\end{aligned}$$

Line 4 follows from $\dim W + \dim W^\perp = \dim V$, and the others from c and d. Hence, we have $\operatorname{rank}(T) = \dim V - \operatorname{null}(t)$, so $\operatorname{rank}(t) + \operatorname{null}(t) = \dim V$.

g) $\operatorname{null} T + \operatorname{rank} T = \dim V$ and $\operatorname{null} T^* + \operatorname{rank} T^* = \dim W$. By c, $\operatorname{rank} T = \operatorname{rank} T^*$, so $\operatorname{null} T^* + \operatorname{rank} T = \dim W$. Then

$$\dim V - \dim W = \operatorname{null} T + \operatorname{rank} T - (\operatorname{null} T^* + \operatorname{rank} T) = \operatorname{null} T - \operatorname{null} T^*$$

Exercise 3. V is a finite dimensional vector space with a corresponding inner product.

a) Show that if $T : V \to V$ is a linear map so that $|T(v)| = |v|$ for all $v \in V$ then T is an isometry.

b) Show that if $\{T_i\}_{i\in\mathbb{N}}$ is a sequence of isometries of V , there is an isometry T of V so that, after passing to a subsequence,

$$\lim_{t\to\inf} T_i(v) = T(v)$$

Solution. a) Suppose T is a linear map such that $|T(\mathbf{v})| = |\mathbf{v}|$ for all $\mathbf{v} \in V$. Fix $\mathbf{v}, \mathbf{w} \in V$. Then $|T\mathbf{v}+T\mathbf{w}|^2 = |\mathbf{v}+\mathbf{w}|^2$. By the distributive property, $|T\mathbf{v}+T\mathbf{w}|^2 = T\mathbf{v}^2+2T\mathbf{v}T\mathbf{w}+T\mathbf{w}^2$ and $|\mathbf{v}+\mathbf{w}|^2 = \mathbf{v}^2+2\mathbf{v}\mathbf{w} = \mathbf{w}^2$. Since $|T\mathbf{v}| = |\mathbf{v}|, T\mathbf{v}^2 = \mathbf{v}^2$ and similarly $T\mathbf{w}^2 = \mathbf{w}^2$, so $T\mathbf{v}^2+2T\mathbf{v}T\mathbf{w}+T\mathbf{w}^2 = |T\mathbf{v}+T\mathbf{w}|^2 = |\mathbf{v}+\mathbf{w}|^2 = \mathbf{v}^2+2\mathbf{v}\mathbf{w} = \mathbf{w}^2 = T\mathbf{v}^2+2\mathbf{v}\mathbf{w}+T\mathbf{w}^2$ and thus $T\mathbf{v}^2+2T\mathbf{v}T\mathbf{w}+T\mathbf{w}^2 = T\mathbf{v}^2+2\mathbf{v}\mathbf{w}+T\mathbf{w}^2$. Subtracting $T\mathbf{v}^2+T\mathbf{w}^2$ and dividng by 2 leaves $T\mathbf{v}T\mathbf{w} = \mathbf{v}\mathbf{w}$, which by definition means T is an isometry.

b) First we will prove that for arbitrary $\mathbf{v} \in V$, a subsequence of $T_i(\mathbf{v})$ converges to some point $\mathbf{w}$. Consider the set $S = \{T_i\mathbf{v}\}_{i\in\mathbb{N}}$. If S is finite, then an infinite number of $T_i(\mathbf{v}) = \mathbf{w}$, so the subsequence converges to $\mathbf{w}$. Otherwise, suppose S has no limit points. Then it is closed, and since S is bounded by $|\mathbf{v}|$, S is compact. Since S has no limit points, every point $\mathbf{p} \in S$ has some ball $B(\mathbf{p}, \delta_{\mathbf{p}})$, $\delta_{\mathbf{p}} > 0$ containing $\mathbf{p}$ such that $B(\mathbf{p}, \delta) \cap S = \{\mathbf{p}\}$. The set

$$G = \{B(\mathbf{p}, \delta_{\mathbf{p}})\}_{\mathbf{p}\in S}$$

is an open cover of S, so since S is compact, G has a finite subset G' with a finite number of balls. Since each ball contains a single element of S, G' has a finite number of elements of S. However, $S \subset G'$, so G' has a infinite number of elements of S, which is a contradiction. Thus if S is infinite then it has some limit point $\mathbf{p}$. Then for any $\delta > 0$, $B(\mathbf{p}, \delta) \cap S$ has an infinite number of elements of $\{T_i(\mathbf{v})\}$, so we can define a subsequence which converges to $\mathbf{p}$.

We will define $T : V \to W$ as $T\mathbf{v} = \lim_{T_i\mathbf{v}}$. We have already proven T is well defined (i.e. $\lim_{T_i\mathbf{v}}$ converges for all $\mathbf{v}$), now we will prove it is a linear transformation. Let $\mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{R}$.

$$\begin{aligned}
\alpha T\mathbf{v} &= \alpha \lim_{i\to\inf} T_i\mathbf{v} \\
&= \lim_{i\to\inf} \alpha T_i\mathbf{v} \\
&= \lim_{i\to\inf} T_i\alpha\mathbf{v} \\
&= T\alpha\mathbf{v} \\
\text{and} & \\
T\mathbf{v} + T\mathbf{u} &= \lim_{i\to\inf} T_i\mathbf{v} + \lim_{i\to\inf} T_i\mathbf{w} \\
&= \lim_{i\to\inf} T_i\mathbf{v} + T_i\mathbf{w} \\
&= \lim_{i\to\inf} T_i(\mathbf{v}+\mathbf{w}) \\
&= T(\mathbf{v}+\mathbf{w})
\end{aligned}$$

Thus T is linear. Since $|\cdot|$ is continuous and $\lim_{i\to\inf} T_i\mathbf{v} = T\mathbf{v}$,

$$|\mathbf{v}| = \lim_{i\to\inf} |\mathbf{v}| = \lim_{i\to\inf} |T_i\mathbf{v}| = |T\mathbf{v}|$$

Then by part a, since $|\mathbf{v}| = |T\mathbf{v}|$ and T is a linear map, T is an isometry.

Exercise 4. V a finite dimensional vector space with dimension 2 or higher.

a) Find T, S linear maps from V to V so that $S \circ T \neq T \circ S$.

b) Show that $T^2 = T$ if and only if $T = \frac{I+B}{2}$ or $T = \frac{I-B}{2}$ where $B^2 = I$ and $T^2 = T \circ T$.

