IBDP Mathematics

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Pure mathematics is, in its way, the poetry of logical ideas. — Albert Einstein, German theoretical physicist. This revision book is to the students of IBDP and if there are something to replenish please contact me by My email (click on "My email")

Contents

1 Sequence and Series

1.1 Arithmetic Sequence

Definition 1.1. *Arithmetic Sequence/Arithmetic Progression (AP) is defined as:*

$$u_n = u_1 + (n-1)d \tag{1}$$

u_n is the general term
u_1 is the first term
n is the number of the term
d is is the common difference.

Note that the u_n in the sum equation of Arithmetic Sequence can be written as $u_1 + (n-1)d$ Thus that the equation can also be written as$S_n = \frac{1}{2}[2u_1 + (n-1)d]$

Theroem 1.1. *The sum of Arithmetic Sequence is :*

$$S_n = \frac{n(u_n + u_1)}{2} \tag{2}$$

Please notice till now we do not learn the Infinite sum equation for Arithmetic Sequence yet.

1.2 Geometric Sequence

Definition 1.2. *Geometric Sequence is defined as :*

$$u_n = u_1 \times (r)^{n-1} \tag{3}$$

where r is the common ratio of this sequence.

In Geometric Sequence there are some theorem which I will list below.

Corollary 1.1. *For the common ratio r it follows the rule:*

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \ldots \tag{4}$$

Theroem 1.2. *The sum of finite Geometric Sequence is :*

$$S_n = \frac{u_1(1-r^n)}{1-r} \tag{5}$$

If r > 1, the sequence is an exponential growth.
If 0 < r < 1, the sequence has an exponential decay.

For the limitation of r , -1 < r < 1 can also be written as r<|1|

Theroem 1.3. *When the $n = \infty$ the sum of Geometric sequence is:*

$$S_\infty = \frac{u_1}{1-r} \; (-1 < r < 1) \tag{6}$$

2 Counting and Binomial Theorem

2.1 Binomial Theorem

In algebra, a binomial is a polynomial that is the sum of two terms, each of which is a monomial. It is the simplest kind of sparse polynomial after the monomials.[1]

[1]

Definition 2.1. *A Binomial is a polynomial which satisfy :*

$$(a+b)^n \tag{7}$$

The method we use to expand the binomial is using the binomial theorem.

The meaning of $\binom{n}{r}$ is the combination.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

.

Theroem 2.1.

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + ... + b^n \quad n \in \mathbb{N} \tag{8}$$

The independent term: The term do not involve x in it since the independent term does not vary as x varies. (constant term)

Using Theorem 2.2 in the case of $(a+x)^n$ which a is not equal to 1

Theroem 2.2.

$$(a+b)^n = a^n (1 + \frac{b}{a})^n \;,\; |\frac{b}{a}| < 1 \;,\; n \in \mathbb{Q} \tag{9}$$

Please notice that when $n \in \mathbb{N}$ (n is a natural number) $(a+b)^n$ have n terms. But in the case which $n \in \mathbb{Q}$ (n is a rational number) the binomial $(a+b)^n$ would have infinite terms.

Problem 1. *Write the first three terms in the expansion of* $(2+x)^{-3}$

Solution:

Using the theorem 2.2 to change $(2+x)^{-3}$ into another form.

$$(2+x)^{-3} = 2^{-3}(1+\frac{x}{x})^{-3}$$

$$= (\frac{1}{8})(1 + (-3)\frac{x}{2} + \binom{-3}{2}(\frac{x}{2})^2 + ...)$$

To calculate the $\binom{-3}{r}$ you can use GDC, or I will discuss this later in the section of Counting theorem.

$$= \frac{1}{8}(1 - \frac{3}{2}x + \frac{12}{4}x^2 + ...)$$

$$= \frac{1}{8} - \frac{3}{16}x + \frac{3}{8}x^2$$

2.2 Counting: permutation and combination

In mathematics, a combination is a selection of items from a set that has distinct members, such that the order of selection does not matter (unlike permutations).[2]

[2]

Definition 2.2. *Combination, choose r from n is :*

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{10}$$

$$n! = n(n-1)(n-2)(n-3)...3 \cdot 2 \cdot 1 \tag{11}$$

The number of ways of arranging n distinct objects in a row is $n!$.

