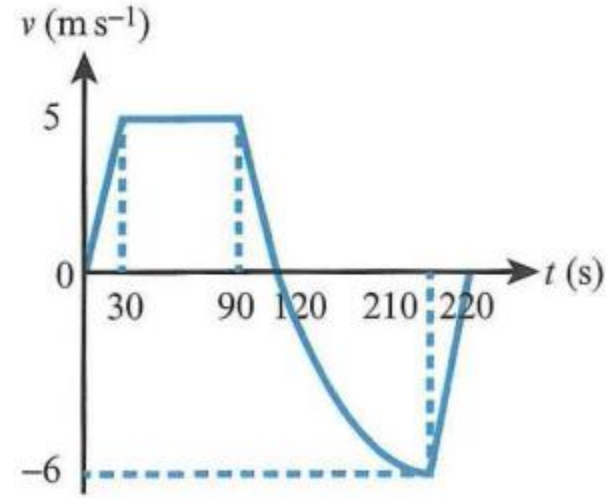
1.5 Velocity–time graphs and multi-stage problems

An athlete starts at rest and gradually increase his speed over the first 30s before maintaining the same speed of 5 m/s for 60s. Then he reduces his speed until coming to rest another 30s later. The athlete then returns to his starting point by increasing his speed quickly at the start and continually trying to increase his speed for 90s, but only managing to increase it by smaller and smaller amounts, peaking at 6m/s. He then slows sown over 10s before coming to rest at his starting points.



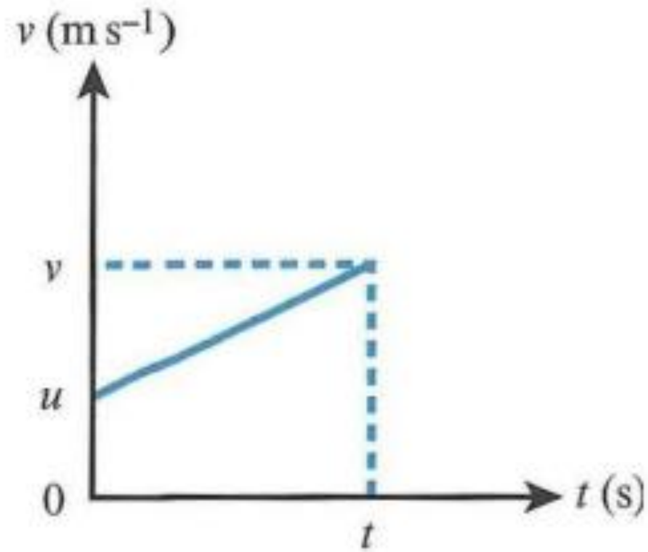
When sketching the v-t graph,

- Always show time on the x-axis and velocity on y-axis
- A horizontal graph line indicates __________.
- The steepness of the line indicates how quickly it is changing – accerlation.
- Show clearly the shape of the graph – line or curve
- Show key points: intercept on vertical/horizontal axis
- if there's more than one stage, indicate the time and displacement at the change in motion.

1.5 Velocity–time graphs and multi-stage problems

KEY POINT 1.10

The gradient of a velocity–time graph is equal to the acceleration of the object.



From this graph you can see that the gradient is $\frac{v-u}{t}$, which is the same as the formula given for acceleration in Section 1.2.

KEY POINT 1.11

The area under the line of a velocity–time graph is the displacement of the object.

1.5 Velocity–time graphs and multi-stage problems

a Arthur travels at a constant speed of $5\,\text{m s}^{-1}$ for $10\,\text{s}$ and then decelerates at a constant rate of $0.5\,\text{m s}^{-2}$ until coming to rest. Sketch the velocity–time graph for his motion.

b Brendan travels at a constant $4\,\text{m s}^{-1}$ starting from the same time and place. Show that Arthur and Brendan are travelling at the same speed after $12\,\text{s}$ and, hence, find the furthest Arthur gets ahead of Brendan.

c Show that for $t > 10$ the gap between them is given by $g(t) = -\frac{1}{4}t^2 + 6t - 25$ and, hence, find the time when Brendan overtakes Arthur.

a Let T be the time spent in deceleration.

