Task 3

a)

$F_X(x) = P(X \leq x) = x$. The pdf of X, $f_X(x) = 1$.

$F_Y^{-1}(X) = Y'$ (F_Y^{-1}: quantile function)

-> Relate CDFs of Y' and X

CDF of Y':

$F_{Y'}(y) = P(Y' \leq y) = P(g(X) \leq y) = P(F_Y^{-1}(X) \leq y)$

$F_Y^{-1}(X) \leq y$ holds only if $X \leq F_Y(y)$

$F_{Y'}(y) = P(X \leq F_Y(y)) = F_X(F_Y(y))$

$= F_Y(y)$

$F_X(x) = x$

=> $F_{Y'}(y) = F_Y(y)$

The pdf: $f_{Y'}(y) = \frac{d}{dy} F_{Y'}(y) = \frac{d}{dy} F_Y(y) = f_Y(y)$

b)

$p_X(x) = \frac{1}{2}x \quad , x \in [0,2]$

$p_Y(y) = -\frac{1}{2}y + 1$

-> Find CDF's

$F_X(x) = \int_0^x \frac{1}{2} u \, du = \frac{1}{4}x^2$

$F_Y(y) = \int_0^y \left(-\frac{1}{2}v + 1\right) dv = -\frac{1}{4}y^2 + y$

$F_X(x) = F_Y(y)$

$\frac{1}{4}x^2 = -\frac{1}{4}y^2 + y$

<=> $0 = y^2 - 4y + x^2$

$y_{1/2} = 2 \pm \sqrt{4 - x^2}$

c)

$(x_1, x_2) = r(\cos\phi, \sin\phi)$

ϕ: uniformly distributed on $[0, 2\pi]$

r: the marginal pdf of r. is: $p(r) = r e^{-\frac{1}{2}r^2}$, $r \geq 0$

-> Generate samples:

$\phi = 2\pi u$, $U_1 \sim$ Uniform $[0,1]$

Compute CDF of r

$F_R(r) = \int_0^r t e^{-\frac{1}{2}t^2} dt = 1 - e^{-\frac{1}{2}r^2}$

Invert

$u = 1 - e^{-\frac{1}{2}r^2} \iff r = \sqrt{-2\ln(1-u_2)}$, $U_2 \sim$ Uniform $[0,1]$

$= \sqrt{-2\ln(u_2)}$

-> Transform back to cartesian coordinates

$x_1 = \sqrt{-2\ln(u_1)}\cos(2\pi u_2)$, $x_2 = \sqrt{-2\ln(u_1)}\sin(2\pi u_2)$

d)

The method works well for univariate distributions, due to the simplicity of computing and inverting the CDF.

In multivariate cases, the complexity of the joint CDF, variable correlations, and dimensionality problems make it unpractical.

e)

$y = h(x)$

-> $dy = \left|\frac{dh}{dx}\right| dx \Rightarrow dx = \frac{dy}{\left|\frac{dh}{dx}\right|}$ *

$P(Y \in [y, y+dy]) = P(X \in [x, x+dx])$

with $x = h^{-1}(y)$

-> $p_Y(y)\,dy = p_X(x)\,dx$

$\overset{*}{\iff}$ $p_Y(y)\,\cancel{dy} = p_X(x)\frac{\cancel{dy}}{\left|\frac{dh}{dx}\right|}$

$$\Leftrightarrow p_Y(y) = \left(p_X(x) \left| \frac{dh}{dx} \right|^{-1} \right) \Big|_{x=h^{-1}(x)}$$