A permutation of a set is, an arrangement of its members into a sequence or linear order, or if the set is already ordered, a rearrangement of its elements.

In permutations, the order matters.In combinations, the order does not matter.

Definition 2.3. *The permutation(Arrangement) of r objects out of n objects is:*

$$_nP_r = \frac{n!}{(n-r)!} \tag{12}$$

3 Exponents and Logarithms

3.1 Basic concepts

The formula of exponents are listed below. In the exam these are the most important formula, therefore you should not forget them.

Theroem 3.1.

$$a^n \cdot a^m = a^{n+m}$$

$$a^n \div a^m = a^{n-m}$$

$$(a^m)^n = a^{mn}$$

$$a^0 = 1$$

$$a^{-m} = (\frac{1}{a})^m = \frac{1}{a^m}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The combine of square root and exp are like this $a^{\frac{m}{n}=(\sqrt[n]{a})^m}$.

Tip: When solving exponential equations, convert them to the same base.

3.2 Logarithms

Logarithms are often defined as the inverse function of exponents, and in IBDP program logarithms are a very common topic. Hence you should remember all the concepts below.

Proof. To proof formula (13) Let $\log_a x = p$, $\log_a y = q$, as a result $a^p = x$, $a^q = y$, hence $a^p \times a^q = a^{p+q} = xy$. Therefore $\log_a xy = p + q = \log_a x + \log_a y$ □

Proof. To proof the formula (14)Let $\log_a x = p$, $\log_a y = q$, as a result $a^p = x$, $a^q = y$, hence $a^p \div a^q = a^{p-q} = \frac{x}{y}$. Therefore $\log_a \frac{x}{y} = p - q = \log_a x - \log_a y$ □

Note that if there are no base number in log function for example $\log 5$ is as same as $\log_1 05$

Definition 3.1. *The function $a = b^x$ can be written as $x = \log_b a$*

The rules of logarithms:

Theroem 3.2.

$$\log_a x^n = n \log_a x$$

$$\log_a 1 = 0$$

$$\log_a a = 1$$

$$-\log_a x = \log_a \frac{1}{x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$\log_a b = \frac{1}{\log_b a}$$

$$\log_a x + \log_a y = \log_a xy \tag{13}$$

$$\log_a x - \log_a y = \log_a \frac{x}{y} \tag{14}$$

4 Function

4.1 Transformation of Function

The transformation vector of $y = f(x+a)$ is $\binom{-a}{0}$, The coordinates will be $(x, y) \to (x - a, y)$.

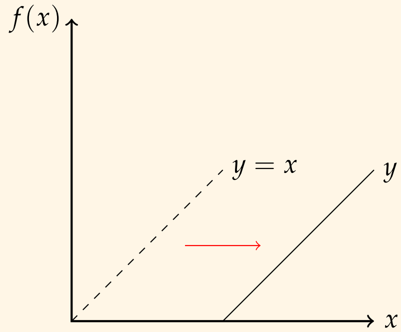
The transformation of function means how the function is moved in graph and how can we use method in algebra to represent it.

Theroem 4.1. ***Horizontal transformation*** *of $y = f(x)$ can be represent in a general form which:*

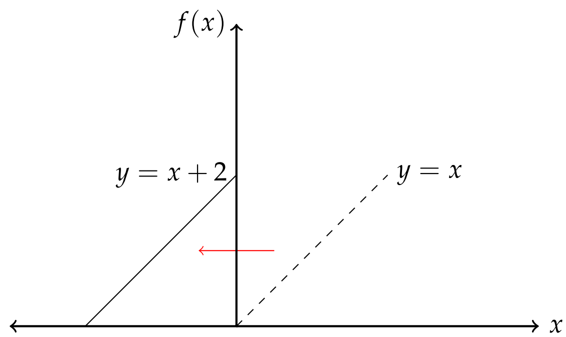
$$y = f(x+a) \tag{15}$$

*where **a** is a constant. When a is negative the graph will shift to right, when a is positive the graph will shift to left.*

Problem 2. *Sketch the graph of Function $y = x$ shift 2 units to right $y = x - 2$.*



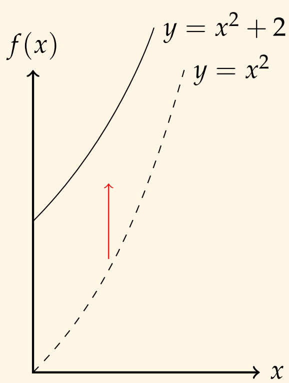
Similarly if $y = x$ is shift to left the function will be $y = x + 2$.



