Exercise 3

Task 1.1

The density of P is:

$$p(x) = \frac{1}{(2\pi)^{n/2} \det \Sigma_a^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_a)^T \Sigma_a^{-1} (x-\mu_a)\right)$$

(analogous for $q(x)$)

$$\Rightarrow \log \frac{p(x)}{q(x)} = \left.\frac{1}{2}[\log \det \Sigma_b - \log \det \Sigma_a] \right\} A$$

$$\left. + \frac{1}{2}\left[(x-\mu_b)^T \Sigma_b^{-1} (x-\mu_b) - (x-\mu_a)^T \Sigma_a^{-1} (x-\mu_a)\right] \right\} B$$

$$\Rightarrow KL(P\|Q) = \mathbb{E}_{x\sim p}[\log p(x) - \log q(x)] = \frac{1}{2}[\log \det \Sigma_b - \log \det \Sigma_a]$$

$$+ \frac{1}{2} \mathbb{E}_{x\sim p}[B]$$

Look at $\mathbb{E}_{x\sim p}[B]$:

$$x - \mu_b = (x - \mu_a) + (\mu_a - \mu_b)$$

$$\mathbb{E}_p[(x-\mu_b)(x-\mu_b)^T] = \Sigma_a + (\mu_a - \mu_b)(\mu_a - \mu_b)^T$$

$$\Rightarrow \mathbb{E}_p[(x-\mu_b)^T \Sigma_b^{-1} (x-\mu_b)] = tr(\Sigma_b^{-1}\Sigma_a)$$

$$+ (\mu_a - \mu_b)^T \Sigma_b^{-1} (\mu_a - \mu_b)$$

And $\mathbb{E}_p[(x-\mu_a)^T \Sigma_a^{-1} (x-\mu_a)] = tr(\Sigma_a^{-1}\Sigma_a) = n$

Putting everything together:

$$KL(P\|Q) = \frac{1}{2}\Big[\log|\Sigma_b| - \log|\Sigma_a|$$

$$+ tr(\Sigma_b^{-1}\Sigma_a) + (\mu_a - \mu_b)^T \Sigma_b^{-1} (\mu_a - \mu_b) - n\Big]$$

Task 1.2

Matrices are diagonal

$\Rightarrow |\Sigma_B| = \Sigma_{i=1} \sigma_{Bi}$, $|\Sigma_a| = \Sigma_{i=1} \sigma_{Ai}$

$\Rightarrow tr(\Sigma_b^{-1} \Sigma_a) = \Sigma_{i=1}^{T} \frac{\sigma A_i}{\sigma B_i}$

Also:

$(\mu_a - \mu_b)^T \Sigma_b^{-1} (\mu_a - \mu_b) = \Sigma_{i=1}^{n} (\mu_a - \mu_b)^2 \sigma_{B_i}$

The KL divergace the simplifies to:

$$KL(P\|Q) = \frac{1}{2}\left(\log\left(\frac{\sum_{i=1}^{T} \sigma_{B_i}}{\sum_{i=1}^{T} \sigma_{A_i}} \right) - n + \sum_{i=1}^{T} \frac{\sigma_{A_i}}{\sigma_{B_i}} - \sum_{i=1}^{T} (\mu_A - \mu_B)^2 \sigma_{B_i} \right)$$

Task 2.

$$z_t = \underset{\text{matrix}}{A} z_{t-1} + \varepsilon \qquad \varepsilon \sim \mathcal{N}(0, \Sigma)$$

$$x_t = \underset{\text{matrix}}{B} z_t + \eta \qquad \eta \sim \mathcal{N}(0, \Gamma)$$

$$x_t \sim \mathcal{N}(Bz_t, \Gamma), \quad z_t = \mathcal{N}(Az_{t-1}, \Sigma)$$

from lecture:

