A Concise Note on Formal Logic

Samena Bahleri
samenabahleri09@gmail.com

November 11, 2025

This page left intentionally blank

Contents

1 Introduction

1.1 Etymology and Terminology

Normally, in the study of logic, the first thing we need to understand is the question: what is logic in terms of its definition? Etymologically, the word logic comes from the Greek word (logos). The word logos carries various meanings, including "word," "speech," "reason," "explanation," and "principle." Over time, this term was adopted into Latin as *logica*, which means the art or science of reasoning. On the other hand, in terms of terminology, logic is the systematic study of valid inference and correct reasoning. Therefore, it can be understood that when we study logic, what we are learning is how to evaluate things rationally and systematically to reach a solid understanding and draw a sound conclusion. One striking statement about logic comes from the philosopher John Locke, who said that "*logic is the anatomy of thought.*" In this metaphorical statement, we can also understand that by studying logic, we are essentially learning about the structure of thought itself.

1.2 Logic in History

1.2.1 Philosophical revolution

In this era, humans began to think not only about how to live in their environment, but also about themselves, truth, and ideas. One figure whose statement represents this revolution is Socrates, with his famous quote: "*The unexamined life is not worth living.*". To understand the systematic development of logic, we can follow the stages of history below:

Pre-Aristotelian Era

In this era, humans tried to understand the world through myth, narratives shaped by imagination and traditional constructions. Thinkers of this era include:

1. Thales

   He believed the world originated from water. This was considered rational at the time and reflected the knowledge of that era. Thales is often regarded as the first philosopher and the first to consider the problem of the one and the many.

2. Anaximander

   He questioned Thales' idea: If everything comes from water, then where does water come from? He introduced the idea of the *Apeiron* (the indefinite/infinite), beginning the philosophical search for a first principle (*archê*).

3. Xenophanes

   As humans began to question truth and divinity, Xenophanes criticized anthropomorphic portrayals of gods. Monotheistic ideas began to appear.

4. Heraclitus

   Famous for the quote: "You cannot step into the same river twice." He believed everything is in constant flux, reality is constant change. He introduced the concept of Logos as the rational structure behind the universe, implying that nature can be understood through patterns. He also introduced dialectical thinking: things are defined by their opposites, a foundation for binary thinking.

5. Parmenides

   Opposing Heraclitus, Parmenides argued that opinions do not necessarily reflect truth. In his view: What is, is; what is not, is not. Truth lies in existence, while change is an illusion.

6. Zeno

   In the realm of logic, Zeno can be considered one of the first figures to raise the issue of the relationship between logic and sensory data. He questioned: Which should we trust, logic or what we see? One of his most famous arguments is the Achilles and the Tortoise paradox.

   In short, in the story, the swift Achilles agrees to give the slower tortoise a head start in a race. However, Zeno argues that, logically, Achilles will never be able to catch up with the tortoise. The reasoning is as follows: every time Achilles reaches the point where the tortoise previously was, the tortoise has already moved a bit farther ahead. To reach this new point, Achilles must again travel a slightly smaller distance, and so on, infinitely. As a result, according to Zeno's logic, Achilles will always remain behind, even if only slightly. This clearly contradicts our sensory experience, as in reality we can observe that Achilles would easily overtake the tortoise. However, logically, Zeno's argument forms a paradox that is difficult to refute within the boundaries of formal logic.

   This paradox became one of the earliest indications of the debate about how we should treat mathematical logic: Is syntax (formal structure) more important than semantics (the meaning or reality we observe)? In this way, Zeno raises an important question about the limits and role of logic in explaining reality.

7. Plato

   Known for distinguishing between the world of ideas and the actual world. He believed that numbers and mathematical concepts do not exist in the physical world, but in the world of forms (ideas).

   Through the early history of the philosophical revolution, we realize that even in the contemporary era, humans are still driven by the same fundamental questions.

   (a) Aren't we still asking the same things today, about the origin of the universe?
   (b) Aren't we still questioning what is the source of all matter?
   (c) Aren't we still contemplating binary concepts, such as only knowing that something is true because something else is false, and vice versa?

And isn't it true that through the development of mathematical logic, Paradoxes arise not from reality itself, but from our conceptual frameworks? The pre-Aristotelian ponderings were clearly questions that mostly belonged to the transcendental realm. Thus, when the Aristotelian era began, these transcendental questions started to be addressed in a more rational manner, by developing a system of thought focused on what can be observed by humans.

Aristotelian era

Aristotle is regarded as the Father of Logic. He developed a system of categories, e.g., Substance, Quantity, Quality, Relation, Place, Time, Position, State, Action, Passion. He also developed syllogistic reasoning: *All men are mortal. Socrates is a man. Therefore, Socrates is mortal.* This illustrates the basics of formal logic.

1.2.2 Modern era

In the modern era, logic evolved in three main phases:

1. Enlightenment

   During the Enlightenment, philosophy grew in popularity, especially in terms of rationality, scientific methods, and freedom of thought. Philosophers and mathematicians like René Descartes, Gottfried Wilhelm Leibniz, and Immanuel Kant began developing more systematic and reflective approaches to logic, emphasizing reason as the means to acquire knowledge.

2. 19 Century

   “Logic” experienced a revolution. Thinkers such as George Boole, Augustus De Morgan, and Gottlob Frege developed symbolic and mathematical logic, which was far more precise than traditional Aristotelian logic. Frege introduced predicate logic, leading to non-classical logic.Georg Cantor founded set theory, now a foundation of modern mathematics. The dominant school of thought: Logicism, which holds that mathematics can be reduced to logic.

3. 20 Century

   “Logic” continued to grow and was applied in mathematics, linguistics, and computer science. This era saw the emergence of modal logic and its branches, such as:

   (a) Alethic logic
   (b) Deontic logic
   (c) Epistemic logic
   (d) Doxastic logic
   (e) Temporal logic
   (f) Dynamic logic
   (g) Action logic
   (h) Intuitionistic logic
   (i) Multi-modal logic
   (j) Provability logic

   Meanwhile, mathematics faced a foundational crisis, initiated by Kurt Gödel. Where the 19th century saw Logicism, the 20th century saw new “isms”:

   (a) Formalism: Logic is symbol manipulation based on formal rules.
   (b) Intuitionism: Mathematical truths are mental constructions.

2 Arguments

In our daily lives, we undoubtedly use logic to analyze things, whether it's someone's statement or a natural phenomenon we observe. Then, in analyzing something, we naturally hope that our analysis is correct, or that we can identify something was wrong, avoid it, or correct it so that it becomes right. In this section, we will specifically learn about arguments: what an argument is, the components that make up an argument, and what makes an argument valid or invalid. Studying arguments is certainly beneficial, whether in the social, academic, or interpersonal realm.

2.1 Component

2.1.1 Statement

A statement or proposition is a declarative sentence that has a definite truth value; it can be either true or false. A mathematical statement is one example of a statement.

Example 1:

1. $\left[2 < 4 \;\wedge\; 4 < 7 \;\rightarrow\; 2 < 7\right]$
2. $s = 2 \rightarrow s^3 \cdot s^5 = s^8$
3. $a = 3 \rightarrow \sqrt[a]{8^2} \equiv 8^{2/3}$
4. $X \cap U = X$
5. $\left[4^{5/1}\right]^6 \vee 4^{30}$
6. $(x \equiv y) \vdash (x \rightarrow y) \wedge (y \rightarrow x)$
7. $\left[6^{1/2}\right]\left[6^{3/2}\right] \vee \neg\neg 6^2$
8. $\log(4 \cdot 5) \equiv \log 4 + \log 5$
9. $m = 3, e = 2 \rightarrow (m + e)^2 \equiv m^2 + 2(me) + e^2$
10. $n = 5, a = 2 \rightarrow (n - a)^2 \equiv n^2 - 2(na) + a^2$

All of the examples above are mathematical statements with truth values that can be verified.

Example 2:

1. $\mathcal{P}(\mathbb{N}) > \mathbb{R}$
2. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$
3. ${}^9\log\ 81 > {}^2\log\ 64 - {}^2\log\ 16$
4. If $b = a^4$, with $a, b > 0$, then ${}^a\log b - {}^b\log a < \dfrac{15}{4}$
5. $a \cup b = a \cap b$

The second set of examples are false statements. However, since each can be evaluated to a definite conclusion, Example 2 is also a statement.

2.1.2 Non-Statement

Consequently, any expression that does not have a definite truth value is not considered a statement.

Example:

1. "What time is it?"
2. "Close the door."
3. "Wow, that's amazing!"
4. "Who is running?"
5. "I don't know"

2.1.3 Open sentence

We also need to understand that an argument can contain open sentences, statements whose truth value is not yet determined.

Example:

1. y is a prime number.
2. $S_n = \frac{n}{2}(a + U_n)$
3. $U_n = a\,r^{\,n-1}$.
4. $x > 10$
5. $\frac{p+4}{\sqrt{p}-\sqrt{4}}$

Until we specify what p, n, x, or y refer to, the truth cannot be judged. Once the variable is assigned, the open sentence becomes a proposition (a definite true/false statement). This distinction is central in predicate logic.

2.1.4 Conclusion

Intuitively, we all understand what a conclusion is. A conclusion is the result derived from a set of premises, or in this case, when premises present facts, the conclusion is what we can logically derive from those facts.

Example 1:

Premise 1: All humans are living beings

Premise 2: Socrates is a human

Conclusion: Therefore, Socrates is a living being

As we can observe, a conclusion is a deduction drawn from the facts. And in this example, the conclusion is clearly correct.

Example 2:

Premise 1: $2, 3 \in \mathbb{R}$

Premise 2: $2 < 3$

Conclusion: Hence, $2 = 3$

In this example, it is true that 2 and 3 are elements of the real numbers, and it is also true that 2 is less than 3. However, the conclusion drawn $2 = 3$ is false because it does not logically follow from the premises.

2.1.5 Inference

Inference is not the same as a conclusion. A conclusion is the specific claim that logically follows from the premises, whereas an inference refers to the entire reasoning process that connects the premises to the conclusion.

Example:

Premise 1: All students must take the final exam

Premise 2: Elica is a student

Conclusion: Therefore, Elica must take the final exam

Here, the conclusion is simply:

"Elica must take the final exam."

The inference is the reasoning as a whole:

Because all students must take the final exam, and Elica is a student,
it follows that Elica must take the final exam.

2.2 Formal and Informal

When analyzing an argument, we can approach it from two perspectives:

2.2.1 Formal Logic

As we will continue to study in this course, formal logic is the discipline that focuses on the form of the argument itself, regardless of the content or meaning of the statements. For example:

Recall:

Premise 1: All humans are living beings

Premise 2: Socrates is a human

Previously, we drew a conclusion by reading the content of these premises (statements in natural language). In formal logic, however, we represent these premises using symbols. For instance, we can represent the entire natural language statement with symbols like p, q, r, s and so on.

Example:

p: All humans are living beings

q: Socrates is a human

From these two premises, we can combine them using the conjunction symbol ($\wedge$), resulting in: $p \wedge q$. If we translate this back into natural language, it becomes: "All humans are living beings and Socrates is a human."

2.2.2 Informal Logic

If formal logic focuses on the form of the premises, informal logic focuses on the content of those premises. We've already seen examples of this above.

Recall:

Premise 1: All humans are living beings

Premise 2: Socrates is a human

Conclusion: Therefore, Socrates is a living being

The focus of informal logic is:

1. Are the premises reasonable?
2. Are the terms used clear?
3. Who is Socrates?

2.2.3 Formal vs Informal

It is important to understand that formal and informal logic each have their own limitations. Specifically, formal logic allows us to draw conclusions systematically and with certainty, because the rules of inference have already been clearly defined. In this kind of reasoning, we do not focus on the content of the statements, we focus solely on the process of drawing conclusions.

Example:

$$\text{If } A \text{ is true and } B \text{ is true, then } A \wedge B \text{ (A and B) is true.}$$

In this case, the conclusion is valid because the rule of conjunction says so. However, we do not know what A or B actually represent. We don't evaluate whether A or B are reasonable or relevant in real-life contexts. On the other hand, informal logic is more contextual, as it reflects what is actually happening in the real world. However, it is more difficult to evaluate objectively and is more vulnerable to bias or logical fallacies. Simply put, informal logic represents a form of reasoning that we often encounter in everyday life.

Example:

"Climate change is real because almost every scientist agrees on it, and we can see unusual weather patterns happening more frequently."

In this case, the reasoning isn't framed in formal symbols like A and B. Instead, it relies on contextual evidence (scientific consensus, observed weather) and appeals to what is happening in the real world. While it can be persuasive, it is also open to subjective interpretation, selective use of evidence, or possible fallacies (e.g., appeal to authority, hasty generalization).

3 Deductive and Inductive

3.1 Deductive

As we have seen in the Introduction, a statement is a declaration that contains truth, whether it is true or false. Specifically, if the premises are false, then the conclusion will also be untrue; and conversely, if the premises are true, then the deduction must be valid. In this case, the conclusion is vital.

Example:

Premise 1: All humans are living beings

Premise 2: Socrates is a human

Conclusion: Therefore, Socrates is a living being

Example:

Premise 1: If A, then B

Premise 2: A occurs

Conclusion: Therefore, B occurs

1. Definition

   Deductive reasoning can also be observed through definitions.

   Example:

   (a) Truth is truth.
   (b) All triangles are flat shapes with three sides.
   (c) A bachelor is an adult man who is not married.

2. Sylogistic

   As we've seen in the Introduction material, syllogistic reasoning is reasoning where the conclusion is necessarily true due to elements such as categories (living beings) and definitions.

   Example:

   Premise 1: All humans are living beings

   Premise 2: Socrates is a human

   Conclusion: Therefore, Socrates is a living being

3.2 Inductive

In terms of etymology, the word "inductive" comes from the Latin verb inducere, meaning "to lead into" or "to bring in." This reflects the process of reasoning that moves from specific observations or examples toward general conclusions or principles. Furthermore, in terms of terminology, inductive reasoning refers to a method of reasoning in which generalizations are formed based on repeated observations or patterns. We have also seen this form of reasoning in the Introduction material, specifically in the section on informal logic. In short, inductive reasoning is a type of reasoning based on probability. If we were to list them, several topics below are forms of reasoning or conclusions that are probabilistic in nature.

1. Logic

   Although inductive reasoning is frequently associated with informal logic or arguments lacking rigid formal structure, in practice, formal logical structures, such as modus ponens, can also be used in inductive reasoning, depending on the content of the premises. Example (Modus Ponens):

   Premise 1: If Paul studies hard, then he will be accepted into a university

   Premise 2: Paul studies hard

   Conclusion: Therefore, he will be accepted into a university

Structurally, this argument takes the form of modus ponens, which is a deductive form. However, the first premise does not deliver logical certainty, but rather a high probability based on past experience or generalization. Thus, while the form is deductive, the content is inductive, meaning the conclusion is not guaranteed, only probable. This shows that inductive reasoning doesn't have to be informal; it can take a formal structure, as long as the truth of the premises is not absolute.

2. Analogy

   One common type of inductive reasoning is reasoning by analogy. In this form, one concludes that because two things share some similarities, they may also share other characteristics. A famous example is the "watchmaker analogy" used by William Paley to support the argument for the existence of God as the designer of the universe.

   Form of Reasoning:

   Premise 1: A watch, with its complexity and order, shows evidence of being designed by a watchmaker

   Premise 2: The universe also shows similar complexity and order

   Conclusion: Therefore, the universe is also likely designed by a creator/designer

   This kind of reasoning is clearly inductive, because it is based on probability. As stated above, just because two things share some features does not necessarily mean they share all features. Such reasoning, while it may sound convincing, is often considered a weak argument.

3. Generalization

   Generalization is a type of inductive reasoning characterized by drawing conclusions from only a few instances or observations.

   Example:

   Premise 1: Swan A is white

   Premise 2: Swan B is white

   Premise 3: Swan C is white

   Conclusion: Therefore, all swans are white

4. Causation

   Although the law of causality feels intuitively persuasive and is widely used in everyday life and science, it is actually a form of inductive reasoning, for several reasons:

   (a) Causality is based on experience and repeated observation of specific patterns.

   The philosopher David Hume is well-known for this view. He reasoned that we never directly experience causality; instead, we only observe that one event regularly follows another. In other words, what we perceive as a causal relationship is merely a constant conjunction of events. Therefore, the link we presume between cause and effect is just a habit of the mind, not something objectively featured of reality.

(b) Causality assumes the consistency of nature, or that the future will resemble the past.

For instance, we believe the sun will rise tomorrow because it has always risen in the past. But this belief is based purely on past experience, not on any inherent truth about the universe itself. Hence, causal reasoning is practically a prediction rooted in experience, and therefore, it is inductive.

3.3 Deduction vs/and Induction

Comprehending this fundamental difference is crucial when producing formal statements, specifically, as we've studied in the distinction between formal and informal logic. Regarding what we call "true", formal logic focuses on the form of the argument, while informal logic focuses on the content of the argument. In this context, deductive reasoning is typically formal, whereas inductive reasoning is generally informal. In practice, however, a conclusion should be evaluated not only based on the form of the argument, but also on the truth of the premises and the relevance of the content (whether the premises correspond to reality). Therefore, we cannot rely solely on the rigid rules we have constructed; we must also take into account our actual experience of reality. Furthermore, apprehending deduction and induction is vital for clearly clarifying the boundaries between what is logically true or false and what is empirically valid or not.

Deductive rules give us logical assurance, since the method is predetermined, conclusions follow necessarily from the premises. Induction, on the other hand, offers probability based on experience. It is also essential to acknowledge that deductive rules are the result of human consensus, agreements we have made, whereas induction reflects reality itself. In other words, we comprehend phenomena only to the extent that we can perceive and interpret them. We assemble patterns to the degree that they can be applied, even if we do not fully understand the underlying reality. This marks the epistemological limits of human knowledge. Essentially, both approaches are equally fundamental, because it is through both that humans can adapt to reality.

