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# ANALYSIS

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## Study Note

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# 1 Words before the text

in the world of mathematics that full of abstract terms and concepts. we know that the real number is the foundation of all the mathematics. the real number is a set that contains all the rational numbers and irrational numbers. the real number is a complete set, which means that it is a set that contains all the limit points of all the sequences. it is important because it is the foundation of all the mathematics. however, the real number is not enough for us to do the analysis. we need to define the limit, the continuity, the derivative, the integral, and so on. Hence the basic structure of this note will be:

1. Define the natural number, the integer, the rational number, and the real number. Explore the properties of them.
2. Define the limit, the continuity, the derivative, the integral, and so on.

## 2 Pre-knowledge

## 3 Numbers

### 3.1 The Natural Numbers

### 3.2 The Peano Axioms

The natural number consist with set  $\mathbb{N}$ , where a distinguished element  $0 \in \mathbb{N}$ , and a function  $\nu : \mathbb{N} \rightarrow \mathbb{N}^\times := \mathbb{N} \setminus \{0\}$  with following properties:

1.  $\nu$  is injective
2. If a subset  $N$  contains 0 and  $\nu(n) \in N$  for all  $n \in N$  then  $N = \mathbb{N}$

And we denote

## 4 Real analysis

### 4.1 Introduce to the topic

In order to construct real numbers we need to use the concept of 'Limit of rational sequence' This method leads to two fresh new terms

- The first is limit
- The second is sequence

And after introduce the method one shall wonder how to ensure the sequences will represent the real numbers It is crucial to introduce the concept of Cauchy sequence, where we can ensure that the sequence will converge to a real number. Which we will talk about later in detail in the section of sequences. Before we start the important concept of Sequences, we need to know the concept of Supremum and Infimum, which are important in the process of defining the limit and real numbers

## 4.2 Construction of the real numbers

We are going to talk about the construction of real number system from rational number system. We know that the rational number system is not complete, which means that there are some points in the rational number system that cannot be represented by a rational number. For example,  $\sqrt{2}$  is an irrational number, which means that it cannot be represented by a rational number. In order to construct the real number system, we need to add some points to the rational number system. These points are called irrational numbers. The real number system is constructed by adding all the irrational numbers to the rational number system. The real number system is a complete number system, which means that it contains all the limit points of all the sequences. The real number system is the foundation of all the mathematics. It is the most important number system in mathematics. The real number system is a complete number system, which means that it contains all the limit points of all the sequences. The real number system is the foundation of all the mathematics. It is the most important number system in mathematics. The real number system is a complete number system, which means that it contains all the limit points of all the sequences. The real number system is the foundation of all the mathematics. It is the most important number system in mathematics. The real number system is a complete number system, which means that it contains all the limit points of all the sequences. The real number system is the foundation of all the mathematics. It is the most important number system in mathematics. The real number system is a complete number system, which means that it contains all the limit points of all the sequences. The real number system is the foundation of all the mathematics. It is the most important number system in mathematics.

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### 4.2.1 Completeness of Real number

Can you prove the "irrationality property" for the following set of numbers?

We know  $\sqrt{2}$  is irrational. We need to prove that the set  $L = \{x \leq 0\} \cup \{x > 0 : x^2 < 2\}$ , together with the set  $R = \{r > 0 : r^2 > 2\}$ , are disjoint subsets of the rational numbers  $\mathbb{Q}$ , and that for every  $r \in R$  there exists a "gap", i.e., there is no number in  $L$  such that it is one less than any number in  $R$ . Formally, this is described as: for every  $x \in (-\infty, \sqrt{2})$  and  $(\sqrt{2}, +\infty)$ , there is no rational number equal to  $\sqrt{2}$ .

Here are the conditions for a set  $A$  and  $B$  to prove the irrationality property, with  $A$  and  $B$  being nonempty subsets of  $\mathbb{Q}$ , disjoint and open:

1. If  $a \in A$ , then there exists an  $a' \in A$  such that  $a' > a$ .

2. If  $b \in B$ , then there exists a  $b' \in B$  such that  $b' < b$ .
3. If  $a \in A$  and  $b \in B$ , then  $a \leq b$ .
4. There are no largest or smallest elements in  $A$  or  $B$ .

Since  $L = A \cap \mathbb{Q}$  and  $R = B \cap \mathbb{Q}$  are nonempty and consist of all rational numbers less than or equal to  $\sqrt{2}$  and greater than  $\sqrt{2}$  respectively, then according to the property above, we can define the set  $A = (-\infty, \sqrt{2}) \cap \mathbb{Q}$ ,  $B = (\sqrt{2}, +\infty) \cap \mathbb{Q}$ .

Therefore, for  $a \in A$  and  $b \in B$ , we have  $a \leq x < \sqrt{2} < b$ . Finally, since  $A = (-\infty, x)$  and  $B = [x, +\infty)$ .

### 4.3 Sequences

### 4.4 Supremum & Infimum

**Definition 4.1.** For a subset  $M \subseteq \mathbb{R}$ ,  $b \in \mathbb{R}$  is called Upper bound iff:

$$\forall x \in M : x \leq b$$

If it is a lower bound, then  $x \geq b$

In simpler terms, an upper bound is a value that is greater than or equal to every element in the set.