Solution. a) Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis of V and

$$T(\sum_{i=1}^{n} \alpha_i \mathbf{v}_i) = \sum_{i=2}^{n} \alpha_{i-1} \mathbf{v}_i$$

and

$$S(\sum_{i=1}^{n} \alpha_i \mathbf{v}_i) = \alpha_n \mathbf{v}_n$$

Then $T \circ S(\sum_{i=1}^{n} \alpha_i \mathbf{v}_i) = T(\alpha_n \mathbf{v}_n) = \mathbf{0}$ but $S \circ T(\sum_{i=1}^{n} \alpha_i \mathbf{v}_i) = S(\sum_{i=2}^{n} \alpha_{i-1} \mathbf{v}_i) = \alpha_{n-1} \mathbf{v}_n$. For $\alpha_{n-1} \neq 0$, these are unequal, so $T \neq S$.

b) Suppose $T = \frac{I+B}{2}$ and $B^2 = I$. Then

$$T^2 = \left(\frac{I+B}{2}\right)\left(\frac{I+B}{2}\right) = \frac{I^2 + IB + BI + B^2}{4} = \frac{I^2 + B + B + I}{4} = \frac{2I + 2B}{4} = \frac{I+B}{2} = T$$

Therefore, $T^2 = T$.
Similarly, suppose $T = \frac{I-B}{2}$. Then

$$T^2 = \left(\frac{I-B}{2}\right)\left(\frac{I-B}{2}\right) = \frac{I^2 - IB - BI + B^2}{4} = \frac{I^2 - B - B + I}{4} = \frac{2I - 2B}{4} = \frac{I-B}{2} = T$$

Therefore $T^2 = T$.

Suppose $T^2 = T$. Let $B = 2T - I$. Then $B^2 = 4T^2 - 4T + I = 4T - 4T + I = I$, so if $T = \frac{I+B}{2}$ then $B^2 = I$. Let $B = I - 2T$ then $B^2 = I - 4T + 4T^2 = I - 4T + 4T = I$. Then $B^2 = I$, so if $T + \frac{I-B}{2}$ then $B^2 = I$.

Exercise 5. a) Show that $|\mathbf{v} \cdot \mathbf{w}| \leq |\mathbf{v}||\mathbf{w}|$ (Cauchy-Schwartz inequality) and equality happens if and only if $\mathbf{v}$ and $\mathbf{w}$ are multiples of each other.

b) Let W be a subspace of V. Show that any vector $\mathbf{v}$ in V can be uniquely expressed as $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ where $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$

c) Using the notation above consider the orthogonal projection $P : V \to V$ so that $P(\mathbf{v}) = \mathbf{w}$. Show that P is linear and that $P(\mathbf{v})$ is the closest point in W to $\mathbf{v}$, i.e., $|P(\mathbf{v}) - \mathbf{v}| \leq |\mathbf{w} - \mathbf{v}|$ for all $\mathbf{w} \in W$.

Solution. a) First we will cover the general case. Let

$$\mathbf{x} = \mathbf{w} - \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{v}^2}\mathbf{v}$$

Then

$$\mathbf{v} \cdot \mathbf{x} = \mathbf{v} \cdot \mathbf{w} - \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{v}^2}\mathbf{v}^2 = \mathbf{v} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w} = 0$$

so $\mathbf{v} \perp \mathbf{x}$. Since $\mathbf{x} + \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}^2}\mathbf{v} = \mathbf{w}$, we can use the pythagorean theorem on a triangle with legs $\mathbf{x}$, $\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}^2}\mathbf{v}$ and hypotenuse $\mathbf{w}$ to see

$$\mathbf{x}^2 + \frac{(\mathbf{v}\cdot\mathbf{w})^2}{\mathbf{v}^4}\mathbf{v}^2 = \mathbf{x}^2 + \frac{(\mathbf{v}\cdot\mathbf{w})^2}{\mathbf{v}^2} = \mathbf{w}^2$$

Since $\mathbf{x}^2 > 0$, we have $\frac{(\mathbf{v}\cdot\mathbf{w})^2}{\mathbf{v}^2} \geq \mathbf{w}^2$ which implies $(\mathbf{v}\cdot\mathbf{w})^2 \geq \mathbf{w}^2\mathbf{v}^2$ and by taking the square root $|\mathbf{v}\cdot\mathbf{w}| \geq |\mathbf{w}||\mathbf{v}|$. In the case that $\mathbf{w} = \alpha\mathbf{v}$, $\alpha \in \mathbb{R}$, then in the solution above,

$$\mathbf{x} = \mathbf{w} - \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}^2}\mathbf{v} = \mathbf{w} - \alpha\frac{\mathbf{v}^2}{\mathbf{v}^2}\mathbf{v} = \mathbf{w} - \mathbf{w} = \mathbf{0}$$

The remainder of the solution remains the same, except we are left at the end with $\frac{(\mathbf{v}\cdot\mathbf{w})^2}{\mathbf{v}^2} = \mathbf{w}^2$ and thus $(\mathbf{v}\cdot\mathbf{w})^2 = \mathbf{v}^2\mathbf{w}^2$ finally leaving $|\mathbf{v}\cdot\mathbf{w}| = |\mathbf{v}||\mathbf{w}|$.

b) Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be a basis of W and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a basis of $W^\perp$. Then $\dim V = n + k$. Suppose for some $\mathbf{u}_i = \sum_{i=1}^n \alpha_i\mathbf{w}_i$. For some $1 \leq i \leq n$ $\alpha_i \neq 0$, so $\mathbf{u}_i \cdot \mathbf{w}_i = \alpha_i \neq 0$, and thus $\mathbf{u}_i \notin W^\perp$, which is a contradiction. Then all $\mathbf{u}_i$ is linear independent to $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$. By similar reasoning, all $\mathbf{w}_i$ is linear independent to $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Thus $\{\mathbf{w}_1, \ldots, \mathbf{w}_n, \mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a linearly independent list of length $n + k$ in V, so it is a basis of $\dim V$. Fix $\mathbf{v} \in V$. Then

$$\mathbf{v} = \sum_{i=1}^n \alpha_i\mathbf{w}_i + \sum_{i=1}^k \beta_i\mathbf{u}_i$$

