IBDP Mathematics Handout

Sam Ren

November 28, 2022

Pure mathematics is, in its way, the poetry of logical ideas. — Albert Einstein

This revision book is to the students of IBDP and if there are something to replenish please contact me by My email (click on "My email")

Contents

1 Sequence and Series

1.1 Arithmetic Sequence

Definition 1.1. *Arithmetic Sequence/Arithmetic Progression (AP) is defined as:*

$$u_n = u_1 + (n-1)d \tag{1}$$

u_n is the general term
u_1 is the first term
n is the number of the term
d is is the common difference.

Note that the u_n in the sum equation of Arithmetic Sequence can be written as $u_1 + (n-1)d$ Thus that the equation can also be written as $S_n = \frac{1}{2}[2u_1 + (n-1)d]$

Theroem 1.1. *The sum of Arithmetic Sequence is :*

$$S_n = \frac{n(u_n + u_1)}{2} \tag{2}$$

Please notice till now we do not learn the Infinite sum equation for Arithmetic Sequence yet.

1.2 Geometric Sequence

Definition 1.2. *Geometric Sequence is defined as :*

$$u_n = u_1 \times (r)^{n-1} \tag{3}$$

where r is the common ratio of this sequence.

In Geometric Sequence there are some theorem which I will list below.

Corollary 1.1. *For the common ratio r it follows the rule:*

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = ... \tag{4}$$

Theroem 1.2. *The sum of finite Geometric Sequence is :*

$$S_n = \frac{u_1(1 - r^n)}{1 - r} \tag{5}$$

If r > 1, the sequence is an exponential growth.

If $0 < r < 1$, the sequence has an exponential decay.

For the limitation of r , -1 < r < 1 can also be written as r<|1|

Theroem 1.3. *When the $n = \infty$ the sum of Geometric sequence is:*

$$S_\infty = \frac{u_1}{1-r} \; (-1 < r < 1) \tag{6}$$

2 *Counting and Binomial Theorem*

2.1 *Binomial Theorem*

In algebra, a binomial is a polynomial that is the sum of two terms, each of which is a monomial. It is the simplest kind of sparse polynomial after the monomials.[1]

[1]

Definition 2.1. *A Binomial is a polynomial which satisfy :*

$$(a+b)^n \tag{7}$$

The method we use to expand the binomial is using the binomial theorem. The meaning of $\binom{n}{r}$ is the combination.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

.

Theroem 2.1.

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + b^n \;\; n \in \mathbb{N} \tag{8}$$

The independent term: The term do not involve x in it since the independent term does not vary as x varies. (constant term) nUsing Theorem 2.2 in the case of $(a+x)^n$ which a is not equal to 1

Theroem 2.2.

$$(a+b)^n = a^n(1+\frac{b}{a})^n \;, \; |\frac{b}{a}| < 1 \;, \; n \in \mathbb{Q} \tag{9}$$

Please notice that when $n \in \mathbb{N}$ (n is a natural number) $(a+b)^n$ have n terms. But in the case which $n \in \mathbb{Q}$ (n is a rational number) the binomial $(a+b)^n$ would have infinite terms.

Problem 1. *Write the first three terms in the expansion of $(2+x)^{-3}$*

Solution:

$$(2+x)^{-3} = 2^{-3}(1+\frac{x}{x})^{-3}$$

$$= (\frac{1}{8})(1 + (-3)\frac{x}{2} + \binom{-3}{2}(\frac{x}{2})^2 + ...)$$

$$= \frac{1}{8}(1 - \frac{3}{2}x + \frac{12}{4}x^2 + ...)$$

$$= \frac{1}{8} - \frac{3}{16}x + \frac{3}{8}x^2$$

Using the theorem 2.2 to change $(2+x)^{-3}$ into another form.

To calculate the $\binom{-3}{r}$ you can use GDC, or I will discuss this later in the section of Counting theorem.