The transformation vector of $y = f(x + a)$ is $\binom{0}{b}$, The coordinates will be $(x, y) \to (x, y + b)$

Theroem 4.2. ***Vertical Transformation*** *of function $y = f(x)$ can be represent by $y = f(x) + b$*

Problem 3. *Sketch the graph of $y = x^2$ moves 2 units upwards :*



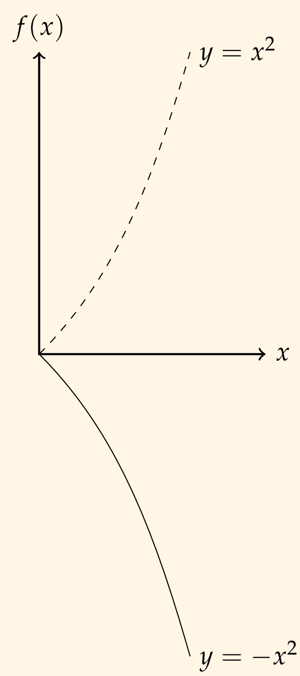
4.2 *Reflection of Function*

As we know reflection needs a line. In the Cartesian coordinate system we can simply divide it into 3 situations:

1. Reflection on $x - axis$
2. Reflection on $y - axis$
3. Reflection on $f(x)$

Theroem 4.3. *Function $y = f(x)$ reflection on x-axis can be represent by $y = -f(x)$*

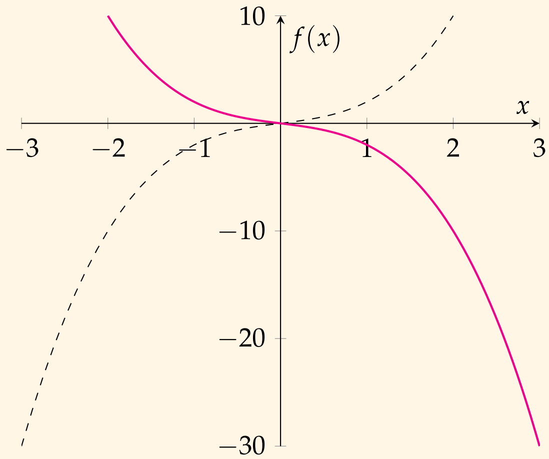
Problem 4. *Sketch the graph of function $y = x^2$ and the reflection on $x - axis \to y = -x^2$*



It is easy to see that the reflection and translation will not change the shape of the function but the position of it.

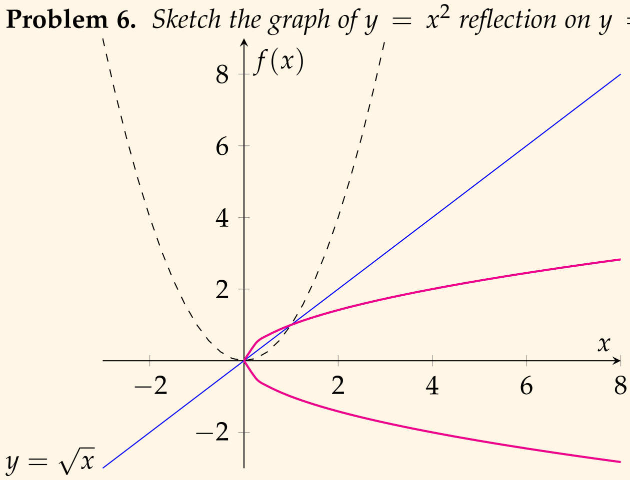
Theroem 4.4. *The **reflection** of function $y = f(x)$ on $y - axis$ can be represent by $y = f(-x)$*

Problem 5. *Sketch the graph of the function $y = x^3 + x$ reflection on the y-axis $y = (-x)^3 + (-x)$.*



