UNSW AUSTRALIA

Coupled pendula

A mechanics experiment

Never Stand Still | Science | School of Physics

SKILLS GAINED

- Computer interfacing.
- Data capture and storage.
- Error analysis
- Data visualisation and presentation
- General lab safety

ASSUMED KNOWLEDGE

- PHYS 1121/1131
- PHYS 1221/1231

1 Experimental aim

Here you will study the motion of a coupled oscillatory systems. In particular you will design an experiment to:

- Characterise a coupling spring.
- Characterise the uncoupled pendula.
- Quantitatively characterise two coupled pendula.
- Qualitatively discuss the effect(s) of more coupled pendula.

2 Background

The motion of two identical coupled pendula has two degrees of freedom: one degree of freedom for each pendulum. Can you think how to quantify the degrees of freedom for a system of coupled pendula?

These pendula exhibit three distinct types of oscillatory motion:

- In phase oscillations: when both pendula oscillate at the same frequency in phase.
- Out of phase oscillations: when both pendula oscillate at the same frequency out of phase by π.
- Beat motion occurs when energy is transferred from one pendulum to the other, resulting in a frequency modulation.

2.1 Theoretical background

The equations of motion for the coupled pendulum will be derived in your prework. Here we include some additional information which may help you derive your equations of motion.

2.1.1 Solving second order inhomogeneous differential equations: a cheat sheet

You may need to solve differential equations of the form:

$$\ddot{\phi}_1(t) + A^2\phi_1(t) = -B^2[\phi_2(t) - \phi_1(t)], \tag{1}$$

$$\ddot{\phi}_2(t) + A^2\phi_2(t) = B^2[\phi_2(t) - \phi_1(t)]. \tag{2}$$

Where A and B are constants; $\phi_1(t)$ and $\phi_2(t)$ are functions of time.

For $\phi_1(t) = \phi_2(t);\ [\phi_1(t) - \phi_2(t)] = 0$ the solution is:

$$\phi_1(t) = \phi_2(t) = \phi_{max}\, cos(A\, t). \tag{3}$$

For $-\phi_1(t) = \phi_2(t);\ [\phi_1(t) - \phi_2(t)] = 2\phi_1(t)$ the solution is:

$$\phi_1(t) = \phi_{max}\, cos(\sqrt{A^2 + 2B^2} \cdot t), \tag{4}$$

$$\phi_2(t) = -\phi_{max}\, cos(\sqrt{A^2 + 2B^2} \cdot t). \tag{5}$$

If both pendula are initially stationary with $\phi_1(0) = \phi_{max}$ and $\phi_2(0) = 0$ the solution is:

$$\phi_1(t) = \phi_{max}\, cos(\frac{\sqrt{A^2 + 2B^2} - A}{2} \cdot t) \cdot cos(\frac{\sqrt{A^2 + 2B^2} + A}{2} \cdot t), \tag{6}$$

$$\phi_2(t) = -\phi_{max}\, sin(\frac{\sqrt{A^2 + 2B^2} - A}{2} \cdot t) \cdot sin(\frac{\sqrt{A^2 + 2B^2} + A}{2} \cdot t) \tag{7}$$

2.2 Useful links

Phys clips[1] about rotation, torques and precession.

Refresher on the motion of a single pendulum[2].

Refresher on basic rotational quantities[3].

3 Prework

3.1 Theoretical pre-work

In this section you will derive the equations of motion for two coupled pendula. You will be working with angular coordinates for convenience. See the Theoretical background for more information (Section 2.1).

[1] http://www.animations.physics.unsw.edu.au/jw/rotation.htm#Newton

[2] http://www.physicsclassroom.com/class/waves/Lesson-0/Pendulum-Motion

[3] http://hyperphysics.phy-astr.gsu.edu/hbase/rotq.html

First label the forces in Figure 1 . Resolve the forces in linear coordinates to start with.