**Definition 4.2.** For a subset  $M \subseteq \mathbb{R}$ ,  $s \in \mathbb{R}$  is called supremum iff:

- $\forall x \in M : x \leq s$
- $\forall \varepsilon > 0, \exists \bar{x} \in M : s - \varepsilon < \bar{x}$

Then we write  $\sup M := s$

**Definition 4.3.** For a subset  $M \subseteq \mathbb{R}$ ,  $l \in \mathbb{R}$  is called infimum iff:

- $\forall x \in M : x \geq l$
- $\forall \varepsilon > 0, \exists \bar{x} \in M : l + \varepsilon > \bar{x}$

Then we write  $\inf M := l$

**remark 4.4.** If  $M$  is not bounded, then we write  $\sup M := \infty, \inf M := -\infty$ .  
If  $M$  is an empty set, then we write  $\sup M := -\infty, \inf M := \infty$

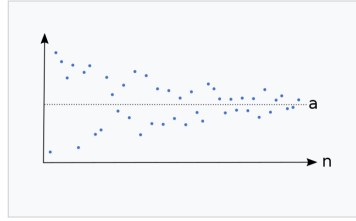
## 4.5 Limits of Sequence

**Definition 4.5.** Limit of Sequence The limit of a sequence  $(a_n)$  is defined as follows:  
 $\forall \varepsilon > 0, \exists N$  such that for all  $n > N$ ,

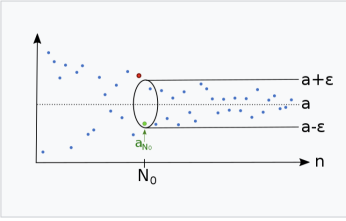
$$|a_n - L| < \varepsilon$$

The limit of the sequence is  $L$ , denoted as  $\lim_{n \rightarrow \infty} a_n = L$ .

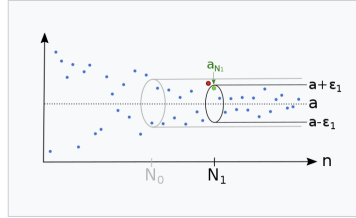
Here is an illustration of this definition:



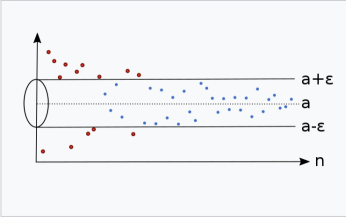
Example of a sequence which converges to the limit  $a$ .



Regardless which  $\varepsilon > 0$  we have, there is an index  $N_0$ , so that the sequence lies afterwards completely in the epsilon tube  $(a - \varepsilon, a + \varepsilon)$ .



There is also for a smaller  $\varepsilon_1 > 0$  an index  $N_1$ , so that the sequence is afterwards inside the epsilon tube  $(a - \varepsilon_1, a + \varepsilon_1)$ .



For each  $\varepsilon > 0$  there are only finitely many sequence members outside the epsilon tube.

If a sequence  $(a_n)$  has a limit, then it is a convergent sequence; otherwise, it is a divergent sequence.

**Theorem 4.6.** The limit of a sequence possesses several important properties, including:

1. The limit of a sequence is unique.
2. If  $a_n \leq b_n$  for all  $n$  greater than some  $N$ , and both sequences have finite limits, then  $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$ .

For further exploration: If  $a_n > 0$  for all  $n > N$ , and the limit of  $a_n$  as  $n$  approaches infinity exists and is finite, then this limit is greater than 0.

## 4.6 Cauchy sequence

**Definition 4.7.** A sequence  $(a_n)_{n \in \mathbb{N}}$  is called Cauchy sequence iff:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N : |a_n - a_m| < \varepsilon$$

In simpler terms, this means that as you move further along in the sequence, the difference between its terms becomes smaller and smaller, eventually becoming as small as you like.

**remark 4.8.** In real numbers, every convergent sequence is a Cauchy sequence, and every Cauchy sequence is convergent. This equivalence is a cornerstone of real analysis.

Cauchy sequence  $\Leftrightarrow$  Convergent sequence

Which is known as completeness axiom.

**Theorem 4.9.** Dedekind completeness:

- If  $M \subseteq \mathbb{R}$  is upper bounded, then  $\sup M \in \mathbb{R}$  exists
- If  $M \subseteq \mathbb{R}$  is lower bounded, then  $\inf M \in \mathbb{R}$  exists

*Proof.* Not yet understand()

□

**Theorem 4.10.** If a sequence is monotonically decreasing and bounded below, then it is a convergent sequence.

## 4.7 Subsequence and Accumulation value

**Definition 4.11.** Let sequence  $(n_k)$  be a strict monotonically increase sequence (that is  $\forall k, n_{k+1} > n_k$ ). Then  $(a_{n_k})_{k \in \mathbb{N}}$  is called a subsequence of  $(a_n)_{n \in \mathbb{N}}$

If the sequence is convergent then every subsequence of it is convergent to the same value.

**Definition 4.12.** Accumulation value of  $(a_n)_{n \in \mathbb{N}}$  is the limit value of subsequence  $(a_{n_k})_{k \in \mathbb{N}}$ .

## 4.8 Bolzano-Weierstrass Theorem

**Theorem 4.13.** If  $(a_n)_{n \in \mathbb{N}}$  is bounded, then it has a accumulation value. In another word, every bounded sequence in real number has a convergent subsequence.



*Proof.* Lets define a new sequence which has lower bound  $c_0$  and upper bound  $d_0$ . Then we can easily divide it into bisection with two same interval.

**Bisection step:** At least one of these subintervals must contain infinitely many terms of the sequence  $(a_n)$ . Label this subinterval as  $[c_1, d_1]$ . Repeat this process for  $[c_1, d_1]$ , dividing it into two and selecting the subinterval, say  $[c_2, d_2]$ , that contains infinitely many terms of  $(a_n)$ . Continue this process indefinitely. In each step, select a subinterval  $[c_k, d_k]$  that contains infinitely many terms of the sequence.

**Limit step** After we get the  $[c_k, d_k]$ , we can see that,  $[c_k, d_k] \subset [c_{k-1}, d_{k-1}] \subset [c_{k-2}, d_{k-2}] \subset \dots \subset [c_0, d_0]$ . Hence,  $d_1 - c_1 = \frac{1}{2}(d_0 - c_0)$ ,  $d_n - c_n = \frac{1}{2^n}(d_0 - c_0)$ . When  $n \rightarrow \infty$ , the difference approach to 0.