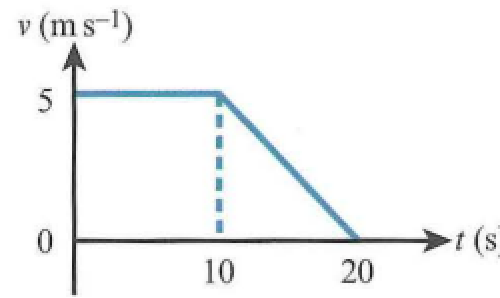
$v = u + at$

$0 = 5 - 0.5T$

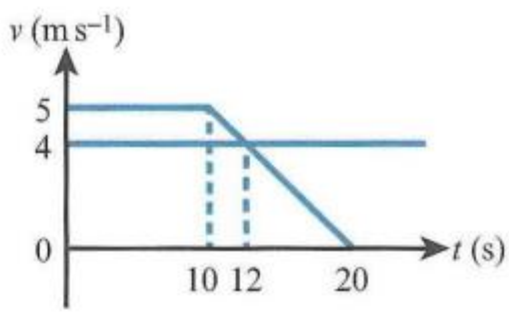
$T = 10\,\text{s}$

$t = T + 10$

$= 20\,\text{s}$



b



$v = 5 - 0.5(t - 10) = 4$

$t = 12\,\text{s}$

$s_A = 5 \times 10 + \frac{1}{2} \times (5 + 4) \times 2 = 59\,\text{m}$

$s_B = 4 \times 12 = 48\,\text{m}$

Therefore, the largest gap is $11\,\text{m}$.

1.5 Velocity–time graphs and multi-stage problems

a Arthur travels at a constant speed of $5\,\mathrm{m\,s^{-1}}$ for 10 s and then decelerates at a constant rate of $0.5\,\mathrm{m\,s^{-2}}$ until coming to rest. Sketch the velocity–time graph for his motion.

b Brendan travels at a constant $4\,\mathrm{m\,s^{-1}}$ starting from the same time and place. Show that Arthur and Brendan are travelling at the same speed after 12 s and, hence, find the furthest Arthur gets ahead of Brendan.

c Show that for $t > 10$ the gap between them is given by $g(t) = -\frac{1}{4}t^2 + 6t - 25$ and, hence, find the time when Brendan overtakes Arthur.

c Starting gap = 0

At time t:

$$s_A = 50 + \frac{1}{2}(t-10)\left(5 + \left(5 - \frac{1}{2}(t-10)\right)\right)$$

$$= 50 + 5(t-10) - \frac{1}{4}(t-10)^2$$

$$s_B = 4t$$

$$g(t) = 0 + 50 + 5(t-10) - \frac{1}{4}(t-10)^2 - 4t$$

$$= -\frac{1}{4}t^2 + 6t - 25$$

Brendan overtakes when the gap is 0, so

$$-\frac{1}{4}t^2 + 6t - 25 = 0$$

$$t^2 - 24t + 100 = 0$$

$t = 5.37$ or 18.6

Since the equations are valid only for $t > 10$, $t = 18.6\,\mathrm{s}$.

1.5 Velocity–time graphs and multi-stage problems

EXERCISE 1E

1 Sketch the velocity–time graphs from the information given. In each case take north to be the positive direction.

a Rinesh starts from rest, moving north with a constant acceleration of $3\,\mathrm{m\,s^{-2}}$ for $5\,\mathrm{s}$.

b Wendi is moving north at $2\,\mathrm{m\,s^{-1}}$ when she starts to accelerate at a constant rate of $0.5\,\mathrm{m\,s^{-2}}$ for $6\,\mathrm{s}$.

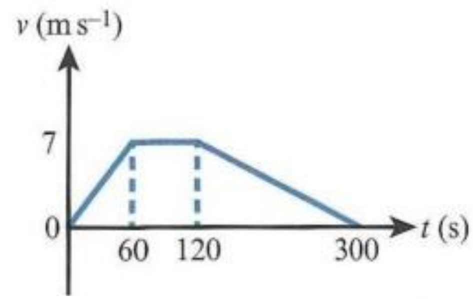
c Dylon is moving south at a constant speed of $4\,\mathrm{m\,s^{-1}}$.

d Susan is moving north at $6\,\mathrm{m\,s^{-1}}$ when she starts decelerating at a constant rate of $0.3\,\mathrm{m\,s^{-2}}$ until she comes to rest.