4 Scope of Deductive and Inductive

4.1 Scope of Deductive

4.1.1 Validity

A valid argument is one that is based on the principles of deduction, in which the conclusion logically follows from the premises. It's important to note that validity here refers not to the truth of the premises, but solely to the logical relationship between the premises and the conclusion.

Example 1:

Premise 1: All humans are living beings

Premise 2: Sam is a human

Conclusion: Therefore, Sam is a living being

The example above is an argument that is structurally, categorically, and definitionally correct. Thus, the truth of the argument is acceptable because both the logical structure and the content of the premises align and support a valid conclusion.

Example 2:

Premise 1: All birds can talk

Premise 2: A pigeon is a bird

Conclusion: Therefore, a pigeon can talk

We can observe that this second example is structurally valid, but factually incorrect in its content. This is because not all birds can talk. Therefore, an argument can be valid when judged solely by the structure of the premises; however, when examining the content, an argument might be valid but not sound. This brings us to the concept of soundness.

4.1.2 Soundness

A sound argument is one that is both valid in structure (logical form) and true in content (factual). Thus, the conclusion of a sound argument can be guaranteed to be true or accurate. In other words, to build a sound argument, we must first ensure that the premises are factually correct, and then organize them into a logically valid structure.

Example 1:

Premise 1: All mammals are warm-blooded creatures

Premise 2: Dolphins are mammals

Conclusion: Therefore, dolphins are warm-blooded creatures

This example is both structurally valid and factually accurate in its premises. Therefore, we can conclude that a sound argument follows this pattern:

$$\text{Sound Argument} = \text{Factual Premises} + \text{Valid Form}$$

From the concepts of validity and soundness, we can summarize the following:

1. An argument can be valid in logical form even if its premises are not factually correct. In this case, even though the logical structure is correct, the conclusion cannot be guaranteed true, because the content of the argument is not factual.
2. A sound argument is an accurate and trustworthy argument because its logical structure is valid, and all its premises are factually true. Therefore, a sound argument always produces a conclusion that is certainly true.

4.2 Scope of Inductive

4.2.1 Strong Argument

An inductive argument is considered strong if its premises provide a high degree of probability in support of the conclusion. The characteristics of a strong inductive argument are similar in structure to deductive arguments, but the conclusion remains uncertain or not absolutely guaranteed.

Example:

Premise 1: 90 percent of the students in this class passed the formal logic exam

Premise 2: Noel is a student in this class

Conclusion: Most likely, Noel passed the formal logic exam

As we can observe, the premises resemble those of a deductive argument. However, the conclusion is not definite; we still need to verify whether Noel indeed passed the exam. This brings us to the next concept: the cogent argument.

4.2.2 Cogent

The term cogent is closely related to strong, but it includes an additional requirement. The characteristics of a cogent argument are:

a. The premises are actually true in reality.

b. The argument is structurally like a deductive argument.

c. Then, the conclusion is plausible and well-supported.

Example:

Premise 1: Cats have whiskers

Premise 2: Cats have whiskers

Conclusion: Therefore, Luna probably has whiskers

The argument above is cogent, because we intuitively agree that cats generally have whiskers. However, given the nature of inductive reasoning, which moves from general observations to specific instances, we still need to confirm whether Luna, the cat in question, indeed has whiskers. In short, a cogent argument is a strong inductive argument with true premises.

4.2.3 Weak Argument

A weak inductive argument is one in which the premises do not provide strong support for the conclusion, or the conclusion does not logically follow from the given premises. Even if the premises are true, they fail to give a high probability that the conclusion is also true.

Example:

Premise 1: Dogs are cute and make good pets

Premise 2: Cats are cute and make good pets

Premise 3: Pandas are cute and make good pets

Conclusion: Therefore, every cute animal makes a good pet

This argument is clearly weak, because the premises only refer to a few cute animals. Moreover, the use of the phrase "every cute animal" in the conclusion is an overgeneralization. Not all cute animals make good pets, since there are many other factors to consider, such as temperament, size, habitat, and potential danger.

Below is a table of indicators that we should pay attention to, both when constructing an argument and when reading or listening to one. The purpose is to help us distinguish between deductive and inductive arguments, whether in social or academic contexts.

5 Symbolization

Before symbolizing statements using formal logical symbols, it is important to clarify two key concepts: First, the Object Language, and second, the Metalanguage:

1. **The Object Language**

   The Object Language is the formal language we analyze, consisting of symbols like $p, q, \wedge, \vee, \neg$, and sentences we form with these symbols.

Category	Deductive Argument	Inductive Argument
Certainty of Conclusion	certainly, always, cannot be false, must, necessarily	possibly, likely, may, could, probably
Common Indicators	therefore, thus, it must be, hence, as a result	likely, most likely, apparently, it could be, tends to
Purpose of Argument	to prove truth with absolute certainty/logically	to show probability or tendency
Logical Structure	if... then..., all..., none..., every...	most..., often..., usually..., on average...
Type of Conclusion	conclusion is definitely true if premises are true	conclusion is a prediction or generalization

2. **The Metalanguage**

   The Metalanguage is the language we use to talk about the Object Language, which in this article is mostly English. We use the Metalanguage to explain the meaning and rules of the Object Language, such as what makes a formula well-formed or how the connectives function.

When talking about Symbolization, we are dealing with the role of symbols, their purpose, and their essence as a formal system in analyzing and representing the truth of what is conveyed through symbols. However, once again, we must remember that in the previous material, we learned that the formal and informal processes, namely deduction and induction, complement each other. Arguments formed from premises that are not logically connected can result in invalid forms, even though the content of those premises may be factually true. Furthermore, sound reasoning is reasoning based on the following points:

1. The initial stage of reasoning is empirical experience.
2. The second stage is representation in natural language.
3. The third stage is symbolic abstraction.

5.1 Role

5.1.1 Proposition

A proposition like "All humans are living beings" can be represented simply by p, and the same applies to other propositions, which can be symbolized as p, q, r, s, etc.

5.1.2 Relation

To express logical relations between propositions in symbolic form, we first need to understand how these relations are conveyed through logical symbols. These symbols represent the connections between propositions and serve as the foundation for constructing arguments in propositional logic (PL). The key types of logical symbols used to represent these connections are called truth-functional connectives. Some commonly used truth-functional connectives include:

1. **Disjunction** ($\vee$)

   In everyday language, the word "or" can represent two meanings: inclusive or, and exclusive or. At this initial stage, we will focus on inclusive or. Note that "or" in formal logic is different from equivalence, which uses $\leftrightarrow$.

   Example:

   Orang utan or Pongo pygmaeus

   Define symbols:

   $$p : 1 + 1 = 2, \quad q : 3 - 2 = 1$$

   Then symbolic form:

   $$p \vee q$$

   Example:

   $$k : \text{Koala}, \quad p : \text{Panda} \quad \rightarrow \quad k \vee p$$

2. **Conjunction** ($\wedge$)

   In everyday language, "and" connects two propositions expected to be true simultaneously. In symbolic logic, conjunction is represented by $\wedge$.

   Example:

   $$x : 1 < 2, \quad y : 3 > 2 \quad \rightarrow \quad x \wedge y$$

   Example:

   $$s : \text{Sun}, \quad m : \text{Moon} \quad \rightarrow \quad s \wedge m$$

3. **Conditional** ($\rightarrow$)

   Conditional logic represents the relation between antecedent and consequent.

   Example:

   $$r : \text{It rains}, \quad w : \text{The ground is wet} \quad \rightarrow \quad r \rightarrow w$$

4. **Negation** ($\neg$)

   Negation represents denial of a proposition.

   Example:

   $$p : \text{All prime numbers are odd} \quad \rightarrow \quad \neg p$$

5. **Biconditional** ($\leftrightarrow$)

   Biconditional logic requires both propositions to have the same truth value.

   Example:

   $$l : \text{Laika is a dog}, \quad m : \text{She is a mammal} \quad \rightarrow \quad l \leftrightarrow m$$

   Example:

   $$p : 3 \text{ is prime}, \quad d : 3 \text{ has exactly two positive divisors} \quad \rightarrow \quad p \leftrightarrow d$$

5.2 Purpose

The purpose of using logical symbols is to represent the truth value of a statement in a formal and systematic way. By using symbols, we can express and analyze the relationships between propositions accurately, consistently, and free from the ambiguities of natural language. To draw logical conclusions from a combination of propositions p, q, r, s, we must know the truth value of each individual proposition. In propositional logic, each proposition has only two possible truth values:

1. True, represented by T or 1
2. False, represented by F or 0

Each proposition represents a single statement and has its own truth value, depending on the context or situation. To represent all possible truth values of one or more propositions and the results of logical operations, we use a tool called a truth table, which will be introduced in the next chapter.

6 Well-Formed Formula

A Well-formed formula, often abbreviated as Wff, is a method for constructing sentences that we refer to as syntax. The formation of a Wff is similar to grammar in natural languages, in that there are specific rules to ensure the language can be understood. While natural languages use alphabets, Wffs in Formal Logic use symbols.

6.1 Expression

A Well-formed formula (Wff) can be formed using the following symbols:

1. Atomic Sentences:

$$a, b, c, \ldots, p, q, r, \ldots, \psi_1, \ldots, \psi_n$$

2. Connectives:

$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$$

3. Brackets:

$$(\; , \;)$$

Using the rules above, we can construct syntax as follows: If A is a Wff and represents a proposition, then $\neg A$ and $\neg\neg A$ are also considered Wffs. Furthermore, if A and B are Wffs, we can combine them using connectives, such as:

1. $a \wedge b$
2. $a \vee b$
3. $a \rightarrow b$
4. $a \leftrightarrow b$

We can also combine two or more atomic sentences, e.g., $\neg x \vee y \to z$. In contexts involving three atomic sentences, to avoid ambiguity, we must use brackets: $(\neg x \vee y) \to z$.

Examples of Compound Wffs:

1. $\neg a$
2. $a \vee b$
3. $(a \vee b) \to c$
4. $((a \wedge b) \vee (a \vee b)) \to c$
5. $(a \to b) \vee z$
6. $\neg a \to b$
7. $\neg x \to \neg y$
8. $(a \to \neg y) \wedge z$
9. $(((a \to \neg y) \wedge \neg\neg z) \vee (a \vee b)) \to c$

6.1.1 Ambiguity in Natural Language

Well-formed formulas (Wffs) help avoid ambiguity often present in natural language. There are two main types:

6.1.2 Lexical Ambiguity

Occurs when a single word has multiple meanings.

Example 1::

"Tail"

1. An animal's tail (rear part of an animal)
2. An airplane's tail (rear part of an aircraft)

Example 2::

"Bank"

1. A financial institution
2. The edge of a river

Example 3::

"I'm sitting at the bank"

1. Are you sitting at a river bank?
2. Or inside a banking institution?

6.1.3 Structural Ambiguity

Occurs when the sentence structure allows multiple interpretations.

Example 1::

"I saw the man with the telescope."

Interpretations: "I used a telescope to see the man" or "I saw a man holding a telescope."

Example 2::

"He watched the dog with one eye."

Interpretations: "He used one eye to watch the dog" or "He watched a dog that had one eye."

Example 3::

Without parentheses:

$$a \lor g \lor h$$

Possible interpretations:

$$(a \lor g) \lor h \quad \text{or} \quad a \lor (g \lor h)$$

With parentheses:

$$a \lor (g \lor h)$$

Variables:

1. a: "It is raining"
2. g: "I will take an umbrella"
3. h: "I will wear a raincoat"

Negation example:

1. $\neg(a \lor b)$: "Not a and not b"
2. $\neg a \lor b$: "Either a is false, or b is true"

Using parentheses properly in formal logic ensures clarity and prevents misinterpretation.

7 Truth Table

As discussed in the section on symbolization, the purpose of using symbolic logic is to express the truth of a statement in a formal way. Then, by using these symbols, we can avoid the ambiguity of natural language. Moreover, the first thing to know is that every proposition, whether simple or compound, only represents either *True* or *False*.

$$p = \begin{cases} 1 \\ 0 \end{cases} \quad \text{and} \quad q = \begin{cases} 1 \\ 0 \end{cases}$$

Furthermore, if we combine these symbols with connectives, the output is also either true or false. In this case, truth tables are very useful for representing the possible values of a proposition.

7.1 Truth Table for Disjunction

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Disjunction is false only when both propositions are false.

7.2 Truth Table for Conjunction

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Conjunction is true only when both propositions are true.

7.3 Truth Table for Implication

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Implication is false only when the premise is true but the conclusion is false.

7.4 Truth Table for Negation

p	$\neg p$
1	0
0	1

Negation negates each proposition.

7.5 Truth Table for Biconditional

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Biconditional is true when both propositions have the same truth value.

7.6 Truth Table for NAND

p	q	$p \uparrow q$
1	1	0
1	0	1
0	1	1
0	0	1

NAND (Not AND) is false only when both p and q are true.

7.7 Truth Table for NOR

p	q	$p \downarrow q$
1	1	0
1	0	0
0	1	0
0	0	1

NOR (Not OR) is true only when both p and q are false.

Another important point to understand is that if we have a number of propositions and want to determine the truth value of each one, every proposition has two possible truth values: true (1) or false (0). Therefore, the total number of possible combinations of truth values for all the propositions is given by 2^n, where n is the number of propositions. For example:

8 Partial Truth Table

The idea of partial truth tables is that we don't always need to list every possible input for a proposition in order to determine whether it is a tautology, contradiction, or something else. Instead, we can strategically test only the cases that matter. One common method is **Reductio ad Absurdum** (RAA), in which we assume the opposite of what we want to prove and then show that this assumption leads to a contradiction.

8.1 Example 1: Tautology

Consider the formula:

$$(a \wedge (a \to b)) \to b$$

To test whether this is a tautology, we assume the formula is false. An implication $x \to y$ is false only when x is true and y is false.

Firstly, we assume the implication is false:

a	$\land$	$(a \to b)$	$\to$	b
			0	

Second, since the implication is false, we can immediately set $b = 0$. Moreover, the implication has two parts:

$$(a \land (a \to b)) \quad \text{and} \quad b$$

By definition of implication, for the whole formula to be false the consequent b must be false and the antecedent $(a \land (a \to b))$ must be true. Therefore, we place *True* under the Conjunction ($\land$) row.

a	$\land$	$(a \to b)$	$\to$	b
	1		0	0

Third, a conjunction is true only when both inputs are true. Thus, a must be true and $(a \to b)$ must also be true. However, since we already assumed b is false, the expression $(a \to b)$ becomes a contradiction.

a	$\land$	$(a \to b)$	$\to$	b
1	1	1	0	0

Finally, as we can see, $(a \to b)$ would only be true when b is true. Yet from the beginning we assumed b is false. This creates a contradiction, so the statement cannot be false under any assignment of truth values. Therefore, the formula is always true, which means it is a tautology.

8.2 Example 2: Contradiction

To show a formula is a contradiction, we try to make the formula true by assigning truth values. If no assignment makes it true, it is a contradiction.

Consider this formula:

$$a \land \neg a$$

First, suppose $a = 1$. Then $\neg a = 0$.

a	$\neg a$	$a \land \neg a$
1	0	0

Second, suppose $a = 0$. Then $\neg a = 1$.

a	$\neg a$	$a \land \neg a$
0	1	0

Ultimately, in both cases, the output of $a \land \neg a$ is *False*. Therefore, the formula $a \land \neg a$ is never true and is a contradiction.

8.3 Example 3: Equivalence

To show two formulas are equivalent, we check whether they always have the same truth value under every possible assignment of inputs.

Consider this equivalence:

$$a \to b \quad \equiv \quad \neg a \vee b$$

Suppose $a = 1$ and $b = 1$.

a	b	$a \to b$	$\neg a$	$\neg a \vee b$
1	1	1	0	1

Also, suppose $a = 1$ and $b = 0$.

a	b	$a \to b$	$\neg a$	$\neg a \vee b$
1	0	0	0	0

Suppose $a = 0$ and $b = 1$.

a	b	$a \to b$	$\neg a$	$\neg a \vee b$
0	1	1	1	1

Finally, suppose $a = 0$ and $b = 0$.

a	b	$a \to b$	$\neg a$	$\neg a \vee b$
0	0	1	1	1

As a result, in every possible case, the columns for $a \to b$ and $\neg a \vee b$ match exactly. Therefore, the two formulas are logically equivalent.

Full truth table:

a	b	$\neg a$	$a \to b$	$\neg a \vee b$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

Note that to show that a formula is an entailment, we must present a complete truth table. We cannot just assume a few simple inputs, as is sometimes done with tautologies or contradictions.

8.4 Example 4: Validity

To check for validity using a partial truth table, we assume the conclusion is false and see if all premises can still be true. If it is impossible, the argument is valid.

Consider the argument:

1. $a \to b$
2. $b \to c$

We want to prove that $a \to c$.

First, assume the conclusion is false. To make $a \to c$ false, recall that a conditional statement $x \to y$ is false only when $x = 1$ and $y = 0$. Thus, we set:

$$a = 1, \quad c = 0$$

Next, we try to keep the premises true. For $a \to b$ to be true with $a = 1$, we must set:

$$b = 1$$

a	b	c	$a \to b$	$b \to c$	$a \to c$
1	1	0	1	0	0

1. $a \to b$ is *True*.
2. $b \to c$ is *False*.
3. $a \to c$ is *False* (as intended).

Recall that for validity we require the premises

$$a \to b \quad \text{and} \quad b \to c \quad \text{to be } \textit{True}$$

However, when we force the conclusion to be false ($a = 1, c = 0$), one of the premises ($b \to c$) inevitably becomes false as well. Thus, it is impossible to make all the premises true while keeping the conclusion false. Therefore, the argument is valid.

8.5 Example 5: Contingency

Consider the formula:

$$a \to b$$

First, suppose $a = 0$ and $b = 0$.

a	b	$a \to b$
0	0	1

Second, suppose $a = 1$ and $b = 0$.

a	b	$a \to b$
1	0	0

Therefore, the formula is contingent.