**Construction step:** It is obvious that  $(c_n)_{n \in \mathbb{N}}$  is monotonically increase and  $(d_n)_{n \in \mathbb{N}}$  is monotonically decrease. That is, they are convergent. (After prove they are convergent we can apply limit on these sequences) Notice:

$$\lim_{n \rightarrow \infty} (d_n - c_n) = 0 = \lim_{n \rightarrow \infty} (d_n) - \lim_{n \rightarrow \infty} (c_n)$$

**Final step:** Then let's define  $(a_{n_k})_{k \in \mathbb{N}}$  as a new sequence where  $a_{n_k} \in [c_k, d_k]$ . That is:

$$c_k \leq a_k \leq d_k$$

By sandwich theorem we know that  $(a_{n_k})_{k \in \mathbb{N}}$  is convergent. □

## 4.9 Limit superior and inferior

$$\lim_{n \rightarrow \infty} a_n = \infty \Leftrightarrow \text{Divergent to } \infty \Leftrightarrow \forall C > 0, \exists N \in \mathbb{N}, \forall n \in N : a_n > C$$

$$\lim_{n \rightarrow \infty} a_n = -\infty \Leftrightarrow \text{Divergent to } -\infty \Leftrightarrow \forall C < 0, \exists N \in \mathbb{N}, \forall n \in N : a_n < C$$

We call this type of accumulation as improper accumulation. When it does not bounded lower, then it has improper value of  $-\infty$ , when it does not bounded above, then it has improper value of  $\infty$ .

**Definition 4.14.** For sequence  $(a_n)_{n \in \mathbb{N}}$ , An element  $a \in \mathbb{R} \cup \{-\infty, \infty\}$  is called:

- Limit superior of  $(a_n)_{n \in \mathbb{N}}$  if  $a$  is the largest (improper) accumulation value of  $(a_n)_{n \in \mathbb{N}}$ . We write:  $a = \limsup_{n \rightarrow \infty} a_n$
- Limit inferior of  $(a_n)_{n \in \mathbb{N}}$  if  $a$  is the smallest (improper) accumulation value of  $(a_n)_{n \in \mathbb{N}}$ . We write:  $a = \liminf_{n \rightarrow \infty} a_n$

At the same time we can write:

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup\{a_k | k \geq n\}$$

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf\{a_k | k \geq n\}$$

There are several reasons for define the limit superior and limit inferior

- For sequences that do not converge, the limit superior and inferior provide a way to describe their behavior. They are particularly useful in handling oscillating sequences or sequences with multiple limit points.
- Unlike the usual limit, the limit superior and limit inferior always exist for any bounded sequence and for many unbounded sequences.
- If a sequence  $a_n$  does converge, then its limit superior and limit inferior are both equal to this limit. This property is often used in proofs to show convergence.
- These concepts are also used in the convergence tests for series, particularly in understanding the behavior of the terms of a series.

**example 4.15.**

$$a_n = (-1)^n \cdot n = -1, 2, -3, \dots$$

It is obvious that this sequence is not convergent however we can find a subsequence where we can easily get the limit superior and inferior:

$$\limsup_{n \rightarrow \infty} a_n = \infty$$

$$\liminf_{n \rightarrow \infty} a_n = -\infty$$

**Theorem 4.16.** Sequence  $a_n$  is convergent if:

$$\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n \notin \{\infty, -\infty\}$$

At the same time, if  $a_n$  is divergent to  $\infty$  then:

$$\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = \infty$$

Some properties which only exist when the value is defined (Not infinity times zero etc.):

1.  $\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$
2.  $\limsup_{n \rightarrow \infty} (a_n \cdot b_n) \leq \limsup_{n \rightarrow \infty} a_n \cdot \limsup_{n \rightarrow \infty} b_n$

For limit inferior, we just change the direction of inequality.

## 4.10 Series

**Definition 4.17.** Series are sequence  $(S_n)$  which is the summation of a sequence  $a_n$  from initial value to infinity, where we denote it as:

$$S_n = \sum_{i=1}^n a_i, n \in \mathbb{N}$$

If  $S_n$  is convergent then we write:

$$\sum_{i=1}^{\infty} a_i := \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$$

**example 4.18.** Harmonic series are special series which write as:

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

This series seems to be convergent however it is not.

*Proof.* Let  $S_n = \sum_{k=1}^n \frac{1}{k}$  where increase monotonically and we are going to prove that this sequence is not bounded above.  $\square$

Here are some properties if series  $\sum a_k, \sum b_k$  are convergent

1.  $\sum (a_k + b_k) = \sum a_k + \sum b_k$  convergent
2.  $\sum (\lambda a_k) = \lambda \sum a_k$

### 4.10.1 Criterion of Series

**Definition 4.19.** Absolute convergence: A series  $\sum_k^{\infty} a_k$  is called absolute convergent iff  $\sum_k^{\infty} |a_k|$  is convergent. When  $\sum_k^{\infty} a_k$  is convergent and  $\sum_k^{\infty} |a_k|$  is divergent then it is called conditional convergence.

**Theorem 4.20.** Cauchy criterion claims that a series in  $\mathbb{R}$  is convergent iff it is a Cauchy sequence and follows that:

$$\sum_k^{\infty} a_k \text{ Convergent} \leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq m \geq N$$

we have:

$$\left| \sum_m^n a_k \right| < \varepsilon$$

which is simply another representation of a Cauchy sequence

**Theorem 4.21.**  $\sum_{k=1}^{\infty} a_k$  is convergent  $\Rightarrow (a_k)_{k \in \mathbb{N}}$  convergent and  $\lim_{k \rightarrow \infty} a_k = 0$

*Proof.*  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall n, m \geq N$  there is  $|s_n - s_m| < \varepsilon \Rightarrow |a_n + a_{n+1} + \dots + a_m| < \varepsilon$ .

