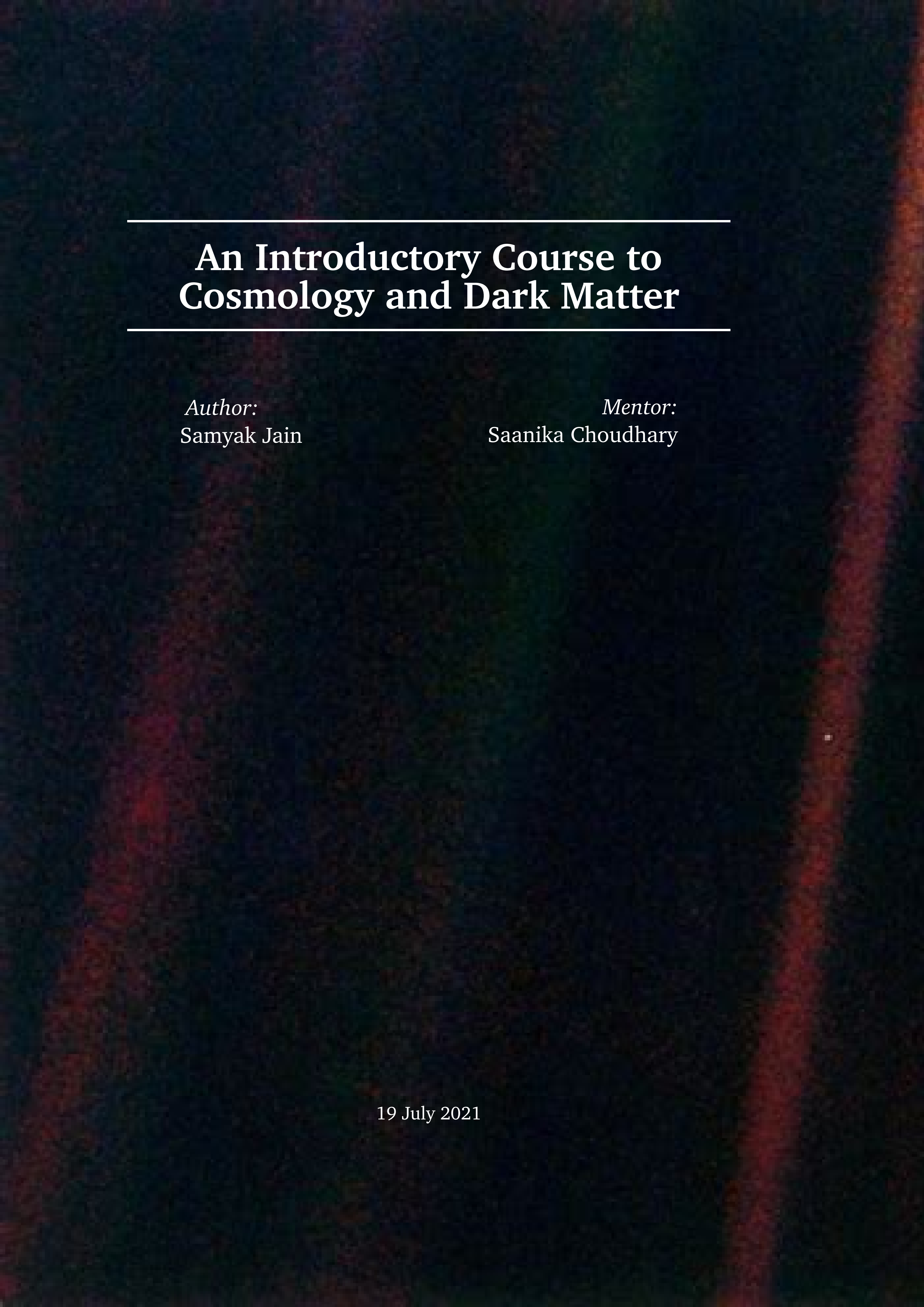

An Introductory Course to Cosmology and Dark Matter

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Contents

Chapter 1

Introduction

Through this report I will try to summarise my findings and the major concepts I covered during the SoS - 2021 programme.. The main texts referred to for this report are :

1. The Feynman's Lectures on Physics Vol. 1
2. A First Course in General Relativity by Bernard Schutz
3. Lectures on cosmology by Leonard Susskind (Lectures 1 through 6)

I would also like to thank my mentor Saanika Choudhary for her guidance throughout the programme.

Chapter 2

Special Theory of Relativity

The fundamental principles of relativity used throughout the report are :
1. The laws of Physics are the same in all inertial frames of reference.
2. This implies one cannot tell if he is in motion or not by any mechanical means.
Consider an observer on the ground, he observes the speed of a photon in the x direction to be c. Using Newtonian mechanics, it follows that the velocity of light seen by an observer in a car moving at velocity u in the x direction is c-u. Scientists tried to use this concept to determine the 'velocity' of the earth, however all the experiments failed, they gave no velocity at all. These were the first observed holes in the Newtonian view, we see that determining relative velocities isn't simply a game of subtraction of velocity vectors.