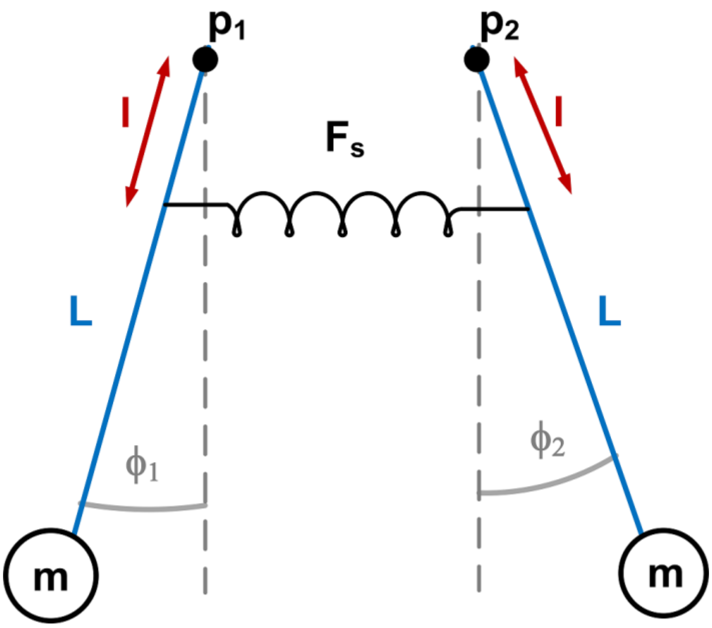


Figure 1. Schematic of a coupled pendulum.

The pendula p_1 and p_2 are set up as follows:

- Both pendula consist of a mass $m = 1\ kg$ suspended by a massless, frictionless, in-extensible rod of length L.
- A spring connects the two pendula and exerts a force F_s (which direction, and where?)
- The coupling length (from the fulcrum to the point where the spring connects to the pendulum) is l for both pendula.
- For any given experiment the maximal angular displacement is $|\phi| = |\phi_1| = |\phi_2|$.

For simplicity concentrate on the motion of one pendulum. The system is symmetric (p_1 and p_2 have identical set-ups) and thus the derivation for p_1 is valid for p_2.

1. For the forces in Figure 1 write a general equation for the total force (Newton's second law of motion) of the system. Your equation should have the general form total torque = summation over individual torques[4].
2. Write the torques (τ) in terms of linear forces (F) and radii (r). This step is important because you can only measure the components of the linear forces[5].
3. Now perform the cross product on the R.H.S to introduce an angular term. The small angle approximation may come in handy for simplification[6]. At this point you will need to think what, if any, limitations this will have on your experiment. This will need to be explicitly addressed in your report.
4. How can you introduce an angular term into the L.H.S? Hint: it should be an angular acceleration $\ddot{\phi}$.
5. Solve for the angular acceleration. Replace the moment of inertia I with known terms. Hint: What is the moment of inertia for a point mass?
6. Solve the differential equations and consider the three initial conditions:
   a. When the motion of the pendula are in phase.
   b. When the motion of the pendula are out of phase.
   c. When the motion of the pendula are beating (the starting condition for beat motion is to maximally displace one pendulum[7] by $\phi_1(t = 0) = \phi_{max}$ while the other pendulum starts at $\phi_2(t = 0) = 0$.)

[4] For short, I will refer to the total torque as the "left hand side" (L.H.S) and the summation over individual torques as the "right hand side" (R.H.S).

[5] For example you will not be working with protractors to measure angles etc.

[6] See, for example, https://en.wikipedia.org/wiki/Small-angle_approximation and http://hyperphysics.phy-astr.gsu.edu/hbase/trgser.html.