Hence  $|a_n - 0| < |a_n + \dots + a_m| < \varepsilon$  therefore  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $n \geq N$  and  $|a_n - 0| < \varepsilon$  it is the same of limit  $\lim_{k \rightarrow \infty} a_k = 0$  exists.  $\square$

## 4.11 Introduction to topology

## 4.12 Open, Close, and Compact sets

We first define what is a **neighborhood**:

**Definition 4.22.**

$$\varepsilon > 0, (x - \varepsilon, x + \varepsilon) := B(x)$$

which we call it  $\varepsilon$ -neighborhood. A neighborhood of  $x$  is:

$$\text{For } M \subseteq \mathbb{R}, \exists \varepsilon > 0, \text{ st. } M \supseteq B(x)$$

**Definition 4.23.**  $M \subseteq \mathbb{R}$  is called **open set** in  $\mathbb{R}$  iff for all  $x \in M$ ,  $M$  is a neighborhood of  $x$ .

$$\forall x \in M, \exists \varepsilon > 0, \text{ st. } N(x) \subseteq M$$

After define what is open set we are going to define close set use the definition of opensets:

**Definition 4.24.** A set  $A$  is close set iff  $A^C$  is open set

**Theorem 4.25.** A set is close can be described by convergent sequence: For  $A \subseteq \mathbb{R}$  if all  $(a_n)_{n \in \mathbb{N}}$  with  $a_n \in A$  and  $\lim_{n \rightarrow \infty} a_n \in A$  then  $A$  is close set.

Then we are going to give a special definition which are so important—compact set

**Definition 4.26.**  $A \subseteq \mathbb{R}$  is called compact if for all  $a_n \in A, \forall n \in \mathbb{N}$  there is a convergent subsequence  $a_{n_k}$  where:

$$\lim_{k \rightarrow \infty} a_{n_k} \in A$$

- $\emptyset$  is compact
- $[c, d], c < d$  is compact
- $\{n\}$  is compact
- $\mathbb{R}$  is not compact

**Theorem 4.27. Heine-Borel theorem:** For  $A \subseteq \mathbb{R}$  is compact iff its bounded and closed

*Proof.* We will prove this equivalent statement in both direction:

$(\Leftarrow)$  : Use the same method in proof of Bolzano-Weierstrass theorem

$(\Rightarrow)$  Assume  $A$  is compact, then there is an convergent sequence  $a_n \in A \subseteq \mathbb{R}$  where has accumulation value  $\bar{a} \in A$ . We first prove that  $A$  is closed—because there is only one accumulation value  $a$ , hence  $\bar{a} = a \in A$ . By theorem 4.25 this means that  $A$  is closed.

We then prove that  $A$  is bounded by contradiction—Assume  $A$  is unbounded, then  $\exists a_n \in A$  where  $|a_n| > n, \forall n \in \mathbb{N}$ , we constructed does not have any convergent subsequence. This is because its terms grow without bound and thus cannot converge to any point in  $\mathbb{R}$ . A sequence that tends to infinity doesn't converge, which contradicts the property of compactness in  $A$ .

□

### 4.13 Pointwise convergence and uniform convergence

**Definition 4.28.** A sequence of function  $(f_n)$  is called Pointwise convergent to function  $f : I \rightarrow \mathbb{R}$  iff:

$$\forall \bar{x} \in I, \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, s.t. |f_n(\bar{x}) - f(\bar{x})| < \varepsilon$$

In a similar way, we can define the uniform convergent of sequence of function:

**Definition 4.29.** A sequence of function  $(f_n)$  is called uniform convergent to function  $f : I \rightarrow \mathbb{R}$  iff:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall \bar{x} \in I, s.t. |f_n(\bar{x}) - f(\bar{x})| < \varepsilon$$

It is easy to see that the difference of definition is simply exchange the order of  $\forall \bar{x} \in I$ , however it will alternate the meaning of the whole definition and cause a totally different understanding.

**Definition 4.30.** Distance of two function  $f(x), g(x)$  are defined as:

$$\sup_{x \in I} |f(x) - g(x)| \equiv \|f - g\|_{\infty}$$

By defining the distance of functions we can rewrite the definition of uniform convergence:

$$n \rightarrow \infty, \|f_n - f\|_{\infty} \rightarrow 0$$

**Theorem 4.31.** The relationship between pointwise convergence and uniform convergence are:

- pointwise convergence  $\nRightarrow$  uniform convergence
- uniform convergence  $\Rightarrow$  pointwise convergence

#### 4.14 Limits of functions

**Definition 4.32.** Let  $f : I \rightarrow \mathbb{R}, x_0 \in I$  If there is  $c \in \mathbb{R}$  and all sequence  $(x_n)$  with  $\lim_{n \rightarrow \infty} x_n = x_0$  we have  $(f(x_n))$  is also convergent with  $\lim_{n \rightarrow \infty} f(x_n) = C$  then we write:

$$\lim_{x \rightarrow x_0, x_0 \in \mathbb{R}} f(x) = C$$

Left and Right limit is an important tool for further exploration.

**Definition 4.33.** (Right limit,  $\lim_{x \rightarrow x_0^+}$ ) For function  $f : I \rightarrow \mathbb{R}$ , and  $x_0 \in I \cap (x_0, +\infty)$  is some adherent point, then we define the right limit of function at  $x_0$  as:

$$\lim_{x \rightarrow x_0, I \cap (x_0, +\infty)} f(x)$$

In short cut we write:

$$\lim_{x \rightarrow x_0, (x_0, +\infty)}$$

In the similar way we define the left limit.

#### 4.15 Compactness and Continuity

**Definition 4.34.** Let  $f : I \rightarrow \mathbb{R}$  be a function with  $I \subseteq \mathbb{R}$ .  $f$  is called continuous at  $x_0 \in I$  if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

If  $f$  is called continuous on  $I$ , if  $f$  is continuous at  $x_0$  for all  $x_0 \in I$ .

