1.

a)

The total transportation cost is:

$\text{total cost} = \sum_{i}^{h} \sum_{j}^{k} T_{ij} c_{ij}$ $\quad T_{ij}$: Represents amount of mass transported from x_i to y_j

-> The objective is: $\quad c_{ij}$: Cost of transporting one unit off mass from x_i to y_j.

$\min_{T_{ij}} \sum_{i}^{h} \sum_{j}^{k} T_{ij} c_{ij}$ $\quad c_{ij} = \|x_i - y_j\|$

Constraints:

- $\sum_{j}^{k} T_{ij} = a_i \quad \forall i$ (Supply)
- $\sum_{i}^{h} a_i = \sum_{j}^{k} b_j$ (mass conservation)
- $\sum_{i}^{h} T_{ij} = b_j \quad \forall j$ (Demand)
- $T_{ij} \geq 0 \quad \forall i,j$ (non negativity)

The transportation cost is linear in distance, therefore the mass needs to be transported in straight lines, because that minimizes the distance traveled by the mass.

b)

Flatten the matrix $\underline{T} = \{T_{ij}\} \in \mathbb{R}^{h \times k}$ into a vector $\underline{x} \in \mathbb{R}^{hk = n}$ with $x_{(i-1)k+j} = T_{ij}$

The total number of variables is $n = mk$

Also flatten $\underline{c} = \{c_{ij}\} \in \mathbb{R}^{h \times k}$ with $c_{(i-1)k+j} = c_{ij}$

The constraints:

$\underline{A} \in \mathbb{R}^{(h+k) \times n}$

$\underline{b} \in \mathbb{R}^{(h+k)} \quad \rightarrow \quad \underline{b} = \begin{pmatrix} a_i \\ \vdots \\ b_j \end{pmatrix} \in \mathbb{R}^{h+k}$

Structure of $\underline{A}$:

$i \in \{1, \dots, h\}$, the total mass leaving i is:

$\sum_{j}^{k} T_{ij} = a_i$

-> Entries for row i: $A_{i,(i-1)k+j} = 1 \quad \forall j \in \{1, \dots, k\}$, 0 elsewhere

$j \in \{1, \dots, k\}$, the total mass arriving sink j is:

$\sum_{i=1}^{m} T_{ij} = b_j$

-> entries for row $k+j$: $A_{k+j,(i-1)k+j} = 1 \quad \forall i \in \{1, \dots, m\}$, 0 elsewhere

$\Rightarrow$ LP formulation $\min \underline{c}^T \underline{x}$ with $\underline{A}\,\underline{x} = \underline{b}$, $\underline{x} \geq 0$

Can be rewritten as: $\underline{A}\,\underline{x} \leq \underline{b}$ and $-\underline{A}\,\underline{x} \leq -\underline{b}$

$$\Rightarrow \begin{cases} \tilde{A} = \begin{bmatrix} A \\ -A \end{bmatrix} \\ \tilde{b} = \begin{bmatrix} b \\ -b \end{bmatrix} \end{cases} \Rightarrow \hat{\underline{A}}\, \underline{x} \leq \hat{\underline{b}}$$

d)

Consider two sources x_1 and x_2, as well as two sinks y_1 and y_2.

Suppose the plan is moving $x_1 \to y_2$ and $x_2 \to y_1$ and this creates an intersecting route.

Alternatively the plan $x_1 \to y_1$ and $x_2 \to y_2$ creates a non-intersecting route

Total cost is

$C_{intersect} = d(x_1, y_2) + d(x_2, y_1)$

$C_{non\text{-}intersect} = d(x_1, y_1) + d(x_2, y_2)$

With the triangle inequality

$d(x_1, y_1) \leq d(x_1, a) + d(a, y_1)$ (with intersection point a)

$d(x_2, y_2) \leq d(x_2, a) + d(a, y_2)$

$\Rightarrow d(x_1, y_1) + d(x_2, y_2) \leq d(x_1, y_2) + d(x_2, y_1)$

$\to$ equality only if $x_1 = x_2$ or $y_1 = y_2$ which eliminates the intersection.

2.

b)

$\psi_t(x) = (1-t)x_0 + t x_1$

at $t = 0$: $\psi_t(x_0) = x_0$

$\Rightarrow L_{CFM} \sim \| v_t(x_0) - (x_1 - x_0) \|^2$

This implies that the optimal velocity field is: $v_0(x_0) = \mathbb{E}_{x_1 \sim q}[x_1 - x_0]$

$\to$ so the optimal velocity field at $t=0$:

- represents the direction of movement from p to q
- smoothly points from regions of high p density to regions of high q density

c)

A centralized velocity field at $t=0$ can lead to curved trajectories, high density bottlenecks and inefficiency. In our case high density.

Minibatch optimal transport helps with these problems with locally optimizing transport, leading to straighter trajectories and more realistic flows.