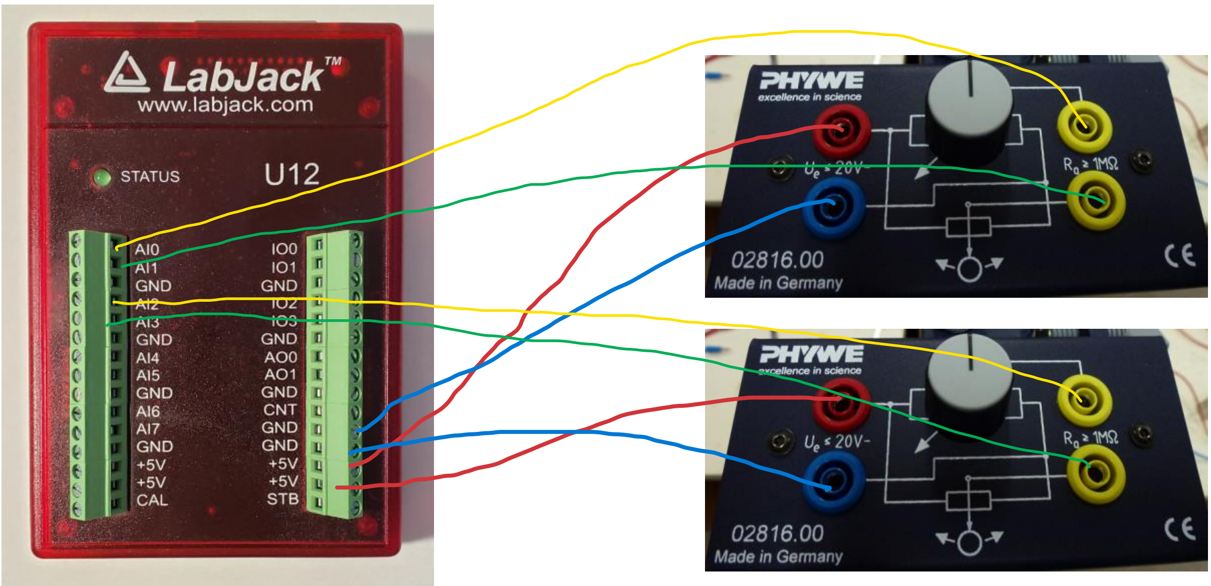
[7] See https://www.youtube.com/watch?v=CguKKl9mX2s at 0:50s for such a starting condition.

3.2 Experimental pre-work

Before completing the experimental pre-work, please read the Operating Instructions for context.

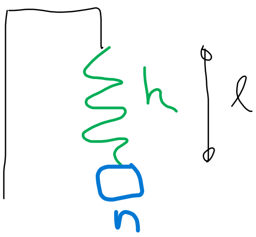
There is a block diagram for the angle reader of one pendulum shown in the Operating Instructions. The angle reader has a voltage supplied by the LabJack via one set of inputs. The other inputs are connected to the LabJack's analog inputs.

1. Using one LabJack, how would you supply the same voltage across two pendula? You can use the images below to draw the connecting wires. Label the components.



2. How will you measure the spring constant with some/all of the equipment below? You are expected to draw and diagram and include physics equations to justify your method.

- A bench clamp (not shown)
- A right angle clamp.
- A hook.
- A rod to suspend weights.
- Five extra weights (10g each).
- A spring (not shown).



Vary m (F=mg) and measure ℓ and get Δx. Use a linear regression.

$F = kx'$ $mg = kx'$

$m = \left(\frac{k}{g}\right) x'$ gradient

3.3 Experimental plan

You will now plan a suitable experiment to investigate the coupling of two pendula. This will be done in several stages: first you will consider how to measure the coupling of the two pendula. Once the core experiment is planned, you will consider if you need to do any initial characterisation or optimization of your system. Finally you will consider how to record, present and analyse your data.

3.3.1 First steps

The first step in experimental planning is to ask yourself:

- What's the hypothesis?
- What is the relevant theory?
- What can I measure?

Some information which could be useful when considering your experimental design:

- Read the operating instructions for the equipment first.
- Do the above pre-work.

It may help to fill in the following table:

Experimental system	
(E.1) List all independent variables and constants in the experimental system:	$m, k, \ldots$
(E.2) List all independent variables you can change in the experimental system:	
(E.3) List all independent variables and constants which are introduced by the coupling spring (in the experimental system):	m
(E.4) Which independent variables occur in both E.2 and E.3	
(E.5) Are there any measureable quantities dependent upon the variable/s in E.4?	
Theoretical derivations	
(T.1) List all independent variables and constants in your derived equations:	$m, k, \ldots$
(T.2) List all independent variables in T.1 which you can change in the experimental system:	
(T.3) List all independent variables and constants in your equations which are introduced by the coupling spring:	
(T.4) List variables which occur in the list T.3 and T.2	
(T.5) Are there any measureable quantities dependent upon the variable/s in T.4?	

