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1. Show that the sliced score matching (SSM) loss can also be written as

$$L_{SSM} = \mathbb{E}_{x\sim p(x)}\mathbb{E}_{v\sim p(v)}\left[\|v^T S(x;\theta)\|^2 + 2v^T\nabla_x(v^T S(x;\theta))\right].$$

Note $L_{SSM} = E_{x\sim p(x)} E_{v\sim p(v)} [(v^T(S(x;\theta) - \nabla_x \log p(x)))^2]$

where $S(x;\theta) = \nabla_x \log p_\theta(x) \in \mathbb{R}^d$

Then, $L_{SSM} = E_{x,v}[(v^T S(x) - v^T \nabla_x \log p(x))^2]$

$\Rightarrow L_{SSM} = E_{x,v}[(v^T S(x))^2 - 2(v^T S(x))(v^T \nabla_x \log p(x)) + (v^T \nabla_x \log p(x))^2]$

$= \overset{①}{E_{x,v}}[(v^T S(x))^2] + \overset{②}{E_{x,v}}[-2(v^T S(x))(v^T \nabla_x \log p(x))] +$

$\underline{E_{x,v}[(v^T \nabla_x \log p(x))^2]}$ not depend on $S(x)$, can regard as constant

① : $\because$ $v^T S(x)$ is scalar

$\therefore (v^T S(x))^2 = \| v^T S(x)\|^2$

② : Consider $T = E_x[(v^T S(x))(v^T \nabla_x \log p(x))]$

Then, $T = \int p(x)(v^T S(x))(v^T \nabla_x \log p(x))\, dx$

$= \int p(x)(v^T S(x))(v^T \frac{\nabla_x p(x)}{p(x)})\, dx$ (Note $\nabla_x \log p(x) = \frac{\nabla_x p(x)}{p(x)}$)

$= \int (v^T S(x))(v^T \nabla_x p(x))\, dx$

By IBP, let $\psi(x) = v^T S(x)$, $F(x) = p(x) v$

For $\nabla_x \cdot F(x) = \nabla_x (p(x) v) = \sum_{i=1}^{k} \frac{\partial (p(x) v_i)}{\partial x_i} = \sum_{i=1}^{k} v_i \frac{\partial p(x)}{\partial x_i}$

$= v^T \nabla_x p(x)$

$\Rightarrow T = \int \psi(x)(\nabla_x \cdot F(x))\, dx$

By Divergence thm., $\int \psi(\nabla_x \cdot F)\, dx = -\int (F \cdot \nabla_x \psi)\, dx$

$F \cdot \nabla_x \psi = p(x) v^T \nabla_x (v^T S(x))$

$$\Rightarrow T = -\int p(x)\, v^T \nabla_x (v^T S(x))\, dx$$

$$= -E_x[v^T \nabla_x (v^T S(x))]$$

Thus, ② $= E_{x,v}[2 v^T \nabla_x (v^T S(x))]$

① + ② : $L_{SSM} = E_{x,v}[\| v^T S(x) \|^2] + E_{x,v}[2 v^T \nabla_x (v^T S(x))]$

$$= E_{x \sim p(x)} E_{v \sim p(v)} [\| v^T S(x) \|^2 + 2 v^T \nabla_x (v^T S(x))]$$

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