Let

$$w = \sum_{i=1}^n \alpha_i\mathbf{w}_i$$

so $w \in W$ and

$$w^\perp = \sum_{i=1}^k \beta_i\mathbf{u}_i$$

so $w^\perp \in W^\perp$ and finally $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$.

c) First we will prove P is a linear map. Fix $\mathbf{v}, \mathbf{u} \in V$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ and $\mathbf{u} = \mathbf{x} + \mathbf{x}^\perp$, $\mathbf{w}, \mathbf{x} \in W$, $\mathbf{w}^\perp, \mathbf{x}^\perp \in W^\perp$. Since $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$, $\alpha\mathbf{v} = \alpha\mathbf{w} + \alpha\mathbf{w}^\perp$. $W, W^\perp$ are subspace,s so $\alpha\mathbf{w} + \mathbf{x} \in W$ and $\alpha\mathbf{w}^\perp + \mathbf{x}^\perp \in W^\perp$. Thus $\alpha\mathbf{v} + \mathbf{u} = (\alpha\mathbf{w} + \mathbf{x}) + (\alpha\mathbf{w}^\perp + \mathbf{x}^\perp)$, so $P(\alpha\mathbf{v} + \mathbf{u}) = \alpha\mathbf{w} + \mathbf{x} = \alpha P(\mathbf{v}) + P(\mathbf{u})$. Thus P is linear.

Now we will prove $|P(\mathbf{v}) - \mathbf{v}| \leq |\mathbf{x} - \mathbf{v}|$ for all $\mathbf{x} \in W$. Let $\mathbf{x} \in W$ and set $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$. Define $\mathbf{u} = \mathbf{w} - \mathbf{x}$. Since $\mathbf{w} \in W$ and $\mathbf{x} \in W$, $\mathbf{u} \in W$. Then $\mathbf{v} - \mathbf{x} = \mathbf{w}^\perp + \mathbf{w} - \mathbf{w} + \mathbf{u} = \mathbf{w}^\perp + \mathbf{u}$.

$$(\mathbf{u} + \mathbf{w}^\perp)^2 = \mathbf{u}^2 + 2\mathbf{u}\mathbf{w}^\perp + \mathbf{w}^2$$

Since $\mathbf{u} \in W$, $\mathbf{u}\mathbf{w}^\perp = 0$, so $(\mathbf{u} + \mathbf{w}^\perp)^2 = \mathbf{u}^2 + \mathbf{w}^2 \geq \mathbf{w}^2$. Therefore $|\mathbf{u} + \mathbf{w}^\perp| \geq |\mathbf{w}^\perp|$. As we previously showed, $\mathbf{v} - \mathbf{x} = \mathbf{w}^\perp + \mathbf{u}$ and since $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ and $P(\mathbf{v}) = \mathbf{w}$, $P(\mathbf{v}) - \mathbf{v} = \mathbf{w}^\perp$. Combining these with the previous inequality gives $|P(\mathbf{v}) - \mathbf{v}| \leq |\mathbf{v} - \mathbf{x}|$ for all $\mathbf{x} \in W$.

Exercise 6. V a finite dimensional vector space with an inner product and V^* the set of all linear maps from V to $\mathbb{R}$, which is itself a finite dimensional vector space.

a) (Riesz Representation Theorem) Given $f \in V^*$, show there is a unique $\mathbf{v} \in V$ so that $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{v}$ for all $\mathbf{x} \in V$. Denote such $\mathbf{v}$ by $\mathbf{v}_f$.

b) Show that the map $Q_V : V^* \to V$, $Q_V(f) = \mathbf{v}_f$ is linear and bijective.

c) Consider W another finite dimensional vector space with another inner product. Given a linear map $T : V \to W$, show that the map below is linear

$$\mathbf{T}^* : W^* \to V^*, T * (f)(\mathbf{v}) = f(T(\mathbf{v})), \mathbf{v} \in V$$

d) Show that $T^* = Q_V \circ \mathbf{T}^* Q_W^{-1}$ where Q_W^{-1} is the inverse of Q_W and T^* is the adjoint of T.

Solution. a) Let $\dim V = n$ Since $\dim \mathbb{R} = 1$, $\operatorname{rank} V \leq 1$ so by Rank + Nullity either $\operatorname{null} f = n$, in which case $f(\mathbf{v}) = 0$ for all $\mathbf{v}$, so $\mathbf{v}_f = \mathbf{0}$, or $\operatorname{null} f = n-1$. In that case, let $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ be an orthonormal basis of $\ker f$ and let $\mathbf{w}$ extend it to an orthonormal basis of V. Let $\mathbf{v}_f = f(\mathbf{w})\mathbf{w}$. Fix $\mathbf{v} \in V$, so $\mathbf{v} = \mathbf{x} + \alpha\mathbf{w}$, where $\mathbf{x} \in \ker f$, $\alpha \in \mathbb{R}$. Then $f(\mathbf{v}) = f(\mathbf{x}) + \alpha f(\mathbf{w}) = \alpha f(\mathbf{w})$. Additionally, since $\mathbf{w}$ is normal to the basis of $\ker f$, $\mathbf{w}$ is normal to every element in $\ker f$ and thus $\mathbf{w} \cdot \mathbf{x} = 0$. Since $|\mathbf{w}| = 1$, $\mathbf{w} \cdot \mathbf{w} = 1$. Using that we see the following:

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{v}_f &= \mathbf{v} \cdot f(\mathbf{w})\mathbf{w} \\
&= f(\mathbf{w})((\mathbf{x} + \alpha\mathbf{w}) \cdot \mathbf{w}) \\
&= f(\mathbf{w})(\mathbf{x} \cdot \mathbf{w} + \alpha\mathbf{w} \cdot \mathbf{w}) \\
&= f(\mathbf{w})(0 + \alpha) \\
&= \alpha f(\mathbf{w})
\end{aligned}$$