2.1 Michealson Morley experiment
The apparatus consists of a light source, a partially silvered glass plate and 2 plane mirrors, all mounted on a rigid base moving at velocity u.

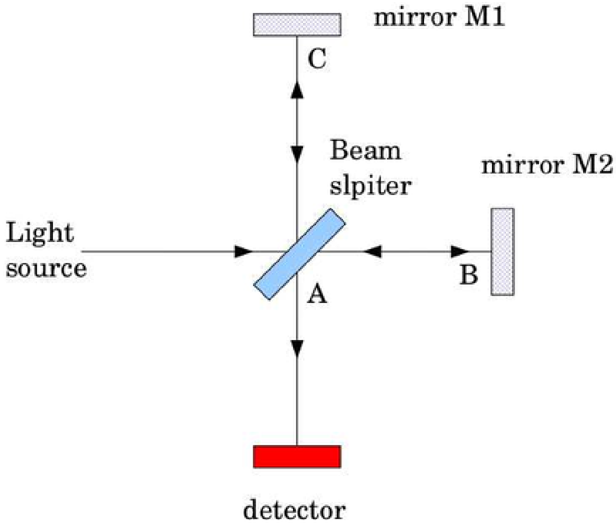

Figure 2.1

We can see that the light travelling to and from the mirror M2 travels a longer distance due to the motion of the apparatus. Therefore we expect to see a corresponding time gap between the light striking the glass plate again, i.e. the time taken to travel to and fro from mirror M1 should have been $\sqrt{1-\frac{u^2}{c^2}}$ times the taken taken to travel to and fro from M2. However no such time gap was found.

Lorentz suggested the concept of 'Length Contraction'. He argued that the length of an object along the direction of it's motion shrinks by a factor of $\sqrt{1-\frac{u^2}{c^2}}$. This explains the absence of any time lag in the Michealson Morley experiment. The reciprocal of this factor is called as the Lorentz factor, this will be indicated by γ throughout the report.

2.2 Time Dilation

Consider a clock that works the following way : We have a 2 plane mirrors at the 2 ends of a rod. A photon oscillates between the 2 mirrors and makes a 'click' each time it strikes a mirror. Imagine if we place such a clock on a spaceship moving at velocity u with respect to an observer O along the x direction. We place the rod perpendicular to the direction of motion, so there is no Length Contraction. Let an observer P be in the spaceship, he observes the time interval between successive clicks to be t'. However, O sees the photon move along a zig-zag path, similar to what we saw in the Michealson Morley experiment. It is elementary to derive that for O, the time interval t is $\frac{t'}{\gamma}$. This implies that any clock on the spaceship, no matter

it's mechanism, would 'slow down' for O. This phenomenon is called as time dilation.
Now consider a new situation : We place a light source at the mid point of the rod, the light source periodically sends out photons in the direction of the mirrors. Let the mirror in the positive x direction of the light source be A and the other B.
P observes that the photons strike their respective mirrors simultaneously. However O observes that the photon travelling towards the mirror A has to cover more distance since the mirror is in motion! Correspondingly, he observes that the photon has to travel a lesser distance to the mirror B!
So O observes a time lag dependent on the x coordinate.

2.3 Lorentz Transformation

Using this experiment, and the Michealson Morley experiment, it is elementary to derive the Lorentz Transformation :
Consider an observer O and observer P moving in the x direction at velocity u with respect to O at velocity u. The Lorentz Transformation relating the coordinates in O's coordinate system (x, y, z, t) and P's coordinate system (x', y', z', t') is

$$x' = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{ux}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

One should note the symmetry of the transformation. Since P sees O moving at -u along x axis, the transformation from the primed coordinates to the un-primed coordinates should be the same as interchanging the primed and un-primed coordinates and replacing u with -u. It is elementary to verify this.
Now we discuss a change of units, we want to get rid of the c, as it would only complicate our equations. Instead, physicists use a system of units in which c = 1. Since now we are measuring time in meters, we define a meter of time as the time taken for light to travel 1 metre, i.e. 1/c = 1. The Lorentz transformation in our new system of units becomes

$$x' = \frac{x - ut}{\sqrt{1 - u^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - ux}{\sqrt{1 - u^2}}$$

2.4 4-vectors and Space - Time

Notation - we reflect coordinates (x, y, z, t) as (x_0, x_1, x_2, x_3) or more generally as x_i.

We now analyse the 4-vector space geometrically. We now define an event as a unique (x, y, z, t). Consider 2 events A$(x_1, y_1, z_1, t_1) and B(x_2, y_2, z_2, t_2)$. Now, since nothing can travel faster than light, if a particle passes through the event A and the event B, we have

$$\Delta x^2 + \Delta y^2 + \Delta z^2 \leq c^2 \Delta t^2$$

In our new unit system we have

$$\Delta x^2 + \Delta y^2 + \Delta z^2 \leq \Delta t^2$$

Therefore, we define the $interval$ between two events as

$$\Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta t^2$$

Which is the same as

$$\eta_{ii}(x_i)^2$$

If the interval between two events is positive, it is called as a $timelike$ interval, a $spacelike$ interval if it is negative and a $null$ interval if it is positive.

