AE6104 HW3

1

a) $(\underline{\underline{A}} \otimes \underline{\underline{B}})^t$

For arbitrary $\underline{\underline{X}}, \underline{\underline{Y}}$,

$\underline{\underline{X}} : (\underline{\underline{A}} \otimes \underline{\underline{B}}) Y$

$\Rightarrow X : A(B:Y)$

$(A:X)(B:Y)$

$\Rightarrow (B(A:X)) : Y$

$\Rightarrow ((B \otimes A)X) : Y$

$= (((A \otimes B)^t X)) : Y$

$\therefore (A \otimes B)^t = B \otimes A$

b) $(A \underline{\otimes} B)^t$

$X : (A \underline{\otimes} B) Y \Rightarrow X : A Y^T B^T = tr(X^T A Y^T B^T)$

$= tr((X^T A Y^T B^T)^T) = tr(A^T X B Y) = B^T X^T A : Y$

$= (B^T \underset{\sim}{\otimes} A) X : Y$

or $(B^T \underline{\otimes} A^T) X : Y$

$\therefore (A \underline{\otimes} B)^t = (B^T \underset{\sim}{\otimes} A) = (B^T \underline{\otimes} A^T)$

c) $(A \bar{\otimes} B)^T$

$$X : (A \bar{\otimes} B) Y = X : A Y B^T = \operatorname{tr}(X B Y^T A^T) = \operatorname{tr}((X B Y^T A^T)^T)$$

$$= \cancel{\operatorname{tr}(A Y B^T X^T) = A Y B^T : X = (A \bar{\otimes} B) Y : X = (A \tilde{\otimes} B^T) Y : X}$$

$$\cancel{\therefore (A \bar{\otimes} B)^T = A \bar{\otimes}}$$

(Note $\operatorname{tr}(AB) = \operatorname{tr}(BA)$)

$$= \operatorname{tr}(B^T X^T A Y) = A^T X B : Y = (A^T \tilde{\otimes} B) X : Y$$

$$= (A^T \bar{\otimes} B^T) X : Y$$

$$\therefore (A \bar{\otimes} B)^T = (A^T \bar{\otimes} B^T) = (A^T \tilde{\otimes} B)$$

d) $(A \tilde{\otimes} B)^T$

$\cancel{X : (A \tilde{\otimes} B)}$

Since $(A \bar{\otimes} B)^T = A^T \tilde{\otimes} B \Rightarrow (A^T \tilde{\otimes} B)^T = A \bar{\otimes} B$

$$\Rightarrow (A \tilde{\otimes} B)^T = \underline{A^T \bar{\otimes} B} = \underline{A^T \tilde{\otimes} B^T}$$

e) $(A \underset{\sim}{\otimes} B)^T$

Since $(A \underline{\otimes} B)^T = B^T \underset{\sim}{\otimes} A \Rightarrow (B^T \underset{\sim}{\otimes} A)^T = A \underline{\otimes} B \Rightarrow$

$$\Rightarrow (A \underset{\sim}{\otimes} B)^T = B \underline{\otimes} A^T = B \underset{\sim}{\otimes} A$$

2.

a) $(A \otimes B)(C \otimes D)$

For arbitrary X,

$$(A \otimes B)(C \otimes D)X = (A \otimes B)\, C\, (D:X) = \underline{A(B:C)(D:X)}$$

$$= (A \otimes C)(B \otimes D)[X]$$

$$\therefore (A \otimes B)(C \otimes D) = (A \otimes C)(B \otimes D)$$

b) $(A \underline{\otimes} B)(C \underline{\otimes} D)$

$$(A \underline{\otimes} B)(C \underline{\otimes} D)X = (A \underline{\otimes} B)\, C X^{T} D^{T} = A D X C^{T} B^{T}$$

$$= (AD \,\bar{\otimes}\, BC)X \quad = (AD \,\tilde{\otimes}\, C^{T}B^{T})X$$

$$\therefore (A \underline{\otimes} B)(C \underline{\otimes} D) = (AD \,\bar{\otimes}\, BC) = (AD \,\tilde{\otimes}\, C^{T}B^{T})$$

c) A

~~c) .... Code generated~~

$$\cancel{(A \bar{\otimes} B)(C \bar{\otimes} D)X}$$

$$\cancel{\Rightarrow (A \bar{\otimes} B)\, C X D^{T} \Rightarrow ACX}$$

Rest is done in code. Refer to q2.py

3.) Separate code file generated

4.