Thus, $f(\mathbf{v}) = \alpha f(\mathbf{w}) = \mathbf{v} \cdot \mathbf{v}_f$. Therefore there exists such a $\mathbf{v}_f$ for every f. Suppose there exists $\mathbf{v}'_f$ such that $f(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}'_f$. Then $f(\mathbf{v}'_f) = \mathbf{v}'_f \cdot \mathbf{v}_f = \mathbf{v}'_f \cdot \mathbf{v}'_f$ and $f(\mathbf{v}_f) = \mathbf{v}_f \cdot \mathbf{v}_f = \mathbf{v}_f \cdot \mathbf{v}'_f$. Then we have $\mathbf{v}_f^2 = \mathbf{v}_f \cdot \mathbf{v}'_f = \mathbf{v}'^2_f$, so $(\mathbf{v}_f \mathbf{v}'_f)^2 = \mathbf{v}'^2_f \mathbf{v}_f^2$, and finally $|\mathbf{v}_f \mathbf{v}'_f| = |\mathbf{v}_f||\mathbf{v}'_f|$. Then by 5.a, $\mathbf{v}'_f = \beta\mathbf{v}_f$, $\beta \in \mathbb{R}$. We already showed that $f(\mathbf{v}'_f) = \mathbf{v}_f \cdot \mathbf{v}'_f = f(\mathbf{v}_f)$, so $f(\mathbf{v}'_f) = f(\beta\mathbf{v}_f) = \beta f(\mathbf{v}_f)$ and thus $f(\mathbf{v}_f) = \beta f(\mathbf{v}_f)$, so $\beta = 1$ and therefore $\mathbf{v}'_f = \mathbf{v}_f$. From this we see that there exists a unique $\mathbf{v}_f \in V$ such that for all $\mathbf{v} \in V$, $f(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_f$.

b) First to prove bijectivity. Fix $\mathbf{v} \in V$ and define $f : V \to \mathbb{R}$ such that $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{v}$. Then $Q_V(f) = \mathbf{v}$. Therefore Q_V is surjective. Fix $f, g \in V^*$ and suppose $Q_v(f) = Q_v(g) = \mathbf{v}$. Then for all $\mathbf{x} \in V$, $f(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x} = g(\mathbf{x})$, so $f = g$ and thus Q_V is injective and therefore bijective.

Now for linearity. Let $\alpha \in \mathbb{R}$ and $f, g \in V^*$. Then $(\alpha f)(\mathbf{v}) = \alpha(\mathbf{v}_f \cdot \mathbf{v}) = (\alpha\mathbf{v}_f) \cdot \mathbf{v}$, so $\mathbf{v}_{\alpha f} = \alpha\mathbf{v}_f$ and thus $Q_V(\alpha f) = \alpha Q_V(f)$. Additionally, $f(\mathbf{v}) + g(\mathbf{v}) = \mathbf{v}_f \cdot \mathbf{x} + \mathbf{v}_g \cdot \mathbf{x} = (\mathbf{v}_f + \mathbf{v}_g) \cdot \mathbf{x}$, so $\mathbf{v}_{f+g} = \mathbf{v}_f + \mathbf{v}_g$, and thus $Q_V(f) + Q_V(G) = Q_V(f + g)$. It follows that Q_V is linear.

c) Fix $\alpha \in \mathbb{R}$, $f, g \in W^*$. Then for ant $\mathbf{v} \in W$, $T^*(\alpha f + g)(\mathbf{v}) = (\alpha f + g)(T(\mathbf{v})) = \alpha f(T(\mathbf{v})) + g(T(\mathbf{v})) = (\alpha T^* f + T^* g)(\mathbf{v})$. Thus $T^*(\alpha f + g)(\mathbf{v}) = (\alpha T^* f + T^* g)(\mathbf{v})$ so $T*$ is linear.

d) Fix $\mathbf{w} \in W$. $Q_W^{-1}(\mathbf{w}) = f$ where for all $\mathbf{x} \in W$, $f(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x}$. $Q_V(\mathbf{T}^*(\mathbf{f})) = \mathbf{v}_{f \circ T}$, so $Q_V \circ \mathbf{T}^* Q_W^{-1}(\mathbf{w}) = \mathbf{v}_{f \circ T}$. For any $\mathbf{v} \in V$, since $f(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x}, f(T\mathbf{v}) = \mathbf{w} \cdot T\mathbf{v} = \mathbf{v} \cdot T^*\mathbf{w}$, so $f \circ T(\mathbf{v}) = \mathbf{v} \cdot T^*\mathbf{w}$ and thus $\mathbf{v}_{f \circ T} = T^*\mathbf{w}$. Therefore, $T^*\mathbf{w} = Q_V \circ \mathbf{T}^* Q_W^{-1}(\mathbf{w})$ so $T^* = Q_V \circ \mathbf{T}^* Q_W^{-1}$.

Exercise 7. Set $\mathcal{P}_n = \{\sum_{i=0}^n a_i t^i : a_i \in \mathbb{R}, i = 1, \ldots, n\}$ the set of all polynomials with degree $\leq n$.Consider the dot product

$$p \cdot q = \int_0^1 p(t)q(t)dt$$

a) Show that $\cdot$ indeed is an inner product.

b) Find an orthonormal basis for $\mathcal{P}_4$.

c) Consider the linear maps $T, R : \mathcal{P}_n \to \mathcal{P}_n$.

$$T(p)(t) = \int_0^1 (t-x)p(x)dx \text{ and } R(p)(t) = \int_0^1 (t-x^2)p(x)dx$$

Show that $T^* = -T$ and compute R^* (for this last case maybe you want to split cases $n \geq 2$ and $n = 1$)

d) Use rank + nullity to conclude that given $q \in \mathcal{P}_n$, $T(p) = q$ has a solution if and only if $q \in \text{span}\{1, t\}$.

Solution. a) Let $p, q, r \in \mathcal{P}_n$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}(\alpha p + q) \cdot r &= \int_0^1 (\alpha p + q)(t)r(t)dt \\ &= \int_0^1 \alpha p(t)r(t)dt + \int_0^1 q(t)r(t)dt \\ &= \alpha \int_0^1 p(t)r(t)dt + \int_0^1 q(t)r(t)dt \\ &= \alpha(p \cdot r) + (q \cdot r)\end{aligned}$$

Since multiplication of polynomials is commutative, $p \cdot q = q \cdot p$. For any $p \in \mathcal{P}_n$ if $p \neq 0$ then $p^2(t) \geq 0$ for all $t \in [0,1]$ and $p(t) > 0$ for some region in $[0,1]$, so $\int_0^1 p^2(t)dt > 0$, and thus $p \cdot p > 0$. Suppose $p = 0$. Then $p(t) = 0$ for all $t \in [0,1]$, so $\int_0^1 p(t)dt = 0$, and thus $p \cdot p = 0$. Since all of these properties hold, $\cdot$ is an inner product.