2.2   Counting: permutation and combination

In mathematics, a combination is a selection of items from a set that has distinct members, such that the order of selection does not matter (unlike permutations).[2]

[2]

Definition 2.2. *Combination, choose r from n is :*

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad (10)$$

$$n! = n(n-1)(n-2)(n-3)...3 \cdot 2 \cdot 1 \qquad (11)$$

A permutation of a set is, an arrangement of its members into a sequence or linear order, or if the set is already ordered, a rearrangement of its elements.

The number of ways of arranging n distinct objects in a row is $n!$.

In permutations, the order matters. In combinations, the order does not matter.

Definition 2.3.   *The permutation(Arrangement) of r objects out of n objects is:*

$$_nP_r = \frac{n!}{(n-r)!} \qquad (12)$$

3   Exponents and Logarithms

3.1   Basic concepts

The formula of exponents are listed below. In the exam these are the most important formula, therefore you should not forget them.

Theroem 3.1.
$$a^n \cdot a^m = a^{n+m}$$
$$a^n \div a^m = a^{n-m}$$
$$(a^m)^n = a^{mn}$$

$$a^0 = 1$$
$$a^{-m} = \left(\frac{1}{a}\right)^m = \frac{1}{a^m}$$
$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The combine of square root and exp are like this $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$.

Tip: When solving exponential equations, convert them to the same base.

3.2   Logarithms

Logarithms are often defined as the inverse function of exponents, and in IBDP program logarithms are a very common topic. Hence you should remember all the concepts below.

Proof. To proof formula (13) Let $\log_a x = p$, $\log_a y = q$, as a result $a^p = x$, $a^q = y$, hence $a^p \times a^q = a^{p+q} = xy$. Therefore $\log_a xy = p + q = \log_a x + \log_a y$ □

Definition 3.1. *The function $a = b^x$ can be written as $x = \log_b a$*

Proof. To proof the formula (14)Let $\log_a x = p$, $\log_a y = q$, as a result $a^p = x$, $a^q = y$, hence $a^p \div a^q = a^{p-q} = \frac{x}{y}$. Therefore $\log_a \frac{x}{y} = p - q = \log_a x - \log_a y$ □

The rules of logarithms:

Note that if there are no base number in log function for example $\log 5$ is as same as $\log_1 05$

Theroem 3.2.
$$\log_a x^n = n \log_a x$$
$$\log_a 1 = 0$$
$$\log_a a = 1$$
$$-\log_a x = \log_a \frac{1}{x}$$
$$\log_a x = \frac{\log_b x}{\log_b a}$$
$$\log_a b = \frac{1}{\log_b a}$$
$$\log_a x + \log_a y = \log_a xy \tag{13}$$
$$\log_a x - \log_a y = \log_a \frac{x}{y} \tag{14}$$

4   Function

4.1   Transformation of Function

The transformation vector of $y = f(x + a)$ is $\begin{pmatrix} -a \\ 0 \end{pmatrix}$, The coordinates will be $(x, y) \to (x - a, y)$.

The transformation of function means how the function is moved in graph and how can we use method in algebra to represent it.

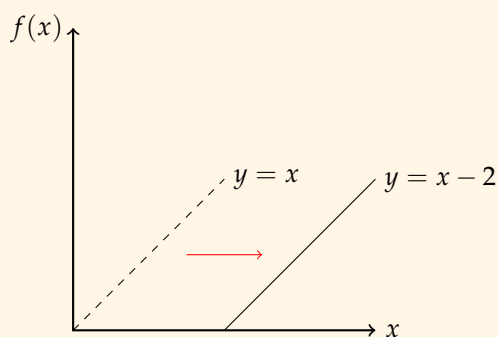**Theroem 4.1.** *Horizontal transformation* of $y = f(x)$ can be rep-

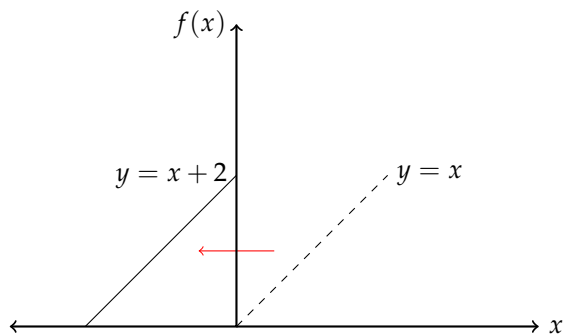*resent in a general form which:*

$$y = f(x + a) \tag{15}$$

*where **a** is a constant. When a is negative the graph will shift to right, when a is positive the graph will shift to left.*