Expectation-Maximization (EM) algorithm

provide initial estimate. $\theta^* = \theta^{(0)}$

E-step: $q^* = \arg\max \text{ELBO}[q, \theta^*]$ given θ^*

$$= \arg\min_q KL[q \,\|\, p_\theta(z|x)]$$

approximate posterior

M-step: $\theta^* = \arg\max_\theta \text{ELBO}[q^*, \theta]$

$$= \arg\max_\theta E_q[\log p_\theta(x, z)]$$

until $|\Delta \text{ELBO}| < \varepsilon$

$\underline{M\text{-step}}$: $\max \; E_q[\log p_\theta(X, Z)]$ w.r.t $\{B, \Gamma, A, \Sigma, \mu_0\}$

$$p_\theta(X, Z) = p(X|Z)\,p(Z)$$

$$= p(x_1|z_1)\,p(z_1) \prod_{t=1}^{T} \overbrace{p(x_t|z_t)}^{\text{obs. model}}\;\overbrace{p(z_t|z_{t-1})}^{\text{latent model}}$$

$$\Rightarrow E_q[\log p_\theta(X, Z)] = E_q[\underline{\log p(z_1)}]$$

$$+ E_q\Big[\sum_{t=2}^{T} \underline{\log p(z_t|z_{t-1})}\Big]$$

$$+ E_q\Big[\sum_{t=1}^{T} \log \underline{p(x_t|z_t)}\Big]$$

gaussian in this case.

M-step → Exps. $\theta^* = \arg\max_\theta \text{ELBO}[q^*, \theta]$, w.r.t $\{B, \Sigma, \Gamma\}$

$$= E_q\Big[-\frac{M}{2}\log(2\pi) - \frac{1}{2}\log|\Sigma| - \frac{1}{2}(z_1 - \mu_0)^T \Sigma^{-1} (z_1 - \mu_0)\Big]$$

$$+ E_q\Big[-\frac{M(T-1)}{2}\log(2\pi) - \frac{T-1}{2}\log|\Sigma| - \frac{1}{2}\sum_{t=2}^{T}(z_t - Az_{t-1})^T \Sigma^{-1}(z_t - Az_{t-1})\Big]$$

$$+ E_q\Big[-\frac{NT}{2}\log(2\pi) - \frac{T}{2}\log|\Gamma| - \frac{1}{2}\sum_{t=1}^{T}(x_t - Bz_t)^T \Gamma^{-1}(x_t - Bz_t)\Big]$$

only last part dep on B

$$\frac{\partial}{\partial B} E_q\Big[\cancel{-\frac{NT}{2}\log(2\pi)} - \cancel{\frac{T}{2}\log|\Gamma|} - \frac{1}{2}\sum_{t=1}^{T}(x_t - Bz_t)^T \Gamma^{-1}(x_t - Bz_t)\Big]$$

$$\frac{\partial}{\partial B} E_q\left[-\frac{1}{2}\sum_{t=1}^{T}\left(x_t^T \Gamma^{-1} x_t - x_t^T \Gamma^{-1} B z_t - z_t^T B^T \Gamma^{-1} x_t + z_t^T B^T \Gamma^{-1} B z_t\right)\right]$$

$$= E_q\left[\frac{\partial}{\partial B} -\frac{1}{2}\sum_{t=1}^{T}\left(\cancel{x_t^T \Gamma^{-1} x_t} - x_t^T \Gamma^{-1} B z_t - z_t^T B^T \Gamma^{-1} x_t + z_t^T B^T \Gamma^{-1} B z_t\right)\right]$$

$$= E_q\left[-\frac{1}{2}\sum_{t=1}^{T}\left(-\Gamma^{-1} x_t z_t^T - \Gamma^{-1} z_t x_t^T + (\Gamma^{-1})^T B z_t z_t^T + \Gamma^{-1} B z_t z_t^T\right)\right]$$