1.5 Velocity–time graphs and multi-stage problems

Homework 1E:

6 The graph shows the journey of a cyclist travelling in a straight line from home to school. Find the distance between her home and the school.



8 A rowing boat accelerates from rest at a constant rate of $0.4\,\mathrm{m\,s^{-2}}$ for 5 s. It continues at constant velocity for some time until decelerating to rest at a constant rate of $0.8\,\mathrm{m\,s^{-2}}$. In total, the boat covers a distance of 30 m. Find how long was spent at constant speed.

12 A car is at rest when it accelerates at $5\,\mathrm{m\,s^{-2}}$ for 4 s. It then continues at a constant velocity. At the instant the car starts moving, a truck passes it, moving at a constant speed of $22\,\mathrm{m\,s^{-1}}$. After 10 s the truck starts slowing at $1\,\mathrm{m\,s^{-2}}$ until coming to rest.

a Show that the velocities are equal after 12 s and, hence, find the maximum distance between the car and the truck.

b Show that the distance covered at a time t s from the start by the car and the truck, for $t > 10$, are given by $40 + 20(t - 4)$ and $220 + 22(t - 10) - \frac{1}{2}(t - 10)^2$, respectively. Hence, find the time at which the car passes the truck.

1.5 Velocity–time graphs and multi-stage problems

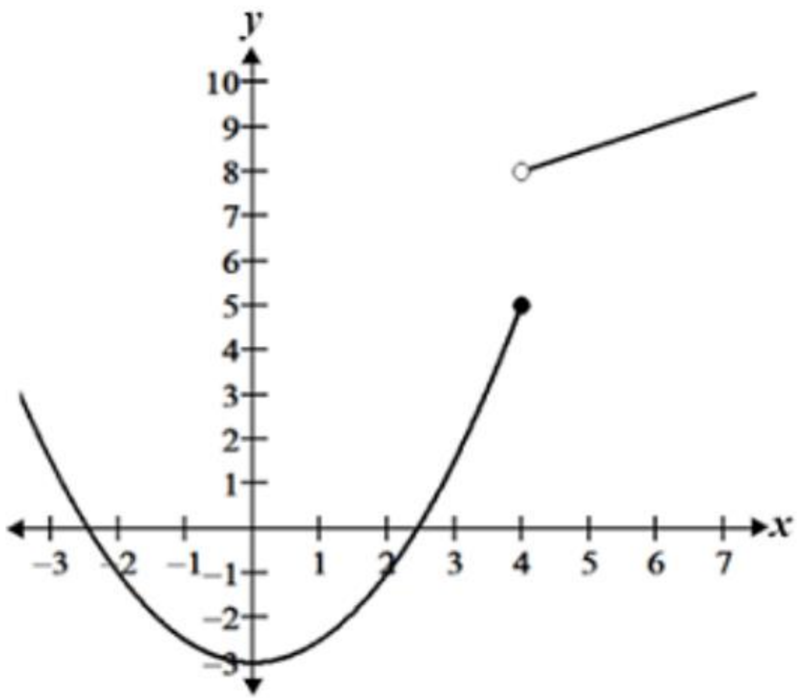
14 A driver travelling at $26\,\mathrm{m\,s^{-1}}$ sees a red traffic light ahead and starts to slow at $3\,\mathrm{m\,s^{-2}}$ by removing her foot from the accelerator pedal. A little later she brakes at $5\,\mathrm{m\,s^{-2}}$ and comes to rest at the lights after $6\,\mathrm{s}$.