Problem 2. *Sketch the graph of Function $y = x$ shift 2 units to right $y = x - 2$.*
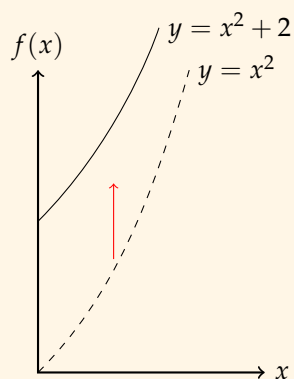

Similarly if $y = x$ is shift to left the function will be $y = x + 2$.

Theroem 4.2. ***Vertical Transformation** of function $y = f(x)$ can be represent by $y = f(x) + b$*

The transformation vector of $y = f(x + a)$ is $\binom{0}{b}$, The coordinates will be $(x, y) \rightarrow (x, y + b)$

Problem 3. *Sketch the graph of $y = x^2$ moves 2 units upwards :*
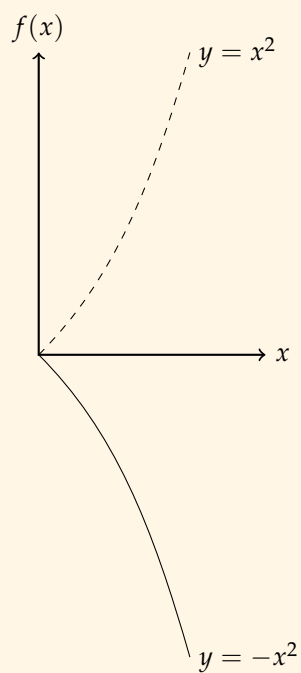
4.2   Reflection of Function

As we know reflection needs a line. In the Cartesian coordinate system we can simply divide it into 3 situations:

1. Reflection on $x - axis$
2. Reflection on $y - axis$
3. Reflection on $f(x)$

**Theroem 4.3.** *Function $y = f(x)$ reflection on x-axis can be represent by $y = -f(x)$*

**Problem 4.** *Sketch the graph of function $y = x^2$ and the reflection on $x - axis \rightarrow y = -x^2$*

It is easy to see that the reflection and translation will not change the shape of the function but the position of it.

Theroem 4.4. *The **reflection** of function $y = f(x)$ on $y - axis$ can be represent by $y = f(-x)$*

Problem 5. *Sketch the graph of the function $y = x^3 + x$ reflection on the y-axis $y = (-x)^3 + (-x)$.*
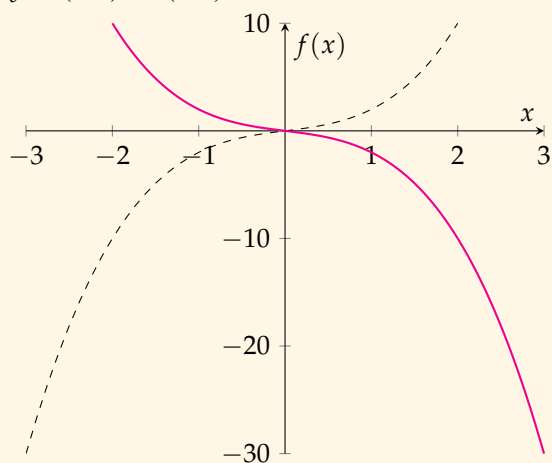
Theroem 4.5. *The reflection of function* $y = f(x)$ *on* $y = x$ *can be represent by* $y = f^{-1}(x)$.

