Task 2.

$$z_t = \underset{\text{matrix}}{A} z_{t-1} + \varepsilon \qquad \varepsilon \sim \mathcal{N}(0, \Sigma)$$

$$x_t = \underset{\text{matrix}}{B} z_t + \eta \qquad \eta \sim \mathcal{N}(0, \Gamma)$$

$$x_t \sim \mathcal{N}(Bz_t, \Gamma), \quad z_t = \mathcal{N}(Az_{t-1}, \Sigma)$$

from lecture:

Expectation - Maximization (EM) algorithm

provide initial estimate. $\theta^* = \theta^{(0)}$

E-step: $q^* = \arg\max \text{ELBO}[q, \theta^*]$ given θ^*

$= \arg\min_q KL[q \,||\, p_\theta(z|x)]$

approximate posterior

M-step: $\theta^* = \arg\max_\theta \text{ELBO}[q^*, \theta]$

$= \arg\max_\theta E_q[\log p_\theta(x, z)]$

until $|\Delta \text{ELBO}| < \varepsilon$

$\underline{M\text{-step}}$: $\max E_q[\log p_\theta(X,Z)]$ w.r.t $\{B, \Gamma, A, \Sigma, \mu_0\}$

$$p_\theta(X,Z) = p(X|Z)\, p(Z)$$

$$= p(x_1|z_1)\, p(z_1) \prod_{t=1}^{T} \overbrace{p(x_t|z_t)}^{\text{obs. model}}\, \overbrace{p(z_t|z_{t-1})}^{\text{latent model}}$$

$$\Rightarrow E_q[\log p_\theta(X,Z)] = E_q[\underline{\log p(z_1)}]$$

$$+ E_q\left[\sum_{t=2}^{T} \underline{\log p(z_t|z_{t-1})}\right]$$

$$+ E_q\left[\sum_{t=1}^{T} \underline{\log p(x_t|z_t)}\right]$$

gaussian in this case.

M-step $\rightarrow$ Exps. $\theta^* = \arg\max_\theta \text{ELBO}[q^*, \theta]$, w.r.t $\{B, \Sigma, \Gamma\}$

$$= E_q\left[-\frac{M}{2}\log(2\pi) - \frac{1}{2}\log|\Sigma| - \frac{1}{2}(z_1-\mu_0)^T \Sigma^{-1}(z_1-\mu_0)\right]$$

$$+ E_q\left[-\frac{M(T-1)}{2}\log(2\pi) - \frac{T-1}{2}\log|\Sigma| - \frac{1}{2}\sum_{t=2}^{T}(z_t - Az_{t-1})^T\Sigma^{-1}(z_t - Az_{t-1})\right]$$

$$+ E_q\left[-\frac{NT}{2}\log(2\pi) - \frac{T}{2}\log|\Gamma| - \frac{1}{2}\sum_{t=1}^{T}(x_t - Bz_t)^T\Gamma^{-1}(x_t - Bz_t)\right]$$

only last part dep on B

$$\frac{\partial}{\partial B} E_q\left[\cancel{-\frac{NT}{2}\log(2\pi)} - \cancel{\frac{T}{2}\log|\Gamma|} - \frac{1}{2}\sum_{t=1}^{T}(x_t - Bz_t)^T\Gamma^{-1}(x_t - Bz_t)\right]$$

$$\frac{\partial}{\partial B} E_q\left[-\frac{1}{2}\sum_{t=1}^{T}\left(x_t^T\Gamma^{-1}x_t - x_t^T\Gamma^{-1}Bz_t - z_t^TB^T\Gamma^{-1}x_t + z_t^TB^T\Gamma^{-1}Bz_t\right)\right]$$

$$= E_q\left[\frac{\partial}{\partial B} -\frac{1}{2}\sum_{t=1}\left(\cancel{x_t^T\Gamma^{-1}x_t} - x_t^T\Gamma^{-1}Bz_t - z_t^TB^T\Gamma^{-1}x_t + z_t^TB^T\Gamma^{-1}Bz_t\right)\right]$$

$$= E_q\left[-\frac{1}{2}\sum_{t=1}\left(-\Gamma^{-1}x_t z_t^T - \Gamma^{-1}z_t x_t^T + (\Gamma^{-1})^T B z_t z_t^T + \Gamma^{-1}Bz_t z_t^T\right)\right]$$

