Global-coarsening multigrid for *hp*-adaptive FEM

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h- vs p-adaptive finite elements

Global refinement:	Guaranteed to drive the error to zero	
Adaptive refinement:	Achieve the same error with less work	
h-adaptation:	dynamic cell sizes	suited for irregular solutions
p-adaptation:	dynamic polynomial degrees	suited for smooth solutions

Figure: Two rectangular cells of which right cell is refined.
Symbols indicate support points on vertices ($\bullet$), lines ($\square$) and quadrilaterals ($\circ$).
Left: h-adaptation with Q_2 elements, *Right:* p-adaptation with Q_2/Q_4 elements

hp-adaptive finite elements

hp-adaptation: Choose both cell size and polynomial degree adaptively

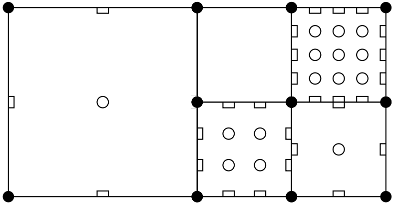

Figure: *hp*-adaptation with $Q_1 \ldots Q_4$ elements.

Superior convergence behavior known since mid-1980s:
(Babuška and Suri 1990; Guo and Babuška 1986)

$$\left\| u - u_{\mathrm{hp}} \right\|_{H^1(\Omega)} \leq C \, \frac{h^p}{p^p} \, \left\| u \right\|_{H^{p+1}(\Omega)}$$

$$\left\| u - u_{\mathrm{hp}} \right\|_{H^1(\Omega)} \leq C(u) \, e^{-b \, N_{\mathrm{dofs}}^{\alpha}}$$

Example: Laplace on L-domain

Elliptic problems have singularities at *non-convex* parts of boundary.

Consider L-shaped domain:

$$\Omega = [-1,1]^2 \setminus ([0,1] \times [-1,0])$$

Manufacture Laplace problem:

$$\begin{aligned} -\nabla^2 u &= 0 \qquad \text{on} \quad \Omega \\ u &= \bar{u} \qquad \text{on} \quad \partial\Omega \end{aligned}$$

$$\begin{aligned} \bar{u} &= r^{2/3} \sin(2/3\ \varphi) \\ \|\nabla \bar{u}\| &= r^{-1/3} \end{aligned}$$

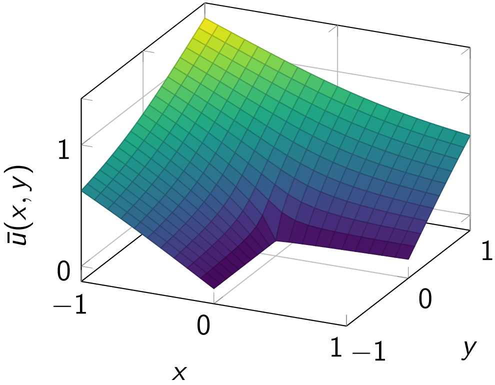

Figure: Manufactured solution.

Laplace: Error convergence

Experiments confirm exponential convergence (Fehling 2020).

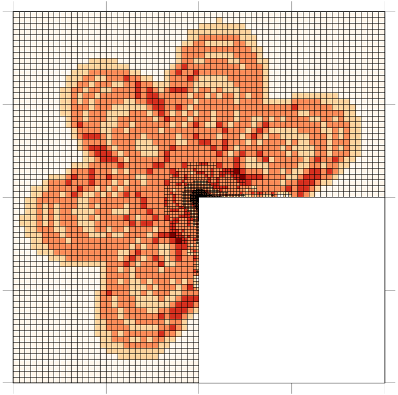

Figure: *hp*-discretization.