If  $f(x)$  is continuous  $\forall x_0 \in I$ , then we have:

$$\lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n)$$

**example 4.35.** There is an important function:

$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$$

You should now that  $\mathbb{Q}$  is dense in the real number field. That is, you can always construct a sequence where it convergent to a real number. You can choose a sequence that convergent to 1 when  $x \rightarrow x_0$  also you can choose another sequence that convergent to 0 when  $x \rightarrow x_0$ . Hence  $\lim_{x \rightarrow x_0} f(x)$  does not exist. The function is not continuous at any points.

This definition of continuity is based on the concept of limit. However, there is an equivalence statement of continuous by using the traditional analytical language— $\varepsilon, \delta$  words.

A function is continuous at  $x_0$  if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Let's proof the equivalence of two definitions:

*Proof.* ( $\Rightarrow$ ) Assume  $\exists \varepsilon > 0, \forall \delta > 0, \exists x \in I : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$ . Let's take  $\delta = \frac{1}{n}, n \in \mathbb{N}$ . We have  $\forall n \in \mathbb{N}, x_n \in I \setminus \{x_0\}$  with  $|x_n - x_0| < \frac{1}{n}$  and  $|f(x_n) - f(x_0)| \geq \varepsilon$ . Therefore it is not continuous at  $x_0$ .

( $\Leftarrow$ ) choose sequence  $(x_n) \subseteq I \setminus \{x_0\}$  with limit  $x_0$ . Let  $\varepsilon > 0$  take  $\delta > 0$ . there is  $N \in \mathbb{N}$  such that for all  $n \geq N$  we have  $|x_n - x_0| < \delta$ , also by assumption we have  $|f(x_n) - f(x_0)| < \varepsilon$  therefore  $f$  is continuous.  $\square$

**Theorem 4.36.** For basic combination of two functions  $f, g : I \rightarrow \mathbb{R}$ : addition, Subtraction, multiplication. If both  $f, g$  are continuous at  $x_0$  then all the result functions are continuous. For division, the denominator should not be 0.

*Proof.* Assume  $f, g : I \rightarrow \mathbb{R}$  is continuous, then we have  $\lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n)$  and  $\lim_{n \rightarrow \infty} g(x_n) = g(\lim_{n \rightarrow \infty} x_n)$ . Now for:

- $f + g = h$ , we have  $\lim_{n \rightarrow \infty} (f(x_n) + g(x_n)) = \lim_{n \rightarrow \infty} (h(x_n)) = f(\lim_{n \rightarrow \infty} x_n) + g(\lim_{n \rightarrow \infty} x_n) = h(\lim_{n \rightarrow \infty} x_n)$ .
- $f \cdot g = h$ , we have  $\lim_{n \rightarrow \infty} (f(x_n) \cdot g(x_n)) = \lim_{n \rightarrow \infty} (h(x_n)) = f(\lim_{n \rightarrow \infty} x_n) \cdot g(\lim_{n \rightarrow \infty} x_n) = h(\lim_{n \rightarrow \infty} x_n)$

$\square$

What about composition of functions  $f(g(x))$ .

**Theorem 4.37.** For two functions  $f : I \rightarrow \mathbb{R}, g : J \rightarrow \mathbb{R}, I, J \subseteq \mathbb{R}$  with  $g[J] \subseteq I$  and  $g$  continuous at  $x_0 \in J$ ,  $f$  continuous at  $g(x_0) \in I$ . Therefore we have  $f(g(x)) : J \rightarrow \mathbb{R}$  continuous at  $x_0 \in J$

*Proof.* Choose sequence  $(x_n) \subseteq J \setminus \{x_0\}$  with limit  $x_0$

$$\lim_{n \rightarrow \infty} (f \circ g)(x_n) = \lim_{n \rightarrow \infty} f(g(x_n)) = f(\lim_{n \rightarrow \infty} g(x_n)) = f(g(\lim_{n \rightarrow \infty} x_n))$$

$\square$

Now let's talk about what is the relationship between compactness and continuous function on a real number field.



**Theorem 4.38.** Let continuous function  $f : I \rightarrow \mathbb{R}$  where  $I$  is compact on  $\mathbb{R}$  (that is  $I$  is bounded and closed by Heine-Borel Theorem). Then the image  $f[I]$  is compact too. With the superior and inferior:

$$f(x^+) := \sup\{f(x)|x \in \mathbb{R}\}$$

$$f(x^-) := \inf\{f(x)|x \in \mathbb{R}\}$$

Now lets look at the property of uniform convergent functions where the sequence of functions are all continuous:

**Theorem 4.39.** Let  $f_n : I \rightarrow \mathbb{R}$  be continuous for all  $n \in \mathbb{N}$  and  $f_n$  uniformly convergent to  $f$ . Then  $f$  is continuous

*Proof.* Let  $\varepsilon > 0, x > 0, x \in I$

Because  $f_n$  is uniform convergent to  $f$  therefore we have:

$$\forall \varepsilon' > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall x \in I, s.t. |f_n(x) - f(x)| < \varepsilon'$$

In this case, we are going to write the continuity condition of  $f_N$  at  $x_0$  down by finding  $\delta > 0$ :

$$|x - x_0| < \delta, \Rightarrow |f_N(x) - f_N(x_0)| < \varepsilon'$$

And now the condition of  $f$  to be continuous is:

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f_N(x) - f(x_0)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \varepsilon' + \varepsilon' + \varepsilon' = \varepsilon \end{aligned}$$

(Explanation: for first and third term, it is given by the uniform convergence and second term is the continuity of  $f_N$ ) Therefore, by choosing  $\varepsilon' = \frac{\varepsilon}{3}$ , we can show that the function  $f$  is continuous.  $\square$

Now we shall introduce a very important theorem which are useful—Intermediate Value Theorem

**Theorem 4.40.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function and  $y \in [f(a), f(b)]$  or  $y \in [f(b), f(a)]$ , the Intermediate Value Theorem tells us there is  $\bar{x} \in [a, b]$  with  $y = f(\bar{x}) \in [f(a), f(b)]$ .