b) Using the Gram-Schmidt process, an orthonormal basis of $\mathcal{P}_4$ is

We will use the basis $\{1, t, t^2, t^3, t^4\}$ as our starting basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, use it to find an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, and then convert it to an orthonormal basis $\{e_1, \ldots, e_n\}$

$\mathbf{u}_1 = \mathbf{v}_1 = t$, so $\epsilon_1 = \frac{1}{|1|} = 1$.
$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{|\mathbf{u}_1|}\mathbf{u}_1 = t - \frac{1}{2}$. $|\mathbf{u}_2| = \frac{\sqrt{3}}{6}$, so $e_2 = 2\sqrt{3}x - \sqrt{3}$.
$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{|\mathbf{u}_1|}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{|\mathbf{u}_2|}\mathbf{u}_2 = t^2 - t + \frac{1}{6}$. $|\mathbf{u}_3| = \frac{1}{6\sqrt{5}}$, so $e_3 = 6\sqrt{5}t^2 - 6\sqrt{5}t + \sqrt{5}$.
Continuing this process for the final 2 elements, we get orthonormal basis $\{e_1, \ldots, e_n\} =$

$\{1, 2\sqrt{3}t - \sqrt{3}, 6\sqrt{5}t^2 - 6\sqrt{5}t + \sqrt{5}, 20\sqrt{7}t^3 - 30\sqrt{7}t^2 + 12\sqrt{7}t - \sqrt{7}, 210t^4 - 420t^3 + 270t^2 - 60t + 3\}$

c) Following from the definition of T, $Tp = p(x) \cdot (t - x) = t(p(x) \cdot 1) - (p(x) \cdot x)$, so

$$\begin{aligned} q(t) \cdot Tp &= q(t) \cdot (t(p(x) \cdot 1) - (p(x) \cdot x)) \\ &= (q(t) \cdot t)(p(x) \cdot 1) - (p(x) \cdot x)(q(t) \cdot 1) \\ &= -((p(x) \cdot x)(q(t) \cdot 1) - (q(t) \cdot t)(p(x) \cdot 1)) \\ &= -(p(x) \cdot (x(q(t) \cdot 1) - (q(t) \cdot t))) \\ &= -p \cdot Tq \end{aligned}$$

For R, we will consider two cases: when $n \geq 2$ and when $n = 1$ ($n = 0$ is impossible because the image of R is a polynomial with degree one and thus is not in $\mathcal{P}_0$). If $n \geq 2$, let $R^*(q)(x) = \int_0^1 (t - x^2)q(t)dt$. Then for $p, q \in \mathcal{P}_n$,

$$\begin{aligned} p \cdot R^*q &= \int_0^1 p(x) \int_0^1 (t - x^2)q(t)dt \\ &= \int_0^1 \int_0^1 p(x)(t - x^2)q(t)dtdx \\ &= \int_0^1 q(t) \int_0^1 p(x)(t - x^2)dxdt \\ &= q \cdot Rp \end{aligned}$$

Thus R^* is the adjoint.

For the $n = 1$ case, define R^* over $p_0(t) = 1$ and $p_1(t) = t$, which are a basis of $\mathcal{P}_1$, as the following: $R^*(p_0) = \frac{2}{3} - t$ and $R^*(p_1) = \frac{5}{12} - \frac{t}{2}$. Fix $p, q \in \mathcal{P}_1$ where $p = \alpha + \beta t$ and $q = \mu + \nu t$. Then

$$\begin{aligned} p \cdot Rq &= (\alpha + \beta t) \cdot R(\mu + \nu t) \\ &= \int_0^1 (\alpha + \beta t) \int_0^1 (t - x^2)(\mu + \nu x)dxdt \\ &= \frac{4\alpha\mu + \beta(r\mu + \nu)}{24} \end{aligned}$$

and since

$$R^*(\alpha + \beta t) = \alpha R^*(p_0) + \beta R^*(p_t) = \alpha(\frac{2}{3} - t) + \beta(\frac{5}{12} - \frac{t}{2}) = \frac{5\beta + 8\alpha}{12} - \frac{t(\alpha + 2\beta)}{2}$$

we have

$$\begin{aligned} q \cdot Rq &= (\mu + \nu t) \cdot R^*(\alpha + \beta t) \\ &= \int_0^1 (\mu + \nu t) \left(\frac{5\beta + 8\alpha}{12} - \frac{t(\alpha + 2\beta)}{2} \right) dt \\ &= \frac{4\alpha\mu + \beta(r\mu + \nu)}{24} \end{aligned}$$

Thus $pRq = qR^*p$, as required.

d) We will first prove that $\ker(T^*)^\perp = \operatorname{span}\{1,t\}$. Suppose $p \in \operatorname{span}\{1,t\}$. Then $p = \alpha + \beta t$ for $\alpha, \beta \in \mathbb{R}$. Fix $q \in \ker T^*$. We know $T^*q = 0$, so $(\alpha + \beta t) \cdot T^*(q) = 0$, and thus $\alpha \cdot T^*(q) = -\beta t \cdot T^*(q)$. It follows that $\int_a^b \alpha(t-x)q(t)dtdx = \int_a^b -\beta(t-x)q(t)dtdx$, so therefore $\alpha q(t) = -\beta t \cdot q(t)$, and thus $(\alpha + \beta t) \cdot q(t) = 0$, so since this holds for all $q(t) \in kerT^*$, we have $\alpha + \beta t = p(t) \in \ker(T^*)^\perp$. Thus $\operatorname{span}\{1,t\} \subset \ker(T^*)^\perp$, so by rank + nullity, $\operatorname{span}\{1,t\} \subset \operatorname{Im} T$. Suppose $q \in \operatorname{Im} T$. Then $q = (t-x) \cdot p(x) = t(1 \cdot p(x)) - x \cdot p(x)$ for some $p \in \mathcal{P}_n$, so $q = \alpha + \beta t$, so $\dim \operatorname{Im} T \leq 2$. Since $\operatorname{span}\{1,t\} \subset \operatorname{Im} T$, $\operatorname{rank} T \geq 2$, so it follows that $\operatorname{rank} T = 2$. Since $\dim \operatorname{span}\{1,2\} = 2$ and $\operatorname{span}\{1,2\} \subset \operatorname{Im} T$, it follows that $\operatorname{Im} t = \operatorname{span}\{1,t\}$.