Using this we can draw space time diagrams, which are incredibly useful for a conceptual understanding. We plot x on the x axis (Consider motion along the x direction only for this case, we will see that the same arguments can be extended to y and z too) and t on the y axis.

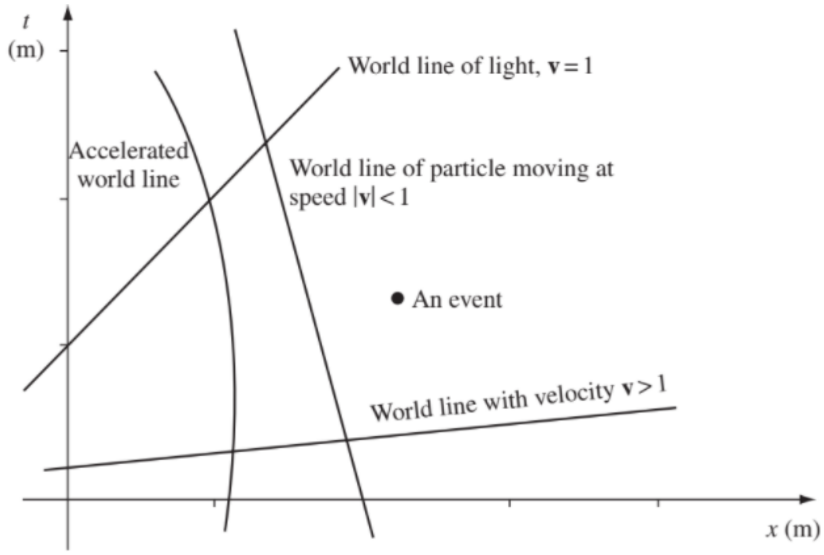

Figure 2.2

We should note the following :
1. A photon always travels at angles of $\frac{\pi}{4}$ (magnitude) to the axis.
2. For any event A, we can designate it's *true past* as the locus of events that have a spacelike interval with A and are on the negative side of the time axis with respect to A. Similarly we can define A's *true future*. All the events in the remaining region cannot affect A or be affected by A.

Chapter 3

Tensor Analysis

3.1 One forms and definition

Before defining a tensor, we first define a one-form, which can be thought of as a function that takes a vector as an input, and returns a scalar. The components of a one-form in a frame can be defined as the outputs obtained when our vector input are the basis vectors of the frame. Consider a tensor A. We define the α^{th} component of A as

$$A^{\alpha} = A(e_{\alpha})$$

. Similarly we can define the basis of a tensor - w_{α} such that

$$A = A^{\alpha} w_{\alpha}$$

Now we define a regular $\binom{M}{N}$ tensor. A $\binom{M}{N}$ tensor can be thought of as a function that takes M one-forms and N vectors as input and returns a scalar. The regular procedure for finding the components and the basis in a frame applies.

3.2 The Metric Tensor

Consider a flat space, with our coordinates being defined using x and y coordinates. We can define the interval between 2 points as

$$ds^2 = dx^2 + dy^2$$

Now, this obviously does not hold for a curved space. We introduce the metric tensor g as a correction to the Pythagoras theorem, to calculate intervals between 2 points in a curved space. We define the components of g to be such that

$$ds^2 = g_{uv} dx^u dx^v$$

Note that for flat space,

$$g_{uv} = \delta^u_v$$

.This also allows us to define the dot product of 2 vectors as

$$A.B = g_{uv} A_u B_v$$

.

3.3 Fluids

Consider a fluid floating around in free space, with a number density n in the MCRF. Now if an observer O observes the dust to be moving at a velocity v_x along the x direction, because of length contraction, the observed number density clearly becomes

$$n' = \frac{n}{\sqrt{1 - v^2}}$$

Now, we can define the flux through a surface as the number of particles crossing a surface per unit time per unit area. If we take the normal to the surface be $\boldsymbol{n}$ (not related to the number density n), we have :

$$\phi = \frac{n * A * (\boldsymbol{v}.\boldsymbol{n})}{A} = nA(\boldsymbol{v}.\boldsymbol{n})$$

If in O's frame, velocities of the dust particles are v_x, y_y, v_z, we have the true velocity, $\boldsymbol{U}$, as

$$(u) = \frac{1}{\sqrt{1 - v^2}}, \frac{v_x}{\sqrt{1 - v^2}}, \frac{v_y}{\sqrt{1 - v^2}}, \frac{v_z}{\sqrt{1 - v^2}}$$

We define the Number Flux-4 vector, $\boldsymbol{N}$, as :

$$\boldsymbol{N} = n\boldsymbol{U}$$

The flux can then be computed as $< \boldsymbol{n}, \boldsymbol{N} >$. This is in sharp contrast to the regular dot product we use in 3 dimensions. We can calculate $\boldsymbol{n}$ for a surface ϕ as

$$\boldsymbol{n} = \frac{\tilde{d}\phi}{|\tilde{\phi}|}$$

We now consider perfect fluids, those that do not conduct heat and are not viscous.