8.6 Example 6: Satisfiability

Consider the formula:

$$a \wedge b \wedge c$$

Suppose $a = 1$, $b = 1$, and $c = 1$.

a	b	c	$a \wedge b \wedge c$
1	1	1	1

Therefore, the formula is satisfiable.

8.7 Example 7: Unsatisfiability

Consider the formula:

$$(a \lor b) \land \neg a \land \neg b$$

First, suppose $a = 1$ and $b = 0$.

a	b	$a \lor b$	$\neg a$	$(a \lor b) \land \neg a \land \neg b$
1	0	1	0	0

Second, suppose $a = 0$ and $b = 1$.

a	b	$a \lor b$	$\neg b$	$(a \lor b) \land \neg a \land \neg b$
0	1	1	0	0

Third, suppose $a = 0$ and $b = 0$.

a	b	$a \lor b$	$\neg a$	$(a \lor b) \land \neg a \land \neg b$
0	0	0	1	0

Fourth, suppose $a = 1$ and $b = 1$.

a	b	$a \lor b$	$\neg a$	$(a \lor b) \land \neg a \land \neg b$
1	1	1	0	0

Therefore, the formula is unsatisfiable.

8.8 Example 8: Non-Equivalence

Consider this claim:

$$a \to b \quad \equiv \quad b \to a$$

Suppose $a = 1$ and $b = 0$.

a	b	$a \to b$	$b \to a$
1	0	0	1

Therefore, the formulas are not equivalent.

8.9 Example 9: Invalidity

Consider the argument:

1. $a \to b$

We want to prove that $b \to a$.
Suppose $a = 0$ and $b = 1$.

a	b	$a \to b$	$b \to a$
0	1	1	0

Therefore, the argument is invalid.

8.10 Example 10: Multiple Premises Validity

Consider the argument:

1. $a \to b$
2. $c \to d$
3. $a \lor c$

We want to prove that $b \lor d$.

First, assume the conclusion is false. Thus, we set:

$$b = 0, \quad d = 0$$

Next, for $a \to b$ to be true with $b = 0$, we must set $a = 0$. For $c \to d$ to be true with $d = 0$, we must set $c = 0$.

a	b	c	d	$a \to b$	$c \to d$	$a \lor c$	$b \lor d$
0	0	0	0	1	1	0	0

Therefore, the argument is valid.

9 Compound Statements

As we have learned in the Well-Formed Formula part, by using connectives such as $\neg$, $\land$, $\lor$, $\to$, $\leftrightarrow$, we can create more complex combinations. Then, we can also calculate their truth values. For example:

Let

$$(a \lor b) \land c$$

Where:

a: 2 is prime

b: 2 is even

c: 2 is divisible by some even number

We check every possible truth value for a, b, and c using a truth table.

a	b	c	$a \lor b$	$(a \lor b) \land c$
1	1	1	1	1
1	1	0	1	0
1	0	1	1	1
1	0	0	1	0
0	1	1	1	1
0	1	0	1	0
0	0	1	0	0
0	0	0	0	0

Since 2 is prime, even, and divisible by some even number, the output $(a \lor b) \land c$ is obviously *True*, corresponding to the first row.

9.1 Example 1

Let

$$a : \left[x = 3, y = 5 \to 2^x \cdot 2^y = 2^5\right]$$
$$b : \left[p = 8, q = 6 \to \frac{2^p}{q^q} < \frac{2^p}{3^9}\right]$$
$$c : \left[\left(3^{1/2}\right)^2 \cdot \left(2^{1/2}\right)^2 = 2^2 + 2\right]$$

We ask for

$$(a \to b) \land c$$

Since

$$a = 1, \quad b = 1, \quad c = 1$$

We conclude

$$\boxed{(a \to b) \land c = \text{True}}$$

9.2 Example 2

Let

$$d : \left[p = 4 \to 2^p \cdot 2^5 = 2^9\right], \qquad f : \left[\frac{2^4 \cdot 3^6}{2^3 \cdot 3^2}\right]^3 = 2^3 \cdot 3^{12}$$
$$e : \left[p = 3 \to (3\pi)^p = 27\pi^3\right], \quad g : \left[\frac{1^{-1}}{5^{-6}}\right] = 5^6$$

We ask for

$$((d \leftrightarrow e) \lor f) \land g$$

Since

$$d = 1, \quad e = 1, \quad f = 1, \quad g = 1$$

Hence

$$\boxed{((d \leftrightarrow e) \lor f) \land g = \text{True}}$$

9.3 Example 3

Let

$$h : \left[x = 2,\ y = 5 \to \frac{8 \cdot x^3 \cdot y^{-4}}{16 \cdot y^{-1/4}}\right] \to \neg\neg \left[\frac{4}{\sqrt[4]{5^{15}}}\right]$$
$$i : \left[\frac{2^5 \cdot 3^{-10}}{2^5 \cdot 3^{-4}}\right]^{1/2} \left[\frac{4^{1/4} \cdot 5^{1/2}}{4^{1/2} \cdot 5^{-1/2}}\right]^{1/2} \to \sqrt{\frac{5}{3^6 \cdot \sqrt{2}}}$$
$$j : \left[p = 3, q = 3 \to 5\sqrt{2^p} \cdot 3\sqrt[q]{3} \leftrightarrow 15 \cdot 2\sqrt{2} \cdot \sqrt[3]{3}\right]$$

We ask for

$$h \lor \neg\neg(i \land j)$$

Since

$$h = 1, \quad i = 1, \quad j = 1$$

Hence

$$\boxed{h \lor \neg\neg(i \land j) = \text{True}}$$

9.4 Example 4

Let

$$k : \sqrt{2} + \sqrt{6} < \sqrt{15}$$
$$l : \sqrt{8} + \sqrt{10} < \sqrt{40}$$
$$m : \sqrt{5} + \sqrt{7} \geq \sqrt{26}$$

We ask for

$$(k \wedge l) \to m$$

Since

$$k = 1, \quad l = 1, \quad m = 0$$

Thus

$$\boxed{(k \wedge l) \to m = \text{False}}$$

9.5 Example 5

Let

$$o : \left({}^{3}\!log\, 24 - {}^{3}\!log\, 8 + {}^{3}\!log\, 9\right) > {}^{8}\!log\, 32$$
$$p : \frac{1}{{}^{3}\!log\, 6} + \frac{1}{{}^{2}\!log\, 6} < \left({}^{3}\!log\, 18 - {}^{3}\!log\, 2\right)$$
$$q : \frac{({}^{5}\!log\, 10)^2 - ({}^{5}\!log\, 2)^2}{{}^{5}\!log\, \sqrt{20}} < \left({}^{6}\!log\, 64 - {}^{2}\!log\, 16\right)$$

We ask for

$$(o \vee p) \wedge q$$

Since

$$n = 1, \quad o = 1, \quad p = 0$$

Thus

$$\boxed{(o \vee p) \wedge q = \text{False}}$$

9.6 Example 6

Recall all the variables:

$$a, b, c, d, e, f, g, h, i, j, k, l, m, n, p$$

$$\begin{array}{lllll} a = 1, & b = 1, & c = 1, & d = 1, & e = 1, \\ f = 1, & g = 1, & h = 1, & i = 1, & j = 1, \\ k = 1, & l = 1, & m = 0, & n = 1, & p = 0 \end{array}$$

We ask for

1. $(\neg a \wedge c) \vee (e \wedge \neg m)$
2. $(i \to k) \leftrightarrow (n \vee p)$
3. $\neg(b \wedge f) \to (d \vee m)$
4. $(g \uparrow l) \wedge \neg(h \downarrow j)$
5. $(p \vee n) \wedge (k \to \neg m)$

Therefore,

1. $(\neg a \wedge c) \vee (e \wedge \neg m)$

a	c	e	m	$(\neg a \wedge c) \vee (e \wedge \neg m)$
1	1	1	0	$(0 \wedge 1) \vee (1 \wedge 1) = 0 \vee 1 = 1$

$$\boxed{(\neg a \wedge c) \vee (e \wedge \neg m) = 1}$$

2. $(i \to k) \leftrightarrow (n \vee p)$

i	k	n	p	$(i \to k) \leftrightarrow (n \vee p)$
1	1	1	0	$(1 \to 1) \leftrightarrow (1 \vee 0) = 1 \leftrightarrow 1 = 1$

$$\boxed{(i \to k) \leftrightarrow (n \vee p) = 1}$$

3. $\neg(b \wedge f) \to (d \vee m)$

b	f	d	m	$\neg(b \wedge f) \to (d \vee m)$
1	1	1	0	$\neg(1 \wedge 1) \to (1 \vee 0) = 0 \to 1 = 1$

$$\boxed{\neg(b \wedge f) \to (d \vee m) = 1}$$

4. $(g \uparrow l) \wedge \neg(h \downarrow j)$

g	l	h	j	$(g \uparrow l) \wedge \neg(h \downarrow j)$
1	1	1	1	$(1 \uparrow 1) \wedge \neg(1 \downarrow 1) = 0 \wedge \neg(0) = 0 \wedge 1 = 0$

$$\boxed{(g \uparrow l) \wedge \neg(h \downarrow j) = 0}$$

5. $(p \vee n) \wedge (k \to \neg m)$

p	n	k	m	$(p \vee n) \wedge (k \to \neg m)$
0	1	1	0	$(0 \vee 1) \wedge (1 \to 1) = 1 \wedge 1 = 1$

$$\boxed{(p \vee n) \wedge (k \to \neg m) = 1}$$

9.7 Example 7

Let

$$
\begin{aligned}
&q : \text{if } S_3 = 9, \text{ and } S_6 = 18, \therefore\ S_n = \frac{n}{3} \cdot 9. \\
&r : \text{if } a = 4, \text{ and } U_4 = 108, \therefore\ r = 3\ (r : \mathit{ratio}) \\
&s : \text{if } a = 16, \text{ and } U_5 = 81, \therefore\ U_3 = 30. \\
&t : \text{if } 5, -2, -9, -16, \cdots, \therefore U_{50} = -239.
\end{aligned}
$$

We ask for

$$(q \wedge r) \vee (s \wedge t)$$

Since

$$q = 1, \quad r = 1, \quad s = 0, \quad t = 0$$

Thus

$$\boxed{(q \wedge r) \vee (s \wedge t) = \text{True}}$$

10 Semantic concepts

In the previous chapter, we learned about symbols such as p, q, $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$, as well as the syntax rules that define well-formed formulas (wff), for example, $a \vee b$. In this chapter, we will explore semantics, which deals with the meaning and interpretation of these formulas.

10.1 Tautology

A formula is a tautology if it is always true, regardless of the truth values of its components. For example, $a \vee \neg a$ is a tautology; it remains true whether a is *True* or *False*. Another example is $a \rightarrow a$, which is always *True* by the definition of implication. The symbol for tautology is usually denoted by $\top$. Hence, we can write tautologies such as:

$$a \vee \neg a \quad (\top) \quad \text{and} \quad a \rightarrow a \quad (\top)$$

Truth Tables:

a	$\neg a$	$a \vee \neg a$	$a \rightarrow a$
1	0	1	1
0	1	1	1

10.2 Contradiction

A contradiction, frequently denoted by $\perp$, is a formula that is always *False*, no matter the truth values assigned. For example, $a \wedge \neg a$ is a contradiction because it can never be true. Another example is $(a \wedge \neg a) \vee (b \wedge \neg b)$, which is also always *False*.

Truth Table:

a	b	$\neg a$	$\neg b$	$a \wedge \neg a$	$b \wedge \neg b$	$(a \wedge \neg a) \vee (b \wedge \neg b)$
1	1	0	0	0	0	0
1	0	0	1	0	0	0
0	1	1	0	0	0	0
0	0	1	1	0	0	0

10.3 Equivalence

In propositional logic, equivalence ($\equiv$ or $\leftrightarrow$) between two well-formed formulas (wffs) means that they have the same truth value for all possible truth assignments to their propositions. If two formulas are logically equivalent, they will always evaluate to the same truth value, regardless of the truth values of the individual propositions.

For example, the formula $a \wedge b$ is equivalent to $\neg(\neg a \vee \neg b)$.

Truth Table:

a	b	$\neg a$	$\neg b$	$\neg a \vee \neg b$	$\neg(\neg a \vee \neg b)$	$a \wedge b$
1	1	0	0	0	1	1
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	1	0	0

Common Forms of Logical Equivalence:

1. Double Negation
$$\neg(\neg a) \equiv a$$

2. De Morgan's Laws
$$\neg(a \wedge b) \equiv \neg a \vee \neg b$$
$$\neg(a \vee b) \equiv \neg a \wedge \neg b$$

3. Commutative Laws
$$a \wedge b \equiv b \wedge a$$
$$a \vee b \equiv b \vee a$$

4. Associative Laws
$$(a \wedge b) \wedge r \equiv a \wedge (b \wedge r)$$
$$(a \vee b) \vee r \equiv a \vee (b \vee r)$$

5. Distributive Laws
$$a \wedge (b \vee r) \equiv (a \wedge b) \vee (a \wedge r)$$
$$a \vee (b \wedge r) \equiv (a \vee b) \wedge (a \vee r)$$

6. Implication Equivalences
$$a \to b \equiv \neg a \vee b$$
$$\neg(a \to b) \equiv a \wedge \neg b$$

7. Biconditional (Equivalence)
$$a \leftrightarrow b \equiv (a \to b) \wedge (b \to a)$$
$$a \leftrightarrow b \equiv (a \wedge b) \vee (\neg a \wedge \neg b)$$

8. Contrapositive
$$a \to b \equiv \neg b \to \neg a$$

10.4 Contingency

A well-formed formula (wff) is called a contingency if it is *True* in some cases and *False* in others. In other words, it is neither a tautology nor a contradiction. For instance, $a \vee b$ is a contingency; it can be *True* or *False* depending on the values of a and b.

Truth Table:

a	b	$a \vee b$
0	0	0
0	1	1
1	0	1
1	1	1

10.5 Satisfiability

Satisfiability refers to whether there exists *at least one* assignment of truth values to the variables in a formula that makes the formula *True*. A formula is satisfiable if it can be true for some assignment of its variables; otherwise, it is unsatisfiable. For example, the formula $a \wedge b$ is satisfiable because there is at least one assignment (when both a and b are *True*) that makes the formula true.

a	b	$a \wedge b$	$Satisfies?$
0	0	0	No
0	1	0	No
1	0	0	No
1	1	1	Yes

10.6 Validity

An argument is called *valid* if its conclusion necessarily follows from its premises, no matter whether the premises are true or false in the actual world. That is, in every case where the premises are true, the conclusion must also be true.

For example, consider the statement:

$$\neg a \to \neg b$$

Truth Table:

a	b	$\neg a$	$\neg b$	$\neg a \to \neg b$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	1
1	1	0	0	1

In the last row, both $\neg a$ and $\neg b$ evaluate to *False*. Since the antecedent ($\neg a$) is *False*, the entire implication $\neg a \to \neg b$ evaluates to *True* according to the truth-table definition of implication. This truth arises because an implication with a *False* antecedent is always *True*, regardless of the consequent.

10.7 Soundness

An argument is sound if:

1. $\Gamma \vdash A$ (syntactic validity - provable)
2. $\Gamma \models A$ (semantic validity - tautological consequence)
3. All premises in Γ are true

Soundness Theorem:

$$\Gamma \vdash A \to \Gamma \models A$$

Example:
Given premises $\Gamma = \{a, b\}$ and conclusion c:

$$(a \wedge b) \to c$$

where:

1. a: all birds have feathers
2. b: a robin is a bird
3. c: a robin has feathers

Truth table:

a	b	c	$a \wedge b$	$(a \wedge b) \to c$
1	1	1	1	1
1	1	0	1	0
1	0	1	0	1
1	0	0	0	1
0	1	1	0	1
0	1	0	0	1
0	0	1	0	1
0	0	0	0	1

In detail:

1. $\{a, b\} \vdash c$ (provable from premises)
2. $\{a, b\} \models c$ (tautological consequence)
3. $\varphi(a) = 1, \varphi(b) = 1, \varphi(c) = 1$ (all true in reality)
4. Therefore: the argument is sound

10.8 Entailment

If validity refers to reasoning that is structurally correct but not necessarily factually accurate, and soundness refers to reasoning that is both valid and factually true, then *entailment* is a logical relationship where the truth of one statement necessarily guarantees the truth of another within a formal system. In other words, entailment shows that a conclusion must be true if the premises are true, based solely on the rules of logic.

Example:

$$a \wedge b \models a$$

This means that the statement $a \wedge b$ (both a and b are true) *entails* a (that a is true).

Truth Table:

a	b	$a \wedge b$	a
1	1	1	1
1	0	0	1
0	1	0	0
0	0	0	0

Notice that in every case where $a \wedge b$ is true (the first row), the conclusion a is also true. This illustrates the entailment: $a \wedge b$ logically entails a.

10.9 Completeness

Definition:

A system is complete if

$$\Gamma \models A \rightarrow \Gamma \vdash A$$

Equivalently:

1. $\text{Cons}(\Gamma) \rightarrow \text{Sat}(\Gamma)$
2. $\models$ then $\vdash$

Completeness Theorem:

$$\Gamma \models A \rightarrow \Gamma \vdash A$$

Example:

Let

$$\Gamma = \{a \rightarrow b,\ a\}, \quad a = b$$

Then

$$\Gamma \models b$$

By (MP):

$$\frac{a \rightarrow b, \quad a}{b}$$

Hence

$$\Gamma \vdash b$$

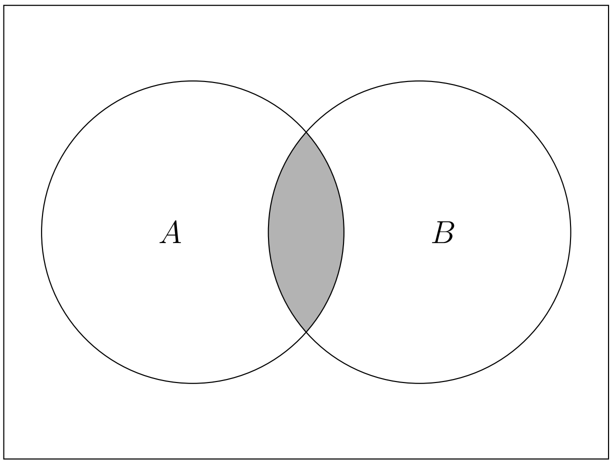
11 Venn Diagram

One illustrative or intuitive way to represent whether a formula is valid or not is by using a Venn diagram. In this case, Venn diagrams can be used to represent the validity of syllogisms and propositions. However, for this chapter, we will only study Venn diagrams for propositions.