Exercise 8. Let T be a self-adjoint linear operator over V.

a) $T(\mathbf{v} + t\mathbf{w}) \cdot (\mathbf{v} + t\mathbf{w}) = T\mathbf{v} + 2tT\mathbf{w} \cdot \mathbf{v} + t^2 T\mathbf{w} \cdot \mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in V$ and $t \in \mathbb{R}$.

b) Arrange the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, where $n = \dim V$ and for each 2-dimensional subspace $P \subset V$ set $\bar{T}(P) = \max T\mathbf{v} \cdot \mathbf{v} : \mathbf{v} \in P, |\mathbf{v}| = 1$. Show that

$$\lambda_2 = \inf \bar{T}(P) : P \text{ is subspace with } \dim(P) = 2.$$

Solution. a) Since T self-adjoint, $\mathbf{v}T\mathbf{w} = \mathbf{w}T\mathbf{v}$, which will be used below

$$\begin{aligned} T(\mathbf{v} + t\mathbf{w}) \cdot (\mathbf{v} + t\mathbf{w}) &= T\mathbf{v} \cdot \mathbf{v} + tT\mathbf{w} \cdot \mathbf{v} + tT\mathbf{v} \cdot \mathbf{w} + t^2 T\mathbf{w} \cdot \mathbf{w} \\ &= T\mathbf{v} \cdot \mathbf{v} + 2tT\mathbf{w} \cdot \mathbf{v} + t^2 T\mathbf{w} \cdot \mathbf{w} \end{aligned}$$

b) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be the orthonormal basis of V with respect to $\lambda_1, \ldots, \lambda_n$. Let P be an arbitrary two dimensional subspace of V, so $P = \operatorname{span}\{\mathbf{v}_a, \mathbf{v}_b\}$ with $b > a$. Let $\mathbf{v} = \alpha\mathbf{v}_a + \beta\mathbf{v}_b \in V$ such that $|\mathbf{v}| = 1$. Then $|\alpha\mathbf{v}_a + \beta\mathbf{v}_b| = 1$, so since $\mathbf{a} \cdot \mathbf{b} = 0$, $\sqrt{\alpha^2\mathbf{v}_a^2 + \beta^2\mathbf{v}_b^2} = 1$. By definition $\mathbf{v}_a^2 = \mathbf{v}_b^2 = 1$, so $\alpha^2 + \beta^2 = 1$ and thus $\alpha^2 = (1 - \beta^2)$. Additionally,

$$\begin{aligned} T\mathbf{v} \cdot \mathbf{v} &= T(\alpha\mathbf{v}_a + \beta\mathbf{v}_b) \cdot (\alpha\mathbf{v}_a + \beta\mathbf{v}_b) \\ &= (\alpha T\mathbf{v}_a + \beta T\mathbf{v}_b)(\alpha\mathbf{v}_a + \beta\mathbf{v}_b) \\ &= (\alpha\lambda_a\mathbf{v}_a + \beta\lambda_b\mathbf{v}_b)(\alpha\mathbf{v}_a + \beta\mathbf{v}_b) \\ &= (\alpha^2\lambda_a\mathbf{v}_a^2 + \beta^2\lambda_b\mathbf{v}_b^2) \\ &= \alpha^2\lambda_a\beta^2\lambda_b \end{aligned}$$

So $T\mathbf{v} \cdot \mathbf{v} = \alpha^2\lambda_a\beta^2\lambda_b$. Then since $\lambda_a \leq \lambda_b$, $(1-\beta^2)\lambda_a \leq (1-\beta^2)\lambda_b$, so $\alpha^2\lambda_a + \beta^2\lambda_b \leq |\lambda_b|$, so $T\mathbf{v} \cdot \mathbf{v} \leq |\lambda_b|$, so finally $|T\mathbf{v} \cdot \mathbf{v}| \leq \lambda_b$. $|\mathbf{b} \cdot T\mathbf{b}| = |\lambda_b|$, so $|\lambda_b| = \max T\mathbf{v} \cdot \mathbf{v} : \mathbf{v} \in P, |\mathbf{v}| = 1$ for all subsets P. Since there is no eigenvalue less than λ_1, for no P can $|\lambda_1| = \bar{T}(P)$. However, $\bar{T}(\operatorname{span} \mathbf{v}_1, \mathbf{v}_2) = |\lambda_2|$, so therefore $\lambda_2 = \inf \bar{T}(P) : P$ is subspace with $\dim(P) = 2$.

Exercise 9. V a finite dimensional vector space with an inner product and $T : V \to V$ an anti self-adjoint linear map ($T^* = -T$).

a) Show that if T is invertible then T has no nonzero vector such that $T(\mathbf{v}) = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$.

b) Compute the dimension of the vector spaces (in terms of $n = \dim V$)

$$V_+ = \{T : V \to V : T^* = T\} \text{ and } V_- = \{T : V \to V : T^* = -T\},$$

where the maps being considered are all linear.

Solution. a) Suppose there exists some non-zero $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$. Then $\mathbf{v} \cdot T(\mathbf{v}) = \lambda\mathbf{v}^2$, so $\mathbf{v} \cdot T^*\mathbf{v} = \mathbf{v} \cdot T\mathbf{v} = \lambda\mathbf{v}^2$. Since $T^* = -T$, $\mathbf{v} \cdot T^*(\mathbf{v}) = \mathbf{v} \cdot -T(\mathbf{v}) = -\lambda\mathbf{v}^2$. Then we have $\lambda\mathbf{v}^2 = -\lambda\mathbf{v}^2$. Since $\mathbf{v} \neq \mathbf{0}$, $\mathbf{v}^2 \neq 0$, so it follows that $\lambda = -\lambda$ and thus $\lambda = 0$. Then we have $T(\mathbf{v}) = 0$, so $\ker T \neq \{0\}$ and therefore $\operatorname{null} T > 0$ and thus T is not invertible. The contrapositive of this is the statement of part a), if T is invertible the T has no nonzero vector such that $T(\mathbf{v}) = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$.

b) Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis of B.