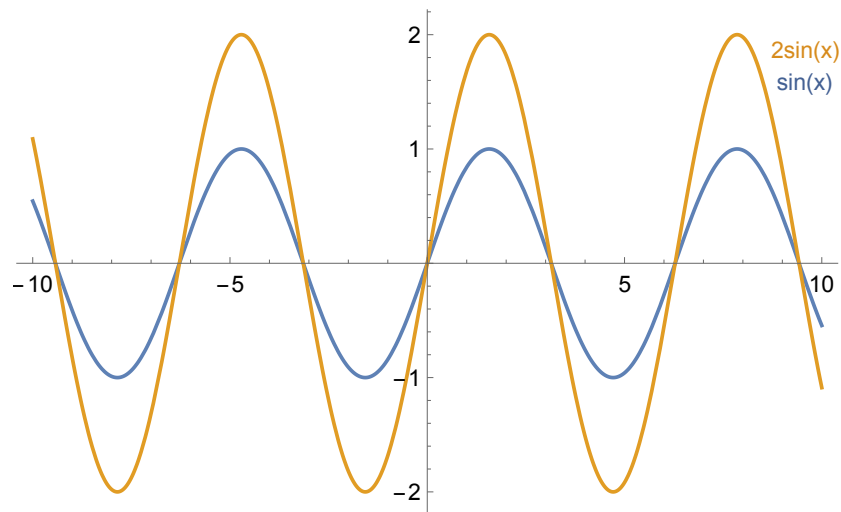**Problem 6.** *Sketch the graph of* $y = x^2$ *reflection on* $y = x$ *which are*
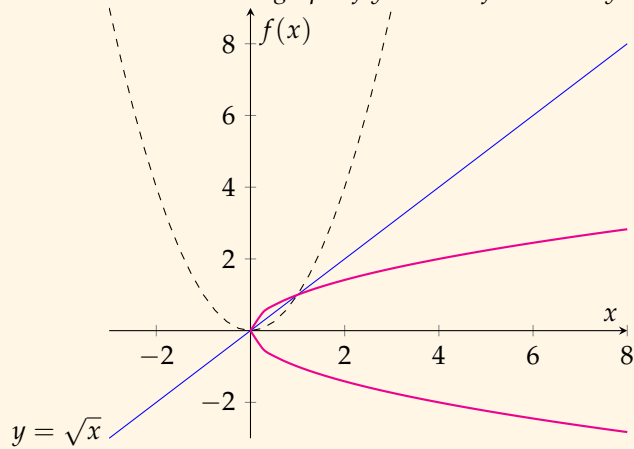
Stretch of Function

The stretch of function means elongate and compress a function. Which is another transformation of function. Let me give an example of this operate.

Note that the stretch of a function will change the shape of the function.

Vertical Stretch
The vertical stretch of a function is like:

Hence we notice the coefficient $f(x)$ is 2. Which means that the

value of y will be multiplied by 2.

Theroem 4.6. *Vertical stretch of a function is:*
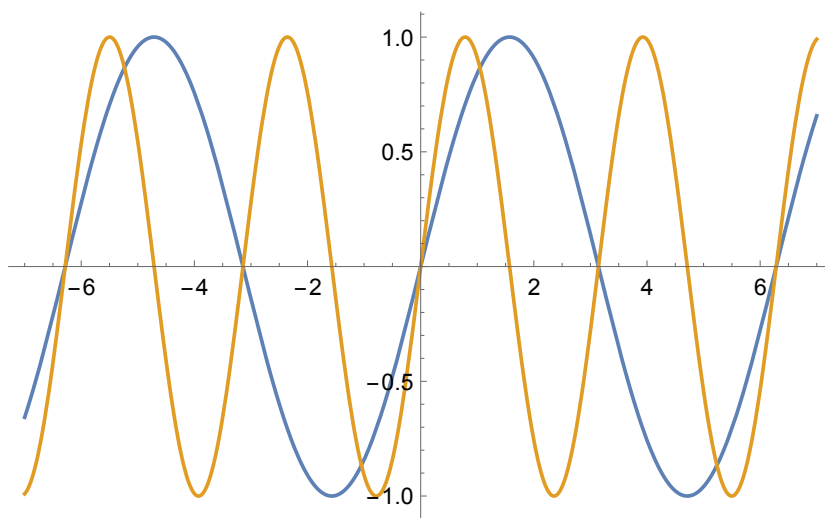
$$a \cdot f(x) \tag{16}$$

Stretch can also be understood by
"every point in the graph are multiplied"

Corollary 4.1. *The points on the graph will change:*

$$(x, y) \Rightarrow (x, a \cdot y)$$

Horizontally Stretch

The horizontally stretch of a function will be:

In this graph we can see that the distance of each point in the
$sin(x)$ is closer to each other.