11.1 Venn Diagram for Conjunction

A Venn diagram for conjunction is used to show the overlap between sets A and B, which means both A and B are true. The overlapping area represents:

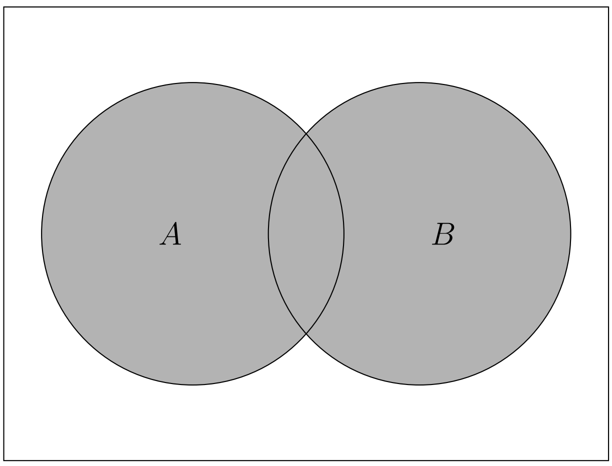
$$A \wedge B$$



11.2 Venn Diagram for Disjunction

A Venn diagram for disjunction is used to show the total area covered by sets A and B, including both individual and overlapping parts. This represents $A \vee B$, meaning either A, B, or both are true.

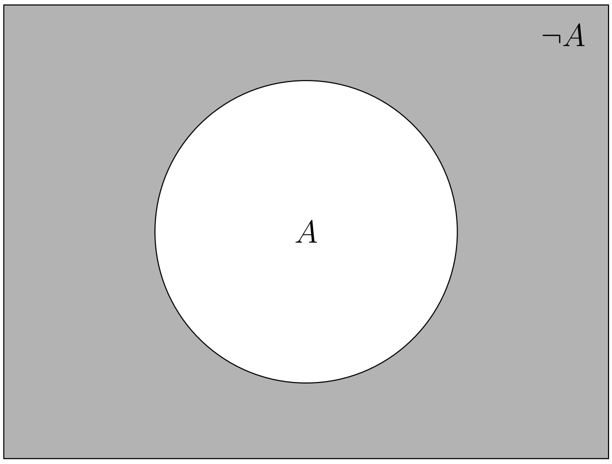
$$A \vee B$$



11.3 Venn Diagram for Negation

A Venn diagram for negation is used to show the area outside of circle A, meaning everything not in A. This represents that A is false.

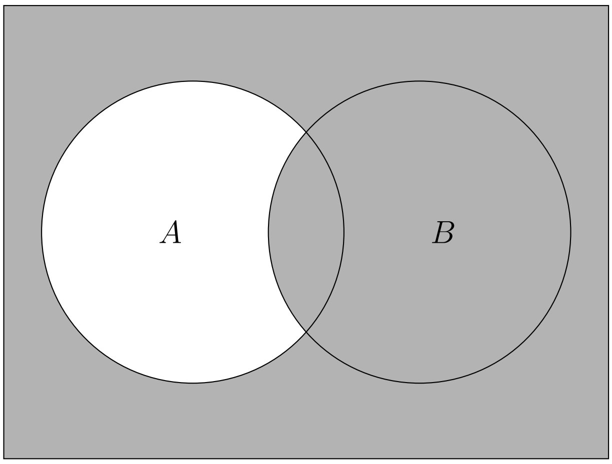
$$\neg A$$



11.4 Venn Diagram for Implication

A Venn diagram for implication is used to show that if A is true, then B must also be true. Visually, this means the part of A that is not in B is excluded, so A is entirely inside B, or the area where A is outside B is empty.

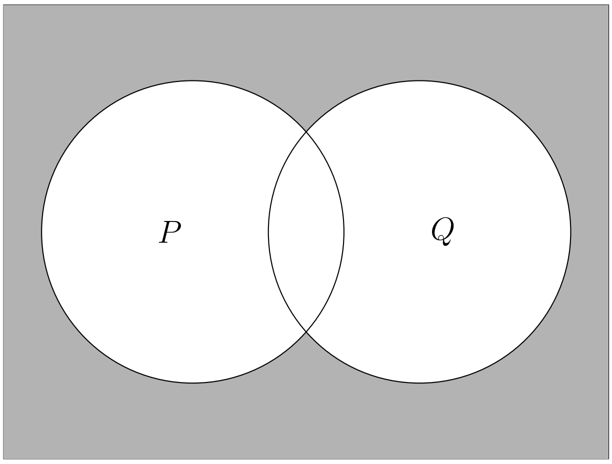
$$A \to B$$



11.5 Venn Diagram for Biconditional

The Venn diagram for $P \leftrightarrow Q$ represents the region where P and Q share the same truth value, that is, both are true or both are false.

$$P \leftrightarrow Q$$



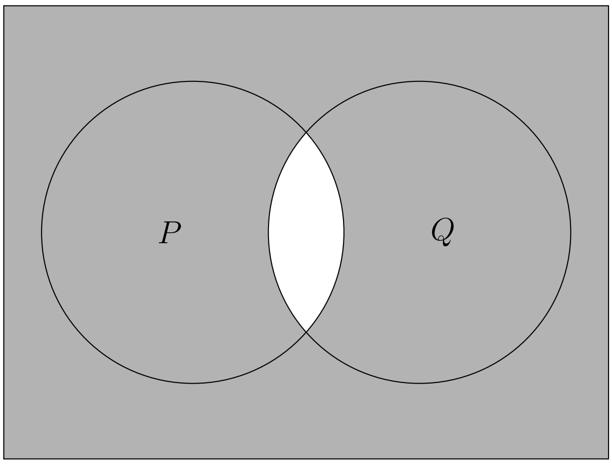
Moreover, with equivalent formulas we can also show the illustration with the Venn Diagram. For example:

11.6 More Examples

Example 1:

We can also represent equivalent formulas with Venn diagrams. For example, by De Morgan's Law: the Venn diagram for $\neg(P \wedge Q)$ shows the area outside the overlap of P and Q. It includes everything except the region where both P and Q are true.

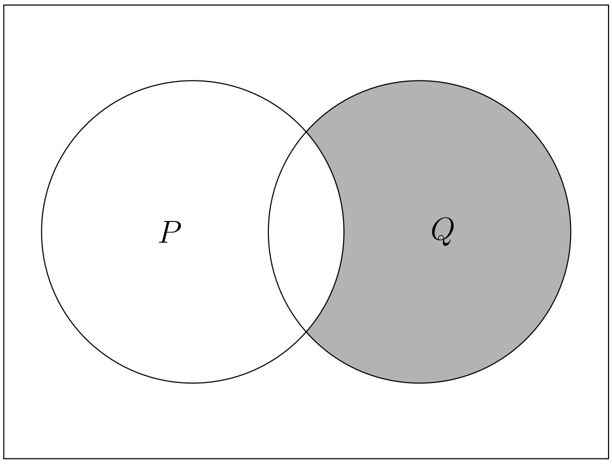
$$\neg(P \wedge Q) \quad \equiv \quad (\neg P) \vee (\neg Q)$$



Example 2:

The Venn diagram for $\neg P \wedge Q$ shows the area where Q is true but P is false. It includes the part inside Q but outside P.

$$(\neg P) \wedge Q$$



12 Rules of Inferences

Rules of inference are a list of rules used for natural deduction. Therefore, it is crucial to learn these rules before entering the natural deduction chapter.

12.1 Modus Ponens (MP)

The modus ponens rule is the process by which we affirm the antecedent.

First, we assume

$$a \to b$$

Then, we consider it true that

$$a$$

As a result, by logical consequence, then

$$b$$

Overall, it can be written as follows:

$$\frac{a \to b, \quad a}{\therefore b}$$

12.2 Modus Tollens (MT)

The modus tollens rule is the process by which we deny the consequent.

First, we assume

$$a \to b$$

Then, we consider it true that b is not true, so

$$\neg b$$

As a result, by logical consequence, then

$$\neg a$$

Overall, it can be written as follows:

$$\frac{a \to b, \quad \neg b}{\therefore \neg a}$$

12.3 Hypothetical Syllogism (HS)

Hypothetical Syllogism is a chain of proof, or often also called Transitive. The process is as follows:

First, we assume that

$$a \rightarrow b$$

Then, we assume

$$b \rightarrow c$$

So logically we can assume that

$$a \rightarrow c$$

can be considered true

Overall, it can be written as follows:

$$\frac{a \rightarrow b, \quad b \rightarrow c}{\therefore a \rightarrow c}$$

12.4 Disjunctive Syllogism (DS)

The reasoning process of DS is by eliminating one of the propositions. For example:

We assume

$$a \vee b$$

Then from that statement, we know that

$$\neg a$$

So, the final result is

$$b$$

Overall, it can be written as follows:

$$\frac{a \vee b, \quad \neg a}{\therefore b}$$

12.5 Constructive Dilemma (CD)

Constructive Dilemma is a form of deductive reasoning that combines two conditional propositions with a disjunction. The pattern is as follows:

1. First premise:
$$a \rightarrow b$$
2. Second premise:
$$c \rightarrow d$$
3. Third premise (disjunction on the antecedent):
$$a \vee c$$
4. Then the conclusion is:
$$b \vee d$$

Overall, the general form of Constructive Dilemma can be written as follows:

$$\frac{a \rightarrow b, \quad c \rightarrow d, \quad a \vee c}{\therefore b \vee d}$$

12.6 Addition (*Add*)

The Addition (*Add*) pattern is one of the simplest forms of inference. Essentially, if we have a true proposition, then we can add another proposition through logical operations (disjunction or conjunction).

For example, suppose we know that:

$$a$$

Then, we can add another proposition and obtain:

1. With disjunction:

   $$a \lor b$$

   Overall, it can be written as follows:

   $$\frac{a}{\therefore a \lor b}$$

2. With conjunction:

   $$a \land c$$

   Overall, it can be written as follows:

   $$\frac{a}{\therefore a \land c}$$

12.7 Simplification (*Simp*)

The Simplification (*Simp*) pattern, also called Conjunction Elimination, is an inference rule that allows us to draw conclusions from a conjunction. If we know that a conjunction is true, then each part of that conjunction is also true.

For example, suppose we have:

$$a \land b$$

Then we can conclude:

$$a$$

or

$$b$$

In general, the Simplification pattern can be written as follows:

$$\frac{a \land b}{\therefore a} \quad \text{or} \quad \frac{a \land b}{\therefore b}$$

12.8 Disjunction Elimination (DE)

The Disjunction Elimination pattern is an inference rule that allows us to draw conclusions from a disjunction. If we know that a disjunction is true, and from each of its alternatives we can conclude the same proposition, then we can conclude that proposition.

For example, suppose we have:

$$a \vee b$$

$$a \to c$$

$$b \to c$$

Then we can conclude:

$$c$$

In general, the Disjunction Elimination pattern can be written as follows:

$$\frac{a \vee b, \quad a \to c, \quad b \to c}{\therefore c}$$

12.9 Resolution (Res)

The Resolution (Res) pattern is an inference rule widely used in formal logic and automated proof. This rule works by eliminating a proposition and its negation from two disjunctions, to then produce a new disjunction.

In general, if we have:

$$(a \vee b), \quad (\neg a \vee c)$$

Then we can conclude:

$$b \vee c$$

In other words, variable a is eliminated because it appears in positive form in the first premise, and in negative form in the second premise.

The Resolution pattern can be written as follows:

$$\frac{a \vee b, \quad \neg a \vee c}{\therefore b \vee c}$$

12.10 Double Negation (DN)

The Double Negation (DN) pattern states that the double negation of a proposition is equivalent to the proposition itself. This means that if we have a proposition preceded by two negation signs, then both can be removed without changing the truth value.

In general:

$$a \equiv \neg\neg a$$

The truth of the statement above can be proven with a truth table:

a	$\neg a$	$\neg\neg a$	$a \equiv \neg\neg a$
1	0	1	1
0	1	0	1

The Double Negation inference rule can be written as follows:

$$\frac{\neg\neg a}{\therefore a}$$

12.11 Commutation (*Comm*)

The Commutation (*Comm*) pattern states that the order of propositions connected by logical operations of conjunction ($\wedge$) and disjunction ($\vee$) can be exchanged without changing the truth value.

In general:

1. For conjunction:
$$a \wedge b \equiv b \wedge a$$

2. For disjunction:
$$a \vee b \equiv b \vee a$$

The Commutation inference rule can be written as follows:

$$\frac{a \wedge b}{\therefore b \wedge a} \quad \text{or} \quad \frac{a \vee b}{\therefore b \vee a}$$

12.12 Association (*Assoc*)

The Association (*Assoc*) pattern states that grouping of propositions connected by conjunction ($\wedge$) and disjunction ($\vee$) does not affect the truth value. In other words, parentheses can be moved without changing the logical meaning.

In general:

1. For conjunction:
$$(a \wedge (b \wedge c)) \equiv ((a \wedge b) \wedge c)$$

2. For disjunction:
$$(a \vee (b \vee c)) \equiv ((a \vee b) \vee c)$$

The Association inference rule can be written as follows:

$$\frac{a \wedge (b \wedge c)}{\therefore (a \wedge b) \wedge c} \quad \text{or} \quad \frac{a \vee (b \vee c)}{\therefore (a \vee b) \vee c}$$

12.13 Distribution (*Dist*)

The Distribution (*Dist*) pattern states that conjunction ($\wedge$) can be distributed over disjunction ($\vee$), and vice versa. This rule allows us to change the form of a logical proposition without changing its truth value.

In general:

1. Conjunction over disjunction:
$$a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$$

2. Disjunction over conjunction:
$$a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$$

The Distribution inference rule can be written as follows:

$$\frac{a \wedge (b \vee c)}{\therefore (a \wedge b) \vee (a \wedge c)} \quad \text{or} \quad \frac{a \vee (b \wedge c)}{\therefore (a \vee b) \wedge (a \vee c)}$$

12.14 De Morgan's Laws (DeM)

De Morgan's Laws (DeM) pattern states the relationship between negation and conjunction ($\wedge$) and disjunction ($\vee$). This law shows how the negation of a conjunction or disjunction can be rewritten in equivalent form.

In general:

1. Negation of conjunction:
$$\neg(a \wedge b) \equiv (\neg a \vee \neg b)$$

2. Negation of disjunction:
$$\neg(a \vee b) \equiv (\neg a \wedge \neg b)$$

The truth of the statement above can be proven with truth tables:

Truth table 1:

a	b	$\neg a$	$\neg b$	$a \wedge b$	$\neg(a \wedge b)$	$\neg a \vee \neg b$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	0	1	1

Truth table 2:

a	b	$\neg a$	$\neg b$	$a \vee b$	$\neg(a \vee b)$	$\neg a \wedge \neg b$
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	1

De Morgan's Laws inference rule can be written as follows:

$$\frac{\neg(a \wedge b)}{\therefore \neg a \vee \neg b} \quad \text{or} \quad \frac{\neg(a \vee b)}{\therefore \neg a \wedge \neg b}$$

12.15 Implication ($Impl$)

The Implication ($Impl$) pattern states that an implication can be rewritten in disjunctive form.

In general:

$$a \to b \equiv \neg a \vee b$$

The truth of the statement above can be proven with a truth table:

a	b	$\neg a$	$\neg b$	$\neg a \vee b$	$a \to b$
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

The Implication inference rule can be written as follows:

$$\frac{a \to b}{\therefore \neg a \vee b}$$

12.16 Exportation (*Exp*)

The Exportation (*Exp*) pattern states that a nested implication is equivalent to an implication from the conjunction of antecedents.

In general (equivalence):

$$(a \to (b \to c)) \equiv ((a \land b) \to c)$$

The truth of the statement above can be proven with a truth table:

a	b	c	$\neg a$	$\neg b$	$\neg c$	$b \to c$	$a \to (b \to c)$	$a \land b$	$(a \land b) \to c$
1	1	1	0	0	0	1	1	1	1
1	1	0	0	0	1	0	0	1	0
1	0	1	0	1	0	1	1	0	1
1	0	0	0	1	1	1	1	0	1
0	1	1	1	0	0	1	1	0	1
0	1	0	1	0	1	0	1	0	1
0	0	1	1	1	0	1	1	0	1
0	0	0	1	1	1	1	1	0	1

The Exportation inference rule can be written in both directions as follows:

$$\frac{a \to (b \to c)}{\therefore (a \land b) \to c} \quad \text{and} \quad \frac{(a \land b) \to c}{\therefore a \to (b \to c)}$$

12.17 Contrapositive (*Contra*)

The Contrapositive rule states that an implication is equivalent to its contrapositive. This means that if a proposition has the form:

$$a \to b$$

then it is logically equivalent to:

$$\neg b \to \neg a$$

The truth of the statement above can be proven with a truth table:

a	b	$\neg a$	$\neg b$	$a \to b$	$\neg b \to \neg a$
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

The Contrapositive inference rule can be written as follows:

$$(a \to b) \equiv (\neg b \to \neg a)$$

12.18 Reductio Ad Absurdum (RAA)

Reductio Ad Absurdum (RAA), often called indirect proof, is an inference rule that states that if by assuming the negation of a proposition we arrive at a contradiction, then that proposition must be true. In other words, if the assumption $\neg a$ produces a contradiction (for example $a \wedge \neg a$), then we can conclude that a is true.