*Proof.*  $\square$

#### 4.15.1 Some continuous functions

Now I am going to list out serval important continuous functions:

**example 4.41.**  $\exp: \mathbb{R} \rightarrow \mathbb{R}$  is continuous function defined by  $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ . And we define  $e$  the constant as  $e := \exp(1)$ . You may know another equivalent definition of this constant given by the limit:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Here are some basic properties:

- $\exp(x + y) = \exp(x) \times \exp(y)$
- $\exp(x) = e^x$
- $\lim_{n \rightarrow \infty} \exp(n) = \infty, \lim_{n \rightarrow -\infty} \exp(n) = 0$
- $\exp: \mathbb{R} \rightarrow (0, \infty)$  is bijection with inverse we called the logarithm function.

We just mentioned Logarithm function, which is the inverse function of  $\exp$  function:

$$\exp(y) = x \Rightarrow y = \log(x)$$

## 4.16 Differentiation

In order to further understand the features of functions, we need to know the rate of change of them. In this case, we shall introduce a new concept—differentiation

**Definition 4.42.** For function  $f: I \rightarrow \mathbb{R}$  where  $I \in \mathbb{R}$  and a limit point  $(x_0)$  of  $I$ . Now we define a function  $f'(x)$  as:

$$\lim_{x \rightarrow x_0, x_0 \in I \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$$

If this limit convergent to  $L \in \mathbb{R}$ . Then we call the function is differentiable at  $x_0 \in I$  with derivative  $L$ . That is  $f'(x_0) := L$

It is obvious that only smooth function are differentiable. Now here are some examples:

**example 4.43.** For function  $f(x) = |x|$  by definition the derivative at 0 is:

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0, x \in \mathbb{R} \setminus \{0\}} \frac{|x|}{x}$$

By take left and right limit of the function at 0 it is obvious that we will get 2 results:  $\{1, -1\}$

## 4.17 Chain Rule

While we are going to calculate the derivative of a composite function, we need to use the chain rule:

**Theorem 4.44** (chain rule). For two function  $f, g$  where  $f$  is differentiable at  $x_0$  and  $g$  is differentiable at  $f(x_0)$ . Then the composite function  $g \circ f$  is differentiable at  $x_0$  with derivative:

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Now we shall give a proof of this theorem:

*Proof of Chain rule.* By definition of derivative, we have:

$$\begin{aligned} & \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \\ &= g'(f(x_0)) \cdot f'(x_0) \end{aligned}$$

□

Chain rule is a powerful tool for us to calculate the derivative of a composite function. It is easy to use and very useful in practice.

## 4.18 Local maximum, minimum

We shall now talk about the local maximum and minimum of a function:

**Definition 4.45** (local maximum). The function  $f : I \rightarrow \mathbb{R}$  has a local maximum at  $x_0 \in I$  if there is  $\varepsilon > 0$  where  $f(x) \leq f(x_0), \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon)$

In a similar way we define the local minimum:

**Definition 4.46** (Local minimum). The function  $f : I \rightarrow \mathbb{R}$  has a local minimum at  $x_0 \in I$  if there is  $\varepsilon > 0$  where  $f(x) \geq f(x_0), \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon)$

Now we shall talk about the relationship between the local maximum and minimum and the derivative of the function:

**Theorem 4.47** (Derivative and local maximum, minimum). For function  $f : I \rightarrow \mathbb{R}$  where  $I \in \mathbb{R}$  and  $x_0 \in I$  is a local maximum or minimum. Then we have  $f'(x_0) = 0$

We shall prove this theorem by using the definition of derivative:

*Proof of Derivative and local maximum, minimum.* Assume  $f$  has a local maximum at  $x_0$ , then we have:

$$\lim_{x \rightarrow x_0, x \in I \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

And we have  $f'(x_0) \leq 0$ . Similarly we can prove the case of local minimum.

□

## 4.19 The mean value theorem

We now shall talk about a baby version of Mean value theorem which named as Rolles theorem:

**Theorem 4.48.** If funtion  $f$  is a funtion where differentiable on  $[a, b]$  and  $f(a) = f(b)$  then there is  $\bar{x} \in (a, b)$  where  $f'(\bar{x}) = 0$

We shall now show a proof of this theorem:

*Proof of Rolles theorem.* By the extreme value theorem, we have  $f$  has maximum and minimum value on  $[a, b]$ . If the maximum and minimum value are the same then we have  $f$  is constant and  $f'(\bar{x}) = 0$ . If they are not the same, then we have  $f$  has maximum and minimum value. Therefore there is  $\bar{x}$  where  $f'(\bar{x}) = 0$ .  $\square$

Now we shall talk about the mean value theorem:

**Theorem 4.49** (Mean value theorem). We shall notice that we can rewrite the Rolles theorem as a special case of mean value theorem. For function  $f$  is a funtion where differentiable on  $[a, b]$  then there is  $\bar{x} \in (a, b)$  where  $f'(\bar{x}) = \frac{f(b)-f(a)}{b-a}$  and we can rewrite it as:  $f'(\bar{x}) = \frac{f(b)-f(a)}{b-a}$

## 4.20 Talor series and Forier series

In this subsection we are going to talk about two important series which are used to approximate the functions by polynomial functions, they are Talor series and Forier series.

**Definition 4.50** (Talor series). Talor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. Where in mathematiacd language, we write the Talor series of a function  $f$  at point  $x_0$  as:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

While the  $n$  gets larger, the approximation of the function gets better. However, the convergence of the series is not guaranteed. It is because the function may not be smooth enough. Now we shall introduce the Forier series:

**Definition 4.51** (Forier series). Forier series is a way to represent a function as the sum of simple sine waves. The Forier series of a function  $f$  is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \quad (1)$$

The reason why we use sine and cosine function is that they are orthogonal to each other. Meanwhile the Fourier series is guaranteed to converge to the function if the function is piecewise smooth. Hence the Fourier series is more powerful than the Taylor series. We can use Fourier series to approximate the function which is not smooth enough.

