Almost Sure Convergence

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16 August 2021

$\exists$ a set A $\subset \Omega$ s.t. P(A)=0, so that given ϵ >0, $\forall \omega \in \Omega \backslash A$ $\exists$ $N = N_\omega$ s.t. $\forall n > N_\omega$,

$$|X_n(\omega) - X(\omega)| < \epsilon$$

This is called almost sure convergence.

Suppose, $\forall \epsilon > 0, \exists A_\epsilon \subset \Omega$ with $P(A_\epsilon) = 0$ and $\forall \omega \in \Omega \backslash A_\epsilon$ $\exists$ $N = N_\omega$ s.t. $\forall n > N_\omega$,

$$|X_n(\omega) - X(\omega)| < \epsilon$$

Clearly, a.e. convergence will guarantee this. What about the converse?

<u>Ans:</u>

Clearly, $A_{\epsilon'} \subseteq A_\epsilon$, whenever $\epsilon' > \epsilon$

Let, $A = \cup_{n=1}^{\infty} A_{\frac{1}{n}}$.

Now $0 \leq P(A_1) \leq P(A_{\frac{1}{2}}) \leq ..... \leq 1$ So $\{P(A_{\frac{1}{n}})\}$ is a non-decreasing sequence of non-negative numbers. Then by upward contnuity of probability,

$$P(A_{\frac{1}{n}}) \uparrow P(\cup_{n=1}^{\infty} A_{\frac{1}{n}})$$

$$\Rightarrow P(A) \downarrow 0$$

$\forall \epsilon > 0, \exists k \in \mathbb{N}$, s.t. $\frac{1}{k} < \epsilon$

$$\Rightarrow A_\epsilon \subseteq A_{\frac{1}{k}} \subset \cup_{n=1}^{\infty} A_{\frac{1}{n}} = A$$

$$\Rightarrow A_\epsilon \subseteq A, \forall \epsilon > 0$$

$$\Rightarrow \Omega \backslash A \subseteq \Omega \backslash A_\epsilon$$

Hence,$\forall \omega \in \Omega \backslash A_\epsilon$ $\exists$ $N = N_\omega$ s.t. $\forall n > N_\omega$, $|X_n(\omega) - X(\omega)| < \epsilon \Rightarrow \forall \omega \in \Omega \backslash A$ $\exists$ $N = N_\omega$ s.t. $\forall n > N_\omega$, $|X_n(\omega) - X(\omega)| < \epsilon$

So the converse is also true.