3.4 The Stress - Energy Tensor Consider a 4 by 4 matrix, whose (u,v) element represents the flux of the u^{th} momenta through a surface of constant x_v. Then in application to general fluids, note that
1. The (0,0) element simply represents the energy density of the fluid.
2. The (0,i) element represents the energy flux through a surface of constant x_i.
3. The (i,0) element represents the i^{th} momenta density.
For an observer O in the MCRF of a small fluid element:
1. All (0,i) elements are 0. Since the fluid does not conduct heat, the only possible energy flow is that of fluid particles themselves, which does not happen in the MCRF.
2. All (i,0) elements are 0 since the observed momenta is 0 in each spatial direction.
3. All the diagonal (i,i) elements must be p, the pressure (The pressure is uniform over a small fluid element), because the flux of the i^{th} momenta over a surface of constant $x_i$is simply the perpendicular pressure along the normal. 4. All the other elements are 0, since we do not have any forces acting on a spatial surface parallel to it.
A tensor whose components are the same as the elements in this matrix is called the stress-energy tensor(**T**). Therefore, we have :

$$\mathbf{T} = \rho U \otimes U + p\mathbf{g}^{-1}$$

where ρ is the energy density. This can be seen by the fact that the RHS has the same components as the matrix defined earlier.

Chapter 4

Curvature

We now come upon the limitations of STR, one the most prominent being that there is no global inertial frame. However, we can define local inertial frames. Consider a small enough volume element that the gravitational field is constant. Then, a frame free-falling in the field is an inertial frame. However, because of the gravitational field not being uniform throughout the universe, this cannot serve as a global inertial frame.

4.1 Redshift experiment

Consider the following process : a mass falls from a height (small enough that g is uniform). At the ground, the energy of the mass is converted into a photon travelling upwards with 100 percent efficiency. When the photon reaches the top, it is converted into a rest mass, again with 100 percent efficiency. To avoid perpetual motion, this rest mass must be same as the original, m_o.
Considering the energy of the photon at top and bottom being $E^{'}$ and E respectively, we have :

$$\frac{E^{'}}{E} = \frac{m_o}{m_o + m_o gh} = \frac{1}{1 + gh}$$

i.e. Hence we see that, from the ground frame, the photon is red-shifted at the top.
Another interesting point to note is that at the bottom the time gap between 2 crests of the light wave is

$$\Delta t^{'} = \frac{1}{\nu}$$

While at the top it is

$$\Delta t = \frac{1}{\nu^{'}}$$

So

$$\frac{\Delta t'}{\Delta t} = 1 + gh$$

. This means that time runs slightly faster at heights than it does on ground. This is a clear indication that gravitational fields have an effect on space time that has not been accounted for in STR.

4.2 Christoffel symbols
For any coordinate system, we have :

$$\boldsymbol{V} = V^{\alpha}\boldsymbol{e}_{\alpha}$$

$$\frac{d\boldsymbol{V}}{dx^{\beta}} = \boldsymbol{e}_{\alpha}V^{\alpha}_{,\beta} + V^{\alpha}\frac{d\boldsymbol{e}_{\alpha}}{dx^{\beta}}$$

We characterise the term $\frac{d\boldsymbol{e}_{\alpha}}{dx^{\beta}}$ as $\Gamma^{u}_{\alpha\beta}\boldsymbol{e}_{u}$.
Rearranging indices, we get:

$$\frac{d\boldsymbol{V}}{dx^{\beta}} = \boldsymbol{e}_{\alpha}(V^{\alpha}_{,\beta} + V^{u}\Gamma^{u}_{\alpha\beta}) = V^{\alpha}_{;\beta}\boldsymbol{e}_{\alpha}$$

We also define the covariant derivative of a vector as a $\begin{pmatrix}1\\1\end{pmatrix}$ tensor which maps $\boldsymbol{e}_{\beta}$ into the vector $\frac{dV}{dx^{\beta}}$.
It's components can be shown to be :

$$(\boldsymbol{\nabla}V^{\alpha}_{\beta}) = V^{\alpha};\beta = V^{\alpha}_{,\beta} + V^{u}\Gamma^{u}_{\alpha\beta}$$

The components of the covariant derivative of a tensor p can be shown to be :

$$\boldsymbol{\nabla}(p)_{\alpha,\beta} = p_{\alpha;\beta} = p_{\alpha,\beta} - p_{u}\Gamma^{u}_{\alpha\beta}$$

Playing around with indices also gives us a relation between the metric and the Christoffel symbols :

$$\Gamma^{\gamma}_{\beta u} = \frac{g^{\alpha u}}{2}(g_{\alpha\beta,u} + g_{\alpha u,\beta} - g_{\beta u,\alpha})$$

