2)

a) $\mathbf{a} = a_i g^j$

$\cancel{\quad}$ $\operatorname{grad} \mathbf{a} = \frac{\partial a}{\partial u^i} \otimes g^i = a_i|_j \, g^i \otimes g^j$

$$\frac{\partial a}{\partial u^i} = \frac{\partial}{\partial u^i}(a_k g^k) = \frac{\partial a_k}{\partial u^i} g^k + a_k \frac{\partial}{\partial u^i} g^k$$

$$= \frac{\partial a_k}{\partial u^i} g^k + a_k \Gamma^m_{ki} g_m$$

$$= \frac{\partial a_k g^k}{\partial u^i} + a_k \Gamma^k_{mi} g^m$$

$$= \underbrace{\left(\frac{\partial a_m}{\partial u^i} + a_l \Gamma^l_{mi}\right)}\, g^m$$

$$\therefore \operatorname{grad} a = \frac{\partial a}{\partial u^i} \otimes g^i = \left(\frac{\partial a_m}{\partial u^i} + a_l \Gamma^l_{mi}\right) g^m \otimes g^i$$

$$\therefore a_m|_i = \frac{\partial a_m}{\partial u^i} + a_l \Gamma^{l}_{mi}$$

$$a^i\big|_j$$

$$a = a^i g_i$$

$$\mathrm{grad}(a) = \frac{\partial a}{\partial u^i} \otimes g^i = a_i\big|_j \, g^i \otimes g^j$$

$$\frac{\partial a}{\partial u^i} = \frac{\partial}{\partial u^i} a^k g_k = \frac{\partial a^k}{\partial u^i} g_k + a^k \frac{\partial g_k}{\partial u^i}$$

$$= \frac{\partial a^k}{\partial u^i} g_k + a^l \Gamma^m_{li} g_m$$

$$= \left(\frac{\partial a^m}{\partial u^i} + a^l \Gamma^m_{li}\right) g_m$$

$$\mathrm{grad}(a) = \frac{\partial a}{\partial u^i} \otimes g^i = \left(\frac{\partial a^m}{\partial u^i} + a^l \Gamma^m_{li}\right) g_m \otimes g^i$$

$$a^m\big|_i = \frac{\partial a^m}{\partial u^i} + a^l \Gamma^m_{li}$$

b) $T^{ij}|_k$

$$\frac{\partial}{\partial u^k}(T^{ij} g_i \otimes g_j) = \left(\frac{\partial}{\partial u^k} T^{ij}\right) g_i \otimes g_j + T^{ij}\left(\frac{\partial g_i}{\partial u^k}\right)\otimes g_j + T^{ij} g_i \otimes \frac{\partial}{\partial u^k} g_j$$

$$= \frac{\partial T^{ij}}{\partial u^k} g_i\otimes g_j + T^{ij}\Gamma^m_{ik}\, g_m\otimes g_j + T^{ij}\Gamma^m_{jk}\, g_i \otimes g_m$$

Exchanging local indices ...

$$= \frac{\partial}{\partial u^k} T^{ij} g_i\otimes g_j + T^{mj}\Gamma^i_{mk}\, g_i\otimes g_j + T^{im}\Gamma^j_{mk}\, g_i\otimes g_j$$

$$= \underbrace{\left(\frac{\partial}{\partial u^k}T^{ij} + T^{mj}\Gamma^i_{mk} + T^{im}\Gamma^j_{mk}\right)}_{T^{ij}|_k} g_i\otimes g_j$$

$T_{ij}|_k \Rightarrow T = T_{ij}\, g^i\otimes g^j$

$$\frac{\partial}{\partial u_k}(T_{ij}\, g^i\otimes g^j) = \left(\frac{\partial}{\partial u_k}T_{ij}\right) g^i\otimes g^j + T_{ij}\frac{\partial g^i}{\partial u_k}\otimes g^j + T_{ij}\, g^i \otimes \frac{\partial g^j}{\partial u_k}$$

$$= \left(\frac{\partial T_{ij}}{\partial u_k}\right) g^i\otimes g^j - T_{ij}\Gamma^i_{km}\, g^m\otimes g^j - T_{ij}\, g^i\otimes g^m\, \Gamma^j_{km}$$

$$\cancel{= \frac{\partial T_{ij}}{\partial u_k} + T_{ij}\Gamma^i_{km}}$$

$$\frac{\partial T_{ij}}{\partial u_k} g^i\otimes g^j - T_{mj}\Gamma^m_{ki}\, g^i\otimes g^j - T_{im}\Gamma^m_{kj}\, g^i\otimes g^j$$

$$= \underbrace{\left(\frac{\partial T_{ij}}{\partial u_k} - T_{mj}\Gamma^m_{ki} - T_{im}\Gamma^m_{kj}\right)}_{T_{ij}|_k} g^i\otimes g^j$$

$$T^{i}_{\cdot j}\Big|_{k}$$

$$T = T^{i}_{\cdot j}\, g_i \otimes g^j$$

$$\frac{\partial T}{\partial u_k} = \frac{\partial T^{i}_{\cdot j}}{\partial u_k}\, g_i \otimes g^j + T^{i}_{\cdot j}\left(\frac{\partial}{\partial_k} g_i\right) \otimes g^j + T^{i}_{\cdot j}\, g_i \otimes \frac{\partial}{\partial_k} g^j$$

$$= \frac{\partial T^{i}_{\cdot j}}{\partial u_k}\, g_i \otimes g^j + T^{i}_{\cdot j}\, \Gamma^{m}_{ik}\, g_m \otimes g^j \cancel{+} - T^{i}_{\cdot j}\, \Gamma^{j}_{km}\, g_i \otimes g^m$$

$$= \frac{\partial T^{i}_{\cdot j}}{\partial u_k}\, g_i \otimes g^j + T^{m}_{\cdot j}\, \Gamma^{i}_{mk}\, g_i \otimes g^j \cancel{+} - T^{i}_{\cdot m}\, \Gamma^{m}_{kj}\, g_i \otimes g^m$$

$$T^{i}_{\cdot j}\Big|_{k} = \frac{\partial T^{i}_{\cdot j}}{\partial u_k} \cancel{g} + T^{m}_{\cdot j}\, \Gamma^{i}_{mk} \cancel{+} - T^{i}_{\cdot m}\, \Gamma^{m}_{kj}$$