AE454/777: Dynamics and Bifurcations
Tutorial 1

1. Consider the system $\dot{x} = x + \mathrm{e}^{-y}$, $\dot{y} = -y$. Plot the direction field numerically. Use the Runge-Kutta method to compute several trajectories, and plot them on the phase plane. In particular, mark the stable and unstable manifolds.

2. Draw the phase portraits of the follwing systems
(a) $\dot{x} = -2\cos x - \cos y$, $\dot{y} = -2\cos y - \cos x$
(b) $\dot{x} = y$, $\dot{y} = -x + y(1 - x^2)$
(c) $\dot{x} = 2xy$, $\dot{y} = y^2 - x^2$
(d) $\dot{x} = y + y^2$, $\dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$
(e) $\dot{x} = y + y^2$, $\dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

3. For each of the following reversible systems, try to sketch the phase portrait by hand. Then use a computer to check your sketch. If the computer reveals patterns you hadn't anticipated, try to explain them.
(a) $\ddot{x} + (\dot{x})^2 + x = 3$
(b) $\dot{x} = y - y^3$, $\dot{y} = x\cos y$
(c) $\dot{x} = \sin y$, $\dot{y} = y^2 - x$

4. Consider the system
$$\dot{r} = r(1 - r^2) + \mu r\cos\theta$$
$$\dot{\theta} = 1$$
(a) Plot the phase portrait for various values of μ.Is there a critical value of μ at which closed orbits cease to exist? If so, estimate it.
(b) Calculate $r(\theta)$ numerically for various values of μ.

5. Consider the equation $\ddot{x} + \epsilon\dot{x}^3 + x = 0$.
(a) Derive the averaged equations
(b) Given the initial conditions $x(0) = a$, $\dot{x}(0) = 0$, solve the averaged equations and thereby find an approximate formula for $x(t, \epsilon)$
(c) Solve this equation numerically for $a = 1$, $\epsilon = 2$, $0 \le t \le 50$, and plot the result in the same graph as your answer to part (b). Do the solutions agree? Notice that ϵ is not small.

6. A system exhibiting homoclinic bifurcation is
$$\dot{x} = y$$
$$\dot{y} = \mu y + x - x^2 + xy$$
Plot the phase portraits for (a) $\mu = -0.92$, $\mu = \mu_c \approx -0.8645$, and $\mu = -0.80$.

7. For each of the following systems , a Hopf bifurcation occurs at the origin when $\mu = 0$. Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical.
(a) $\dot{x} = y + \mu x$, $\dot{y} = -x + \mu y - x^y$
(b) $\dot{x} = \mu x + y - x^3$, $\dot{y} = -x + \mu y - 2y^3$
(c) $\dot{x} = \mu x + y - x^2$, $\dot{y} = -x + \mu y - 2x^2$