For example, suppose we assume:

$$\neg a$$

and from that assumption we can derive a contradiction:

$$a \wedge \neg a$$

Then we can conclude that:

$$a$$

In general, the Reductio Ad Absurdum pattern can be written as follows:

$$\frac{\neg a \quad \vdash \quad (a \wedge \neg a)}{\therefore a}$$

12.19 Conditional Proof (CP)

Conditional Proof (CP) is an inference rule used to prove implications. Essentially, if by assuming premise a we can derive conclusion b, then we can conclude that $a \to b$ is true.

For example, suppose we want to prove:

$$a \to b$$

From the following complex series of statements:

1. $a \to (c \wedge d)$
2. $c \to e$
3. $d \to f$
4. $e \wedge f \to b$

Steps using Conditional Proof:

Step	Statement	Justification
1.	a	Assumption (for CP)
2.	$c \wedge d$	Modus Ponens: Step 1, Premise 1 ($a \to (c \wedge d)$)
3.	c	Simplification: Step 2
4.	d	Simplification: Step 2
5.	e	Modus Ponens: Step 3, Premise 2 ($c \to e$)
6.	f	Modus Ponens: Step 4, Premise 3 ($d \to f$)
7.	$e \wedge f$	Conjunction: Step 5, Step 6
8.	b	Modus Ponens: Step 7, Premise 4 ($e \wedge f \to b$)
9.	$a \to b$	Conditional Proof: Steps 1-8

If from assumption a we can derive b, then we can conclude $a \to b$.

In general, the Conditional Proof pattern can be written as follows:

$$\frac{a \quad \vdash \quad b}{\therefore a \to b}$$

13 Natural Deduction Rules

After comprehending the rules of inference, we will have a better grasp of what natural deduction is. The most well-known idea of natural deduction is the work of Frederic Fitch. He created a system known as the *Fitch-style calculus*. With this system, we can mechanically derive propositions to reach a valid conclusion, based on the initial premises or statements.

13.1 Conjunction ($\wedge$)

13.1.1 Conjunction Introduction

$$\begin{array}{r|l l}
1 & A & \\
2 & \underline{B} & \\
3 & A \wedge B & \wedge\mathrm{I},\ 1,\ 2
\end{array}$$

13.1.2 Conjunction Elimination

$$\begin{array}{r|l l}
1 & \underline{A \wedge B} & \\
2 & A & \wedge\mathrm{E},\ 1 \\
3 & B & \wedge\mathrm{E},\ 1
\end{array}$$

13.2 Disjunction ($\vee$)

13.2.1 Disjunction Introduction

$$\begin{array}{r|l l}
1 & \underline{A} & \\
2 & A \vee B & \vee\mathrm{I},\ 1
\end{array}$$

13.2.2 Disjunction Elimination

$$\begin{array}{r|l l}
1 & \underline{P \vee Q} & \\
2 & \quad \vert\ \underline{P} & \\
3 & \quad \vert\ R & \mathrm{R},\ \ldots \\
4 & \quad \vert\ \underline{Q} & \\
5 & \quad \vert\ R & \mathrm{R},\ \ldots \\
6 & R & \vee\mathrm{E},\ 1,\ 2\text{–}3,\ 4\text{–}5
\end{array}$$

13.3 Implication ($\to$)

13.3.1 Implication Introduction

$$
\begin{array}{r|l l}
1 & \quad \begin{array}{|l} A \\ \hline \end{array} & \\
2 & \quad \begin{array}{|l} B \end{array} & \\
3 & A \to B & \Rightarrow\text{I}, 1\text{–}2
\end{array}
$$

13.3.2 Implication Elimination (Modus Ponens)

$$
\begin{array}{r|l l}
1 & A \to B & \\
2 & A & \\ \cline{2-2}
3 & B & \Rightarrow\text{E}, 1, 2
\end{array}
$$

13.4 Negation ($\neg$)

13.4.1 Negation Introduction

$$
\begin{array}{r|l l}
1 & \quad \begin{array}{|l} A \\ \hline \end{array} & \\
2 & \quad \begin{array}{|l} \bot \end{array} & \\
3 & \neg A & \neg\text{I}, 1\text{–}2
\end{array}
$$

13.4.2 Negation Elimination

$$
\begin{array}{r|l l}
1 & \neg A & \\
2 & A & \\ \cline{2-2}
3 & \bot & \neg\text{E}, 1, 2
\end{array}
$$

13.5 Bottom ($\bot$)

13.5.1 Bottom Elimination (Ex Falso Quodlibet)

$$
\begin{array}{r|l l}
1 & \bot & \\ \cline{2-2}
2 & A & \bot\text{E}, 1
\end{array}
$$

13.6 Reiteration (R)

$$
\begin{array}{r|l l}
1 & A & \\ \cline{2-2}
2 & A & \text{R}, 1
\end{array}
$$

13.7 Reductio Ad Absurdum (RAA)

1	│ $\neg A$	
2	│ $\bot$	
3	$\neg\neg A$	$\neg$I, 1–2
4	A	$\neg$E, 3

14 Semantic Tableaux

14.1 Conjunction Decomposition

$$\begin{array}{c} A \wedge B \\ | \\ A \\ | \\ B \end{array}$$

14.2 Double Negation

$$\begin{array}{c} \neg\neg A \\ | \\ A \end{array}$$

14.3 Negated Disjunction Decomposition

$$\begin{array}{c} \neg(A \vee B) \\ | \\ \neg A \\ | \\ \neg B \end{array}$$

14.4 Negated Conditional Decomposition

$$\begin{array}{c} \neg(A \to B) \\ | \\ A \\ | \\ \neg B \end{array}$$

14.5 Disjunction Decomposition

$$\begin{array}{ccc} & A \vee B & \\ / & & \backslash \\ A & & B \end{array}$$

14.6 Negated Conjunction Decomposition

$$\begin{array}{ccc} & \neg(A \wedge B) & \\ / & & \backslash \\ \neg A & & \neg B \end{array}$$

14.7 Conditional Decomposition

$$\begin{array}{ccc} & A \rightarrow B & \\ / & & \backslash \\ \neg A & & B \end{array}$$

14.8 Biconditional Decomposition

$$\begin{array}{ccc} & A \leftrightarrow B & \\ / & & \backslash \\ A,\ B & & \neg A,\ \neg B \end{array}$$

14.9 Negated Biconditional Decomposition

$$\begin{array}{ccc} & \neg(A \leftrightarrow B) & \\ / & & \backslash \\ A,\ \neg B & & \neg A,\ B \end{array}$$

15 First Order Logic

Basically, reasoning rules such as Double Negation (DN), Modus Ponens (MP), and Reductio ad Absurdum (RAA) are the most basic forms of reasoning. However, as we have learned, these three rules more often only work on logical patterns between statements. In other words, although deductively valid, their application is not yet strong enough to capture deeper mathematical structures.

As an example, consider the following statements:

p: 2 is prime

q: $2 \in \mathbb{N}$

If we convert this into symbolic form, we get:

$$p \wedge q$$

However, it should be noted that p and q here are merely representations of complete sentences in natural language. This symbolization has not yet touched the predicate level inherent in the number 2, namely the property of being prime and its membership in the set $\mathbb{N}$. In other words, propositional logic only "packages statements", but has not yet "dissected" the mathematical content of the statement itself. Therefore, a richer framework is needed to capture such expressions. This is where First-Order Logic (FOL) becomes crucial, as it allows us to express properties, relations, and quantification over mathematical objects more precisely.

15.1 Sentence of FOL

Normally, there are six kinds of symbols in FOL, and most of them are similar to those in propositional logic:

1. Constant symbols
$$a, b, c, \ldots, r, a_1, b_{224}, h_7, m_{32}, \ldots$$
2. Variable symbols
$$s, t, u, v, w, x, y, z, x_1, y_1, z_1, x_2, \ldots$$
3. Function symbols
$$f, g, h, f_1, g_2, h_{37}, \ldots$$
4. Predicate symbols
$$A, B, C, \ldots, Z, A_1, B_1, Z_1, A_2, A_{25}, J_{375}, \ldots, =$$
5. Logical connectives
$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$$
6. Brackets
$$(\;,\;)$$
7. Quantifiers
$$\forall, \exists$$

15.2 Terms and formulas

1. Every atomic formula is a formula.
2. If a and b are formulas, then so are:
    1. $\neg a$
    2. $(a \wedge b)$
    3. $(a \vee b)$
    4. $(a \rightarrow b)$
    5. $(a \leftrightarrow b)$
3. If a is a formula and x is a variable, then:
    1. $\forall x\, a$
    2. $\exists x\, a$

15.3 Bracketing Conventions in FOL

1. Outer parentheses may be omitted.

$$a \wedge b \quad \text{instead of} \quad (a \wedge b)$$

2. Negation binds most tightly.

$$\neg a \vee b \quad \text{means} \quad (\neg a) \vee b$$

3. Quantifiers extend as far to the right as possible.

$$\forall x\, a \vee b \quad \text{means} \quad (\forall x\, a) \vee b$$

4. Other connectives associate to the right unless parentheses indicate otherwise.

$$a \vee b \vee c \quad \text{means} \quad a \vee (b \vee c)$$

15.4 Superscripts on Predicates in FOL

1. Predicate symbols may be written with superscripts to indicate their arity.

$$P^1 \text{ (unary)}, \quad Q^2 \text{ (binary)}, \quad R^3 \text{ (ternary)}$$

2. In general, if P^n is an n-place predicate and $t_1, \ldots, t_n$ are terms, then

$$P^n(t_1, \ldots, t_n) \text{ is an atomic formula.}$$

15.5 Name

Names, in general, function to indicate both the existence of an entity and serve as a reference for a particular object or place. For example, the word "Gottlob Frege" refers specifically to an individual, a great logician who was instrumental in the development of predicate logic. However, it must be recognized that if someone has never heard the name "Gottlob Frege" before, they may not know who is being referred to. In fact, that person might even imagine that "Gottlob Frege" is not the name of a human being, but merely a term or particular object. Such conditions are known in philosophy of language as a problem inherent in proper names, namely the potential ambiguity in reference.

Within the framework of FOL, such ambiguity cannot be tolerated. The reason is simple, when we engage in formal reasoning, we require that the expressions used be clear, syntactically valid, and also sound in meaning. Therefore, naming in FOL must have definite reference and leave no room for double interpretation. In FOL, constant symbols usually function like proper names, pointing specifically to an object within the domain. For example:

1. Constant a refers to object 2 in $\mathbb{N}$.
2. Constant b refers to object -5 in $\mathbb{Z}$.

So when we write the predicate $Prime(a)$, it must be understood definitively as "2 is a prime number." There can be no ambiguity about whether a refers to the number 2, to a person named "Gottlob Frege" or even to a chair, because in FOL, the interpretation of every symbol must be clear within the established domain.

15.6 Predicate

In everyday language, a predicate is the part of a statement that provides information or describes the subject. For example, in the sentence *Snow is white* the phrase "is white" is the predicate.

Consider the following example:

Premise 1: All humans are mortal.

Premise 2: Socrates is a human.

Conclusion: Therefore, Socrates is mortal.

As we can see, expressions such as *"are mortal"* and *"is a human"* are predicates. Through such structures, predicate logic enables us to express relationships and properties of objects in greater depth than standard propositional logic. Moreover, it is important to understand that a predicate like *"mortal"* is not a standalone proposition but a component of one. A predicate can be thought of as a framework or function with an empty slot waiting to be filled by a subject. For example:

$Mortal(\dots)$

This structure may feel counterintuitive, since we are accustomed to complete sentences in natural language. However, there are historical and logical reasons why this functional form is preferred in formal logic.

Following the example above, if we fill the blank with *Socrates*, we obtain:

$$Mortal(Socrates)$$

This indicates that *"Socrates"* is placed in the subject position of the predicate *"mortal"*, forming a complete proposition. For simplicity and efficiency, logicians often replace predicate names with a single capital letter. Thus, $Mortal(Socrates)$ is simplified to:

$M(Socrates)$, where M represents the predicate *"mortal."*

We can simplify even further by representing the name *"Socrates"* with the constant S. Therefore, the whole expression becomes:

$$M(S)$$

Furthermore, to express *"Socrates is human and mortal"*, we can combine the two predicates into a single expression:

$$H(S) \wedge M(S)$$

Thus, the expression $H(S) \wedge M(S)$ states that Socrates satisfies both predicates: he is a human and he is mortal. Accordingly, we can also represent general statements using formulas with *free variables*. The idea is that the variable serves as a placeholder, or a "blank," that can later be filled by any element from the domain of discourse. For example:

$$P(x), \quad O(x)$$

15.7 Quantifiers

Shortly speaking, when discussing predicate logic, one immediately thinks of Gottlob Frege (1848–1925). Frege introduced the concept of quantifiers, namely $\forall$ and $\exists$. These concepts were later popularized in the early 20th century by Bertrand Russell and Alfred North Whitehead through their seminal work *Principia Mathematica*.

The use of these symbols is essential to represent not only a single variable but also many variables. For example:

Everyone is happy

In this case, we cannot represent the statement merely as $H(e)$, as we did in the Socrates" example. Instead, we represent it as:

$$\forall x\, H(x)$$

In this case, we can represent anything with the predicate *Happy*. The variable acts as a placeholder, so whatever we choose from the domain can be tested against the predicate. This shows that the predicate *Happy* is not tied to a single individual but can be applied universally to any object in the chosen domain of discourse.

Similarly, consider the example:

Someone is unique

We can represent this statement using the existential quantifier as follows:

$$\exists x\, U(x)$$

Here: $\exists x$ intuitively means *"there exists at least one x"* in the domain.

$U(x)$ states that this x has the property of being unique.

Thus, the formula $\exists x\, U(x)$ expresses that there is at least one object in the domain such that the predicate *Unique* applies to it.

Everyone loves someone

$$\forall x \exists y\, L(x, y)$$

This formula uses two variables: x represents the lover and y represents the beloved. The formula states that for every person x in the domain, there exists some person y such that x loves y.

Someone is loved by everyone

$$\exists y \forall x\, L(x, y)$$

This formula states that there exists a person y such that for all persons x, x loves y, meaning there is one person who is universally loved.

If someone is a teacher, then everyone respects them

$$\exists x(T(x) \land \forall y\, R(y, x))$$

This formula combines both quantifiers with a conditional structure. It states that there exists a person x who is a teacher, and for all persons y, y respects x.

Everyone who teaches logic is respected by all students

$$\forall x((T(x) \land L(x)) \to \forall y(S(y) \to R(y, x)))$$

This more complex formula involves multiple predicates, $T(x)$ means "x teaches", $L(x)$ means "x teaches logic", $S(y)$ means "y is a student", and $R(y, x)$ means "y respects x". The formula demonstrates how multiple variables can interact within nested quantifier scopes.

15.8 Domains

It is also important to comprehend the domain of our reasoning in predicate logic. The critical point arises when we begin to express more complex statements, we must be precise about how variables are used within predicates.

Consider the formula:

$$\forall x\, H(x)$$

which we might translate as "Everyone is happy." But who exactly counts as "everyone"? In everyday English, we do not usually mean literally every person who has ever lived or ever will live. We typically mean something more limited: everyone in this building, everyone in a class, everyone on a team, and so on. Predicate logic removes this ambiguity by requiring us to specify a domain of discourse. The domain is the set of objects under discussion.

For example, suppose we set our symbolization key as follows:

1. Domain: people in Finland
2. $H(x)$: x is happy

Now the quantifiers range only over that domain.

$\forall x\, H(x)$ means "Every person in Finland is happy."
$\exists x\, H(x)$ means "Some person in Finland is happy."

In FOL, the domain must contain at least one member, and each name must refer to exactly one object in that domain. So if S names Sam, then from $H(S)$ ("Sam is happy") we can infer $\exists x\, H(x)$ ("Someone is happy").

The choice of domain is crucial. Suppose we instead use this symbolization key:

For example, suppose we set our symbolization key as follows:

1. Domain: the Eiffel Tower
2. $P(x)$: x is in Paris

Now the quantifiers range only over that domain. Since the domain has only one member (the Eiffel Tower), the universal and existential quantifiers have the same meaning in this specific context.

$$\exists x\, P(x) \quad \text{means "The Eiffel Tower is in Paris."}$$

Then $\forall x\, P(x)$ would translate as "Everything is in Paris." But since the domain contains only the Eiffel Tower, what the sentence really says is simply: "The Eiffel Tower is in Paris."

Another example, suppose we want to say:

There is exactly one king in the palace.

If we only write:

$$\exists x\, King(x)$$

this means *"there exists at least one x such that x is a king."* But it does not prevent the possibility of there being two or more kings. To capture the idea of uniqueness, we need to express that there is *one and only one*. This is often written with the uniqueness quantifier $\exists!$

$$\exists! x\, King(x)$$

Alternatively, we can express uniqueness using standard quantifiers:

$$\exists x(King(x) \land \forall y(King(y) \to x = y))$$

This formula states: "There exists an x such that x is a king, and for all y, if y is a king, then x equals y", ensuring there is exactly one king.

15.9 Quantifiers and Scope

When we introduce quantifiers into formal logic, understanding their scope becomes crucial for accurate translation and interpretation. The scope of a quantifier determines which variables it binds and how far its influence extends in a formula.

1. **Example 1 :** :

   If anyone is a teacher, they are respected

   $$\overbrace{\forall x\,(T(x) \to R(x))}^{\text{Scope of } \forall x}$$

   The universal quantifier scopes over the conditional, creating a universal generalization. This statement holds vacuously true in domains without teachers. *All teachers are respected* yields the same logical form.