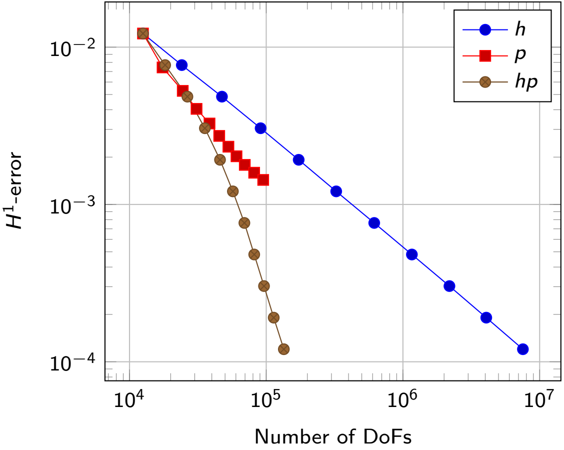

Figure: Error convergence for different refinement types.

Examples in `deal.II`

`step-27` solves Laplace eq.
with successive *hp*-refinement

simple – only 275 lines of code
(measured with `cloc`)

Useful in other applications:

- `step-46`: multi-physics problems
- `step-85`: CutFEM
- `step-90`: Trace FEM

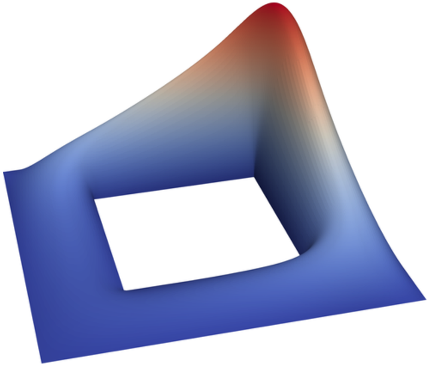

Figure: `step-27` solution.

Implementation challenges for parallelization

- Algorithms to decide between *h*- and *p*- refinement
- Unique and replicable enumeration of degrees of freedom
- Weighted load balancing on decomposed domain
- Choice of robust and efficient solver

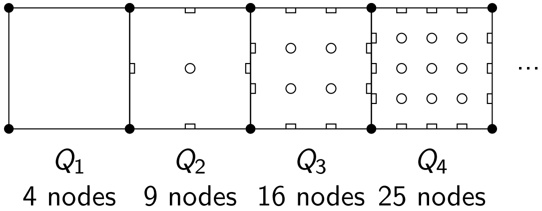

Figure: Different finite elements and their number of nodes in 2D

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First try: AMG

Use what has proven successful before: Algebraic Multigrid (AMG).

Solver performance drops with increasing fragmentation of polynomial degrees.

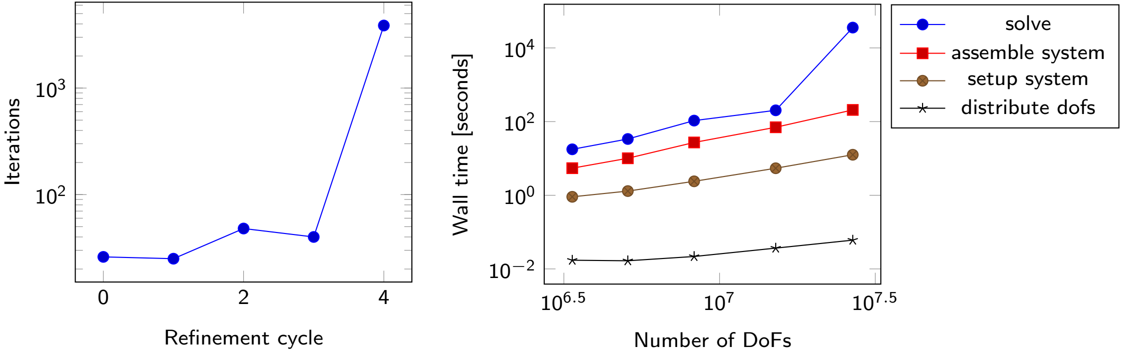


Figure: Iterations and scaling for consecutive refinements using `PreconditionAMG`.