Γ sym. $\Rightarrow$ Γ^{-1} symmetric $\Gamma^{-1} = (\Gamma^{-1})^T$

$$\Rightarrow \quad = E_q\left[\Gamma^{-1} x_t z_t^T - \Gamma^{-1} B z_t z^T\right] \overset{!}{=} 0$$

using linearity of exp val and B indep of q can be pulled out

$$\Rightarrow \quad E_q\left[\sum_{t=1}^{T}\Gamma^{-1} x_t z_t^T\right] - B\, E_q\left[\sum_{t=1}^{T}\Gamma^{-1} z_t z_t^T\right] \overset{!}{=} 0$$

$$B = \sum_{t=1}^{T} x_t E_q[z_t^T]\left(\sum_{t=1}^{T} E_q[z_t z_t^T]\right)^{-1}$$

fuzo first part dep on Σ

$$= \frac{\partial}{\partial \Sigma}\Big(E_q\Big[\cancel{-\frac{M}{2}\log(2\pi)} - \frac{1}{2}\log|\Sigma| - \frac{1}{2}(z_1-\mu_0)^T\Sigma^{-1}(z_1-\mu_0)\Big]$$

$$+ E_q\Big[\cancel{-\frac{M(T-1)}{2}\log(2\pi)} - \frac{T-1}{2}\log|\Sigma| - \frac{1}{2}\sum_{t=2}^{T} (z_t - Az_{t-1})^T\Sigma^{-1}(z_t - Az_{t-1})\Big]\Big)$$

cookbook

$$-\frac{1}{2}\frac{\partial}{\partial\Sigma}\log|\Sigma| = -\frac{1}{2}\cancel{|\Sigma|^{-1}}\cancel{|\Sigma|}(\Sigma^{-1})^T = (\Sigma^{-1})^T \xrightarrow{\text{sym}} -\frac{1}{2}\Sigma^{-1}$$

$$-\frac{T-1}{2}\frac{\partial}{\partial\Sigma}\log(\Sigma) = -\frac{T-1}{2}\Sigma^{-1}$$

Σ is sym

$$\frac{\partial}{\partial\Sigma}\Big(-\frac{1}{2}\big(z_1^T\Sigma^{-1}z_1 - z_1^T\Sigma^{-1}\mu_0 - \mu_0^T\Sigma^{-1}z_1 + \mu_0^T\Sigma^{-1}\mu_0\big)\Big)$$

$$-\frac{1}{2}\Big(-\Sigma^{-1}z_1z_1^T\Sigma^{-1} + \Sigma^{-1}z_1\mu_0^T\Sigma^{-1} + \Sigma^{-1}\mu_0z_1^T\Sigma^{-1} - \Sigma^{-1}\mu_0\mu_0^T\Sigma^{-1}\Big)\Big)$$

$$\frac{\partial}{\partial\Sigma}\Big(-\frac{1}{2}(z_t - Az_{t-1})^T\Sigma^{-1}(z_t - Az_{t-1})]\Big) = -\frac{1}{2}\frac{\partial}{\partial\Sigma}\Big(z_t^T\Sigma^{-1}z_t - z^T\Sigma^{-1}Az_{t-1} - z_{t-1}^TA^T\Sigma^{-1}z_t + z_{t-1}^TA^T\Sigma^{-1}Az_{t-1}\Big)$$

$$-\frac{1}{2}\Big(-\Sigma^{-1}z_tz_t^T\Sigma^{-1} + \Sigma^{-1}z_tz_{t-1}^TA^T\Sigma^{-1} + \Sigma^{-1}Az_{t-1}z_t^T\Sigma^{-1} - \Sigma^{-1}Az_{t-1}z_{t-1}^TA^T\Sigma^{-1}\Big)$$

$$E_q\Big[-\frac{1}{2}\Sigma^{-1} - \frac{T-1}{2}\Sigma^{-1} - \frac{1}{2}\Sigma^{-1}\overbrace{\big(z_1\mu_0^T + \mu_0z_1^T - z_1z_1^T - \mu_0\mu_0^T\big)}^{A'}\Sigma^{-1}$$