Γ sym. $\Rightarrow$ Γ^{-1} symetric $\Gamma^{-1} = (\Gamma^{-1})^T$

$$\Rightarrow \quad = E_q\left[\Gamma^{-1}x_t z_t^T - \Gamma^{-1}Bz_t z^T\right] \overset{!}{=} 0$$

using linearity of exp val and B indep of q can be pulled out

$$\Rightarrow \quad E_q\left[\sum_{t=1}^{T}\Gamma^{-1}x_t z_t^T\right] - B\,E_q\left[\sum_{t=1}^{T}\Gamma^{-1}z_t z_t^T\right] \overset{!}{=} 0$$

$$\underline{B = \sum_{t=1}^{T} x_t E_q[z_t^T]\left(\sum_{t=1}^{T} E_q[z_t z_t^T]\right)^{-1}}$$

two first part dep on Σ

$$= \frac{\partial}{\partial \Sigma}\Big(E_q\big[-\frac{M}{2}\cancel{\log(2\pi)} - \frac{1}{2}\log|\Sigma| - \frac{1}{2}(z_1-\mu_0)^T\Sigma^{-1}(z_1-\mu_0)\big]$$

$$+ E_q\big[-\frac{M(T-1)}{2}\cancel{\log(2\pi)} - \frac{T-1}{2}\log|\Sigma| - \frac{1}{2}\sum_{t=2}^{T}(z_t-Az_{t-1})^T\Sigma^{-1}(z_t-Az_{t-1})\big]\Big)$$

cookbook

$$-\frac{1}{2}\frac{\partial}{\partial \Sigma}\log|\Sigma| = -\frac{1}{2}\cancel{|\Sigma|^{-1}}\cancel{|\Sigma|}(\Sigma^{-1})^T = (\Sigma^{-1})^T \xrightarrow{\text{sym}} -\frac{1}{2}\Sigma^{-1}$$

$$-\frac{T-1}{2}\frac{\partial}{\partial \Sigma}\log(\Sigma) = -\frac{T-1}{2}\Sigma^{-1}$$

Σ is sym

$$\frac{\partial}{\partial \Sigma}\Big(-\frac{1}{2}\big(z_1^T\Sigma^{-1}z_1 - z_1^T\Sigma^{-1}\mu_0 - \mu_0^T\Sigma^{-1}z_1 + \mu_0^T\Sigma^{-1}\mu_0\big)\Big)$$

$$-\frac{1}{2}\Big(-\Sigma^{-1}z_1z_1^T\Sigma^{-1} + \Sigma^{-1}z_1\mu_0^T\Sigma^{-1} + \Sigma^{-1}\mu_0z_1^T\Sigma^{-1} - \Sigma^{-1}\mu_0\mu_0^T\Sigma^{-1}\Big)$$

$$\frac{\partial}{\partial \Sigma}\Big(-\frac{1}{2}(z_t-Az_{t-1})^T\Sigma^{-1}(z_t-Az_{t-1})\Big) = -\frac{1}{2}\frac{\partial}{\partial \Sigma}\Big(z_t^T\Sigma^{-1}z_t - z^T\Sigma^{-1}Az_{t-1} - z_{t-1}^TA^T\Sigma^{-1}z_t + z_{t-1}^TA^T\Sigma^{-1}Az_{t-1}\Big)$$

$$-\frac{1}{2}\Big(-\Sigma^{-1}z_tz_t^T\Sigma^{-1} + \Sigma^{-1}z_tz_{t-1}^TA^T\Sigma^{-1} + \Sigma^{-1}Az_{t-1}z_t^T\Sigma^{-1} - \Sigma^{-1}Az_{t-1}z_{t-1}^TA^T\Sigma^{-1}\Big)$$

$$E_q\Big[-\frac{1}{2}\Sigma^{-1} - \frac{T-1}{2}\Sigma^{-1} - \frac{1}{2}\Sigma^{-1}\Big(\overbrace{z_1\mu_0^T + \mu_0z_1^T - z_1z_1^T - \mu_0\mu_0^T}^{A'}\Big)\Sigma^{-1}$$