Alternative: Geometric Multigrid (GMG)

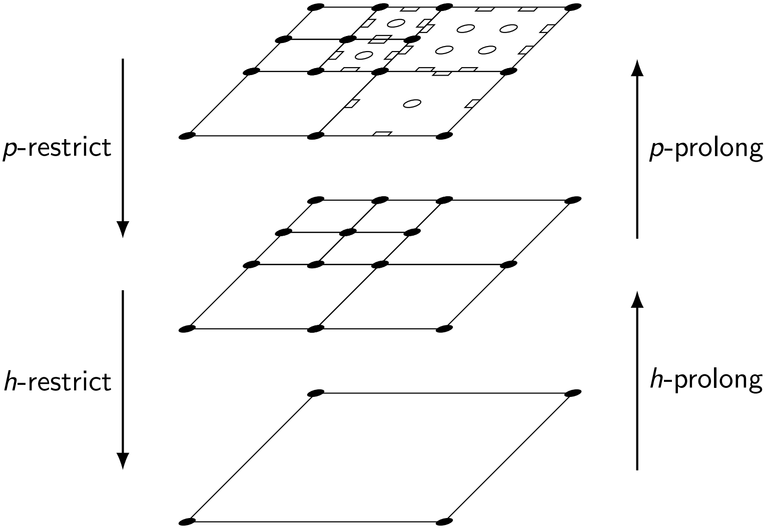
Idea: Use advanced methods: Geometric Multigrid (GMG)

- build hierarchy in different polynomial degrees and mesh sizes (Mitchell 2010)
  - reduce maximum polynomial degree first, then coarsen mesh
- solve coarse system with AMG (Fehn et al. 2020) $\rightarrow$ *Hybrid*-GMG

Matrix-free

Allows for use with *matrix-free* methods!

Hybrid-GMG



Configurations that lead to bad conditioning

Using diagonal as smoother yields high eigenvalues.

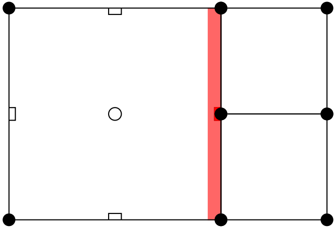


Figure: Coarse Q_2 cell neighbors fine Q_1 cells.

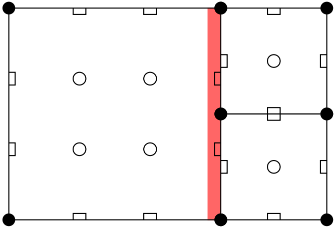


Figure: Coarse Q_3 cell neighbors fine Q_2 cells.

Continuous elements

faces on a coarse, high p cell neighboring a fine, low p cell are problematic

Eigenvalues

Use `LAPACK` to investigate eigenvalues, see `xSYGV`.
Eigenvalue distribution looks odd. Discussion on mg-ev-estimator/#4.

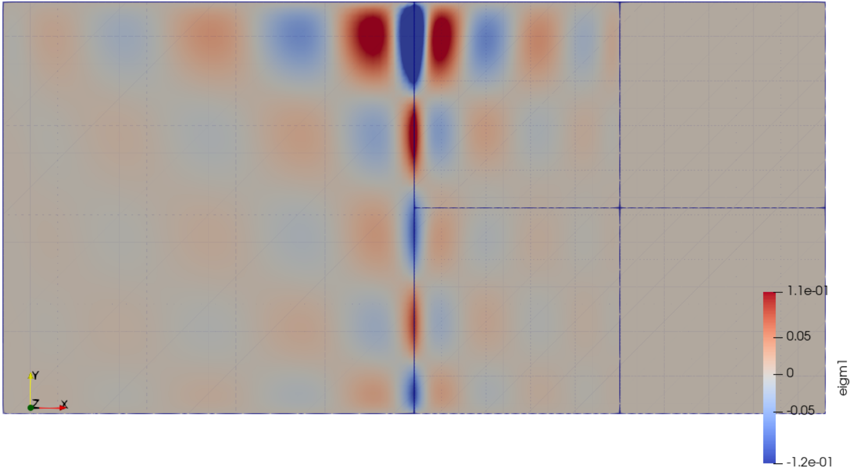


Figure: Eigenvector of largest eigenvalue with diagonal smoother.

Additive Schwarz Methods (ASM)

Constrained DoFs are problematic. We need more information than just the diagonal.

Use block-Jacobi method.

Requires some `SparseMatrix` to extract cell-local matrices as blocks.