Theroem 4.7. *The Horizontally Stretch is :*

$$f(b \cdot x) \tag{17}$$

Corollary 4.2. *The points on the graph will change :*

$$(x, y) \Rightarrow (\frac{1}{b}x, y)$$

4.4 *Reciprocal Function*

Reciprocal Function is a special operator which turn $f(x)$ into $\frac{1}{f(x)}$ or
$[f(x)]^{-1}$

4.5   *Rational Function*

> **Definition 4.1.** *Rational function is* $y = \frac{1}{x}$

4.6   *Further Rational function*

> **Theroem 4.8.** *There three types of rational functions*
>
> - $f(x) = \frac{ax+b}{cx+d} \Rightarrow \frac{linear}{linear}$
>
> - $f(x) = \frac{ax+b}{cx^2+dx+e} \Rightarrow \frac{linear}{quadratic}$
>
> - $f(x) = \frac{ax^2+bx+c}{dx+e} \Rightarrow \frac{quadratic}{linear}$

For the type 1 we have learned it, hence we will only discuss the type 2 and type 3.In the case while we are discussing the rational function we have to talk about the horizontal asymptotes and vertical asymptotes.

Let's first give an example of type 2:

> **Problem 7.** *Draw the graph of* $\frac{x-1}{(x+3)(x-2)}$
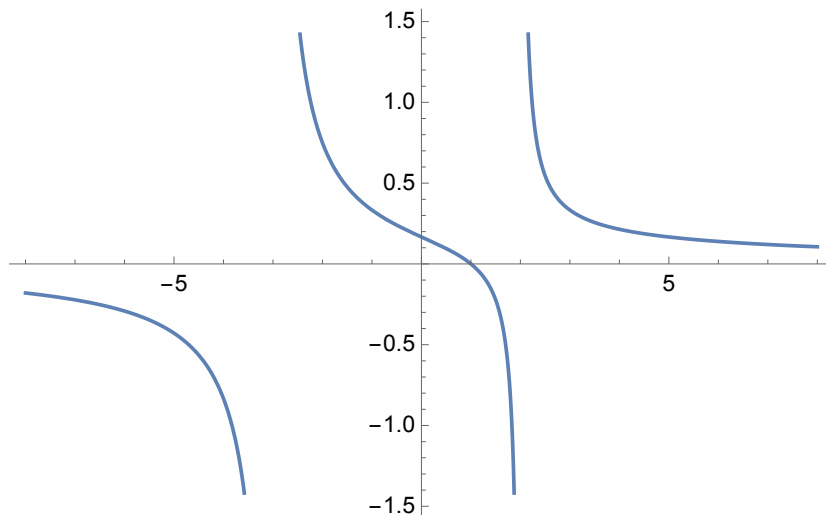
5   *Further Function*

5.1   *Even and Odd function*

In mathematics, even functions and odd functions are functions which satisfy particular symmetry relations, with respect to taking
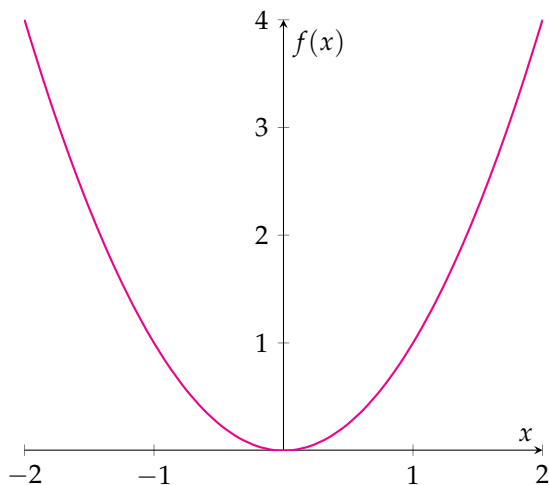
additive inverses.