a) $(A \circledast B)^{-1}$

Let $Y = (A \circledast B) \cdot X$

$\Rightarrow Y = A\,\underbrace{(B : X)}_{\text{Non invertible}}$

$\therefore (A \circledast B)^{-1}$ doesn't exist

b) $(A \underline{\otimes} B)^{-1}$

$Y = (A \underline{\otimes} B)X \Rightarrow Y = AX^T B^T \Rightarrow A^{-1} Y B^{-T} = X^T$

$\Rightarrow X = B^{-1} Y^T A^{-T} = (B^{-1} \underline{\otimes} A^{-1}) Y \Rightarrow X = \underbrace{(B^{-1} \underline{\otimes} A^{-1})(A \underline{\otimes} B)}_{I} X$

$\therefore (A \underline{\otimes} B)^{-1} = B^{-1} \underline{\otimes} A^{-1}$

c) $(A \bar{\otimes} B)^{-1}$

$Y = (A \bar{\otimes} B) X \Rightarrow Y = AXB^T \Rightarrow A^{-1} Y B^{-T} = X$

$X = (A^{-1} \bar{\otimes} B^{-1}) Y \Rightarrow \therefore (A \bar{\otimes} B)^{-1} = (A^{-1} \bar{\otimes} B^{-1})$

d) $(A \tilde{\otimes} B)^{-1}$ $\quad Y = (A \tilde{\otimes} B) X \Rightarrow AXB = Y \Rightarrow X = A^{-1} Y B^{-1} \Rightarrow X = (A^{-1} \tilde{\otimes} B^{-1}) Y$

$\Rightarrow (A \tilde{\otimes} B)^{-1} = (A^{-1} \tilde{\otimes} B^{-1})$

e) $(A \underset{\sim}{\otimes} B)^{-1} \Rightarrow \quad Y = (A \underset{\sim}{\otimes} B) X \Rightarrow AX^T B = Y \Rightarrow X^T = A^{-1} Y B^{-1}$

$\Rightarrow X = B^{-T} Y^T A^{-T} \Rightarrow X = (B^{-T} \underset{\sim}{\otimes} A^{-T}) Y$

$\therefore (A \underset{\sim}{\otimes} B)^{-1} = (B^{-T} \underset{\sim}{\otimes} A^{-T})$

by

1

grad $(v \cdot w)$ =

$$\operatorname{grad}(v \cdot w)[x] = [\operatorname{grad}(v)][x] \cdot w + v \cdot [\operatorname{grad}(w)][x]$$

$$\Rightarrow \operatorname{grad}(v)^T w \cdot x + \operatorname{grad}(w)^T v \cdot x$$

$$\Rightarrow \operatorname{grad}(v \cdot w) = \operatorname{grad}(v)^T w + \operatorname{grad}(w)^T v$$

2. Wrong?

3

grad $(S^T v)$ =

$$\operatorname{grad}(S^T v)[x] = (\operatorname{grad}(S^T)[x])[v] + S^T(\operatorname{grad}(v)[x])$$

$$= [v \cdot \operatorname{grad}(S^T)][x] + [S^T \operatorname{grad}(v)][x]$$

$$\Rightarrow \operatorname{grad}(S^T v) = v \cdot \operatorname{grad}(S^T) + S^T \operatorname{grad}(v)$$

4)
$$\operatorname{div}(S^T v) = \operatorname{grad}(S^T v) : g$$

$$= v \cdot \underbrace{\operatorname{grad}(S) : g}_{\text{div}(S) \text{ as grad}(S) \text{ is third order tensor}} + S^T \operatorname{grad}(v) : g$$

$$= v \cdot \operatorname{div}(S) + \operatorname{tr}(S^T \operatorname{grad}(v))$$

$$= v \cdot \operatorname{div}(S) + S : \operatorname{grad}(v)$$

5) $\mathrm{div}(\phi S) = \mathrm{grad}(\phi S) : g$

$= (S \otimes \mathrm{grad}\,\phi + \phi\,\mathrm{grad}(S)) : g$

$= S\,\mathrm{grad}\,\phi + \phi(\mathrm{grad}(S) : g)$

$= S\,\mathrm{grad}\,\phi + \phi\,\mathrm{div}(S)$

6) $\mathrm{div}(S^T a) = S : \underbrace{\mathrm{grad}(a)}_{0} + a \cdot \mathrm{div}(S)$