$$-\frac{1}{2}\Sigma^{-1}\sum_{t=2}^{T}\underbrace{\big(z_tz_{t-1}^TA^T + Az_{t-1}z_t^T - z_tz_t - Az_{t-1}z_{t-1}^TA^T\big)}\Sigma^{-1}\Big]$$

$$B' \qquad \stackrel{!}{=} 0 \quad \Big| \cdot 2$$

$$\Sigma^{-1}(-1 - T + 1) - \Sigma^{-1}A'\Sigma^{-1} - \Sigma^{-1}B'\Sigma^{-1} \stackrel{!}{=} 0 \quad \Big| \cdot \Sigma$$

$$-T = \Sigma^{-1}A' + \Sigma^{-1}B' \quad \Big| \cdot \Sigma$$

$$-\Sigma T = A' + B'$$

$$\Sigma = -\frac{1}{T}(A' + B')$$

$$\Rightarrow \Sigma = -\frac{1}{T}\Big\{-E_q(z_1\mu_0^T) - E_q[\mu_0 z_1^T] + E_q[z_1 z_1^T] + E[\mu_0\mu_0^T]$$

$$+ \sum_{t=2}^{T}\Big(-E_q[z_t z_{t-1}^T]A^T - A\,E_q[z_{t-1}z_t^T] + E_q[z_t z_t] + A\,E_q[z_{t-1}z_{t-1}^T]A^T\Big\}$$

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only last part dep on Γ.

$$\frac{\partial}{\partial \Gamma} E_q\left[-\frac{NT}{2}\log(2\pi) - \frac{T}{2}\log|\Gamma| - \frac{1}{2}\sum_{t=1}^{T}(x_t - Bz_t)^T\Gamma^{-1}(x_t - Bz_t)\right]$$

$$-\frac{T}{2}\frac{\partial}{\partial\Gamma}\log|\Gamma| = -\frac{T}{2}\Gamma^{-1}$$

$$-\frac{1}{2}\sum_{t=1}^{T}(x_t - Bz_t)^T\Gamma^{-1}(x_t - Bz_t) = -\frac{1}{2}\,\mathrm{tr}\Big(\Gamma^{-1}\underbrace{(x_t - Bz_t)(x_t - Bz_t)^T}_{A'}\Big)$$

$$\Rightarrow -\frac{1}{2}\frac{\partial}{\partial\Gamma}\mathrm{tr}(\Gamma^{-1}A') = -\frac{1}{2}\left(-\Gamma^{-1}A'\Gamma^{-1}\right)^T$$

$$\Gamma \text{ sym} \Rightarrow \frac{1}{2}\Gamma^{-1} A^{T} \Gamma^{-1}$$

$$\Rightarrow \frac{1}{2}\Gamma^{-1}(x_t - Bz_t)(x_t - Bz_t)^T \Gamma^{-1}$$

$$\frac{1}{2}\Gamma^{-1}(x_t x_t^T - x_t z^T B^T - Bz_t x_t^T + Bz_t z_t^T B^T)\Gamma^{-1}$$

$$\Rightarrow E_q\left[-\frac{T}{2}\Gamma^{-1} + \frac{1}{2}\sum_{t=1}^{T}\Gamma^{-1}\underbrace{(x_t x_t^T - x_t z_t^T B^T - Bz_t x_t^T + Bz_t z_t^T B^T)}_{Y}\Gamma^{-1}\right] \overset{!}{=} 0$$

$\cdot 2$

$$\Rightarrow T\Gamma^{-1} = \sum_{t=1}^{T}\Gamma^{-1}E_q(Y)\Gamma^{-1} \quad | \cdot \Gamma$$

$$T = \sum_{t=1}^{T}\Gamma^{-1}E_q(Y) \quad | \cdot \Gamma$$

$$\Gamma = \frac{1}{T}\sum_{t=1}^{T} E_q(Y)$$

$$\Gamma = \frac{1}{T}\sum_{t=1}^{T}\left\{x_t x_t^T - x_t E[z_t^T]B^T - BE_q[z_t]x_t^T + BE_q[z_t z_t^T]B^T\right\}$$