Theroem 4.5. *The reflection of function $y = f(x)$ on $y = x$ can be represent by $y = f^{-1}(x)$.*

Problem 6. *Sketch the graph of $y = x^2$ reflection on $y = x$ which are*

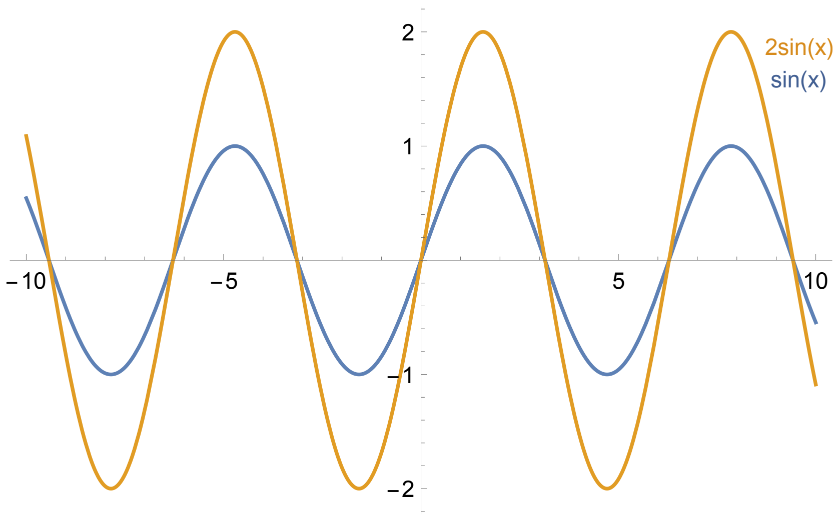


4.3 *Stretch of Function*

The stretch of function means elongate and compress a function. Which is another transformation of function. Let me give an example of this operate.

Note that the stretch of a function will change the shape of the function.

Vertical Stretch
The vertical stretch of a function is like:



Hence we notice the coefficient $f(x)$ is 2. Which means that the value of y will be multiplied by 2.

Theroem 4.6. *Vertical stretch of a function is:*

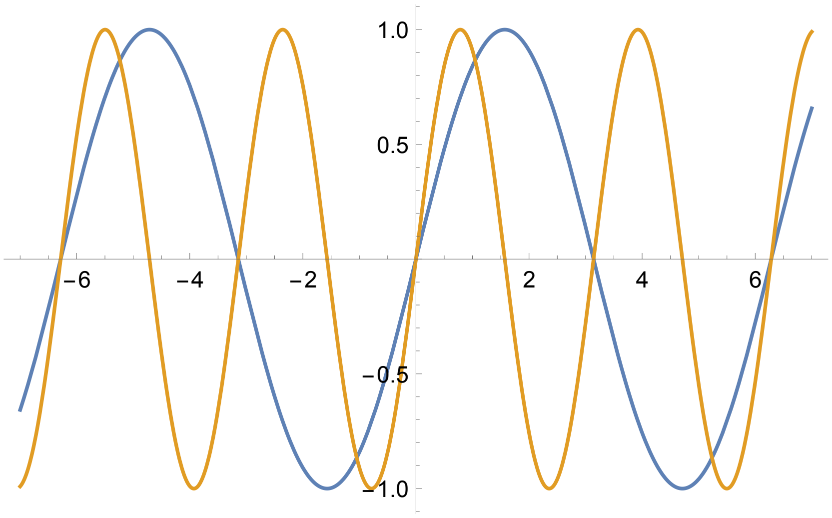
$$a \cdot f(x) \tag{16}$$

Stretch can also be understood by *"every point in the graph are multiplied"*

Corollary 4.1. *The points on the graph will change:*

$$(x, y) \Rightarrow (x, a \cdot y)$$

Horizontally Stretch
The horizontally stretch of a function will be:



In this graph we can see that the distance of each point in the $sin(x)$ is closer to each other.