Current design:

1. Build patches on locally owned cells with locally active DoFs.
2. Build sparse matrix with patch DoFs only.
3. Build patch matrices with `SparseMatrixTools::restrict_to_full_matrices`.
4. In `vmult`, apply patch matrices on patch DoFs, and inverse diagonal on others.

Eigenvalues with patch

Block-diagonal smoother relaxes eigenvalues thus improves conditioning.

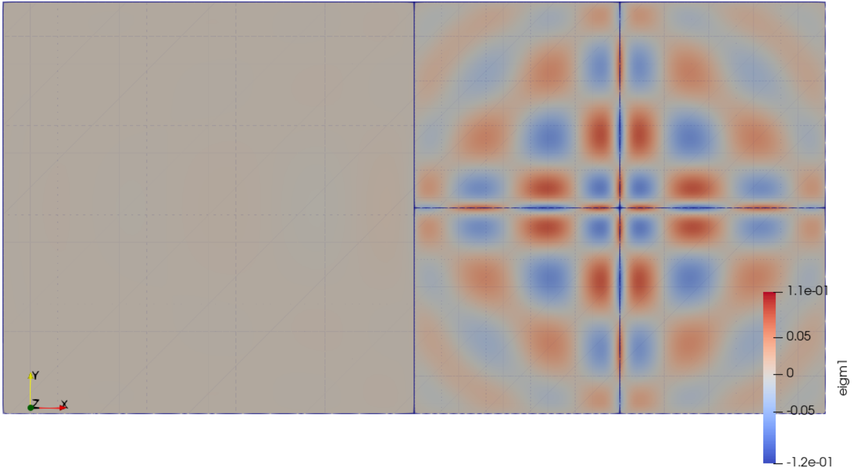


Figure: Eigenvector of largest eigenvalue with block-diagonal smoother.

Constraints

On distributed 3D domains, constraints on locally active DoFs *might* differ.

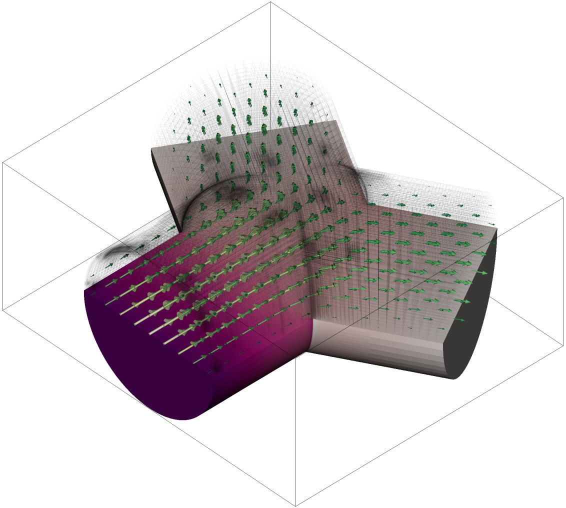
Negative effect on solver convergence (iteration count).

Make sure your constraints are consistent in *parallel hp*–applications!

Useful functions:

- to check: `AffineConstraints::is_consistent_in_parallel`
- to correct: `AffineConstraints::make_consistent_in_parallel`

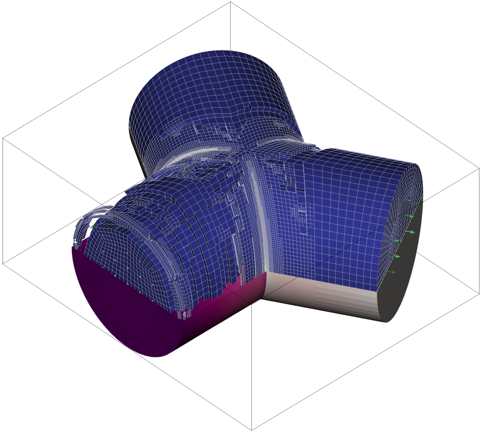
Example: Stokes on Y-pipe



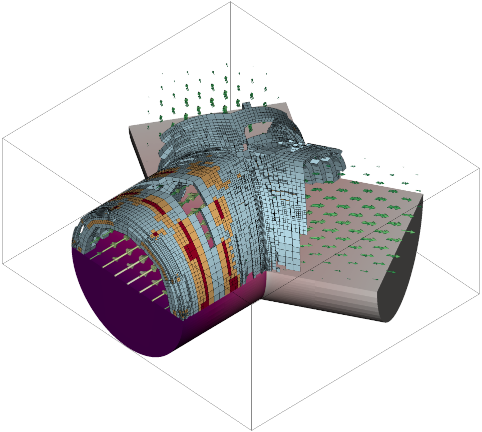
Construct 3d test problem with solution both smooth and singular in different parts of the domain.

