1.1

Any AR(p) can be reexpressed as a p-variate VAR(1). (see lecture)

$$AR(4): \quad x_t = a_0 + \sum_{i=1}^{4} a_i x_{t-i} + \varepsilon_t \,, \quad \varepsilon_t \sim N(0,\sigma^2)$$

-> Express as VAR(1):

$$\begin{pmatrix} x_t \\ x_{t-1} \\ x_{t-2} \\ x_{t-3} \end{pmatrix} = \begin{pmatrix} a_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ x_{t-2} \\ x_{t-3} \\ x_{t-4} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{x_t} = \underline{a_0} + \underline{\underline{A}} \; \underline{x_{t-1}} + \underline{\varepsilon_t}$$

$$p \times 1 \qquad p \times 1 \qquad p \times p \qquad p \times 1 \qquad p \times 1$$

-> use multivariate least squares to estimate the parameters

$$\underline{x}_{t+4} = [x_5, x_6, \dots, x_T]^T, \quad \underline{x}_{t+3} = [x_4, x_5, \dots, x_{T-1}]$$

$$\hat{\underline{\underline{A}}} = (\underline{x}_{t+4} \cdot \underline{x}_{t+3}^T)(\underline{x}_{t+3} \cdot \underline{x}_{t+3}^T)^{-1}$$

1.2.

AR(4) model for $t = 5, \dots, T$

$$x_t = a_0 + a_1 x_{t-1} + a_2 x_{t-2} + a_3 x_{t-3} + a_4 x_{t-4} + \varepsilon_t \,, \quad \varepsilon_t \sim N(0,\sigma^2)$$

$$\hat{\varepsilon}_t = x_t - (a_0 + a_1 x_{t-1} + a_2 x_{t-2} + a_3 x_{t-3} + a_4 x_{t-4}) \,, \quad t = 5, \dots, T$$

ε_t are independent Gaussians, so the conditional joint density of $x_5, \dots, x_T$ is:

$$L(a_0, \dots, a_4, \sigma^2) = \prod_{t=5}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\hat{\varepsilon}_t^2}{2\sigma}\right)$$

$$\log L(\dots) = -\frac{T-4}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=5}^{T} \hat{\varepsilon}_t^2$$

AR coefficients

-> hold σ^2 fixed and maximize $\log L$, solve OLS

Variance σ^2

Treat $\log L$ as a function of σ^2 alone and set $\partial \log L / \partial \sigma^2 = 0$

$$\Rightarrow \sigma^2 = \frac{1}{T-4} \sum_{t=5}^{T} \hat{\varepsilon}_t^2$$

3.1.

$$c_t = \begin{pmatrix} c_{1t} \\ \vdots \\ c_{kt} \end{pmatrix} \;;\; t = 1, \dots, T \;;\; i = 1, \dots, k$$

$$c_{it} \mid c_{t-1}, \dots, c_{t-p} \sim \text{Poisson}(\lambda_{it})$$

$$P(c_{it} \mid \lambda_{it}) = \frac{\lambda_{it}^{c_{it}}}{c_{it}!} e^{-\lambda_{it}}$$

$$\underbrace{\log}_{\text{link function}} \lambda_t = a_0 + \sum_{j=1} A_j c_{t-j}$$

The link function is used to ensure positivity of λ_t. Also we get a linear relationship on the log scale.