

1 Math Fundamentals

Formulae

Asymptotics Let f and g be non-negative functions, then $f(n)$ is...

	...if and only if...	if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}$
$O(g(n))$	$\exists c, N : f(n) \leq c \cdot g(n)$ for $n \geq N$	$\neq \infty$
$o(g(n))$	$\forall c, \exists N : f(n) \leq c \cdot g(n)$ for $n > N$	$= 0$
$\Omega(g(n))$	$\exists c, N : f(n) \geq c \cdot g(n)$ for $n \geq N$	$\neq 0$
$\omega(g(n))$	$\forall c, \exists N : f(n) \geq c \cdot g(n)$ for $n > N$	$= \infty$
$\Theta(g(n))$	$f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$	$\neq 0, \infty$

Implications	Equivalences
$f = o(g) \Rightarrow f = O(g)$	$f = O(g) \Leftrightarrow g = \Omega(f)$
$f = \omega(g) \Rightarrow f = \Omega(g)$	$f = o(g) \Leftrightarrow g = \omega(f)$
	$f = O(g)$ and $f = \Omega(g) \Leftrightarrow f = \Theta(g)$

Generalizations: For all $a, b > 0$, $k > 1$, and $n \geq 1$

$$\begin{cases} (\log n)^b = o(n^a) \\ n^b = o((1+a)^n) \\ a^{\sqrt{\log n}} = o(n^b) \\ k^n > n! > n^{n^b} > n^a > \log n > n^{1/k} > c \approx \text{comparisons} \\ n! \approx n^n \cdot e^{-n} \cdot \sqrt{2\pi n} \end{cases} \quad \text{Stirling's Formula}$$

Master Theorem: Let $T(n) = aT(n/b) + cn^k$ for $a \geq 1$, $b \geq 2$, $c, k \geq 0$. Then $T(n)$ is

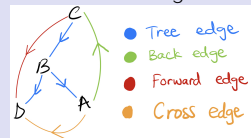
$$\begin{cases} \Theta(n^k) & \text{if } a < b^k \\ \Theta(n^k \log n) & \text{if } a = b^k \\ \Theta(n^{\log_b a}) & \text{if } a > b^k \end{cases}$$

2 Graph Algorithms

Depth First Search Algorithm

DFS Algorithm **Runtime:** $O(|V| + |E|)$

- Goes as deep as possible in the graph then backtrack.
- (In-class algorithm) marks preorder and postorder numbers for every node.
- In a directed graphs, DFS will label edges into (tree, forward, back, cross). No cross edges in undirected graphs.



Applications:

- Detecting cycles
- Topological Sort on **DAGs**
- Strongly Connected Components
- Check for a back edge
- Descending post-order
- Hint: DFS on reversed Graph.

Single Source Shortest Paths

BFS Algorithm **SSSP for unweighted graphs only.**

- Goes level by level in the graph.
- Uses a queue to process nodes.

Runtime: $O(|V| + |E|)$

Dijkstra's Algorithm: **only +ve weighted graphs**

- Let $Dist[v] = \infty$ and $Prev[v] = \text{NIL}$ for all vertices v .
- Let $Dist[s] = 0$ and initialize MinHeap with $(s, 0)$.
- Keep popping the vertex with the smallest distance.
- For each neighbor of the popped vertex, update the distance if it is smaller and add it to the heap. Repeat 2. until heap is empty.

Runtime: $O(|V| * \text{popMin} + |E| * \text{Insert})$

Bellman-Ford Algorithm: **Both +/ -ve weighted graphs**

- Let $Dist[v] = \infty$ and $Prev[v] = \text{NIL}$ for all vertices v .
- Let $Dist[s] = 0$.
- Repeat the following $|V| - 1$ times:
 - For each edge (u, v) with weight w , if $Dist[u] + w < Dist[v]$, update $Dist[v] = Dist[u] + w$ and $Prev[v] = u$.
- Checking for negative weight cycles: repeat the above step once. If any distance can still be improved, a negative weight cycle exists.