4.3 Ri emann Tensor
We know try to analyse the curvature of a surface quantitatively. Consider 3 points A, B, C on a sphere, B at the pole, the other 2 on the equator with a longitudinal difference of $\frac{\pi}{2}$. Consider a vector $\boldsymbol{v}$ at A pointing along the equator. Divide the 'curved' triangle into many points along it's circumference and parallel transport a vector (at each point draw a vector parallel

to the vector at the previous point). On circling back to A we find that the vector is not the same as the original! This does not happen in a flat space. If we do the same thing on a differential area, in our case a parallelogram ABCD of sides dx_1, dx_2, we can quantize the curvature of any surface in terms of the change of the vector which is parallel transported. We define the Riemann tensor as the $\binom{1}{3}$ tensor which when fed the vector V, the side da of the parallelogram parallel to $\boldsymbol{e_{x_\alpha}}$, the side db of the parallelogram parallel to $\boldsymbol{e_{x_\beta}}$ and the basis one form w_α , gives the change in the α^{th} component of vector V on a complete parallel transport around the parallelogram. It can be showed that

$$R^\alpha_{\beta\mu v} = \Gamma^\alpha_{\beta v,\mu} - \Gamma^\alpha_{\beta\mu,v} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta v} - \Gamma^\alpha_{\sigma v}\Gamma^\sigma_{\beta\mu}$$

We define the Ricci tensor as the contraction of the Riemann tensor on the first and third indices :

$$R_{\mu v} = R^\alpha_{\mu\alpha v}$$

Based on this we define the curvature scalar :

$$R = g^{\mu v} R_{\mu v}$$

However, given that these may seem intimidating, in a qualitative sense, one can treat the Riemann and the Ricci tensor as the same : they both uniquely describe the curvature of a surface quantitatively. The curvature scalar is just as an extension of this : a measure of the curvature of a surface. Given my own weak grasp of the mathematics here, I will qualitatively describe the results of this.

4.4 Geodesics

A geodesic is simply the shortest line (not necessarily straight) which represents the shortest distance between 2 points on a curved surface, along the surface. It can be shown that along a geodesic :

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma$$

Where Γ represents an expression of a christoffel symbols with some indices (Remember, we're doing a qualitative discussion now). We know propose a parallel between this and the relation :

$$F = m\frac{d^2x}{dt^2}$$

The conclusion here is that We can draw a parallel between Γ and a force. This will help in our qualitative discussion on Einstein's field equations.

Chapter 5

Einstein's Field Equations

In essence, we now want to relate gravity to curvature, we want an equation which can describe the curvature of a surface based on some aspect of gravity at a point. Consider the Newtonian world, it is essentially an approximation of Einstein's world when the gravitational forces are small enough that they cannot induce relativistic speeds. We have :

$$\Gamma = \frac{g_{ad}}{2}(\frac{dg_{dc}}{dx_b} + \frac{dg_{db}}{dx_c} + \frac{dg_{bc}}{dx_d}$$

This can be derived using the results in the section 4.2 (again, we're doing a qualitative discussion now). Now, in the Newtonian world, the derivative of the metric wrt a spatial coordinate is 0. Therefore, in a Newtonian world, this reduces to :

$$\Gamma^{\gamma}_{\beta u} = \frac{g^{00}}{2}$$

Now we make use of our parallel between force and the christoffel symbol we proposed in the previous chapter. Using $F = -\frac{d\phi}{dx}$ we see that :

$$g_{00} = -2\phi$$

We ignore the integration constant here. Phi is a potential here.
Now, in gravitational forces :

$$\nabla^2\phi = 4\pi G\rho$$

So we have :

$$\nabla^2 g_{00} = 8\pi G\rho$$

Now, since we're using units with c = 1, we can generalise this to (not a mathematically rigorous argument) :

$$G_{uv} = 8\pi G T_{uv}$$

Here, we define G as the Einstein tensor. Now, for the LHS, we must have $\nabla G_{uv} = 0$ for energy conservation. Using $\nabla R_{uv} = \frac{g_{uv}}{2} R$, we can try

$$R_{uv} - \frac{g_{uv}}{2} R = \frac{8\pi G T_{uv}}{c^4}$$

We use c^4 to make the equation dimensionally consistent.
Now, years before his work on GR, Einstein had proposed a scalar known as the cosmological constant to explain large scale movement of galaxies. At the moment, he considered it a blunder, but while working on GR, he saw it's application in his field equations, to account for the fact that $\nabla g_{uv} = 0$. Therefore he proposed :

$$R_{uv} - \frac{g_{uv}}{2} R + \Lambda g_{uv} = \frac{8\pi G T_{uv}}{c^4}$$

This represents the Einstein's Field Equations. Note that because of symmetry of g_{uv} and R_{uv} we have 10 different equations instead of 16.
Using a lot of daunting mathematics, we can derive the weak-field Einstein's equations, i.e. where the gravitational fields are weak. The key result is :

$$ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2)$$

Chapter 6

Application to black holes

We now qualitatively discuss the key results one obtains when the field equations are applied in context of a black hole. The schwarzschild metric can be defined as :

$$ds^2 = -(1 + 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2 d\omega^2$$

where

$$d\omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$$

6.1 Orbits

Using this one can derive the trajectory of a particle and a photon in a black hole's gravitational field.