$= a \cdot \mathrm{div}(S)$

7) ~~$\mathrm{div}(S^T a)$~~

$\mathrm{div}(u \otimes w) = \mathrm{grad}(u \otimes w) : g$

$= (\mathrm{grad}(u) \otimes w + u \otimes \mathrm{grad}(w)) : g$

$= \mathrm{grad}(u)\, w + u\,\mathrm{div}(w)$

8. $\int_{\partial D} u \otimes n \, dA$

$\Rightarrow \left[\int_{\partial D} u \otimes n \, dA\right] a$ ← constant vector $= \int_{\partial D} u(n \cdot a)\, dA = \int_{\partial D} (u \otimes a)\, n \, dA$

$= \int_D \mathrm{div}(u \otimes a)\, dV = \int_D (\underbrace{u\,\mathrm{div}(a)}_{\substack{0 \\ \text{as } \mathrm{div}(a) = 0}} + \mathrm{grad}(u)[a])\, dV$

$\Rightarrow \left[\int_{\partial D} u \otimes n \, dA\right][a] = \int_D \mathrm{grad}(u)\,[a]\, dV = \left[\int_D \mathrm{grad}(u)\, dV\right][a]$

$\Rightarrow \boxed{\int_{\partial D} u \otimes n \, dA = \int_D \mathrm{grad}(u)\, dV}$

9)

$$\int_{\partial D} v \cdot Sn\, dA$$

$$\int_{\partial D} v \cdot Sn\, dA = \int_{\partial D} S^T v \cdot n\, dA = \int_{D} \operatorname{div}(S^T v)\, dV$$

$$= \int_{D} (S : \operatorname{grad} v + v \cdot \operatorname{div}(S))\, dV$$

10) ~~10)~~

11) $\int_{\partial D} u(w \cdot n)\, dA$

$$\Rightarrow \int_{\partial D} (u \otimes w) n\, dA = \int_{D} \operatorname{div}(u \otimes w)\, n\, dA$$

$$= \int_{D} (u \operatorname{div}(w) + \operatorname{grad}(u) w)\, dV$$

10)

$$\int_{\partial D} Sn \otimes u \, dA$$

$$a \cdot \left[\int_{\partial D} Sn \otimes u \, dA \right] b$$

$$\Rightarrow \int_{\partial D} a \cdot Sn \, (u \cdot b) \, dA = \int_{\partial D} (u \cdot b) \, (S^T a \cdot n) \, dA$$

$$= \int_{D} \operatorname{div}\big(\underbrace{(u \cdot b)}_{\phi} \underbrace{(S^T a)}_{w} \big) \, dV = \underbrace{S^T a \cdot \operatorname{grad}(u \cdot b)}_{\mathrm{I}} + \underbrace{(u \cdot b) \operatorname{div}(S^T a)}_{\mathrm{II}}$$

I $\quad S^T a \cdot \operatorname{grad}(u \cdot b) = S^T a \left((\operatorname{grad}(u))^T b + \underbrace{\operatorname{grad}(b)^T}_{0} w \right)$

$$\Rightarrow \quad S^T a \cdot \operatorname{grad}(u)^T b = \cancel{b \cdot \operatorname{grad}(u) S^T} a$$

$$= a \cdot S \operatorname{grad}(u)^T b$$

II $\quad u \cdot b \operatorname{div}(S^T a) = (u \cdot b) \, (a \cdot \operatorname{div}(S))$

$$= a \cdot (\operatorname{div}(S) \otimes u) \, b$$

$$\therefore \quad a \left[\int_{\partial D} Sn \otimes v \, dA \right] b = a \cdot \left(S \operatorname{grad}(u)^T + \operatorname{div}(S) \otimes u \right) b$$

$$\therefore \quad \int_{\partial D} Sn \otimes v \, dA = S \operatorname{grad}(u)^T + \operatorname{div}(S) \otimes v$$