Runtime: $O(|V| * |E|)$

Linear Runtime: **only for Directed Acyclic Graphs (DAGs)**

- Run DFS to get topological sort.
- Visit all edges coming out of nodes in the topological order. If multiple edges coming out, pick the shortest.

Runtime: $O(|V| + |E|)$

Min Heaps

Representation: Visually a complete tree. Implementationwise, let $A[0..n-1]$ be a list where $A[0]$ is the root and for any i th element, its parent, left, and right children are at $\lfloor i/2 \rfloor$, $2i$, $2i+1$ respectively.

Heap Property: The parent element is smaller than its children.

Operations	Description
$Insert(a)$	let $A[n] = a$ and $HeapifyUp(n)$
$PopMin$	let $A[0] = A[n-1]$ and $HeapifyDown(0)$
$HeapifyUp(i)$	repeatedly swap $A[i]$ with its parent until the heap property is restored
$HeapifyDown$	repeatedly swap $A[i]$ with its smallest child until the heap property is restored

Heaps Operations and Runtimes: Both $O(\log n)$ with binary heaps.

Note: Can do better with Fibo-Heaps (amortized $O(1)$ for PopMin.)

Minimum Spanning Trees

Basic Properties:

- a Tree** is connected, acyclic, and has $|V| - 1$ edges (any two implies the third).
- Cut Property** states that for any cut of a connected, undirected graph, the minimum weight edge that crosses the cut belongs to the MST.
- Only for connected, undirected, and weighted (non-negative) graphs.

Prim's Algorithm: **Runtime:** $O(|E| \log |V|)$ with Fibo-heaps.

- Start with a single vertex and add edges greedily.
- Implemented in a similar fashion as Dijkstra's, where we add vertex v to the MST when we pop it from the heap.

Kruskal's Algorithm: **Runtime:** $O(|E| \log^* |V|)$ amortized

- Sort edges in ascending order of weight.
- Repeatedly add the lightest edge that does not create a cycle until we have $|V| - 1$ edges.

Notes: Implemented using a union-find data structure.

Disjoint Forest Data Structure

Maintain disjoint sets that can be combined ("unioned") efficiently. Operations $\text{MakeSet}(x)$, $\text{Find}(x)$, and $\text{Union}(a, b)$

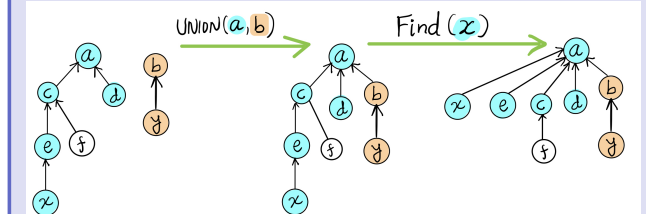
Runtime: m UNION $+ n$ FIND operations take $O((m+n) \log^* n)$.

Note: $\log^* n$ is the number of times we can $\log_2 n$ until we get ≤ 1 .

Heuristics:

- Union by rank. When performing a union operation, we prefer to merge the shallower tree into the deeper tree.
- Path compression. After performing a find operation, attach all the nodes touched directly onto the root of the tree.

Example:



3 Greedy Algorithms

Main idea: At each step, make a locally optimal choice in hope of reaching the globally optimal solution.

Horn Formula

Algorithm: Set all variables to false and greedily set ones to be true when forced to.

Runtime: linear time in the length of the formula (i.e., the total number of appearances of all literals).

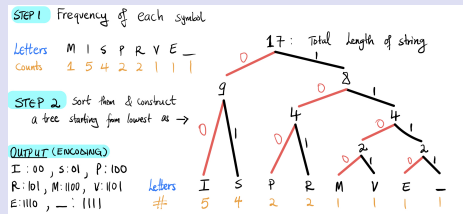
Notes: Only works for SAT instances where in each clause, there is at most one positive literal.

Huffman Coding

Algorithm: **Runtime:** $O(n \log n)$

Find the best encoding by greedily combining the two least frequently items. Optimal in terms of encoding one character at a time.