$$
\begin{aligned}
-\nabla^2 \mathbf{u} + \nabla p &= 0 \\
-\nabla \cdot \mathbf{u} &= 0
\end{aligned}
$$

Stokes: *hp*-Discretization



Low polynomial degrees: In the interior (solution ≈ Poiseuille flow) and at non-convex parts of the boundary.



High polynomial degrees: In smooth parts of the flow.

Stokes: Preconditioner design

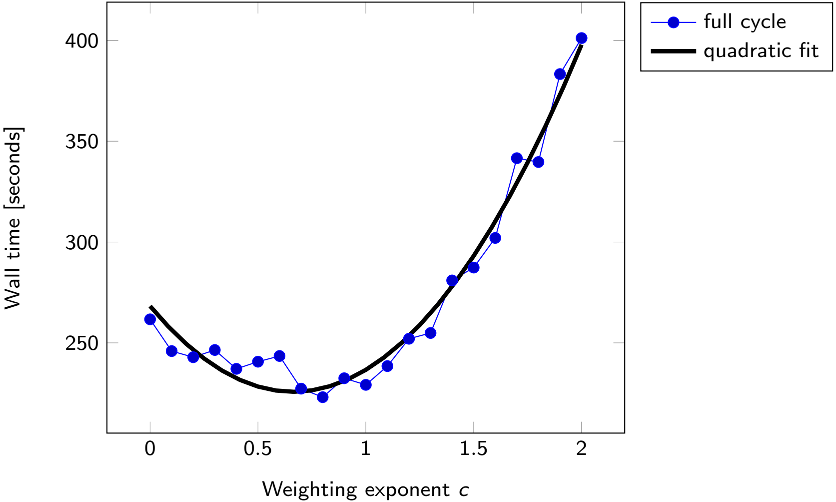
Silvester–Wathen type preconditioner P^{-1} on system matrix M:
(for details see `step-31` and `step-32`)

$$P^{-1}M = \begin{pmatrix} A^{-1} & 0 \\ S^{-1}BA^{-1} & -S^{-1} \end{pmatrix} \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

- Approximate Schur complement $S = BA^{-1}B^T$ by mass matrix $S^{-1} \approx M_p^{-1}$
- Approximate action of A^{-1} by multigrid preconditioner $\bar{A}$
- Solve linear system with `FGMRES`

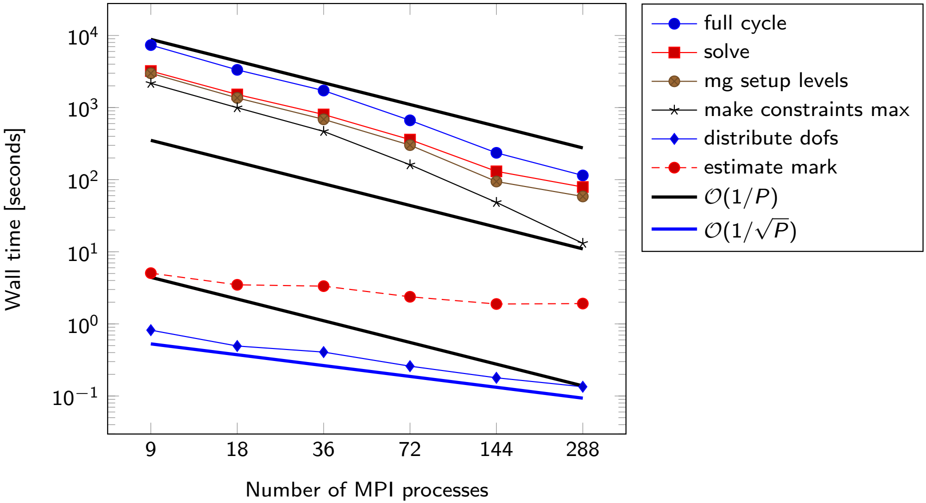
Stokes: Weighted load balancing

Balance sum of weights among processes, with weights n_{DoFs}^{c} per cell.
One fixed problem (≈74M DoFs, 144 MPI processes), variable weighting exponent c.



