Exercise 3

Task 1.1

The density of P is:

$$p(x) = \frac{1}{(2\pi)^{n/2} \det \Sigma_a^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_a)^T \Sigma_a^{-1} (x-\mu_a)\right)$$

(analogous for $q(x)$)

$$\Rightarrow \log \frac{p(x)}{q(x)} = \left.\frac{1}{2}[\log\det\Sigma_b - \log\det\Sigma_a]\right\} A$$

$$\left.+\frac{1}{2}[(x-\mu_b)^T \Sigma_b^{-1}(x-\mu_b) - (x-\mu_a)^T \Sigma_a^{-1}(x-\mu_a)]\right\} B$$

$$\Rightarrow KL(P\|Q) = \mathbb{E}_{x\sim p}[\log p(x) - \log q(x)] = \frac{1}{2}[\log\det\Sigma_b - \log\det\Sigma_a]$$

$$+\frac{1}{2}\mathbb{E}_{x\sim p}[B]$$

Look at $\mathbb{E}_{x\sim p}[B]$:

$$x-\mu_b = (x-\mu_a) + (\mu_a - \mu_b)$$

$$\mathbb{E}_p[(x-\mu_b)(x-\mu_b)^T] = \Sigma_a + (\mu_a-\mu_b)(\mu_a-\mu_b)^T$$

$$\Rightarrow \mathbb{E}_p[(x-\mu_b)^T \Sigma_b^{-1}(x-\mu_b)] = tr(\Sigma_b^{-1}\Sigma_a)$$

$$+(\mu_a-\mu_b)^T\Sigma_b^{-1}(\mu_a-\mu_b)$$

And $\mathbb{E}_p[(x-\mu_a)^T\Sigma_a^{-1}(x-\mu_a)] = tr(\Sigma_a^{-1}\Sigma_a) = n$

Putting everything together:

$$KL(P\|Q) = \frac{1}{2}[\log|\Sigma_b| - \log|\Sigma_a|$$

$$+ tr(\Sigma_b^{-1}\Sigma_a) + (\mu_a-\mu_b)^T\Sigma_b^{-1}(\mu_a-\mu_b) - n]$$

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Task 1.2

Matrices are diagonal

$\Rightarrow |\Sigma_B| = \Sigma_{i=1} \sigma_{Bi} \; , \; |\Sigma_a| = \Sigma_{i=1} \sigma_{Ai}$

$\Rightarrow tr(\Sigma_b^{-1} \Sigma_a) = \Sigma_{i=1}^{T} \frac{\sigma_{Ai}}{\sigma_{Bi}}$

Also:

$(\mu_a - \mu_b)^T \Sigma_b^{-1} (\mu_a - \mu_b) = \Sigma_{i=1}^{n} (\mu_a - \mu_B)^2 \sigma_{Bi}$

The KL divergace the simplifies to:

$$KL(P \| Q) = \frac{1}{2} \left(\log \left(\frac{\sum_{i=1}^{T} \sigma_{Bi}}{\sum_{i=1}^{T} \sigma_{Ai}} \right) - n + \sum_{i=1}^{T} \frac{\sigma_{Ai}}{\sigma_{Bi}} - \sum_{i=1}^{T} (\mu_A - \mu_B)^2 \sigma_{Bi} \right)$$