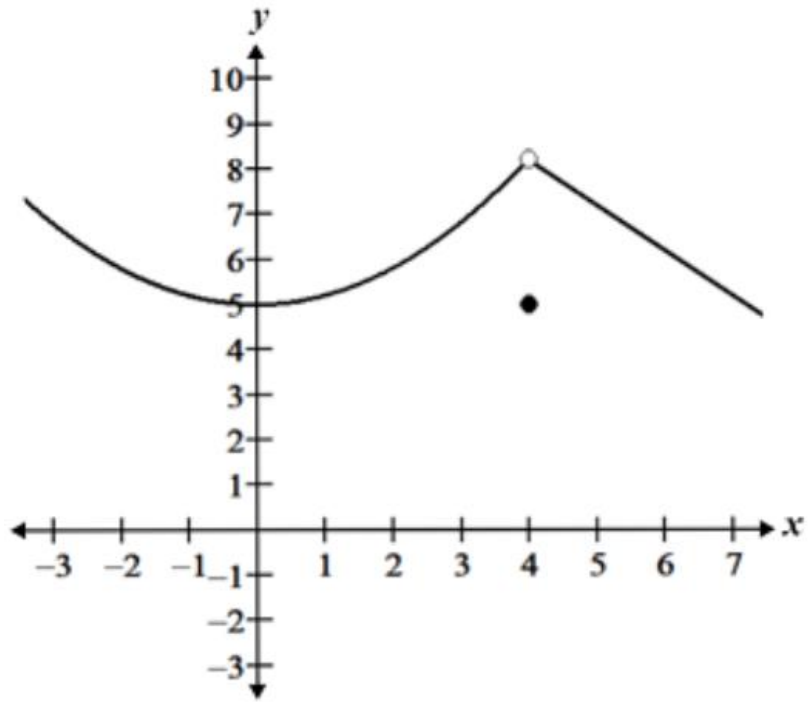
a Sketch the velocity–time graph of the motion.

b Find the equations of the two sections of the graph.

c Hence, find the time when the driver needs to start braking.

1.6 Graphs with discontinuities





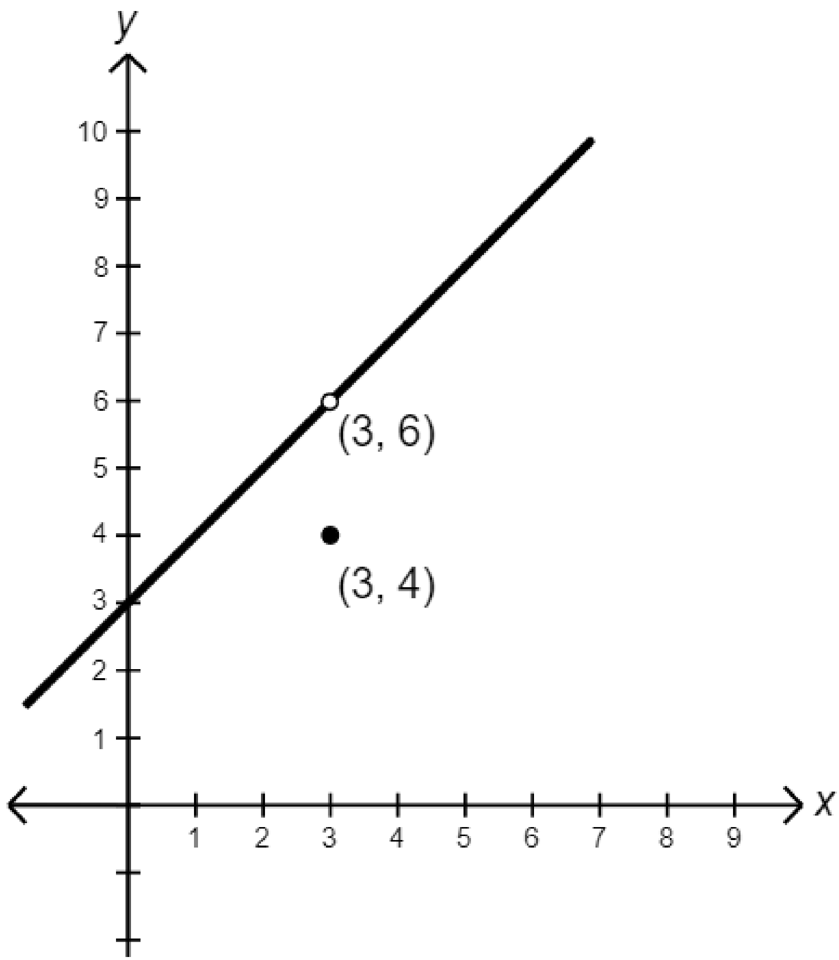
- Functions are classified as continuous or discontinues.
- Formal definition of discontinuity requires the use of limits.
- A function f(x) as a discontinuity at a point x = a if any of the following is true:

1. f(a) is undefined.
2. $\lim_{x \to a} f(x)$ does not exist.
3. f(a) is defined and the limit exists, but $\lim_{x \to a} f(x) \neq f(a)$.

