Advantages of Using Quaternions in 3-D Geometric Rotation

Over Euler Angles and Matrices

Sam

12^{th} Nov. 2021

Introduction to Quaternions

- Quaternions provide a mathematical notation for representing orientations and rotations of objects in three dimensions.
- Historically developed by Hamilton as an extension to complex numbers.
- Used in the solution of cubic and quartic equations.
- Led to the development of four-dimensional complex numbers.

Hamilton's Breakthrough

- Hamilton's quest to extend complex numbers to higher dimensions.
- The challenge of defining multiplication and division in three dimensions.
- His realization of the need for a four-dimensional approach.
- The historic moment of discovery: the inscription on Brougham Bridge.

Basic Concepts of Quaternions

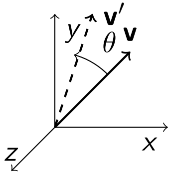
- A quaternion is a four-dimensional complex number expressed as $q = w + xi + yj + zk$.
- Components:
  - Real part: w
  - Imaginary parts: xi, yj, zk where i, j, k are the fundamental quaternion units.
- Basic rules:

$$
\begin{aligned}
i^2 = j^2 = k^2 = ijk &= -1 \\
ij = k, \quad ji &= -k \\
jk = i, \quad kj &= -i \\
ki = j, \quad ik &= -j
\end{aligned}
$$

Applications in Mathematics and Physics

- Quaternions are extensively used in three-dimensional rotations and orientation problems, particularly in computer graphics, robotics, and aerospace engineering.
- In mathematics, quaternions provide a way to represent rotations that is more robust and less susceptible to the problems of gimbal lock, unlike Euler angles.

- They are also employed in the field of physics, especially in areas involving angular momentum, orbital mechanics, and the representation of rotations/spins of particles.
- Quaternions are advantageous for interpolation in 3D space, known as "slerp" (spherical linear interpolation), offering smoother and more stable transitions than traditional methods.



Quaternion Operations

- **Addition:** For quaternions $q_1 = w_1 + x_1 i + y_1 j + z_1 k$ and $q_2 = w_2 + x_2 i + y_2 j + z_2 k$, the addition is given by:

$$q_1 + q_2 = (w_1 + w_2) + (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$$

- **Multiplication:** The product of q_1 and q_2 is:

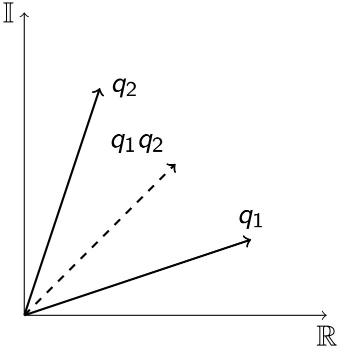
$$q_1 q_2 = w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 + (w_1 x_2 + x_1 w_2 + y_1 z_2 - z_1 y_2)i +$$

$$(w_1 y_2 - x_1 z_2 + y_1 w_2 + z_1 x_2)j + (w_1 z_2 + x_1 y_2 - y_1 x_2 + z_1 w_2)k$$

- **Conjugation:** The conjugate of $q = w + xi + yj + zk$ is:

$$\bar{q} = w - xi - yj - zk$$

In visualize:

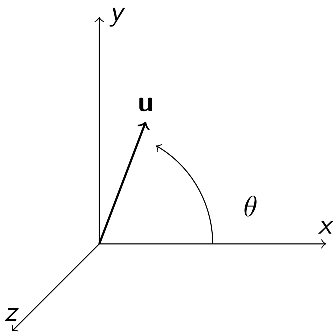


Geometric Meaning of Quaternions

- Quaternions offer a powerful way to represent and compute rotations in 3D space.
- Each quaternion corresponds to a rotation around a vector in three-dimensional space.
- The rotation is defined by the direction of the vector and the angle of rotation.

- **Geometric Interpretation:** A quaternion $q = w + xi + yj + zk$ can be seen as a rotation by an angle θ around a unit vector $\mathbf{u} = (x, y, z)$.
- The relationship between quaternion components and rotation:

$$q = \cos\left(\frac{\theta}{2}\right) + \mathbf{u}\sin\left(\frac{\theta}{2}\right)$$



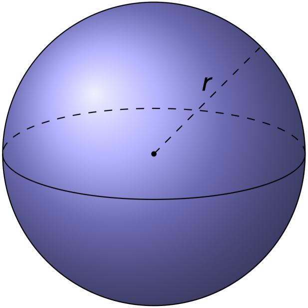
Quaternions and the Hypersphere

- Quaternions extend the concept of rotation from three dimensions to four dimensions.
- They can be visualized as points on the surface of a 4D hypersphere, called a 3-sphere or S^3.
- This 4D extension allows for smooth, continuous rotation in 3D space without singularities.

- **Mathematical Representation:** A unit quaternion (or versor) represents a point on the 3-sphere:

$$q = w + xi + yj + zk \quad \text{where} \quad w^2 + x^2 + y^2 + z^2 = 1$$

- The 3-sphere is a higher-dimensional analogue of the 2-sphere (surface of a globe) in 4D space.



Euler Angles vs. Quaternions

- **Euler Angles:** Represent rotation using three angles (roll, pitch, yaw). Prone to gimbal lock, a condition where two axes align and lose one degree of rotational freedom.
- **Quaternions:** Provide a more robust solution for representing rotations. Not susceptible to gimbal lock due to their four-dimensional nature.
- **Advantages of Quaternions:**
  - Smoother and more stable interpolation (slerp).
  - More efficient computations and less memory usage.
  - Intuitive representation of combined rotations.
- Quaternions are increasingly preferred in 3D computer graphics, robotics, and aerospace engineering for their reliability and efficiency in handling complex rotations.

Conclusion and Future Applications

- **Summary of Key Points:**
  - Quaternions provide an efficient and robust way to represent 3D rotations.
  - They overcome the limitations of Euler angles, such as gimbal lock.
  - Quaternions find extensive applications in computer graphics, robotics, aerospace, and physics.
- **Future Applications:**
  - As technology advances, the use of quaternions is expected to grow, especially in virtual reality, advanced robotics, and 3D simulation.
  - The mathematical properties of quaternions might lead to new discoveries in higher-dimensional space representations and quantum computing.
- Quaternions remain a vibrant and evolving field of study, offering exciting opportunities for innovation and discovery.

THANK YOU