2. **Example 2:** :

Every professor knows a student who failed

$$\overbrace{\forall y \left(\text{Professor}(y) \rightarrow \overbrace{\exists x \left(\text{Student}(x) \wedge \text{Failed}(x) \wedge \text{Knows}(y, x)\right)}^{\text{Scope of } \exists x}\right)}^{\text{Scope of } \forall y}$$

3. **Example 3:** :

Every student has submitted some assignment to each professor

$$\overbrace{\forall x \left(\text{Student}(x) \rightarrow \overbrace{\forall z \left(\text{Professor}(z) \rightarrow \overbrace{\exists y \left(\text{Assignment}(y) \wedge \text{Submitted}(x, y, z)\right)}^{\text{Scope of } \exists y}\right)}^{\text{Scope of } \forall z}\right)}^{\text{Scope of } \forall x}$$

15.10 The Order of Quantifiers

1. **Example 1:**

   Consider the sentence *Everyone admires some Logician.* This is ambiguous and can be represented in two different ways:

   Interpretation 1: For every person x, there exists some Logician y such that x admires y:

$$\overbrace{\forall x \left(\overbrace{\exists y \left(Person(x) \wedge Logician(y) \wedge Admires(x, y)\right)}^{\text{Scope of } \exists y}\right)}^{\text{Scope of } \forall x}$$

   Interpretation: Everyone may admire different Logician.

   Interpretation 2: There exists some particular Logician y such that every person x admires y:

$$\overbrace{\exists y \left(\text{Logician}(y) \wedge \overbrace{\forall x \left(\text{Person}(x) \wedge \text{Admires}(x, y)\right)}^{\text{Scope of } \forall x}\right)}^{\text{Scope of } \exists y}$$

   Interpretation: There's one particular Logician who everyone admires.

2. **Example 2:**

   Consider the sentence *Every student passed some exam.* This has two distinct interpretations:

Interpretation 1: For each student, there exists at least one exam they passed

$$\overbrace{\forall x \left(\text{Student}(x) \to \overbrace{\exists y \, (\text{Exam}(y) \land \text{Passed}(x, y))}^{\text{Scope of } \exists y} \right)}^{\text{Scope of } \forall x}$$

Interpretation: Each student passed at least one exam (possibly different exams).

Interpretation 2: There exists one particular exam that every student passed

$$\overbrace{\exists y \left(\text{Exam}(y) \land \overbrace{\forall x \, (\text{Student}(x) \to \text{Passed}(x, y))}^{\text{Scope of } \forall x} \right)}^{\text{Scope of } \exists y}$$

Interpretation: There's one specific exam that all students passed.

Consider the sentence *Every patient has some symptom*:

Interpretation 1: Each patient exhibits at least one symptom

$$\overbrace{\forall x \left(\text{Patient}(x) \to \overbrace{\exists y \, (\text{Symptom}(y) \land \text{Has}(x, y))}^{\text{Scope of } \exists y} \right)}^{\text{Scope of } \forall x}$$

Interpretation: Every patient shows some symptoms (different symptoms for different patients).

Interpretation 2: There's one symptom that all patients have

$$\overbrace{\exists y \left(\text{Symptom}(y) \land \overbrace{\forall x \, (\text{Patient}(x) \to \text{Has}(x, y))}^{\text{Scope of } \forall x} \right)}^{\text{Scope of } \exists y}$$

Interpretation: There's a common symptom shablack by all patients.

3. **Example 3:**

   Consider the sentence *Every company hiblack some consultant*:

   Interpretation 1: Each company hiblack at least one consultant

$$\overbrace{\forall x \left(\text{Company}(x) \to \overbrace{\exists y \, (\text{Consultant}(y) \land \text{Hiblack}(x, y))}^{\text{Scope of } \exists y} \right)}^{\text{Scope of } \forall x}$$

   Interpretation: Each company hiblack consultants (possibly different ones).

   Interpretation 2: There's one consultant hiblack by all companies

$$\overbrace{\exists y \left(\text{Consultant}(y) \land \overbrace{\forall x \, (\text{Company}(x) \to \text{Hiblack}(x, y))}^{\text{Scope of } \forall x} \right)}^{\text{Scope of } \exists y}$$

Interpretation: One super-consultant was hiblack by every company.

Consider the sentence *For every number, there exists a larger number*:

Interpretation 1: For each number, we can find a larger one

$$\overbrace{\forall x\, \overbrace{\exists y\, (x < y)}^{\text{Scope of } \exists y}}^{\text{Scope of } \forall x}$$

Interpretation: There's no largest number (true statement).

Interpretation 2: There exists one number larger than all numbers

$$\overbrace{\exists y\, \overbrace{\forall x\, (x < y)}^{\text{Scope of } \forall x}}^{\text{Scope of } \exists y}$$

Interpretation: There's a supremely large number (false statement).

4. **Example 4:**

   Consider the sentence *Everyone trusts someone*:

   Interpretation 1: Each person trusts at least one person

$$\overbrace{\forall x\left(\text{Person}(x) \to \overbrace{\exists y\, (\text{Person}(y) \land \text{Trusts}(x, y))}^{\text{Scope of } \exists y}\right)}^{\text{Scope of } \forall x}$$

   Interpretation: Nobody is completely distrustful; everyone trusts someone.

   Interpretation 2: There's one person trusted by everyone

$$\overbrace{\exists y\left(\text{Person}(y) \land \overbrace{\forall x\, (\text{Person}(x) \to \text{Trusts}(x, y))}^{\text{Scope of } \forall x}\right)}^{\text{Scope of } \exists y}$$

   Interpretation: There's a universally trusted person.

5. **Example 5:**

   Consider the sentence *Every device connects to some network*:

   Interpretation 1: Each device connects to at least one network

$$\overbrace{\forall x\left(\text{Device}(x) \to \overbrace{\exists y\, (\text{Network}(y) \land \text{Connects}(x, y))}^{\text{Scope of } \exists y}\right)}^{\text{Scope of } \forall x}$$

   Interpretation: All devices have network connectivity (possibly to different networks).

Interpretation 2: There's one network that all devices connect to

$$\overbrace{\exists y \left(\text{Network}(y) \land \overbrace{\forall x \, (\text{Device}(x) \to \text{Connects}(x, y))}^{\text{Scope of } \forall x}\right)}^{\text{Scope of } \exists y}$$

Interpretation: There's a universal network that connects all devices.

These examples demonstrate how the *quantifier shift fallacy* can dramatically alter meaning. The pattern $\forall x \exists y$ (distributive) versus $\exists y \forall x$ (collective) represents fundamentally different logical relationships, making quantifier order crucial for precise logical reasoning.

15.11 Free and Bound Variables

In the rules of predicate logic, we must also understand the concepts of free variables and bound variables. In short, a free variable is an object being referred to but is not within the scope of a universal or existential quantifier. Whereas a bound variable is an object that is expressed within the scope of a quantifier, either universal or existential. Understanding this distinction is essential for correctly interpreting logical formulas and determining whether a formula is closed (has no free variables) or open (contains free variables).

1. **Free variable**

   Consider the formula:

   $$\forall x \left(P(x) \land \overbrace{Q(y)}^{\text{Free}}\right)$$

   In this formula, x is bound by the universal quantifier $\forall x$, but y is free because it is not within the scope of any quantifier. The truth of this formula depends on the value of y, while x is quantified over all possible values.

2. **Bound variable**

   Consider the formula:

   $$\overbrace{\exists x \left(P(x) \land Q(x)\right)}^{\text{Scope of } \exists x}$$

   In this formula, x is bound by the existential quantifier $\exists x$. The truth of the formula does not depend on any external value of x, because it is fully specified within the scope of the quantifier.

3. **Mixed free and bound variables**

   Consider the formula:

   $$\forall x \left(P(x, \overbrace{y}^{\text{Free}}) \to \exists z \; \overbrace{Q(x, z)}^{\text{Both bound}}\right)$$

   Here:

(a) x is bound by $\forall x$

(b) z is bound by $\exists z$

(c) y is free (appears in $P(x, y)$ but isn't quantified)

The truth value depends on what y represents, while x and z are internally determined.

4. **Nested quantifiers with name collision**

Consider the formula:

$$\forall x \left(P(x) \to \overbrace{\exists x\, R(x, y)}^{\text{Inner } x \text{ bound}} \right)$$

Here we have two different variables both named x:

(a) The outer x is bound by $\forall x$ in $P(x)$

(b) The inner x is bound by $\exists x$ in $R(x, y)$

(c) y is free throughout

The inner quantifier shadows the outer one within its scope.

5. **Multiple free variables**

Consider the formula:

$$\overbrace{P(a, b)}^{\text{Both free}} \land \forall x\, \overbrace{Q(x, c)}^{c \text{ is free}}$$

Here:

(a) a, b, and c are all free variables

(b) x is bound by $\forall x$

The truth depends on the specific values assigned to a, b, and c.

6. **Complex nesting**

Consider the formula:

$$\exists x\, \forall y \left(\overbrace{P(x, y)}^{\text{Both bound}} \to \exists z \left(\overbrace{Q(x, z)}^{\text{Both bound}} \land \overbrace{R(w)}^{w \text{ free}} \right) \right)$$

Here:

(a) x is bound by $\exists x$ (outermost)

(b) y is bound by $\forall y$

(c) z is bound by the inner $\exists z$

(d) w is free (not quantified anywhere)

7. **No quantifiers (all free)**

   Consider the formula:

$$\overbrace{P(a) \land Q(b,c) \to R(a,d)}^{\text{All variables free}}$$

   Since there are no quantifiers, all variables a, b, c, and d are free. The truth value depends entirely on the interpretation of these variables.

8. **Sequential quantifiers**

   Consider the formula:

$$\forall x\, \exists y\, \forall z\, \overbrace{P(x,y,z)}^{\text{All bound}} \land \overbrace{Q(w)}^{w \text{ free}}$$

   Here:

   (a) x is bound by $\forall x$
   (b) y is bound by $\exists y$
   (c) z is bound by $\forall z$
   (d) w is free

   The quantifiers create a chain: "for all x, there exists a y, such that for all z..."

9. **Partial binding**

   Consider the formula:

$$\exists x \left(\overbrace{P(x)}^{\text{Bound}} \land \overbrace{Q(x,y)}^{x \text{ bound, } y \text{ free}} \right) \to \overbrace{R(y,z)}^{\text{Both free}}$$

   Here:

   (a) x is bound by $\exists x$ within the antecedent of the $\to$.
   (b) y is free (it appears within the scope of $\exists x$ but is not quantified by it, and it is also free in $R(y,z)$).
   (c) z is free throughout.

   This shows how a variable like y can be free across different parts of a formula. (Note: The original text had a slight error in describing the status of y in $R(y,z)$, which is definitely free.)

15.12 Quantifier Equivalences

We understand that one logical symbol can often be rewritten in terms of others. In propositional logic, for instance, the implication symbol ($\to$) can be expressed using disjunction ($\lor$) and negation ($\neg$). For example:

$$p \to q \quad \equiv \quad \neg p \lor q$$

A similar relationship exists in picate logic between the universal quantifier ($\forall$) and the existential quantifier ($\exists$), connected through negation:

$$\forall x\, c(x) \quad \equiv \quad \neg\exists x\, \neg c(x)$$

This means: *"For all x, $c(x)$ holds"* is equivalent to *"There does not exist an x such that $c(x)$ does not hold."*

Conversely:

$$\exists x\, c(x) \quad \equiv \quad \neg\forall x\, \neg c(x)$$

This means: *"There exists an x such that $c(x)$ holds"* is equivalent to *"It is not the case that $c(x)$ fails for all x."*

These equivalences also extend naturally to statements beginning with negation:

$$\neg\forall x\, c(x) \quad \equiv \quad \exists x\, \neg c(x)$$

This means: *"It is not true that $c(x)$ holds for all x"* is logically equivalent to saying *"There exists at least one x such that $c(x)$ does not hold."*

And similarly:

$$\neg\exists x\, c(x) \quad \equiv \quad \forall x\, \neg c(x)$$

This means: *"It is not true that there exists an x such that $c(x)$ holds"* is logically equivalent to saying *"For every x, $c(x)$ does not hold."*

16 Multi Place Predicates

As we have seen in the previous chapter, a formula like this is called a *One-Place*:

$$\forall x((A(x) \to B(x)))$$

However, not every mathematical statement deals with only one variable and predicate. That's why we use *Multi-Place Predicates*. With multi-place predicates, we can create more expressive formulas. For example:

16.1 Two-Place Predicates

$$R(x, y)$$

Expresses a relation between two objects.

1. If there exists a number greater than x, then there exists a number greater than $x + 1$
$$\forall x\, (\exists y\, G(y, x) \to \exists z\, G(z, x + 1))$$
2. If there exists a number y equal to x, then x equals y
$$\forall x\, (\exists y\, E(x, y) \to E(y, x))$$
3. If $x = y$ and there exists z such that $y = z$, then $x = z$
$$\forall x \forall y\, ((E(x, y) \land \exists z\, E(y, z)) \to \exists z\, E(x, z))$$
4. If x divides y, and there exists z divisible by y, then x divides z
$$\forall x \forall y\, (D(x, y) \land \exists z\, D(y, z) \to \exists z\, D(x, z))$$

16.2 Three-Place Predicates

$$R(x, y, z)$$

Expresses a relation among three objects.

1. For every number x, there exists a number y such that x is less than y.

$$\forall x\, \exists y\, L(x, y)$$

2. For all numbers p, if p is prime, then there exists a number q such that $q = p + 2$.

$$\forall p\, (\mathbb{P}(p) \to \exists q\, (q = p + 2))$$

3. For all numbers n, if n is even, then there exists a number m such that $m = n/2$.

$$\forall n\, (\text{Even}(n) \to \exists m\, (m = n/2))$$

4. For all x, if x is a natural number, then there exists a y in the integers such that x is not equal to y.

$$\forall x\, (x \in \mathbb{N} \to \exists y\, (y \in \mathbb{Z} \wedge x \neq y))$$

16.3 Four-Place Predicates

$$Q(w, x, y, z)$$

Expresses a relation among four objects.

1. For every a and b, there exists a c such that $a + b + c$ is even

$$\forall a \forall b\, \exists c \exists d\, (Q(a, b, c, d) \wedge \text{Even}(d))$$

2. If a, b, c are positive, then there exists $d > 0$ such that $a + b + c = d$

$$\forall a \forall b \forall c\, ((a > 0 \wedge b > 0 \wedge c > 0) \to \exists d\, Q(a, b, c, d) \wedge d > 0)$$

3. For every x, y, there exists z such that $x \times y + z$ is even

$$\forall x \forall y\, \exists z \exists w\, (P(x, y, z, w) \wedge \text{Even}(w))$$

4. If $x, y, z > 0$, then there exists $w > 0$ such that $x \times y + z = w$

$$\forall x \forall y \forall z\, ((x > 0 \wedge y > 0 \wedge z > 0) \to \exists w\, P(x, y, z, w) \wedge w > 0)$$

17 Identity and Quantity

17.1 Identity

Simply put, in logic, something is considered identical if a is the same as b, symbolically written as $a = b$. However, we must be careful in understanding these two constants. Specifically, the term "identical" here does not refer to two objects that are very similar, but rather to one and the same object. Empirically, two different objects can never be truly identical. Identity does not speak of similarity, but of sameness, that a and b are not merely alike, but actually refer to the exact same entity. Consider the classic example from the philosophy of language, "Morning Star" and "Evening Star", these are both names that refer to the same object, Venus. Another example, consider this expression, $\frac{1}{2} = 0.5$, logically we agree that this expression is same.

For example, Consider the "Morning Star" and "Evening Star", we can symbolize these statements like this:

Let:

1. m = the Morning Star
2. e = the Evening Star

Then the identity claim is:

$$m = e$$

This states that both names refer to one and the same object.

If we treat them as predicates:

1. $MS(x)$ = "x is the Morning Star"
2. $ES(x)$ = "x is the Evening Star"

Then we can express:

$$\exists x\, (MS(x) \wedge ES(x))$$

Meaning: *there exists an object x such that x is both the Morning Star and the Evening Star.*

Moreover, identity is not just a symbol but follows specific rules that govern how it works.

Example:

1. **Reflexivity**

   Every object is identical to itself. This is written as

   $$\forall x(x = x)$$

   which expresses the obvious fact that no matter what object we choose, it is always equal to itself.

   For example, let $P(x)$ mean "x is a prime number." If $x = 7$, then it must also be true that $7 = 7$. In predicate form, we can write this as

$$\forall x\,(x = 7 \to x = x)$$

And if we combine reflexivity with a property, we get

$$\forall x\,(x = 7 \to (P(x) \leftrightarrow P(7)))$$

This means that if $x = 7$, then whatever property P applies to x must also apply to 7. For instance, if $P(x)$ means "x is prime," then since $x = 7$, $P(x)$ holds if and only if $P(7)$ holds.

2. **Leibniz's Law**

   Leibniz's Law, also called the principle of substitutivity of identicals. This states that if $x = y$, then whatever is true of x is also true of y, symbolized as

$$x = y \to (P(x) \leftrightarrow P(y))$$

   If two names refer to the same object, then they are interchangeable in any true statement. For example, if $m = e$ (Morning Star = Evening Star), and the statement "m is visible at dawn" is true, then it must also be true that "e is visible at dawn."

17.2 Quantity

In everyday language, we often encounter vagueness, even ambiguity, when interpreting someone's statements. For example, consider the sentence: *"All humans love some dogs."* The word *some* here is vague, because it is unclear how many dogs are being referred to, or whether all humans love the same dogs or different ones. This ambiguity can lead to different interpretations. For instance, the sentence can be understood in two distinct ways:

Interpretation 1: All humans love at least one dog. In predicate logic, this is written as

$$\forall x\,(Human(x) \to \exists y\,(Dog(y) \land Loves(x, y)))$$

This means: every human loves at least one dog, though the specific dog may differ from person to person.

Interpretation 2: There is some dog that all humans love. In predicate logic, this is written as

$$\exists y\,(Dog(y) \land \forall x\,(Human(x) \to Loves(x, y)))$$

This means: there exists a particular dog that is loved by all humans.