And now we can talk about the theorem of these series:

**Theorem 4.52.** Fourier series is convergent to  $f$  if  $f$  is piecewise smooth.

*Proof.* We can prove this theorem by using the Dirichlet's theorem. Where we can show that the partial sum of the Fourier series is bounded.  $\square$

## 4.21 Uniform convergence and differentiability

In the previous topics we have already talked about the pointwise convergence and uniform convergence of a sequence of functions.

$$\|f_n - f\|_\infty \rightarrow 0$$

when  $n \rightarrow \infty$ . And there is the fact that if  $f_n$  is continuous and  $\|f_n - f\|_\infty \rightarrow 0$  then  $f$  is continuous.

**Definition 4.53.**  $f : I \rightarrow \mathbb{R}$  is called differentiable if  $f$  is differentiable at all  $x_0 \in I$ . In this case,  $f'(x) : I \rightarrow \mathbb{R}$  defined by  $x \rightarrow f'(x)$  is called derivative of  $f$ .

Let's look at an example:

**example 4.54.** For function  $f_n(x) = x^n$  where  $x \in [0, 1]$ . It is easy to see that  $f_n(x)$  is continuous and differentiable on  $[0, 1]$ . However, the limit of the derivative of  $f_n$  is not continuous.

**Theorem 4.55.** Let  $(f_1, f_2, f_3, \dots)$  be a sequence of functions  $f_n : I \rightarrow \mathbb{R}$ . Assume  $f_n$  is differentiable for all  $n$ . Then there is  $g : I \rightarrow \mathbb{R}$  with  $\|f'_n - g\|_\infty \rightarrow 0$  when  $n \rightarrow \infty$ .

This theorem tells us that a sequence of differentiable functions are convergent to a function  $g$ . However this will not lead to uniform convergence of this sequence of functions. We need another additional condition for that.

That  $f_n$  are pointwise convergent to  $f$  and  $f'_n$  are uniformly convergent to  $g$ . Where  $g = f'$ .

*Proof.* We shall use inequality to prove this theorem.  $\square$

## 4.22 Power series and derivatives

**Definition 4.56** (Power series). Power series is a series of the form:

$$\sum_{k=0}^{\infty} a_k(x)^k \quad (2)$$

Where when we differentiate the power series, we get:

$$\sum_{k=0}^{\infty} k \cdot a_k(x)^{k-1} \quad (3)$$

We already know the original series is convergent  $f : (r, -r) \rightarrow \mathbb{R}$  with convergence radius  $r$ . However we want to test if this apply for general

## 4.23 Riemann integration

we need to describe how one can partition a large interval into smaller intervals.

**Definition 4.57.** A space  $X \subseteq \mathbb{R}$  is called connected iff for any  $x, y \in X$  and  $x < y$  the bounded interval  $[x, y]$  is always a subset of  $X$ . That is, in another word, every element in the interval is in  $X$ .

It is obvious that: Connected and bounded space in  $\mathbb{R} \equiv$  bounded interval in  $\mathbb{R}$ . Now we shall define the length of an interval:

**Definition 4.58.** (Length of interval), the length of interval  $I$ , such as  $I = (a, b), (a, b], [a, b], [a, b)$  where  $a < b$ . We define the length of the interval  $|I|$  as  $b - a$ . If  $I = \emptyset$  or a point, then  $|I| = 0$

Now we shall define the partitions:

**Definition 4.59.** Let  $I$  be bounded interval. A partition  $\mathbf{P}$  of  $I$  is a collection of bounded intervals contained in  $I$ . Where every  $x \in I$  lies exactly one of the bounded interval  $J$  in  $\mathbf{P}$

Now we shall discuss some property of length based on the partition:

**Theorem 4.60.** (Length is finitely additive) for interval  $I$  and its partition we have:

$$|I| = \sum_{J \in \mathbf{P}} |J|$$

*Proof.* You can prove this theorem by using induction □

Now we wonder how can we compare two partition of same interval:

**Definition 4.61.** For interval  $I$  we have  $\mathbf{P}$  and  $\mathbf{P}'$  as partition of it. We say  $\mathbf{P}$  is finer than  $\mathbf{P}'$  if  $\forall J \in \mathbf{P}, \exists K \in \mathbf{P}', s.t. J \subseteq K$

This definition is very intuitive of the idea 'finer'.

**Definition 4.62.** We define the common refinement( $\mathbf{P} \# \mathbf{P}'$ ) of who paritions  $\mathbf{P}, \mathbf{P}'$  as the set of intersection of partitions of an interval:

$$\mathbf{P} \# \mathbf{P}' := \{K \cap J | K \in \mathbf{P}, J \in \mathbf{P}'\}$$

**Definition 4.63.** A function is called piecewised function if for the function  $f : I \rightarrow \mathbb{R}$ ,  $I$  is bounded interval, the parition of  $I$  is  $P$ , where for every  $J \in P$ , the  $f$  is a constant.