First we will consider V_+. We will prove $\{T_i\}_{1\leq i\leq n}$ is a basis of V_+ where

$$T_i\left(\sum_{i=1}^{n} \alpha_i\mathbf{v}_i\right) = \alpha_i\mathbf{v}_i$$

Fix $T \in V_+$. Since T is self-adjoint, it is diagonalizable. Set $\lambda_i = T\mathbf{v}_i$. Fix $\mathbf{v} \in V$. Then

$$\begin{aligned}
T\mathbf{v} &= T\left(\sum_{j=1}^{n} \alpha_j\mathbf{v}_j\right) \\
&= \sum_{j=1}^{n} \alpha_j T\mathbf{v}_j \\
&= \sum_{j=1}^{n} \alpha_j \lambda_j \mathbf{v}_j \\
&= \sum_{j=1}^{n} \lambda_j T_j \left(\sum_{j=1}^{n} \alpha_j\mathbf{v}_j\right) \\
&= \sum_{j=1}^{n} \lambda_j T_j \mathbf{v}
\end{aligned}$$

Thus $\{T_i\}_{1\leq i\leq n}$ spans V_+. Suppose there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \alpha_i T_i = \mathbf{0}$. Then for $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n$, $\sum_{i=1}^{n} \alpha_i T_i(\mathbf{v}) = \mathbf{0}$ which implies that $\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \mathbf{0}$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ and thus $\{T_i\}_{1\leq i\leq n}$ is linearly independent and therefore is a basis. It clearly has length n, so $\dim V_+ = n$.

Now we will consider V_-. Consider the set

$$V' = \{T : V \to V : T \text{ has at least one eigenvector}\}$$

. Without loss of generality, let $\mathbf{v}_1$ be an eigenvector for all $T \in V'$. Let $\{T_1, \ldots, T_{(n-1)^2}\}$ be a basis of $\mathcal{L}(\operatorname{span} \mathbf{v}_1^{\perp})$. Define $\{S_1, \ldots, S_{(n-1)^2}\}$ such that

$$S_i \sum_{j=1}^{n} \alpha_j\mathbf{v}_j = \alpha_1\mathbf{w}_1 + T_1 \sum_{i=2}^{n} \alpha_j\mathbf{v}_j$$

In other words, $S_i\mathbf{v}_1 = \mathbf{w}_1$ and for any other $\mathbf{v}$, $S_i\mathbf{v} = T_i\mathbf{v}$ projected into V. $\{S_1, \ldots, S_{(n-1)^2}\}$ is a basis of V' since all of its elements are linearly independent (due to the linear independence of $\{T_1, \ldots, T_{(n-1)^2}\}$) and it can sum to any linear map with at least one eigen vector. $\{S_1, \ldots, S_{(n-1)^2}\}$ can be extended with $\{S_{(n-1)^2+1}, \ldots, S_{n^2}\}$ to a basis of $\mathcal{L}(V)$. It follows that $\operatorname{span}\{S_{(n-1)^2+1}, \ldots, S_{n^2}\}$ are the set of all maps without an eigen vector. Then by a) $\operatorname{span}\{S_{(n-1)^2+1}, \ldots, S_{n^2}\} = V_-$, so V_- has dimension $n^2 - (n-1)^2 = 2n - 1$.

Exercise 10. V a finite dimensional vector space of $\dim(V) = n$ with an inner product.

a) Show that if $T : V \to V$ has k nonzero vectors $v_1, \cdots, v_k$ so that $T(v_i) = \lambda_i v_i$ and $\lambda_i \neq \lambda_j$ if $i \neq j$, then $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ are all linearly independent.

b) Show that if λ is an eigenvalue of $T : V \to V$, then it is also an eigenvalue of T^*.

c) If v is an eigenvector of $T : V \to V$, is it also an eigenvector of T^*? Prove or give a counterexample.

d) Assuming $n \geq 2$, find $T : V \to V$ so that $T^2 = \mathbf{0}$ but $T \neq \mathbf{0}$.

e) Show that if $T : V \to V$ linear then $\ker(T^* \circ T) = \ker T$.

f) Conclude that if T is self-adjoint, then $T^k = \mathbf{0}$ implies $T = \mathbf{0}$, for any $k \in \mathbb{N}$.

g) Show that if T is diagonalizable, i.e., V has an orthonormal basis made from eigenvectors of T, then T is self-adjoint.

Solution. a) Clearly if T has 1 nonzero vector $\mathbf{v}_1$, it is linearly independent. Suppose there exists some greatest $n \in \mathbb{N}$ such that T has n nonzero vectors $v_1, \cdots, v_n$ so that $T(v_i) = \lambda_i v_i$ and $\lambda_i \neq \lambda_j$ if $i \neq j$, then $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ are all linearly independent. Then for $n+1$ with vectors $v_1, \cdots, v_n$ and corresponding eigenvalues there exists $\alpha_1, \cdots, \alpha_n \neq 0$ such that $\sum_{i=1}^{j} \alpha_i \mathbf{v}_i = \mathbf{v}_{n+1}$. Then by applying T, we have

$$\sum_{i=1}^{j} \alpha_i \lambda_i \mathbf{v}_i = \lambda_{n+1}\mathbf{v}_{n+1} = \lambda_{n+1}\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \sum_{i=1}^{n} \lambda_{n+1}\alpha_i \mathbf{v}_i$$

Since $\sum_{i=1}^{n} \alpha_i \lambda_i \mathbf{v}_i = \sum_{i=1}^{n} \lambda_{n+1}\alpha_i \mathbf{v}_i$, so $\sum_{i=1}^{n} \alpha_i(\lambda_i - \lambda_{n+1})\mathbf{v}_i = \mathbf{0}$. For some $1 \leq j \leq n$, $\alpha_j \neq 0$ but since $\{\mathbf{v}_1, \cdot, \mathbf{v}_n\}$ are linearly independent, $\alpha_1(\lambda_1 - \lambda_{n+1}) = \cdots = \alpha_n(\lambda_n - \lambda_{n+1}) = 0$, so $\alpha_j(\lambda_j - \lambda_{n+1}) = 0$ and thus $\lambda_j - \lambda_{n+1} = 0$, so $\lambda_j = \lambda_{n+1}$ which is a contradiction.