To avoid ambiguity, we can express the specific quantity of objects we are talking about. Predicate logic allows us to distinguish between *at least*, *at most*, and *exactly* a certain number of objects satisfying a condition $A(x)$.

1. **At least (minimum number).** This means there are at least n different objects that satisfy $A(x)$, possibly more.

(a) At least one:

$$\exists x\, A(x)$$

This means: *"There exists at least one x such that $A(x)$ is true."*

Example:

Let $A(x)$ = "x is a prime number." Then the statement says: *"There is at least one prime number."* This is true, since $x = 2$ works.

(b) At least two:

$$\exists x \exists y\, (A(x) \wedge A(y) \wedge x \neq y)$$

This means: *"There exist at least two different objects such that $A(x)$ holds."*

Example:

Let $A(x)$ = "x is a prime number." Then the statement says: *"There are at least two prime numbers."* This is true, because $x = 2$ and $y = 3$ both satisfy $A(x)$, and $2 \neq 3$.

(c) At least three:

$$\exists x \exists y \exists z\, (A(x) \wedge A(y) \wedge A(z) \wedge x \neq y \wedge x \neq z \wedge y \neq z)$$

This means: *"There exist at least three different objects such that $A(x)$ holds."*

Example:

Let $A(x)$ = "x is a prime number." Then the statement says: *"There are at least three prime numbers."* This is true, because $x = 2$, $y = 3$, and $z = 5$ all satisfy $A(x)$, and they are pairwise distinct.

2. **At most (maximum number).** This means there are no more than n different objects satisfying $A(x)$.

(a) At most one:

$$\forall x \forall y\, ((A(x) \wedge A(y)) \rightarrow x = y)$$

This means: *"If x and y both satisfy A, then they must be the same object."*

Example:

Let $A(x)$ = "x is the even prime number." Then the statement says: *"There is at most one even prime."* This is true, since 2 is the only even prime, and no two distinct numbers can both satisfy $A(x)$.

(b) At most two:

$$\forall x \forall y \forall z\, ((A(x) \wedge A(y) \wedge A(z)) \rightarrow (x = y \vee x = z \vee y = z))$$

This means: *"If x, y, and z all satisfy A, then at least two of them are the same."* So there cannot be three distinct objects satisfying $A(x)$.

Example:

Let $A(x)$ = "x is a square root of 4." Then the statement says: *"There are at most two square roots of* 4.*"* This is true, because the only solutions are $x = 2$ and $x = -2$.

(c) At most three:

$$\forall x \forall y \forall z \forall w \; ((A(x) \land A(y) \land A(z) \land A(w)) \\ \to (x = y \lor x = z \lor x = w \lor y = z \lor y = w \lor z = w))$$

This means: *"If x, y, z, and w all satisfy A, then at least two of them must be the same."* So there cannot be four distinct objects satisfying $A(x)$.

Example:

Let $A(x)$ = "x is a primary color of light." Then the statement says: *"There are at most three primary colors of light."* This is true, since the only options are red, green, and blue.

3. **Exactly (precise count).** This means there are exactly n distinct objects satisfying $A(x)$, not more, not less.

(a) Exactly one:

$$\exists x \, (A(x) \land \forall y (A(y) \to y = x))$$

This means: *"There exists exactly one object such that $A(x)$ holds."*

Example:

Let $A(x)$ = "x is the even prime number." Then the statement says: *"There is exactly one even prime number."* This is true, since 2 is the only even prime.

(b) Exactly two:

$$\exists x \exists y \, (A(x) \land A(y) \land x \neq y \land \forall z (A(z) \to (z = x \lor z = y)))$$

This means: *"There exist exactly two distinct objects such that $A(x)$ holds."*

Example:

Let $A(x)$ = "x is a square root of 4." Then the statement says: *"There are exactly two square roots of 4."* This is true, since the solutions are 2 and -2, and no others.

(c) Exactly three:

$$\exists x \exists y \exists z \; (A(x) \land A(y) \land A(z) \land x \neq y \land y \neq z \land x \neq z \\ \land \forall w (A(w) \to (w = x \lor w = y \lor w = z)))$$

This means: *"There exist exactly three distinct objects such that $A(x)$ holds."*

For example:

Let $A(x)$ = "x is a primary color of light." Then the statement says: *"There are exactly three primary colors of light."* This is true, since the primary colors are red, green, and blue, no more, no less.

18 Definite Descriptions

18.1 Russell's Analysis

The idea of definite descriptions was introduced by Bertrand Russell in his classic paper *On Denoting*. A definite description is a phrase of the form *"the F"*, intended to pick out a *unique object* with property F.

Russell analyzed such phrases not as names, but as logical constructions:

$$\text{“The } F \text{ is G”} \quad \equiv \quad \exists x \left(F(x) \wedge \forall y \left(F(y) \rightarrow y = x \right) \wedge G(x) \right)$$

This formula breaks down the statement into three clauses: Existence ($\exists x$), Uniqueness ($\forall y$), and Predication ($G(x)$).

Russell's aim was to handle cases where a description refers to something that does not actually exist. For example, consider the chapter 2, about the mathematical statement, that clearly we can evaluate the value whether its True or False:

1. $2 < 5$
2. "The smallest prime number is 2":

$$\exists! x \left((Prime(x) \wedge \forall y \left(Prime(y) \wedge y < x \ \rightarrow \ y = x \right)) \ \wedge \ x = 2 \right)$$

Clearly, "number" does not exist in physical reality. To see this more clearly, consider the following statement:

Agent A: "The Eiffel Tower exists."

When we are in discussion with someone who claims that such a tower exists, the natural response is to ask further questions about its *location*. By verifying its position, we can reach agreement that the tower truly exists in reality. Similarly, we can examine the claim about the number 2: that $2 < 5$ and that 2 is the smallest prime number. The question is: does 2 exist in the same way as the Eiffel Tower? Can we point to its location, or provide map coordinates to prove the position of "number 2" in the world? It is obvious that such a number does not exist empirically. Numbers are not objects in space, but abstract entities whose existence is logical rather than physical.

In this case, Russell's analysis is adequate to treat *non-existent objects*. It allows us to symbolize descriptions like *"the smallest prime number"* or even *"the current king of France"* without collapsing into nonsense. However, the real problem lies in the *conclusion*: should we pay attention at all to things that do not exist?

After all, expressions such as:

$$\forall x \, A(x)$$

are perfectly valid in terms of *syntax*. But in the end, the crucial question is what x is meant to represent. In other words, when we evaluate a claim, we are not only concerned with the syntactic form, but also with its *semantic content*. Russell's analysis handles the logical structure, but the question of existence pulls us back toward interpretation and meaning.

For example, recall Russell's formalization:

$$\text{“The } F \text{ is G”} \quad \equiv \quad \exists x \left(F(x) \wedge \forall y \left(F(y) \to y = x\right) \wedge G(x)\right)$$

Nevertheless, suppose that there are no Fs in the domain. On Russell's analysis, both *"the F is G"* and *"the F is non-G"* turn out to be false.

Therefore, The **negation** of Russell's analysis is:

$$\neg\exists x \left(F(x) \wedge \forall y(F(y) \to y = x) \wedge G(x)\right)$$

Which is equivalent to:

$$\forall x \, \neg\left(F(x) \wedge \forall y(F(y) \to y = x) \wedge G(x)\right)$$

18.2 P. F. Strawson's Critic

In response to this tricky expression, P. F. Strawson suggested that such sentences should not be regarded as *false* exactly, but rather as involving *presupposition failure*, and thus need to be treated as neither true nor false.

For example, consider the sentence:

"The present king of France is bald."

Russell's analysis would say this is simply false, since there is no such king. But Strawson argued that the problem is different: the sentence presupposes that there *is* a present king of France. Since that presupposition fails, the sentence is not properly true or false at all.

Another example:

"The smallest prime greater than 10 is even."

This presupposes that there exists a *smallest prime greater than 10.* But since there are infinitely many primes, no such smallest prime exists. So, according to Strawson, the sentence is neither true nor false, but suffers from presupposition failure.

18.3 Keith Donnellan's Idea

If P. F. Strawson focuses only on intuitions, then there is room to disagree with his idea. For example, how can we ever be completely certain about what we see or mean in context? In response to this kind of concern, Keith Donnellan introduces the role of intention, and offers a famous example involving two men drinking:

1. A very tall man drinking what looks like a gin martini.

2. A very short man drinking what looks like a pint of water.

Seeing them, Malika says: "The gin-drinker is very tall!"

Now suppose that the very tall man is actually drinking water from a martini glass; whereas the very short man is drinking a pint of neat gin. By Russell's analysis, Malika has said something false. But don't we want to say that Malika has said something true?

We can agree that, from Malika's point of view, she intended to pick out a particular man and say something true of him (that he was tall). Donnellan distinguished between two uses of definite descriptions:

1. Attributive Use: The speaker intends to refer to whatever unique object fits the description.

2. Referential Use: The speaker intends to refer to a specific object, regardless of whether it actually fits the description.

On Russell's purely attributive analysis, Malika actually picked out a different man (the short one, who was factually the gin-drinker) and consequently said something false about him.

So, should we judge Malika's sentence as true because her intention (referential use) was to refer to the tall man? Or should we judge it as false because her logical description (attributive use) did not match the facts?

19 Semantic Concepts for FOL

19.1 Validity

A formula is valid if it's true in every interpretation

Example:

$$\forall x \in \{1, 2, 3\}\,(P(x) \to P(x))$$

This is valid because *if 1 has property P, then 1 has property P* is always true, regardless of what P represents.

19.2 Satisfiability

A formula is satisfiable if there exists at least one interpretation that makes it true

Example:

$$\exists x \in \{1, 2, 3\}\,(x > 1 \land x < 3)$$

This is satisfiable because we can choose $x = 2$, making $2 > 1$ and $2 < 3$ true.

19.3 Contradiction

A formula is unsatisfiable if no interpretation can make it true

Example:

$$\exists x \in \{1, 2, 3\}\,(x > 3 \land x < 1)$$

This is unsatisfiable because no number in our domain can be both greater than 3 and less than 1.

19.4 Entailment

$\Gamma \models \phi$ if ϕ is true whenever all formulas in Γ are true

Example:

$$\{\forall x \in \{1, 2, 3\}\,(x < 3), 2 \in \{1, 2, 3\}\} \models (2 < 3)$$

The premises *all numbers in our set are less than* 3 and 2 *is in our set* entail 2 *is less than* 3.

19.5 Non-Entailment

$\Gamma \not\models \phi$ if there exists an interpretation where Γ is true but ϕ is false

Example:

$$\{\exists x \in \{1,2,3\}\,(x > 1)\} \not\models \forall x \in \{1,2,3\}\,(x > 1)$$

Some number is greater than 1 doesn't entail *all numbers are greater than* 1 because 1 itself isn't greater than 1.

19.6 Joint Satisfiability

Formulas are jointly satisfiable if some interpretation makes them all true simultaneously

Example:

$$\{P(1), \neg P(2), \exists x\, P(x)\} \text{ where domain } = \{1,2\}$$

We can make P true for 1, false for 2, and *something has property P* true, all consistent.

19.7 Logical Equivalence

$\phi \equiv \psi$ if ϕ and ψ have the same truth value in every interpretation

Example:

$$\forall x \in \{1,2,3\}\,\neg(x > 2) \equiv \forall x \in \{1,2,3\}\,(x \leq 2)$$

No number is greater than 2 is equivalent to *every number is less than or equal to* 2.

19.8 Contingency

A formula is contingent if it's true in some interpretations and false in others

Example:

$$\exists x \in \{1,2,3\}\,(x = 2)$$

This is contingent, true when our domain includes 2, false if we change the domain to $\{4,5,6\}$.

19.9 Expressibility

A property is expressible if there exists a formula that captures exactly that property

Example:

$$\text{Even}(x) := (x = 2) \vee (x = 4) \text{ over domain } \{1,2,3,4\}$$

The property *being even* is expressible because we can write a formula true exactly for even numbers.

19.10 Definability

A relation is definable if it can be expressed using existing predicates and logical operators

Example:

$$\text{Between}(x, y, z) := (x < y < z) \lor (z < y < x) \text{ over } \{1, 2, 3, 4, 5\}$$

Using the less-than relation, we can define *y is between x and z* without introducing new predicates.

19.11 Implicit Definability

A relation is uniquely determined by certain conditions without explicit construction

Example:

$$\forall x, y\, (\text{Succ}(x, y) \leftrightarrow (x < y \land \neg\exists z\, (x < z < y)))$$

The successor relation in $\{1, 2, 3\}$ is implicitly defined as the unique immediate-next relation.

19.12 Finite Expressibility

Properties expressible only over finite domains of specific size

Example:

$$\exists x \exists y \exists z\, (x \neq y \neq z \land \forall w\, (w = x \lor w = y \lor w = z))$$

Having exactly 3 elements can be expressed for domain $\{1, 2, 3\}$ but doesn't work for arbitrary domains.

19.13 Cardinality Constraints

Expressing numerical constraints on elements satisfying properties

Example:

$$\exists x \exists y\, (x \neq y \land \text{Prime}(x) \land \text{Prime}(y)) \text{ over } \{2, 3, 4, 5, 6\}$$

This expresses *at least two distinct elements are prime* using existential quantification with inequality.

20 Properties of Relations

20.1 Reflexivity

A relation R is reflexive if every element in the set A is related to itself.

$$\forall x \in A,\ xRx$$

Example:
The relation $\leq$ (less than or equal to) on the set of real numbers ($\mathbb{R}$) is reflexive because $3 \leq 3$.

20.2 Irreflexivity

A relation R is irreflexive if no element in the set A is related to itself.

$$\forall x \in A,\ \neg(xRx)$$

Example:

The relation $<$ (less than) on the set of natural numbers ($\mathbb{N}$) is irreflexive because no number is less than itself: $5 < 5$ is false, so $\neg(5 < 5)$.

20.3 Asymmetry

A relation R is asymmetric if whenever an element x is related to y, y is never related back to x.

$$\forall x, y \in A,\ xRy \to \neg(yRx)$$

Example:

The relation $<$ on $\mathbb{N}$ is asymmetric because $2 < 3$, but $3 < 2$ is false. There is no pair such that both xRy and yRx hold. Any asymmetric relation is automatically irreflexive.

20.4 Symmetry

A relation R is symmetric if whenever an element x is related to y, y is always related back to x.

$$\forall x, y \in A,\ xRy \to yRx$$

Example:

Let $A = \{1, 2, 3\}$ and $R = \{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle\}$. This relation is symmetric because:

$$\text{If } 1R2 \text{ holds, then } 2R1 \text{ also holds.}$$

Every pair has its mirror image in the relation.

20.5 Antisymmetry

A relation R is antisymmetric if the only way for x to be related to y and y to be related back to x is if x and y are the same element.

$$\forall x, y \in A,\ (xRy \wedge yRx) \to x = y$$

Example:

Let $R =\leq$ on $\mathbb{N}$. If $x \leq y$ and $y \leq x$, then $x = y$. Therefore, $\leq$ is antisymmetric.

20.6 Transitivity

A relation R is transitive if whenever x is related to y and y is related to z, then x must be related to z.

$$\forall x, y, z \in A,\ (xRy \wedge yRz) \to xRz$$

Example:
The relation $<$ on $\mathbb{N}$ is transitive:

$$2 < 4 \land 4 < 5 \to 2 < 5$$

20.7 Connectivity

A relation R is connected (or total) if, for every pair of distinct elements x and y in the set A, one element must be related to the other.

$$\forall x, y \in A,\ x \neq y \to (xRy \lor yRx)$$

Example:
The relation $\leq$ on $\mathbb{N}$ is connected because for any distinct x, y, either $x \leq y$ or $y \leq x$ must hold.

Importanly, in evaluating binary relations, order matters. If $L(x, y)$ is a binary predicate, then $L(a, b)$ and $L(b, a)$ represent distinct logical claims unless the semantics of L itself makes them equivalent.

Example:
If $L(x, y)$ means "x loves y," then $L(a, b)$ ("a loves b") and $L(b, a)$ ("b loves a") are logically distinct. We use the ordered pair notation $\langle a, b \rangle$ or the predicate notation xRy precisely to stress that the order is significant.

21 Relation on a Set

Firstly, a set is a collection of objects, considered as a single object. The objects making up a set are called its *elements* or *members*. For example, if x is an element of a set U, we write:

$$x \in U$$

Conversely, if x is not an element of U, we write:

$$x \notin U$$

We can specify a set either by listing its elements explicitly or by defining a property that determines membership. In fact, sets may be specified in several different ways.

1. **Roster (Tabular) Method**

   This method lists all the elements of the set explicitly:

   $$A = \{1, 2, 3, 4, 5\}$$

   Example:

   The first five natural numbers are

   $$\mathbb{N}_5 = \{1, 2, 3, 4, 5\}$$

2. **Set-Builder Notation**

   This method defines a set by specifying a property that its members must satisfy:

   $$B = \{x \in \mathbb{N} \mid x \text{ is even}\}$$

   Example:

   All positive even natural numbers:

   $$E = \{x \in \mathbb{N} \mid x \equiv 0 \pmod{2}\}$$

3. **Descriptive Method**

   This method describes the set in words:

   $$C = \{\text{all real numbers greater than zero}\}$$

4. **Predicate-Logic Method**

   This is a formal version of set-builder notation using logical expressions:

   $$D = \{x \in \mathbb{Z}^+ \mid x^2 < 10\}$$

   Explicitly, this set is

   $$D = \{1, 2, 3\}$$

5. **Recursive Definition**

   Sets can be defined recursively, where each stage is built from the previous one:

   $$S_0 = \varnothing, \quad S_{n+1} = \mathcal{P}(S_n)$$

   Example:

   The von Neumann ordinals:

   $$0 = \varnothing, \quad 1 = \{0\}, \quad 2 = \{0, 1\}, \quad 3 = \{0, 1, 2\}, \ldots$$

6. **Definition via Set Operations**

   Sets can also be defined in terms of other sets, using operations like union, intersection, and difference:

   Example 1:

   $$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6\}$$

   Then:

   $A \cup B = \{1, 2, 3, 4, 5, 6\}$

   $A \cap B = \{3, 4\}$

$A \setminus B = \{1, 2\}$

Example 2:

The set of rational numbers:

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z},\ q \in \mathbb{Z} \setminus \{0\} \right\}$$

Some rational numbers are

$$\left\{ -\frac{3}{2},\ -1,\ 0,\ \frac{1}{2},\ \frac{2}{3},\ 1,\ \frac{5}{4} \right\}.$$

Recall that defining a set means to say what it is by giving a membership condition or description. Notice that a definition does not actually produce the set; it only states the property that characterizes its members. On the other hand, to construct a set means to actually build it out of existing sets, using the axioms of set theory such as union, intersection, et cetera. The difference between defining and constructing is crucial in set theory. When we define a set, we are assuming its existence. But to construct a set, we must first establish its existence from the axioms. In short, to define is to describe a set, while to construct is to produce it.