Definition 5.1. *A function $f(x)$ is Even Function if it satisfy:*
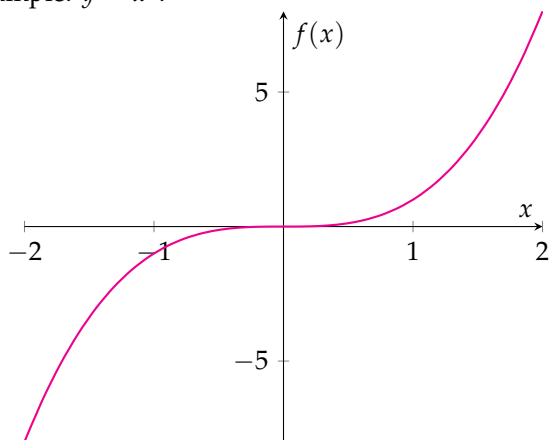
$$f(-x) = f(x) \tag{18}$$

In this case we can know that the even function is symmetrical about $y - axis$. For example: $y = x^2$.

Definition 5.2. *A function is an Odd Function if it satisfy:*

$$f(-x) = -f(x) \tag{19}$$

The graph of odd function is not symmetry about $y - axis$. For example: $y = x^3$.

Theroem 5.1. *The addition of two odd function is an even function.*

5.2   *The graph of $y = [f(x)]^2$*

The section we will talk about the square of a function.

$f(x)$		cell3
cell4	cell5	cell6
cell7	cell8	cell9
cell7	cell8	cell9
cell7	cell8	cell9
cell7	cell8	cell9
cell7	cell8	cell9

5.3   *Absolute value function*

We know that the **Absolute value** of any number is positive. For example the absolute value of -5 is 5. Therefore we give out the definition of absolute value.

Definition 5.3. *The absolute value of x is:*

$$|x| = \begin{cases} x & if \ x \geq 0 \\ -x & if \ x < 0 \end{cases} \tag{20}$$
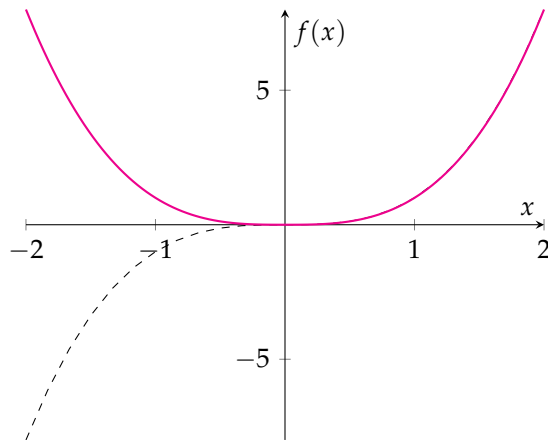
Hence we are going to find out the absolute value of function we need first to see that how the points on the graph changes.

For function $|f(x)|$ The *a, b* are both positive value.

$$(a, b) \Rightarrow (a, b)$$

$$(-a, b) \Rightarrow (-a, b)$$

$$(a, -b) \Rightarrow (a, b)$$

$$(-a, -b) \Rightarrow (-a, b)$$

Therefore we can see that all the points on the function $|f(x)|$ which have negative y value will change to positive value. So the graph which under $y - axis$ of $f(x)$ will reflection.