3.3.2 Planning your core experiment

- What independent variable can you vary to investigate coupling of two pendula? Hint, see: T.4 and E.4.
- What dependent variable can you measure to investigate the coupling of two pendula? Hint, see: T.5, E.5 and look at your derived equations of motion.
- Can you compare your measured results to theory or calculations?
- How can you investigate the different types of motion of the coupled pendula in your experiment? Can you relate this to variables you can measure?

3.3.3 Initial characterisation and optimisation

Now you need to think about the initial characterisation of your set up. You may not know everything about your system before you start your core experiment, and thus may need to measure some values. Furthermore, you system may not be optimised.

For optimisation tips, see the Experimental Hints and Tips section.

The table below, may help with planning your initial characterisation, some values are filled out to get you started.

Experimental system	
(Ex.1) List all fixed variables in the experimental system:	$m, k, \ldots$
(Ex.2) List all fixed variables you know:	$m = 1\ kg,$
(Ex.3) List all fixed variables you don't know:	$k = ?$
(Ex.4) Write down the dependent variable/s you will measure (from previous section):	
(Ex.5) Write down the independent variables which effect the dependent variable you want to measure: For example if $F = ma$ and you measure F, the independent variables are m and a.	k
(Ex.6) Write down the variables common to Ex.5 and Ex.3	k

- What variables or constants do you need to measure before your core experiment? Hint look at Ex.6.
- Recall your equations of motion are derived under the assumption that the coupled pendula are identical. How can you demonstrate this experimentally?

Verify same F.T. ω when operated uncoupled

3.3.4 Collecting data and calculating errors

Now you know what you need to measure, how will you record your data? In a table? Think about what you will (approximately) need to vary, how many times you will vary this parameter and how you will record the data.

It is a good idea to roughly sketch out how you plan to record your data. For example, if measuring dependent variable D and idepenedent variable IV for two oscillating modes of the coupled pendulum, the table below may be one way to record your data:

Dependent variable D → Independent variable IV ↓	**Mode 1 [units]**	**Mode 2 [units]**
$IV = \ I_1\ [units]$		
$IV = \ I_2\ [units]$		
$IV = \ I_3\ [units]$		

Don't forget to think about how you will record your errors for your raw data.

3.3.5 Analysis and errors

Think about:

- How you will analyse your data. What do you need to extract from your data?
- If you need to compare your results to theory.
- How you will present your final results (graph, table?)
- Error propagation.
- How to sensibly use significant figures.

3.3.6 Constructing your experimental plan

Using the information above, you can construct your experimental plan in a logical manner. An example experimental plan could be:

1. Set-up equipment
2. Optimise measurement
    a. Try different measurement configurations
    b. Record data
    c. Choose best configuration.
3. Initial characterisation
    a. Measure something
    b. Record data
    c. Record error
    d. Analysis
4. Core measurement
    a. …
    b. …
5. Analysis of main results.
6. Conclusion

4 Experimental hints and tips

A key part of experimental physics is signal optimization. In this section we will discuss how you can optimize your signal.

Note, all your signal optimization can be performed with only one (uncoupled) pendulum.

In this experiment mechanical motion (amplitude) is converted to an electrical signal. You will be recording this electrical signal.

You will be expected to discuss how you optimized your signal, and you will need show evidence of your optimization. Consider the following:

- Think about what changing parameter of the pendulum is “causing” the electrical signal variation. Why not vary this experimental parameter and see what happens to your signal?
- In your tests, can you identify what is your signal? What feature is the noise? You want the biggest (reasonable) signal for the smallest noise. Which configuration has the largest ratio of signal-to-noise? For this experiment, you can provide a qualitative explanation, without quantitative analysis.

5 Analysis hints and tips

Fourier analysis of all output signals will be crucial to understanding the physics underlying coupled pendula. You need to familiarise yourself with basic Fourier theory.

Consider addressing the following in your report:

- Can you compare the uncoupled pendulum to the coupled pendulum? What do you expect?
- Can you compare measured and theoretical results?
- Can you qualitatively explain your results with reference to your equations?
- Based on your results of a coupled system, how would you define a “strongly coupled system”? Thus what configuration results in a strong or weakly coupled system?

6 References

6.1 Background references

[1] http://www.thefouriertransform.com/#introduction