For a particle :

$$(\frac{dr}{d\tau})^2 = E^2 - (1 - \frac{2M}{r})(1 + \frac{L^2}{r^2})$$

For a photon :

$$(\frac{dr}{d\tau})^2 = E^2 - (1 - \frac{2M}{r})(\frac{L^2}{r^2})$$

Leaving aside these intimidating equations, the key result we can derive is that for a particle only 2 stable circular orbits exist, and only for $L < 12M^2$. The radii are :

$$R = \frac{L^2}{2M}(1 \pm (1 - \frac{12M^2}{L^2})^{\frac{1}{2}}$$

For a photon we obtain only one stable circular orbit of radius :

$$R = 3M$$

We define the schwarzschild radius as the radius inside which no photon can escape the gravitational field. We get it as :

$$R = 2M$$

This is coincidentally the same as what Newton's laws would predict.

6.3 Free fall of a body to the singularity
An interesting implication is that if we analyse the motion of a freely falling body towards the black hole, in terms of the true time of the freely falling of the observer, we see that the time taken to reach the schwarzschild radius is finite. However, in terms of another observer's time (the coordinate time), we see that the expression diverges, i.e. the falling observer takes an infinite time to reach the singularity.

Gravitational lensing
Another interesting application is the gravitational bending of light. It can be derived that the change in the direction (change in angle) for a photon whose radius of closest approach is b is :

$$\Delta\phi = \frac{4M}{b}$$

This bending of light is equivalent to gravitational bodies acting as lenses. For instance, this phenomena can produce 2 different images of a star behind the sun for an observer on earth. Lensing can also magnify images, by focusing light on them : this is called magnification. One can also map the mass distributions of clusters of galaxies using this : a key result is that the galaxies have a mass far greater than the mass of it's luminous stars : this is accounted for by dark matter.

Chapter 7

Cosmology

6.1 Geometry of the Universe

We assume the following postulates to be true :
1) The universe is isotropic : it is radially symmetric from any point at large enough scales.
2) The universe is homogeneous : at large enough scales, the universe looks same at all points.
3) The relative speeds between galaxies are non-relativistic. Obviously, these do not apply for smaller scales, say that of a solar system, or even that of a galaxy. The mass density in any differential volume would be of the form of Dirac-Delta functions, with particular points (stars, nebulae etc) representing a finite non zero density. However at large enough, the distribution evens out.
Based on this, we consider various geometries which satisfy these properties.

6.2 Flat geometry

Newton imagined the universe as a grid, with galaxies at particular fixed points on a grid. The grid can either expand or contract, inducing changes in the distances between galaxies, but the coordinates (those defined by the grid) of the galaxies in would remain the same. This is akin to imagining galaxies as marked dots on an expanding balloon.
This idea is validated by the Hubble's law : the velocity of separation between galaxies is proportional to the distance between them. There are obviously exceptions to this : the Milky Way and the Andromeda galaxies in fact are on a collision course. However at the scales of the expansion of the universe, this registers as mere noise.
Let the grid coordinates be x, y, z. We can define the grid distance r using the Pythagorean metric (since the speeds between galaxies are non-

relativistic). Now, we scale this interval by a value a to get the actual distance between galaxies (d). Note that this value changes with time, increasing with expansion of the universe, decreasing with contraction.
Differentiating, we get :

$$v = a'r$$

Using Hubble's law $v = Hd$, we get :

$$\frac{a'}{a} = H$$

Now, consider a sphere of radius d around any point (say the origin), say the earth. The net gravitational acceleration on any mass on the sphere is :

$$A = \frac{GM}{a^2r^2}$$

Since the universe is isotropic around any point, the mass in the sphere is radially symmetric. So we can simply use Newton's gravitation here. Double differentiating our previous equations, we get :

$$A = a^{"}$$

Implying :

$$\frac{a"}{a} = \frac{GM}{a^3r^3}$$

Here, $\frac{M}{a^3r^3}$ is a measure the density ρ of the sphere (by extension, the universe). This however is variable. However the density of a grid cell is constant since the galaxies are fixed. So we have :

$$\rho = \frac{\nu}{a^3}$$

Here ν is a constant. So we have :

$$\frac{a"}{a} = -\frac{4\pi}{3}G\rho = -\frac{4\pi}{3a^3}\nu$$

Another important result is that for r = 1, using energy conservation, we get :

$$\frac{a'^2}{a^2} - \frac{8\pi G\nu}{3a^3} = \frac{c}{a^2}$$

Here c is $\frac{E}{2m}$, where E is the total energy of the mass (conserved) An important result is that for E = 0 (by extension, the total energy of the entire universe is 0),on solving the resulting differential equation we get that :