The graph of $|x^3|$

In this case we have find out the graph of function $|f(x)|$. In the next we are trying to find out the graph of $f(|x|)$. Using the same method we can list the points of $f(|x|)$.

$$(a,b) \Rightarrow (a,b),(-a,b)$$

$$(-a,b) \Rightarrow \textit{Does not exist}$$

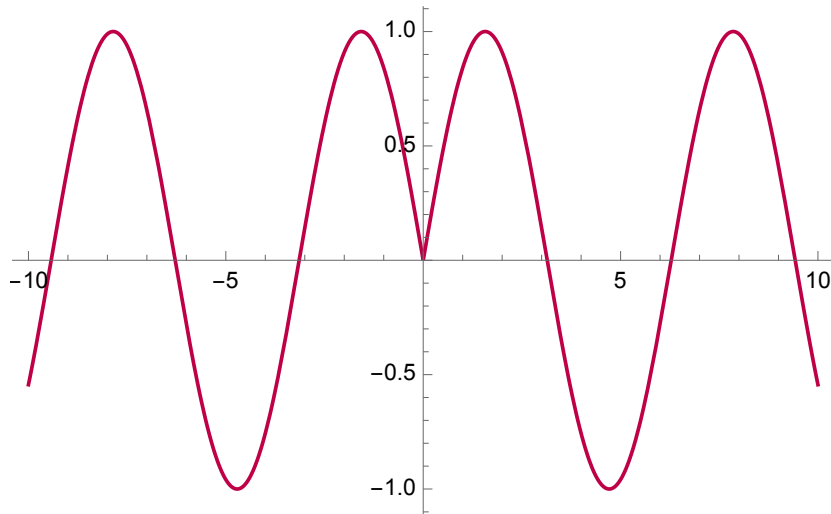$$(a,-b) \Rightarrow (a,-b),(-a,-b)$$

$$(-a,-b) \Rightarrow \textit{Does not exist}$$

Therefore we can say that all the graph of the left side of $y-axis$ of $f(x)$ does not exist in the graph of $f(|x|)$. And the points at the right side of $f(x)$ will be reflected by the $y-axis$. Although it seems different with $|f(x)|$. But do not worry I will give you few examples.
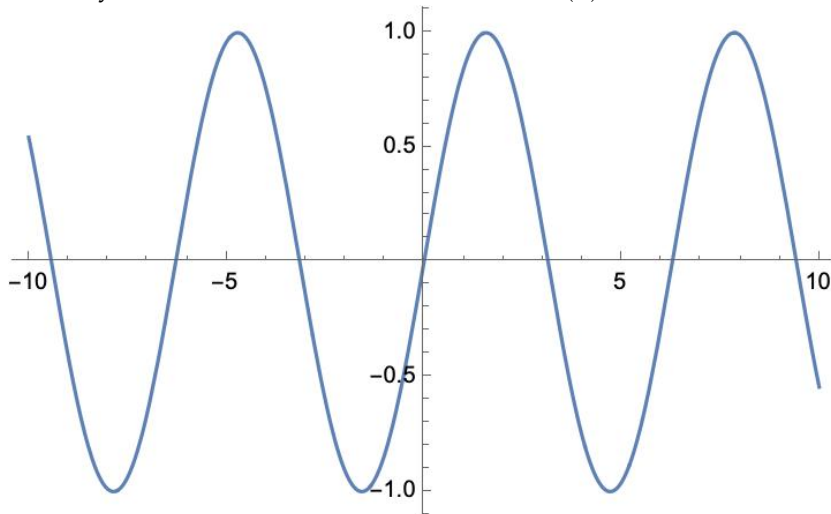
Problem 8. *Draw the graph of $f(|x|)$ where $f(x) = \sin(x)$*

Hence we first draw the function of $\sin(x)$

The graph of $f(|x|) = \sin|x|$

It is easy to find out the difference between $\sin(x)$

The graph of $f(x) = \sin(x)$

- All the points in the left side of the $y-axis$ disappeared.