Theroem 4.7. *The Horizontally Stretch is :*

$$f(b \cdot x) \tag{17}$$

Corollary 4.2. *The points on the graph will change :*

$$(x, y) \Rightarrow (x, \frac{1}{b}y)$$

4.4 Reciprocal Function

4.5 Rational Function

Definition 4.1. *Rational function is* $y = \frac{1}{x}$

5 Further Function

5.1 Even and Odd function

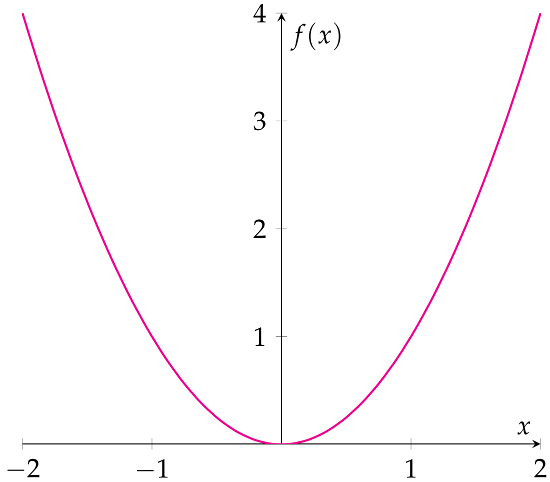
In mathematics, even functions and odd functions are functions which satisfy particular symmetry relations, with respect to taking additive inverses.

Note that there are some functions is neither odd or even, such as $y = \sqrt{x}$

Definition 5.1. *A function* $f(x)$ *is Even Function if it satisfy:*

$$f(-x) = f(x) \tag{18}$$

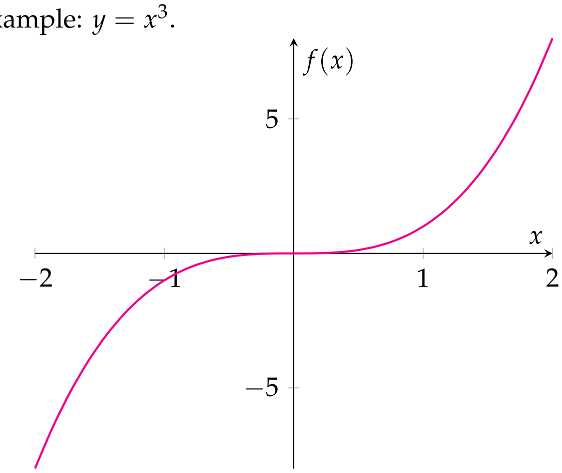
In this case we can know that the even function is symmetrical about $y - axis$. For example: $y = x^2$.



Definition 5.2. *A function is an Odd Function if it satisfy:*

$$f(-x) = -f(x) \tag{19}$$

The graph of odd function is not symmetry about $y - axis$. For example: $y = x^3$.



Theroem 5.1. *The addition of two odd function is an even function.*

5.2 *The graph of $y = [f(x)]^2$*

The section we will talk about the square of a function.

$f(x)$		cell3
cell4	cell5	cell6
cell7	cell8	cell9
cell7	cell8	cell9
cell7	cell8	cell9
cell7	cell8	cell9
cell7	cell8	cell9

5.3 *Absolute value function*

We know that the **Absolute value** of any number is positive. For example the absolute value of -5 is 5. Therefore we give out the definition of absolute value.

Definition 5.3. *The absolute value of x is:*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{20}$$

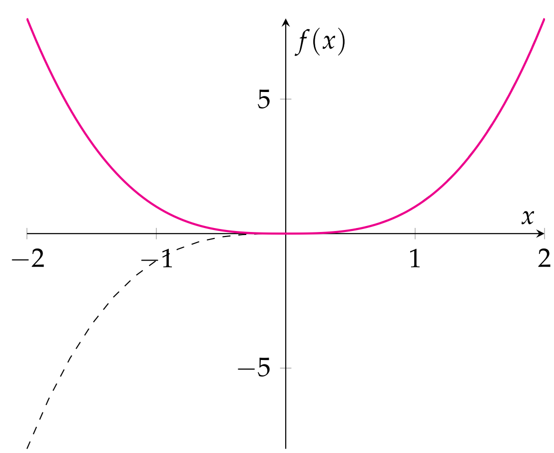
Hence we are going to find out the absolute value of function we need first to see that how the points on the graph changes.