$$a \propto t^{\frac{2}{3}}$$

For positive E, we get that the universe expands as in the case of zero energy at first, but later it's expansion velocity tapers off to a constant value. For negative E, the universe expands, then the expansion comes to a halt, then the universe contracts. Using the cosmic redshift, one can derive that for photons :

$$\frac{a'^2}{a^2} - \frac{8\pi G\nu}{3a^4} = \frac{c}{a^2}$$

Here, for c = 0, it can be derived that the expansion is proportional to $t^{\frac{3}{2}}$. Since our universe consists of both radiation and matter, we have :

$$\frac{a'^2}{a^2} = \frac{c_m}{a^3} + \frac{c_R}{a^4}$$

This equation is a form of energy conservation, with the terms of $\frac{c_m}{a^3}$ and $\frac{c_R}{a^4}$ representing the change in energy in the matter and radiation components, while $\frac{a'^2}{a^2}$ represents the kinetic energy of the expansion of the universe.

6.3 Spherical geometry

We define a *1-sphere* as simply a circle, a *2-sphere* as an aggregate of these 1-spheres. The surface of the earth is a 2 sphere, it is a 2 dimensional surface. Similarly one can define a 3-sphere. While we human beings cannot visualize a 3-sphere graphically, we can do so with equations.

Before we discuss the details of the spherical geometry, we define the metrics of these surfaces.

For a 1 sphere we have :

$$ds^2 = dr^2 + r^2 d\theta^2$$

We define the $d\theta^2$ as the metric *on* the circle, and write it as $d\Omega_1^2$.

For a 2 sphere, consider the origin as say the south pole. We define each point on the sphere (consider the radius of the 2 sphere to be 1) by the angle subtended by it and the origin at the centre - r. The locus of the points with the same r is a 1-sphere or a circle, we uniquely identify any point here by the angle θ of the 1-sphere.

So we have :

$$ds^2 = dr^2 + \sin^2 r d\Omega_1^2$$

Or

$$d\Omega_2^2 = dr^2 + \sin^2 r d\Omega_1^2$$

$d\Omega_2^2$ represents the metric of the 2 sphere.
Similarly for a 3-sphere, we have :

$$d\Omega_3^2 = dr^2 + \sin^2 r d\Omega_2^2$$

For a 2-dimensional universe, consider that the entire universe lies on a 2 sphere of unit radius. Consider yourself as the origin. Now look out from your position at a distance of say 0.2. Graphically, this is equivalent to drawing an circle of radius 0.2 from a point O on the circumference of a circle of unit radius. Note that you can only have a limited angular vision. Now as you increase the radius of your vision to 2, you see that the size of the one-sphere you draw on the 2-sphere increases, then decreases to 0. Specifically, the radius of the 1-sphere is a $\sin$ function, $\sin r$. Consider a galaxy of diameter $d << 1$. The angular diameter of it can be calculated from the metric of the 2-sphere, we get it as $\frac{d}{\sin r}$. So we see that as r increases from 0 to infinity, the angular diameter of the same galaxy decreases from infinity, then blows up again. This is a key feature of the spherical-geometry of the universe.
These properties can be extrapolated to a 3-sphere, which is a candidate geometry for our universe.

Stereo graphic projection of a spherical universe
Consider a 2-dimensional universe again. We place our 2-sphere on an infinite plane and consider the origin as the south pole. For any point on the sphere, draw a line connecting it to the north pole and extend it backwards to meet the plane. The intersection point is a unique representation of the point on the sphere.
Here we can once again see how a galaxy's apparent angular diameter decreases and later increases with r. The corresponding points for the points close to the north pole tend towards infinity.

6.4 Hyperbolic geometry
The concept here is the same as the spherical geometry, except that we imagine our universe being on a 3-dimension hyperboloid. The metric for a 2-D hyperboloid is :

$$dH_2^2 = dr^2 + \sinh^2 r d\Omega_1^2$$

For a 3-D hyperboloid, this becomes :

$$dH_3^2 = dr^2 + \sinh^2 r d\Omega_2^2$$

Considering a galaxy of diameter d as before, we get the angular dispersion as $\frac{d}{\sinh r}$. Here, we see that the angular dispersion of the galaxy decreases exponentially for large r.
We can discuss the stereo graphic projection here too. Consider the hyperboloid $z^2 - x^2 - y^2 = 1$. We draw the tangent plane z = 1. For any point on the hyperboloid, connect it to the origin and mark it's intersection with the tangent plane. This point uniquely represents the point on the hyperboloid. Note that now objects at infinity on the hyperboloid lie on the ring $x^2 + y^2 = 1, z = 1$.