b) First we will prove that for any linear operator $T : V \to V$ and any $\lambda \in \mathbb{R}$, $T - \lambda I$ is not invertible if and only if λ is an eigenvalue of T. Suppose λ is an eigenvalue with eigenvector $\mathbf{v} \neq \mathbf{0}$. Then $(T - \lambda I)(\mathbf{v}) = \lambda\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$, so since $T\mathbf{0} = \mathbf{0}$, T is not injective and hence not inevertible. Suppose $(T - \lambda I)$ is not invertible. Then since T is a linear operator, $\operatorname{null}(T - \lambda I) < \dim V$, so there exists $\mathbf{v} \neq 0 \in \ker(T - \lambda I)$. Then $(T - \lambda I)\mathbf{v} = T\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$, so $\mathbf{v}$ is an eigenvector with eigenvalue λ. Next we will prove that if $(T - \lambda I)$ is not invertible then $(T - \lambda I)^*$ is not invertible. Suppose $(T - \lambda I)$ is not invertible and $(T - \lambda I)^*$ is invertible. Then let $S = ((T - \lambda I)^*)^{-1}$. It follows that S^* is the inverse of $(T - \lambda I)^{*}* = (T - \lambda I)$, which is a contradiction. Thus if $(T - \lambda I)$ is not invertible then $(T - \lambda I)^*$ is not invertible. If λ is an eigenvalue of T then $T - \lambda I$ is not invertible, so $(T - \lambda I)^* = T^* - (\lambda I)^* = T^* - \lambda I$ $(\mathbf{v} \cdot (\lambda I)\mathbf{w} = \lambda\mathbf{v} \cdot \mathbf{w} = \mathbf{w}(\lambda I)\mathbf{v}$, so $(\lambda I)^* = \lambda I)$ is not invertible and thus λ is an eigenvalue of T^*.

c) Consider the linear operator over $\mathbb{R}^2$ $T(a,b) = (a+b, b)$. First I will show that $S(c,d) = (c, c+d)$ with the canonical inner product is the adjoint of T. For $(a,b),(c,d) \in \mathbb{R}^2$, $(c,d)\cdot T(a,b) = c(a+b)+db = ca+(c+d)b = (a,b)\cdot S(c,d)$. Thus $T^*(c,d) = (c, c+d)$. $T(1,0) = (1,0)$, so $(1,0)$ is an eigenvector of T but $T^*(1,0) = (1,1)$ so $(1,0)$ is not an eigenvector.

d) Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ be a basis of V and define $T\sum_{i=1}^n \alpha_i\mathbf{v}_i = \alpha_1\mathbf{v}_n$. $T\sum_{i=1}^n \mathbf{v}_i = \mathbf{v}_n \neq \mathbf{0}$ so $T \neq \mathbf{0}$. However, $T\circ T(\sum_{i=1}^n \alpha_i\mathbf{v}_i) = T(\alpha_1\mathbf{v}_n) = \mathbf{0}$.

e) By 2.e), if $\mathbf{v} \neq \mathbf{0} \in \operatorname{Im} T$ then $\mathbf{v} \in \ker(T^*)^\perp$, so $\mathbf{v} \notin \ker(T^*)$. Suppose $\mathbf{v} \in \ker T$. Then $T\mathbf{v} = \mathbf{0}$ so $T^*\circ T\mathbf{v} = \mathbf{0}$, leaving $\mathbf{v} \in \ker T^*\circ T$. Suppose $\mathbf{v} \in \ker T^*\circ T$. Then either $T\mathbf{v} = \mathbf{0}$, in which case $\mathbf{v} \in \ker T$ or $T^*\circ T\mathbf{v} = \mathbf{0}$ but $T\mathbf{v} \neq \mathbf{0}$. Then $T\mathbf{v} \in \operatorname{Im} T$ and $T\mathbf{v} \in \ker(T^*)$, which is a contradiction. Therefore, if $\mathbf{v} \in \ker(T^*\circ T)$ then $\mathbf{v} \in \ker(T)$. It follows that $\ker(T^*\circ T) = \ker(T)$.

f) Let T self-adjoint, and $P(n), n \in \mathbb{N}$ be the propisition that $(T^*)^n T(\mathbf{v}) = \mathbf{0}$ implies $T\mathbf{v} = \mathbf{0}$. By the previous exercise, $P(1)$ holds. Suppose for some $k \in \mathbb{N}$ that $P(k)$ holds. Then $(T^*)^{n+1}T\mathbf{v} = T^*((T^*)^nT\mathbf{v})$ then either $(T^*)^nT\mathbf{v} = \mathbf{0}$ or $(T^*)^nT\mathbf{v} \neq \mathbf{0}$ but $T^*((T^*)^nT\mathbf{v}) = \mathbf{0}$. The first case, by $P(n)$, implies that $T\mathbf{v} = \mathbf{0}$. In the second case, since T is self-adjoint, $T^* = T$, so $(T^*)^nT\mathbf{v} = T^{n+1}\mathbf{v}$, so $T^{n+1}\mathbf{v} \in \operatorname{Im} T$. Then by the same argument as in the previous exercise, a contradiction is reached. Thus the first case is the only possibility, so $T\mathbf{v} = \mathbf{0}$.

g) Suppose $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is an orthonormal eigenvector basis of V with corresponding eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$. Fix $\mathbf{v} = \sum_{i=1}^n \alpha_i\mathbf{v}_i$ and $\mathbf{w} = \sum_{i=1}^n \beta_i\mathbf{v}_i$. Then

$$\begin{aligned}
\mathbf{w}\cdot T\mathbf{v} &= \sum_{i=1}^n \alpha_i\mathbf{v}_i \cdot T\sum_{i=1}^n \beta_i\mathbf{v}_i \\
&= \sum_{i=1}^n \alpha_i\mathbf{v}_i \cdot \sum_{i=1}^n \beta_i\lambda_i\mathbf{v}_i \\
&= \sum_{i=1}^n \alpha_i\beta_i\lambda_i
\end{aligned}$$

And similarly

$$\begin{aligned}
\mathbf{v}\cdot T\mathbf{v} &= \sum_{i=1}^n \beta_i\mathbf{v}_i \cdot T\sum_{i=1}^n \alpha_i\mathbf{v}_i \\
&= \sum_{i=1}^n \beta_i\mathbf{v}_i \cdot \sum_{i=1}^n \alpha_i\lambda_i\mathbf{v}_i \\
&= \sum_{i=1}^n \beta_i\alpha_i\lambda_i
\end{aligned}$$

Hence $\mathbf{w}\cdot T\mathbf{v} = \mathbf{v}\cdot T\mathbf{w}$, so T is self-adjoint.