1.6 Graphs with discontinuities

Types of discontinuities

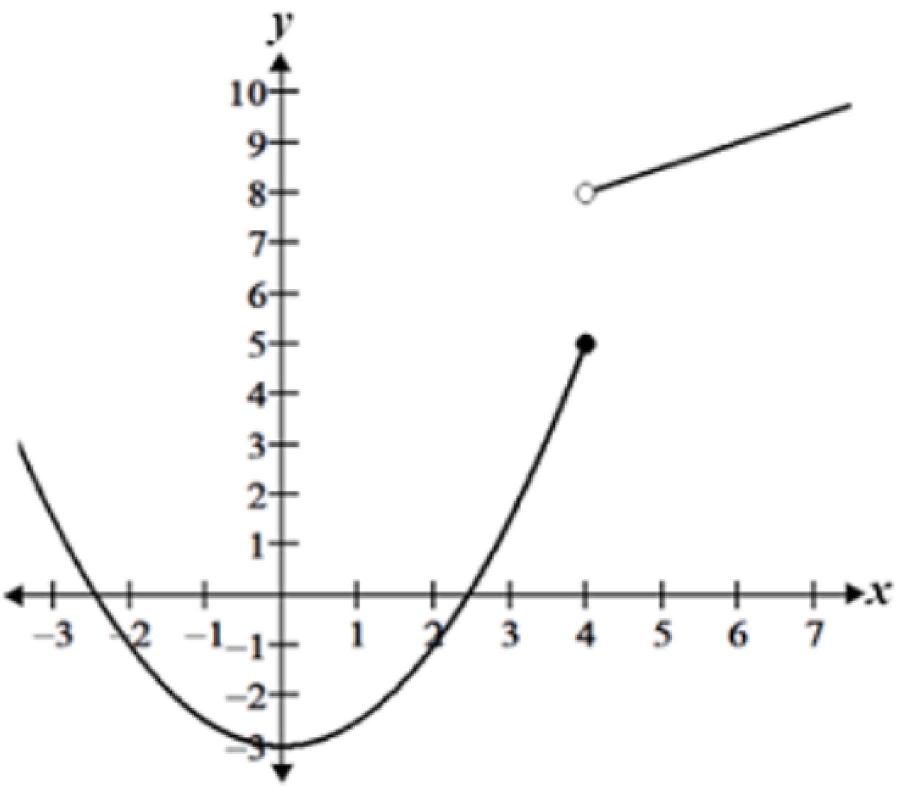
Removable discontinuity



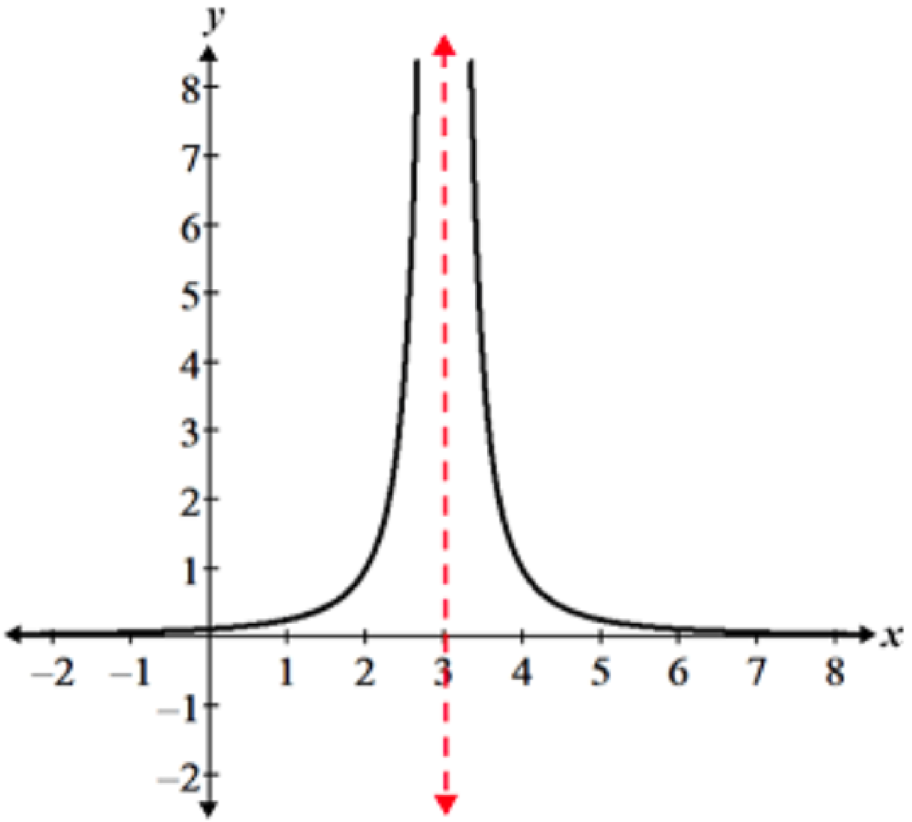
1.6 Graphs with discontinuities

Types of discontinuities

Non-removable discontinuity



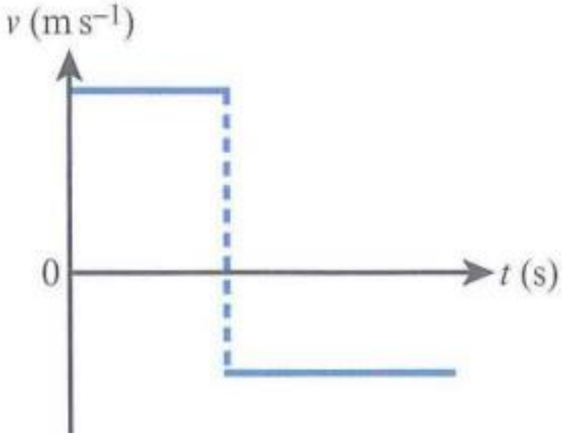
Jump discontinuity



Infinite discontinuity

1.6 Graphs with discontinuities

In practice, when the change in velocity happens over a tiny amount of time that it is reasonable to ignore, we can assume the change is instantaneous.



Velocity-time graph will have discontinuity as the velocity instantaneously changes.

Displacement-time graph cannot have discontinuity, but the gradient will instantaneously change.

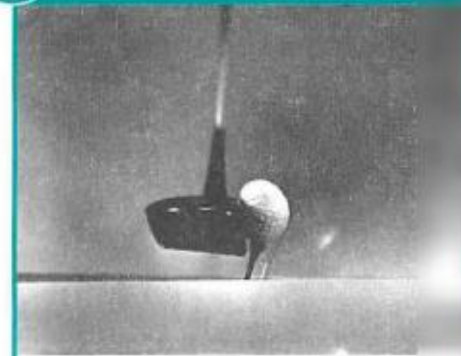
KEY POINT 1.12

On the velocity–time graph of an object that instantaneously changes velocity by bouncing or being struck, the change is represented by a vertical dotted line from the velocity before impact to the velocity after impact.

1.6 Graphs with discontinuities

By modeling the objects as particles, we can assume the objects do not lose shape and the time in contact is sufficiently small to be negligible.

DID YOU KNOW?



Golf balls look and feel solid, but in the instant after impact from a golf club moving at around 200 km h^{-1} the ball appears to squash so that its length is only about 80% of its original diameter and its width increases.

1.6 Graphs with discontinuities

a A ball is travelling at a constant speed of $10\,\mathrm{m\,s^{-1}}$ for 2 s until it strikes a wall. It bounces off the wall at $5\,\mathrm{m\,s^{-1}}$ and maintains that speed until it reaches where it started. When it passes that point it decelerates at $1\,\mathrm{m\,s^{-2}}$. Find the times and displacements when each change in the motion occurs.

The distance to the wall is

$s = 10 \times 2 = 20\,\mathrm{m}$

The time between hitting the wall and returning to the starting point is, therefore,