For function $|f(x)|$

The a, b are both positive value.

$$(a,b) \Rightarrow (a,b)$$
$$(-a,b) \Rightarrow (-a,b)$$
$$(a,-b) \Rightarrow (a,b)$$
$$(-a,-b) \Rightarrow (-a,b)$$

Therefore we can see that all the points on the function $|f(x)|$ which have negative y value will change to positive value. So the graph which under $y-axis$ of $f(x)$ will reflection.



The graph of $|x^3|$

In this case we have find out the graph of function $|f(x)|$. In the next we are trying to find out the graph of $f(|x|)$. Using the same method we can list the points of $f(|x|)$.

$$(a,b) \Rightarrow (a,b),(-a,b)$$

$$(-a,b) \Rightarrow \textit{Does not exist}$$

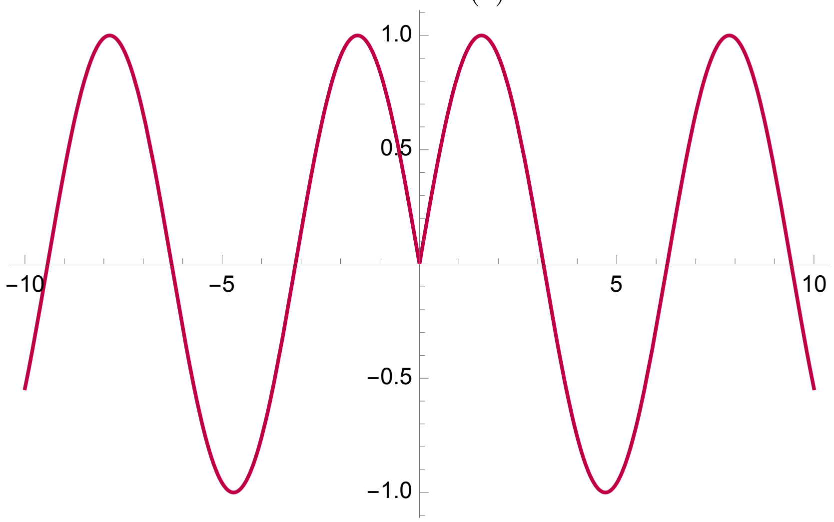
$$(a,-b) \Rightarrow (a,-b),(-a,-b)$$

$$(-a,-b) \Rightarrow \textit{Does not exist}$$

Therefore we can say that all the graph of the left side of $y - axis$ of $f(x)$ does not exist in the graph of $f(|x|)$. And the points at the right side of $f(x)$ will be reflected by the $y - axis$. Although it seems different with $|f(x)|$. But do not worry I will give you few examples.

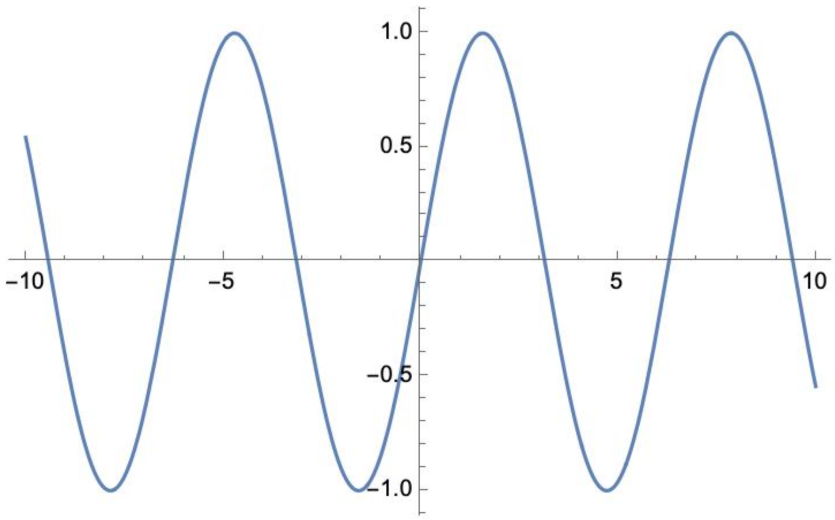
Problem 7. *Draw the graph of $f(|x|)$ where $f(x) = \sin(x)$*