6.5 Metrics in special relativity
Extrapolating our results from STR we have :
Flat space-time :

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

Spherical space-time :

$$ds^2 = -dt^2 + a^2(t)(d\Omega_3^2)$$

Hyperbolic space-time :

$$ds^2 = -dt^2 + a^2(t)(dH_3^2)$$

6.6 Friedmann equation
On solving the 00 component of the Einstein's field equations by writing the LHS in terms of derivatives of the metric, we get :

$$(\frac{a'}{a})^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}$$

where

$$k = \begin{cases} 1 & \text{for a spherical universe} \\ 0 & \text{for a flat universe} \\ -1 & \text{for a hyperbolic universe} \end{cases}$$

This is surprisingly the same as what Newton predicted, this is because at small scales any piece of space looks flat. This important result is known as the *Friedmann equation*.
Considering Newton's results, we have that $k = 0$ represents a universe at just the escape velocity for it to forever expand, $k = 1$ represents a universe that expands and then contracts and $k = -1$ represents a universe that forever expands.

6.7 The equation of state for the universe
Assuming that the universe follows the standard equation of state $P = W\rho$, using basic thermodynamic principles we can derive that

$$\rho \propto \frac{1}{a^{3(1+W)}}$$

For a matter dominated universe, we have $W = 0$, for a radiation dominated universe we have $W = \frac{1}{3}$.
6.8 Dark energy
This represents the energy density present in vacuum. We will see later that the existence of this was needed to explain the redshift observations in our universe. The very fact that this energy is present in vacuum is the reason it is called *dark* energy. By definition it is constant, it does not change with the expansion of the universe. We represent it as

$$\rho_o = \frac{3}{8\pi G}\Lambda$$

where Λ is the same cosmological constant that showed up in Einstein's field equations. We now see why Einstein added it to his equations, it was necessary to account for the energy density inherent to the vacuum of space.
We will later see how exactly one can prove this energy exists, we right now just assign the vacuum energy density a number, which would simply be 0 if there was no such energy density.

6.9 Devising a model for the universe
We have the Friedmann equation as :

$$(\frac{a'}{a})^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}$$

Here, ρ is the energy density of the universe. This could be due to matter (both luminous and dark matter are treated as the same), radiation, or the vacuum energy density.

$$(\frac{a'}{a})^2 = \frac{8\pi G}{3}(\rho_M + \rho_R + \rho_o) - \frac{k}{a^2}$$

Using $\rho_M \propto a^{-3}$, $\rho_R \propto a^{-4}$ and $\rho_o = \frac{3}{8\pi G}\Lambda$ we have :

$$H^2 = \frac{c_M}{a^3} + \frac{c_R}{a^4} + \Lambda - \frac{k}{a^2}$$

Dividing by H^2 we have :

$$\Omega_M + \Omega_R + \Omega_\Lambda + \Omega_k = 1$$

These unique constants tell us about the proportion of each form of energy of the universe. A unique set of values of these defines the universe, it's past and it's future.
Lets say we have the values of $\Omega_M, \Omega_R, \Omega_\Lambda, \Omega_k$ at the present time. We also have the value of H^2 at the present time. We scale the ratio numbers by H^2 to get the values of $\frac{c_M}{a^3}, \frac{c_R}{a^4}, \Lambda, \frac{-k}{a^2}$ at the present time. We use the value of a at the present time to calculate c_M, c_R, k. Then solving the IVP :

$$\frac{c_M}{a^3} + \frac{c_R}{a^4} + \Lambda - \frac{k}{a^2} = (\frac{a'}{a})^2$$

$$a(0) = a(\text{today})$$

We take the present time to be t = 0.
This allows us to uniquely define a universe using the 4 numbers $\Omega_M, \Omega_R, \Omega_\Lambda, \Omega_k$.

6.10 Checking if a model of the universe is correct
Now we use 2 variables to check if a given model of the universe is correct : number of galaxies dn with redshift between Z and $Z + dZ$, and the average luminosity of galaxies with redshift between Z and $Z + dZ$.
We can calculate $\frac{dn}{dz}$ for a universe using the metric of the geometry of the universe, the density of galaxies in space, a at the present time, all of which can be observed. Similarly we can calculate the average luminosity of galaxies at distance r using the metric of the geometry of space. We match these results with the observed values of these quantities in our universe.
Using this algorithm, the present values of $\Omega_M, \Omega_R, \Omega_\Lambda, \Omega_k$ have been calculated to be :

$$\Omega_R = 0$$

$$\Omega_M = 0.31$$

$$\Omega_\Lambda = 0.69$$

$$\Omega_k = 0 \pm 0.01$$

6.11 Dark Matter
Consider a spherical galaxy. Most of the luminous matter is aggregated at the centre. Therefore, following Newton's laws of gravitation, one would expect to see the tangential velocity (magnitude) fall off as $\frac{1}{\sqrt{r}}$. However it was observed that the velocity more or less remains the same. Therefore

one would expect that the mass distribution varies like $M(r)$ (the mass inside a sphere of radius r) to vary as r. However, since this 'extra mass' does not aggregate towards the centre of the galaxy, one would expect it to be very weakly interacting. This hard-to-detect form of matter is called as *dark matter* and outweighs the luminous matter roughly six to one.