$\frac{20}{5} = 4\,\mathrm{s}$ so $t = 6\,\mathrm{s}$

The time from starting to decelerate until it stops is

$\frac{0-(-5)}{1} = 5\,\mathrm{s}$ so $t = 11\,\mathrm{s}$

The distance covered is

$5 \times 5 + \frac{1}{2} \times (-1) \times 5^2 = 12.5\,\mathrm{m}$

so displacement is

$s = -12.5\,\mathrm{m}$

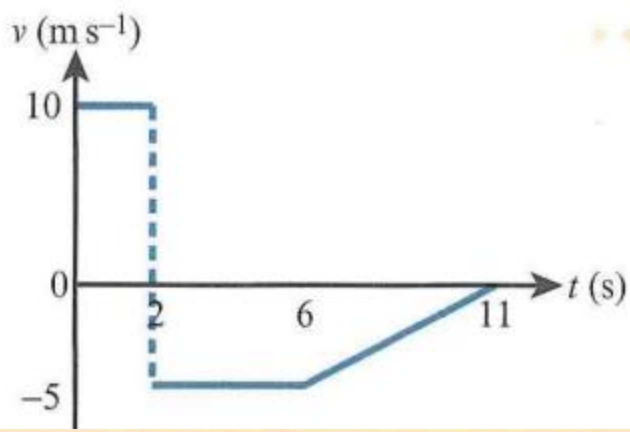
Note that displacements are measured from the starting position, taking the original direction as positive.

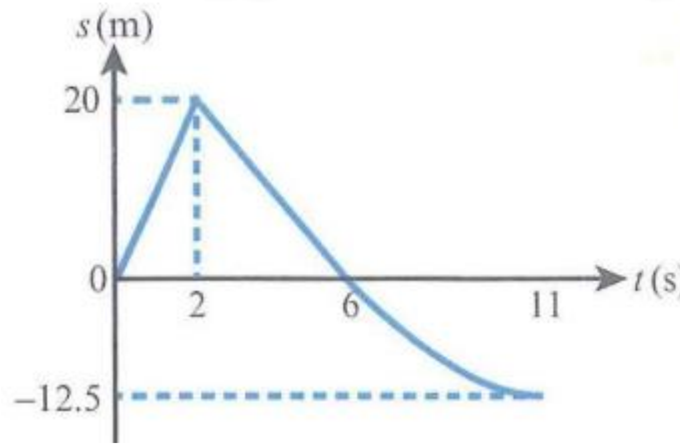
Although decelerating, the acceleration is positive because the velocity is negative.

1.6 Graphs with discontinuities

a A ball is travelling at a constant speed of $10\,\text{m s}^{-1}$ for 2 s until it strikes a wall. It bounces off the wall at $5\,\text{m s}^{-1}$ and maintains that speed until it reaches where it started. When it passes that point it decelerates at $1\,\text{m s}^{-2}$. Find the times and displacements when each change in the motion occurs.

b Sketch a velocity–time graph and a displacement–time graph for the motion. Measure displacements as distances from the starting point and the original direction of motion as positive.





Notice the graph is discontinuous at $t = 2$.

Although the ball is decelerating after $t = 6$, the gradient is positive because the velocity is negative.

Notice that at $t = 2$ the gradient is different on either side of the cusp. This indicates a discontinuity in the velocity.

1.6 Graphs with discontinuities

EXERCISE 1F

1 An ice hockey puck slides along a rink at a constant speed of $10\,m\,s^{-1}$. It strikes the boards at the edge of the rink 20 m away and slides back along the rink at $8\,m\,s^{-1}$ until going into the goal 40 m from the board. Sketch a velocity–time graph and a displacement–time graph for the motion, measuring displacement from the starting point in the original direction of motion.

Homework 1F:

4 A ball is dropped from rest 20 m above the ground. It accelerates towards the ground at a constant rate of $10\,m\,s^{-2}$. It bounces on the ground and leaves with a speed that is half the speed it struck the ground originally. The ball is then caught when it reaches the highest point of its bounce. Sketch a velocity–time graph and a displacement–time graph for the motion, measuring displacement above the ground.

6 A billiard ball is on the centre spot of a 6 m long table and is struck towards one of the cushions with initial speed $3.1\,m\,s^{-1}$. It slows on the table at $0.2\,m\,s^{-2}$. When it bounces off the cushion its speed reduces to 70% of the speed with which it struck the cushion. The ball is left until it comes to rest.

a Sketch the velocity–time and displacement–time graphs for the ball, taking the centre of the table as the origin for displacement and the original direction of motion as positive.

b What assumptions have been made in your answer?

Checklist of learning and understanding

- The equations of constant acceleration are:

$$v = u + at$$

$$s = \frac{1}{2}(u + v)t$$

$$s = ut + \frac{1}{2}at^2$$

$$s = vt - \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

- A displacement–time graph shows the position of an object at different times. The gradient is equal to the velocity.
- A velocity–time graph shows how quickly an object is moving at a given time. The gradient is equal to the acceleration. The area under the graph is equal to the displacement.