Hence we first draw the function of $\sin(x)$



The graph of $f(|x|) = \sin|x|$

It is easy to find out the difference between $sin(x)$



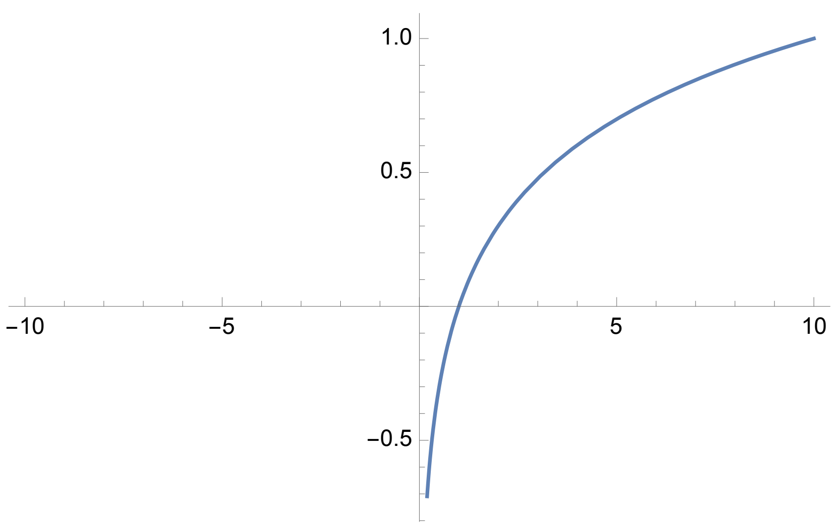
The graph of $f(x) = \sin(x)$

- All the points in the left side of the $y - axis$ disappeared.
- The graph at the right side of the $y - axis$ have reflect to the left side.

You can only use two step: 1. Erase the left side of $f(x)$. 2. Reflect the right side to left side. Where left side and right side means the two side of $y - axis$.

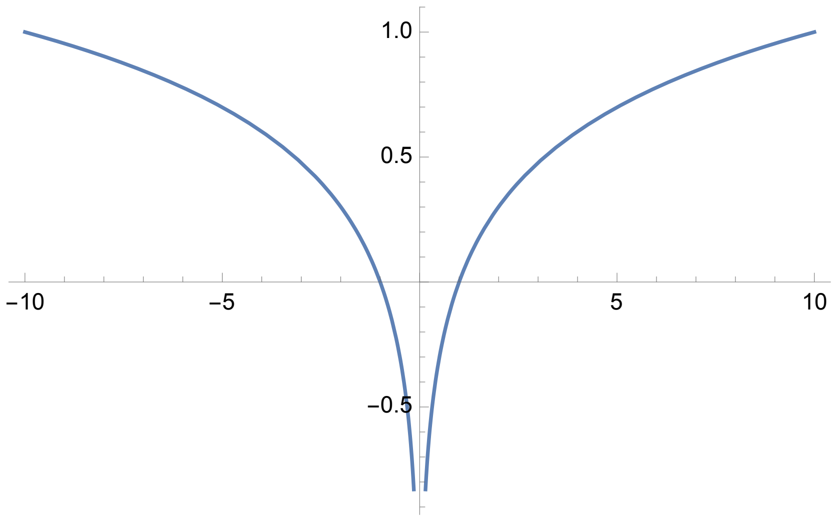
Let's give more examples.

Problem 8. *Draw the graph of* $f(|x|)$ *where* $f(x) = log(x)$



The graph of $f(x) = \log x$

We draw the graph of $f(x)$ first.



The graph of $f(x) = \log|x|$

5.4 *Modulus function*

In FP we have learned that the modulus function is a function like $|f(x)|$. Therefore let's review the concept modulus(absolute value)

Definition 5.4. *The absolute value of x is:*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{21}$$

But how do we solve the equation which contain modulus?

Theroem 5.2. *The*

- $|x| \geq 0$
- $|-x| = |x|$
- $|x|^2 = x^2$
- $|x \cdot y| = |x| \cdot |y|$
- $|\frac{x}{y}| = \frac{|x|}{|y|}$

Corollary 5.1.
- $|x| = a \Rightarrow x = +_-a$
- $|x| = |b| \Rightarrow x = +_-b$
- $|x| = |a| \Rightarrow x^2 = a^2$

- $|x| \leq a \Rightarrow -a \leq x \leq a$
- $|x| \geq a \Rightarrow x \leq -a \; and \; x \geq a$