Example: A Huffman tree for string "Mississippi River"



Set Cover

Given $X = \{x_1, \dots, x_n\}$, and a collection of subsets S of X such that $\bigcup_{S \in \mathcal{S}} S = X$, find the subcollection $\mathcal{T} \subseteq \mathcal{S}$ such that the sets of \mathcal{T} cover X .

Algorithm: **Runtime:** $O(|U|)$

1. Greedily choose the set that covers the most number of the remaining uncovered elements at the given iteration.

claim: Let k be the size of the smallest set cover for the instance (X, S) . Then the greedy heuristic finds a set cover of size at most $k \ln n$.

Note:

Not always optimal; achieves $O(\log n)$ approximation ratio.

4 Divide and Combine

Main Idea: Divide the problem into smaller pieces, recursively solve those, and then combine their results to get the final result.

Famous Examples w/ Runtimes

Mergesort	$O(n \log n)$
Min and Max on a line	$\frac{3}{2}n - 2$ comparisons; $O(n)$ runtime.
Closest Pair of Points	$O(n \log^2 n)$

n -digit Integer Multiplication

standard Multiplication	$\Theta(n^2)$
3 products on $n/2$ digits	$\Theta(n^{\log_2 3}) = \Theta(n^{1.59})$
5 products on $n/3$ digits	$\Theta(n^{\log_3 5}) = \Theta(n^{1.46})$

$n \times n$ Matrix Multiplication

Strassen's Algorithm: **Runtime:** $O(n^{\log_2 7})$

Divide into four submatrices, each of size $n/2$ by $n/2$.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

Find: $P_1 = A(F - H)$, $P_2 = (A + B)H$, $P_3 = (C + D)E$, $P_4 = D(G - E)$, $P_5 = (A + D)(E + H)$, $P_6 = (B - D)(G + H)$, $P_7 = (C - A)(E + F)$, then:

$AE + BG = -P_2 + P_4 + P_5 + P_6$ and $AF + BH = P_1 + P_2$
 $CE + DG = P_3 + P_4$ and $CF + DH = P_1 - P_3 + P_5 + P_7$

5 Dynamic Programming

Main Idea: Maintain a lookup table of correct solutions to sub-problems and build up this table towards the actual solution.

Steps:

1. Define subproblems and recurrence to solve subproblems.
2. Combine with **reuse**.
3. Runtime and space analysis.

String Reconstruction/Word Break

Find the shortest sequence of words that make up a given string.

Algorithm: **Runtime:**

Edit Distance

Find the minimum number of operations required to transform one string, $A[1 \dots n]$, into another, $B[1 \dots m]$.

Algorithm: **Runtime and Space:** $O(nm)$

1. Subproblem: let $D(i, j)$ represent the edit distance between $A[1 \dots i]$ and $B[1 \dots j]$.
2. Recurrence is:
Base cases: $D(i, 0) = i$, $D(0, j) = j$.
 $D(i, j) = \min[D(i - 1, j) + 1, D(i, j - 1) + 1, D(i - 1, j - 1) + (1 \text{ if } i = j, 0 \text{ otherwise})]$.
3. return $D(n, m)$.

All Pairs Shortest Paths

Given a graph G with n vertices and m edges, calculate distances of the shortest paths between every pair of nodes.

Floyd-Warshall Algorithm: **Runtime:** $O(n^3)$

1. Subproblem: let $D_k[i, j]$ represent the shortest path between i and j using only nodes in $[1 \dots k]$.
2. Recurrence is:
$$\begin{cases} D_0[i, j] = d_{ij} \text{ if } i \text{ and } j \text{ are connected, } \infty \text{ otherwise.} \\ D_k[i, j] = \min(D_{k-1}[i, j], D_{k-1}(i, k) + D_{k-1}[k, j]). \end{cases}$$
3. return $D(i, j, n)$.

Notes: Does not work for cyclic graphs.

More DP Examples

Problems	SubProblem
LongestIncrSubseq	$\frac{3}{2}n - 2$ comparisons; $O(n)$ runtime.
Knapsack Problem	
DP on trees: Dominating Set	
Traveling salesman problem	