**Theorem 4.64.** We now shall introduce some properties of integrationn of p.c. functions

- $p.c. \int_I (f + g) = p.c. \int_I f + p.c. \int_I g$
- $p.c. \int_I c \cdot f = c \cdot p.c. \int_I f$

Now lets get into the topic, which 'Integration' the concept first appear in this note:

**Definition 4.65.** Let  $I$  be a bounded interval, let  $P$  be a partition of  $I$ . Let  $f : I \rightarrow \mathbb{R}$  be a function which is piecewise constant with respect to  $P$ . Then we deine the piecewise constnt integral p.c.  $\int_{[P]} f$  of  $f$  with respect to the partition  $P$  by the formula:

$$p.c. \int_{[P]} f := \sum_{J \in P} c_J |J|$$

Where for each  $J$  in  $P$ , we let  $c_J$  be the constant value of  $f$  on  $J$

**Definition 4.66.** For

$$I = [a, b], a < b$$

we define the Riemann integral of a function  $f : I \rightarrow \mathbb{R}$  as:

$$\int_a^b f(x) dx := \sup \{ p.c. \int_{[P]} f | P \text{ is partition of } I \} \quad (4)$$

#### 4.24 Introduction to measure theory

#### 4.25 Lebesgue integration

#### 4.26 Lebesgue measure

#### 4.27 Relation to complex analysis

### 5 Complex analysis

#### 5.1 Complex numbers and geometric interpretation

#### 5.2 Complex functions

#### 5.3 Riemann sphere

#### 5.4 Complex Differentiation

#### 5.5 The Logarithmic Function

#### 5.6 Riemann surface

#### 5.7 Complex Integration

#### 5.8 Riemann Integral

In the previous sections we have talked about the integral in general, now we are going to talk about the Riemann integral which is a special type of integral.

**Definition 5.1** (Riemann integrable). A function  $f : I \rightarrow \mathbb{R}$  is called Riemann integrable on  $I$  if there is a number  $I$  such that for every  $\varepsilon > 0$ , there is a partition  $P$  of  $I$  such that for every refinement  $P'$  of  $P$  we have  $|I - \int_{[P']} f| < \varepsilon$

This definition tells us that the Riemann integral is a limit of the piecewise constant integral. where we can esaly prove that the Riemann integral is a linear operator. Meanwhile the Riemann integral is a continuous operator. The derivarive as we already know is the reverse operation of the integral. Hence Derivative is also a linear operator. This definition tells us the integrability of the function is based on the limit of the piecewise constant integral.

Now we are going to talk about some property of the Riemann integral:

**Theorem 5.2** (Riemann integral is linear). Riemann integral is a linear operator. That is for  $f, g$  are Riemann integrable on  $I$  and  $a, b \in \mathbb{R}$  then we have:

$$\int_I (af + bg) = a \int_I f + b \int_I g$$

We shall now prove this theorem:



*Proof of Riemann integral is linear.* For  $f, g$  are Riemann integrable on  $I$  and  $a, b \in \mathbb{R}$  then we have:

$$\begin{aligned}
\int_I (af + bg) &= \sup\{p.c. \int_{[P]} (af + bg) | P \text{ is partition of } I\} \\
&= \sup\{p.c. \int_{[P]} (af) + p.c. \int_{[P]} (bg) | P \text{ is partition of } I\} \\
&= a \sup\{p.c. \int_{[P]} f | P \text{ is partition of } I\} + b \sup\{p.c. \int_{[P]} g | P \text{ is partition of } I\} \\
&= a \int_I f + b \int_I g
\end{aligned}$$

□

## 6 Introduction to category theory

Category theory is a very important branch of mathematics which is used to study the structure of mathematics. We shall now introduce the basic concept of category theory.

**Definition 6.1** (Category). A category  $\mathcal{C}$  consists of: Objects  $A, B, C, \dots$  and morphisms  $f, g, h, \dots$ . In addition, for each morphism  $f$  there are two objects  $A, B$  called the domain and codomain of  $f$ . Where we write  $f : A \rightarrow B$  to denote the morphism  $f$  with domain  $A$  and codomain  $B$ .

## Axioms

### 6.1 The Peano axioms

The Peano axioms are a set of axioms for the natural numbers presented by the 19th century Italian mathematician Giuseppe Peano. These axioms have been used nearly unchanged in a number of metamathematical investigations, including research into fundamental questions of the foundations of mathematics.

In mathematical language, the Peano axioms read as follows:

- 0 is a natural number.
- Every natural number has a successor, which is also a natural number.
- 0 is not the successor of any natural number.
- Two natural numbers with the same successor are equal.
- (Induction axiom) If a property is possessed by 0 and also by the successor of every natural number by which it is possessed, then it is possessed by all natural numbers.

SSSS

## 6.2 set theory axioms

- (Sets are objects). If  $A$  is a set, then  $A$  is also an object. In particular, given two sets  $A$  and  $B$ , it is meaningful to ask whether  $A$  is also an element of  $B$ .
- (Empty set). There exists a set  $\emptyset$ , known as the empty set, which contains no elements, i.e., for every object  $x$  we have  $x \notin \emptyset$ .
- (Singleton sets and pair sets). If  $a$  is an object, then there exists a set  $\{a\}$  whose only element is  $a$ , i.e., for every object  $y$ , we have  $y \in \{a\}$  if and only if  $y = a$ ; we refer to  $\{a\}$  as the singleton set whose element is  $a$ . Furthermore, if  $a$  and  $b$  are objects, then there exists a set  $\{a, b\}$  whose only elements are  $a$  and  $b$ ; i.e., for every object  $y$ , we have  $y \in \{a, b\}$  if and only if  $y = a$  or  $y = b$ ; we refer to this set as the pair set formed by  $a$  and  $b$ .

## 6.3 Axiom of choice

The Axiom of Choice is a mathematical axiom that is used in set theory. It was formulated in 1904 by Ernst Zermelo. The axiom states that given a collection of sets, it is possible to choose exactly one element from each set, even if the collection is infinite. In other words, the axiom of choice allows for the construction of a set from an infinite number of other sets, each of which has exactly one element. In mathematical terms, the axiom of choice states that for any set  $S$  of non-empty sets, there exists a function  $f$  such that for every set  $A$  in  $S$ ,  $f(A)$  is an element of  $A$ . It is not that intuitive, but it is a very important axiom in set theory, and it has many applications in mathematics, including in the study of infinite sets and in the construction of mathematical models.