- The graph at the right side of the $y-axis$ have reflect to the left side.
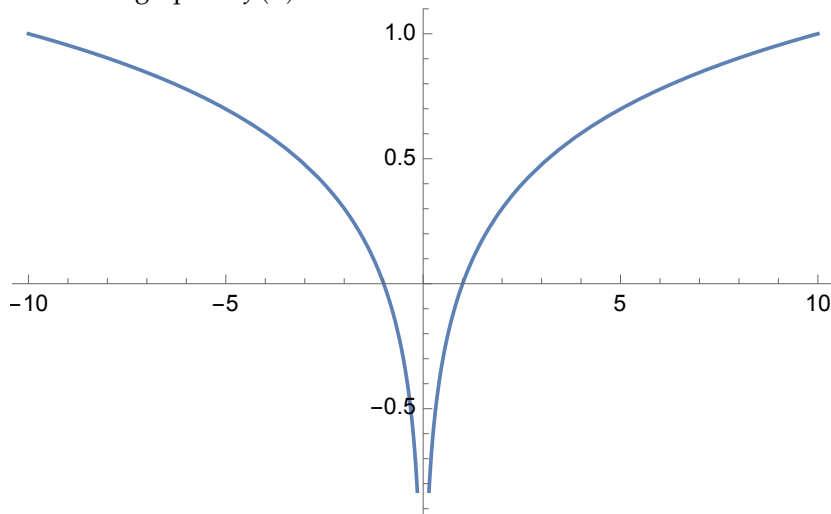
Let's give more examples.

You can only use two step: 1. Erase the left side of $f(x)$. 2. Reflect the right side to left side. Where left side and right side means the two side of $y-axis$.

Problem 9. *Draw the graph of $f(|x|)$ where $f(x) = \log(x)$*
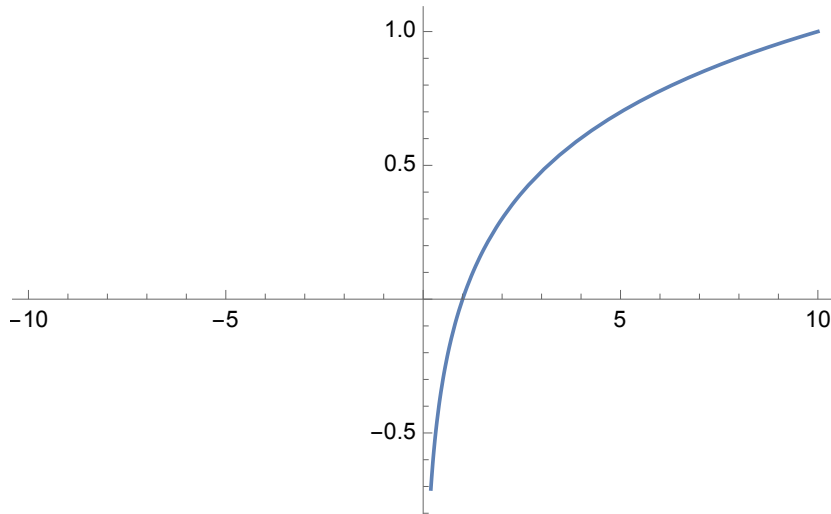
The graph of $f(x) = \log x$

We draw the graph of $f(x)$ first.

The graph of $f(x) = \log |x|$

5.4   *Modulus function*

In FP we have learned that the modulus function is a function like $|f(x)|$. Therefore let's review the concept modulus(absolute value)

Definition 5.4. *The absolute value of x is:*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{21}$$

But how do we solve the equation which contain modulus?

Theroem 5.2. *The*

- $|x| \geq 0$
- $|-x| = |x|$
- $|x|^2 = x^2$
- $|x \cdot y| = |x| \cdot |y|$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

Corollary 5.1. • $|x| = a \Rightarrow x = +_- a$

- $|x| = |b| \Rightarrow x = +_- b$
- $|x| = |a| \Rightarrow x^2 = a^2$
- $|x| \leq a \Rightarrow -a \leq x \leq a$
- $|x| \geq a \Rightarrow x \leq -a \text{ and } x \geq a$

6 Geometry

6.1 Space geometry

7 Complex number

Definition 7.1. *The complex number is numbers:*

$$c = a + ib \tag{22}$$

8   Polynomials

8.1   Factor theorem

Theroem 8.1. *If a polynomial has a factor $(x - a)$ then when $x = a$ the $P(x) = 0$.*

Problem 10. *Find a given that:*

- *a) $(x - 1)$ is a factor of $2x^3 - 3x^2 + ax - 4$*

Solution is: bring $x = 1$ into, $P(1) = 0$ hence we can find out the value of a.