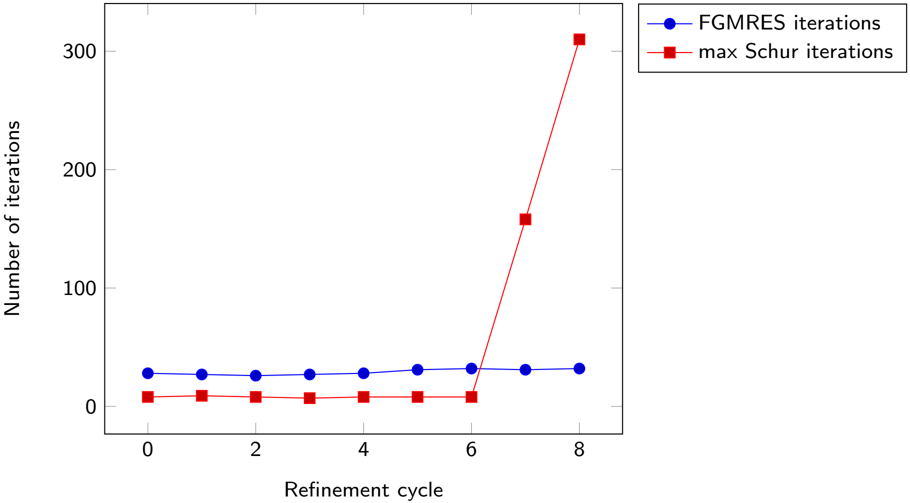
Stokes: Strong scaling

One fixed problem (≈74M DoFs), variable number of MPI processes.



Stokes: Consecutive refinements

Fixed number of processes (=144), successive adaptive refinements.



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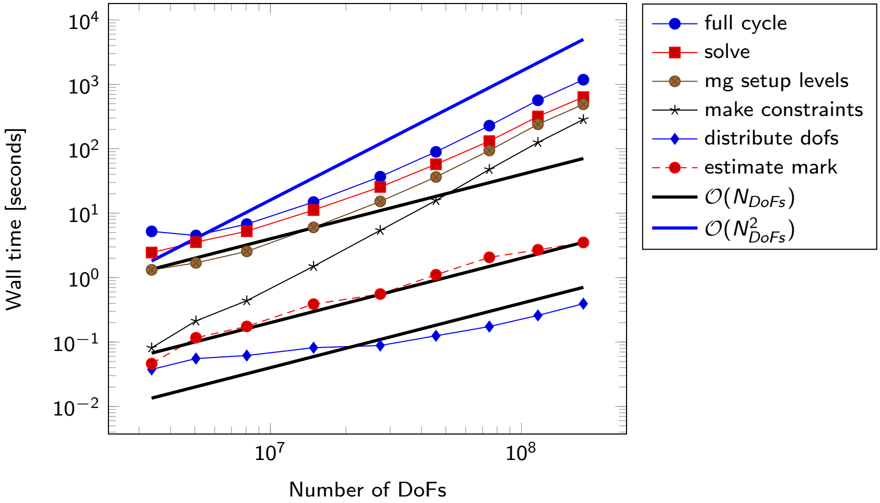

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Outlook

All algorithms available in `hpbox`.

Next steps:

- Adjust `step-75` to incorporate ASM.
- Manuscript for publication.

Remaining problems:

- More refinements lead to convergence issues in solving the Schur Block.
  - Also with h-refinement. Also in `aspect`?
- Performance of `AffineConstraints::make_consistent_in_parallel`?
- Multi-point constraints as alternative to ASM?
- Issues with PR #14905 Merge strategy on make consistent in parallel.
  - Try to find failing example in this workshop.

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