$$-\frac{1}{2}\Sigma^{-1}\sum_{t=2}^{T}\Big(\underbrace{z_tz_{t-1}^TA^T + Az_{t-1}z_t^T - z_tz_t - Az_{t-1}z_{t-1}^TA^T}\Big)\Sigma^{-1}\Big]$$

$$B' \qquad \overset{!}{=} 0 \quad | \cdot 2$$

$$\Sigma^{-1}(-1 - T + 1) - \Sigma^{-1}A'\Sigma^{-1} - \Sigma^{-1}B'\Sigma^{-1} \overset{!}{=} 0 \quad | \cdot \Sigma$$

$$-T = \Sigma^{-1}A' + \Sigma^{-1}B' \quad | \cdot \Sigma$$

$$-\Sigma T = A' + B'$$

$$\Sigma = -\frac{1}{T}(A' + B')$$

$$\Rightarrow \Sigma = -\frac{1}{T}\Big\{ -E_q(z_1 \mu_0^T) - E_q[\mu_0 z_1^T) + E_q[z_1 z_1^T] + E[\mu_0 \mu_0^T]$$

$$+ \sum_{t=2}^{T} \Big(-E_q[z_t z_{t-1}^T] A^T - A\, E_q[z_{t-1} z_t^T] + E_q[z_t z_t) + A E_q[z_{t-1} z_{t-1}^T] A^T \Big\}$$

---

only last part dep on Γ.

$$\frac{\partial}{\partial \Gamma} E_q\Big[-\frac{NT}{2}\log(2\pi) - \frac{T}{2}\log|\Gamma| - \frac{1}{2}\sum_{t=1}^{T}(x_t - Bz_t)^T \Gamma^{-1}(x_t - Bz_t)\Big]$$

$$-\frac{T}{2}\frac{\partial}{\partial \Gamma}\log|\Gamma| = -\frac{T}{2}\Gamma^{-1}$$

$$-\frac{1}{2}\sum_{t=1}^{T}(x_t - Bz_t)^T \Gamma^{-1}(x_t - Bz_t) = -\frac{1}{2}\operatorname{tr}\Big(\Gamma^{-1}\underbrace{(x_t - Bz_t)(x_t - Bz_t)^T}_{A'}\Big)$$

$$\Rightarrow -\frac{1}{2}\frac{\partial}{\partial \Gamma}\operatorname{tr}(\Gamma^{-1}A') = -\frac{1}{2}\left(-\Gamma^{-1}A'\Gamma^{-1}\right)^T$$

$$\Gamma \text{ sym} \Rightarrow \frac{1}{2}\Gamma^{-1}A^{T}\Gamma^{-1}$$

$$\Rightarrow \frac{1}{2}\Gamma^{-1}(x_t - Bz_t)(x_t - Bz_t)^T\Gamma^{-1}$$

$$\frac{1}{2}\Gamma^{-1}\left(x_t x_t^T - x_t z^T B^T - Bz_t x_t^T + Bz_t z_t^T B^T\right)\Gamma^{-1}$$

$$\Rightarrow E_q\left[-\frac{T}{2}\Gamma^{-1} + \frac{1}{2}\sum_{t=1}^{T}\Gamma^{-1}\underbrace{\left(x_t x_t^T - x_t z_t^T B^T - Bz_t x_t^T + Bz_t z_t^T B^T\right)}_{Y}\Gamma^{-1}\right] \overset{!}{=} 0 \quad | \cdot 2$$

$$\Rightarrow T\Gamma^{-1} = \sum_{t=1}^{T}\Gamma^{-1}E_q(Y)\Gamma^{-1} \quad | \cdot \Gamma$$

$$T = \sum_{t=1}^{T}\Gamma^{-1}E_q(Y) \quad | \cdot \Gamma$$

$$\Gamma = \frac{1}{T}\sum_{t=1}^{T} E_q(Y)$$

$$\Gamma = \frac{1}{T}\sum_{t=1}^{T}\left\{x_t x_t^T - x_t E\left[z_t^T\right]B^T - BE_q[z_t]\,x_t^T + BE_q\left[z_t z_t^T\right]B^T\right\}$$