Because we will mostly be dealing with sets whose elements are mathematical objects, it is important to keep in mind the following sets:

1. $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, *the set of natural numbers*
2. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, *the set of integers*
3. $\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0 \right\}$, *the set of rationals*
4. $\mathbb{R} = (-\infty, \infty)$, *the set of real numbers (the continuum)*

21.1 Union

The symbolic notation for union is:

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Example 1:
Let

$$A = \{1, 2, 3\}, \quad B = \{3, 4, 5\}.$$

Then the union of A and B is

$$A \cup B = \{1, 2, 3, 4, 5\}.$$

Notice that the element 3, which is in both sets, only appears once in the union.
Example 2:
Let

$$A = \{1, 2, 3, 4, 5\}, \quad B = \{3, 4, 5, 6, 7\}, \quad C = \{5, 6, 7, 8, 9\}.$$

We ask for

$$U = A \cup B \cup C.$$

First,

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}.$$

Second,

$$(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Hence,

$$\boxed{U = A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}}$$

Example 3:
Let

$$A = \left\{ x \in \mathbb{N}^+ \;\middle|\; \frac{x^2}{10} < 1 \right\},$$
$$B = \{x \in \mathbb{Z} \mid -7 < x \leq -6\}.$$

Since,

$$A = \{1, 2, 3\}, \quad \text{and} \quad B = \{-6\},$$

Therefore,

$$\boxed{A \cup B = \{-6, 1, 2, 3\}}$$

Example 4:
Let $\mathcal{A}$ be a set whose elements are themselves sets. Then the union of all sets in $\mathcal{A}$, denoted $\bigcup \mathcal{A}$, is defined as:

$$\bigcup \mathcal{A} = \{x \mid \exists B \in \mathcal{A} \text{ such that } x \in B\}.$$

Examples:

1. $\mathcal{A}_1 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$
$$\bigcup \mathcal{A}_1 = \{1, 2, 3, 4\}$$

2. $\mathcal{A}_2 = \{\{\{1\}, \{2\}\}, \{\{2\}, \{3\}\}\}$
$$\bigcup \mathcal{A}_2 = \{\{1\}, \{2\}, \{3\}\}$$

3. $\mathcal{A}_3 = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}$
$$\bigcup \mathcal{A}_3 = \{a, b, c, d\}$$

Example 5:
Let

$$A = \left\{ x \in \mathbb{R} \;\middle|\; -8 - x > -\frac{x}{4} - 12 \right\},$$
$$B = \left\{ x \in \mathbb{R} \;\middle|\; 30 - 2x < -\frac{x}{2} + 1 - 2x \right\}.$$

Since,

$$A = \left(-\infty, \frac{16}{3}\right), \quad B = (-\infty, -58),$$

Therefore,

$$\boxed{A \cup B = \left(-\infty, \frac{16}{3}\right)}.$$

21.2 Intersection

The symbolic notation for intersection is:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Moreover, if two sets have no common element, then the two sets are *disjoint*. Formally:

$$A \cap B = \varnothing$$

Example 1:
Let

$$P = \{a, b, c, d\}, \quad Q = \{b, d, e, f\}.$$

The intersection of P and Q contains only the elements that are in both sets:

$$\boxed{P \cap Q = \{b, d\}}$$

Notice that a, c, e, and f are excluded because they are not in both sets.
Example 2:
Consider three sets:

$$M = \{1, 3, 5, 7\}, \quad N = \{3, 4, 5, 6\}, \quad O = \{0, 2, 5, 8\}.$$

We want the elements that all three sets share:
Only the number 5 appears in M, N, and O:

$$M \cap N \cap O = \{5\}.$$

Hence, the intersection is:

$$\boxed{M \cap N \cap O = \{5\}}$$

Example 3:
Let

$$A = \{x \in \mathbb{Z} \mid 0 \leq x \leq 10\},$$
$$B = \{x \in \mathbb{Z} \mid 7 \leq x \leq 12\}.$$

Since the set A is

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},$$

and the set B is

$$B = \{7, 8, 9, 10, 11, 12\},$$

we have

$$\boxed{A \cap B = \{7, 8, 9, 10\}}$$

Example 4:

Let

$$\begin{aligned}
C &= \left\{x \in \mathbb{N}^+ \;\middle|\; |x-5| < 4\right\}, \\
D &= \{x \in \mathbb{Z} \mid 3x - 7 = 2\}, \\
E &= \left\{x \in \mathbb{N}^+ \;\middle|\; |2x-10| \leq 6\right\}, \\
F &= \{x \in \mathbb{Z} \mid |x+1| = 7\}, \\
G &= \left\{x \in \mathbb{N}^+ \;\middle|\; |x-12| > 5\right\}.
\end{aligned}$$

Since,

$$\begin{gathered}
C = \{2,3,4,5,6,7,8\} \\
D = \{3\} \\
E = \{2,3,4,5,6,7,8\} \\
F = \{-8,6\}
\end{gathered}$$

For G, since $x \in \mathbb{N}^+$,

$$G = \{1,2,3,4,5,6\} \cup \{18,19,20,\ldots\}.$$

Hence

1. $C \cap D = \{3\}$
2. $C \cap E = \{2,3,4,5,6,7,8\}$ (same set)
3. $C \cap F = \{6\}$
4. $E \cap F = \{6\}$
5. $C \cap G = \{2,3,4,5,6\}$
6. $E \cap G = \{2,3,4,5,6\}$
7. $D \cap G = \{3\}$

We conclude that,

$$\boxed{C \cap D \cap E \cap F \cap G = \varnothing}$$

Example 5:
Let

$$\begin{aligned}
A &= \left\{x \in \mathbb{N}^+ \;\middle|\; \frac{3x-4}{5} \leq 7\right\}, \\
B &= \left\{x \in \mathbb{Z} \;\middle|\; \frac{2x+9}{4} = \frac{5x-1}{6}\right\}.
\end{aligned}$$

Since the set A is

$$A = \{1,2,3,4,5,6,7,8,9,10,11,12\},$$

and the set B is

$$B = \varnothing \quad (\text{since } \frac{29}{4} \notin \mathbb{Z}),$$

we have

$$\boxed{A \cap B = \varnothing}$$

Example 6:
Suppose

$$A = \{\{a, b\}, \{a, d, e\}, \{a, d\}\}$$

In this case, we will answer by using intersection over the set. Therefore, the intersection over all subsets of A is:

$$\bigcap A = \bigcap_{X \in A} X$$

The element common to all subsets is a, so:

$$\boxed{\bigcap A = \{a\}}$$

Example 7:
We can also define union and intersection over an indexed collection of sets $A_1, A_2, \ldots$:

$$\bigcup_i A_i = \{x \mid x \in A_i \text{ for some } i\}$$
$$\bigcap_i A_i = \{x \mid x \in A_i \text{ for all } i\}$$

If I is an index set and we are considering A_i for each $i \in I$, then:

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$
$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

Let the index set be:

$$I = \{1, 2, 3\}$$

Define a family of sets $\{A_i\}_{i \in I}$:

$$A_1 = \left\{ x \in \mathbb{Z} \;\middle|\; \frac{x-1}{x} = \frac{6}{3} \right\}$$
$$A_2 = \left\{ y \in \mathbb{R} \;\middle|\; \frac{y+8}{12} = \frac{5y+6}{4} \right\}$$
$$A_3 = \left\{ z \in \mathbb{N} \;\middle|\; -11(z+9) = -6 + 5\left[-z - \frac{99}{5}\right] \right\}$$

Then we compute:

$$\bigcup_{i \in I} A_i = A_1 \cup A_2 \cup A_3, \quad \bigcap_{i \in I} A_i = A_1 \cap A_2 \cap A_3$$

Since the sets are

$$A_1 = \{-1\}, \quad A_2 = \left\{-\frac{5}{7}\right\}, \quad A_3 = \{1\},$$

Hence,

$$\boxed{\bigcup_{i \in I} A_i = \left\{-1, -\frac{5}{7}, 1\right\}, \quad \bigcap_{i \in I} A_i = \varnothing}$$

Example 8:
Case 1:
Let the index set be:

$$I = \mathbb{N}$$

Define a family of sets:

$$A_i = \{i, i+1, \ldots, i^2\}$$

Therefore, the first few sets are:

$$\begin{aligned} A_1 &= \{1\}, \\ A_2 &= \{2, 3, 4\}, \\ A_3 &= \{3, 4, 5, 6, 7, 8, 9\}, \\ &\vdots \end{aligned}$$

Finally, the intersection remains:

$$\bigcap_{i \in \mathbb{N}} A_i = \varnothing$$

while the union of all sets in the family is:

$$\bigcup_{i \in \mathbb{N}} A_i = \mathbb{N}$$

Case 2:
Let the index set be:

$$I = \mathbb{Z}$$

Define a family of sets:

$$A_i = (i-1, i)$$

Therefore, the first few sets are:

$$
\begin{aligned}
A_{-2} &= (-3,-2),\\
A_{-1} &= (-2,-1),\\
A_0 &= (-1,0),\\
A_1 &= (0,1),\\
A_2 &= (1,2),\\
A_3 &= (2,3),\\
&\vdots
\end{aligned}
$$

Finally, the intersection of all sets is:

$$\bigcap_{i\in\mathbb{Z}} A_i = \varnothing$$

and the union of all sets is:

$$\bigcup_{i\in\mathbb{Z}} A_i = \mathbb{R}.$$

21.3 Difference

The difference of sets A and B, written $A \setminus B$ or $A - B$ is the set of all elements of A that are not in B:

$$A \setminus B = \{x \mid x \in A \land x \notin B\}$$

Example 1:
Let

$$A = \{apple, banana, cherry, date\}, \quad B = \{banana, date, fig\}.$$

The difference $A - B$ consists of elements in A that are not in B:

$$\boxed{A - B = \{apple, cherry\}}$$

Here, *banana* and *date* are removed because they are present in B.
Example 2:
Consider three sets:

$$X = \{red, blue, green, yellow\}, \quad Y = \{blue, yellow, purple\}, \quad Z = \{yellow, orange\}.$$

We want the elements in X that are not in Y or Z. Remove elements of Y from X first:

$$X - Y = \{red, green\}.$$

Then remove elements of Z (no change since *red* and *green* are not in Z):

$$(X - Y) - Z = \{red, green\}.$$

Hence, the difference is:

$$\boxed{X - Y - Z = \{red, green\}}$$

Example 3:
Let

$$A = \{x \in \mathbb{Z} \mid -1 \le x \le 6\},$$
$$B = \{x \in \mathbb{Z} \mid x \ge 2\}.$$

Since the set A is

$$A = \{-1, 0, 1, 2, 3, 4, 5, 6\},$$

and the set B is

$$B = \{2, 3, 4, 5, 6, 7, 8, \dots\},$$

we have

$$\boxed{A \setminus B = \{-1, 0, 1\}}$$

Example 4:
Let

$$\mathbb{Z} = \bigcup_{k=-\infty}^{\infty} \{k\}, \quad \mathbb{N}^+ = \bigcup_{k=0}^{\infty} \{k+1\}.$$

Then the difference is

$$\mathbb{Z} \setminus \mathbb{N}^+ = \{z \in \mathbb{Z} \mid z \le 0\}.$$

So the result is the set of all non-positive integers:

$$\boxed{\bigcup_{k=-\infty}^{0} \{k\} = (-\infty, 0] \cap \mathbb{Z}}$$

Example 5:
Let the index set be:

$$I = \{1, 2, 3\}.$$

Define the family of sets $\{A_i\}_{i \in I}$:

$$A_1 = \left\{ x \in \mathbb{N} \;\middle|\; \frac{3x - 7}{4} = \frac{6x}{2} + \frac{49}{7} \right\},$$
$$A_2 = \left\{ y \in \mathbb{Z} \;\middle|\; \frac{2y + 9}{3} \le 6 \right\},$$
$$A_3 = \left\{ z \in \mathbb{R} \;\middle|\; \frac{5z - 11}{2} > 4 \right\}.$$

Since:

$$A_1 = \varnothing, \quad A_2 = \{x \in \mathbb{Z} \mid x \le 4\}, \quad A_3 = \left\{ x \in \mathbb{R} \mid x > \frac{19}{5} \right\},$$

therefore:

$$\bigcup_{i\in I} A_i = \left\{x \in \mathbb{R} \mid x > \frac{19}{5}\right\} \cup \{x \in \mathbb{Z} \mid x \leq 4\}, \quad \bigcap_{i\in I} A_i = \varnothing.$$

We conclude:

$$\boxed{\left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcap_{i\in I} A_i\right) = \{x \in \mathbb{R} \mid x > \frac{19}{5}\} \cup \{x \in \mathbb{Z} \mid x \leq 4\}}$$

Example 6:
Let

$$\begin{aligned} A &= \left\{x \in \mathbb{R} \;\middle|\; \frac{x}{2} + \frac{1}{3} < \frac{2x}{5}\right\} \\ B &= \left\{y \in \mathbb{N} \;\middle|\; 2 < \frac{y-1}{3} \leq 6\right\} \\ C &= \{z \in \mathbb{Z} \mid -5 \leq z \leq 10\} \end{aligned}$$

Since

$$A = \left(-\infty, -\frac{10}{3}\right), \quad B = (7, 19] \cap \mathbb{N}, \quad C = [-5, 10] \cap \mathbb{Z}$$

Thus,

$$A \cap B = \varnothing$$

We conclude that,

$$\boxed{(A \cap B) \setminus C = \varnothing}$$

21.4 Symmetric Difference

The symmetric difference of sets A and B, written $A \triangle B$, is the set of all elements that are in either A or B but not in both:

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

Example 1:
Let

$$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6\}.$$

Therefore

$$\boxed{A \,\Delta\, B = \{1, 2, 5, 6\}}$$

Notice that 3 and 4 are excluded because they appear in both sets.

Example 2:
Consider three sets:

$$X = \{1, 2, 3, 4\}, \quad Y = \{3, 4, 5, 6\}, \quad Z = \{4, 5, 6, 7\}.$$

The symmetric difference of all three sets can be computed step by step:

First, compute $X \,\Delta\, Y$:

$$X \,\Delta\, Y = \{1, 2, 5, 6\}.$$

Then, compute $(X \,\Delta\, Y) \,\Delta\, Z$:

$$\{1, 2, 5, 6\} \,\Delta\, \{4, 5, 6, 7\} = \{1, 2, 4, 7\}.$$

Hence, the symmetric difference of X, Y, and Z is:

$$\boxed{X \,\Delta\, Y \,\Delta\, Z = \{1, 2, 4, 7\}}$$

Example 3:
Let

$$\begin{aligned} A &= \left\{ (p, q) \in \mathbb{N}^2 \;\middle|\; \frac{3p + 2q}{p} \geq 9, \quad p \neq 0 \right\}, \\ B &= \left\{ x \in \mathbb{Z}^+ \;\middle|\; \frac{x - 1}{x} = x + 1 \right\}. \end{aligned}$$

Since

$$A = \bigcup_{p=1}^{\infty} \{p\} \times [3p, \infty) \cap \mathbb{N},$$

and

$$B = \varnothing.$$

Consequently,

$$A \cap B = \varnothing.$$

Since the sets are disjoint, we have $A \triangle B = A \cup B = A$.
We conclude that,

$$\boxed{A \triangle B = \bigcup_{p=1}^{\infty} \{p\} \times [3p, \infty) \cap \mathbb{N}}$$

Example 4:
Let

$$\begin{aligned} A &= \{p \in \mathbb{R} \mid -4(p - 4) \geq -2(p + 1)\}, \\ B &= \{q \in \mathbb{R} \mid -46 < 4q - 6 \leq -26\}, \\ C &= \{r \in \mathbb{R} \mid -(1 + 2r) > -6(r - 4) - 1\}. \end{aligned}$$

We want to solve for

$$(A \cup B) \triangle (C \setminus A).$$

Since

$$A = (-\infty, 9], \quad B = (-10, -5], \quad C = (6, \infty),$$

we have

$$A \cup B = (-\infty, 9] \cup (-10, -5] = (-\infty, 9],$$

and

$$C \setminus A = (6, \infty) \setminus (-\infty, 9] = (9, \infty).$$

Since $(-\infty, 9]$ and $(9, \infty)$ are disjoint sets, their symmetric difference is:

$$(A \cup B) \triangle (C \setminus A) = (-\infty, 9] \cup (9, \infty) = (-\infty, \infty).$$

Therefore

$$\boxed{(A \cup B) \triangle (C \setminus A) = (-\infty, \infty)}$$

Example 5:
Let

$$A = \bigcup_{n \in \mathbb{N}} \{f^k(n) : k \geq 0\}, \quad B = \bigcup_{k \in \mathbb{N}} \{k^2 - k + 1\},$$

where

$$f(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even,} \\ 3n + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Since $f^0(n) = n$, we have $n \in \{f^k(n) : k \geq 0\}$ for every n. Hence

$$A = \mathbb{N}.$$

Therefore

$$A \cup B = \mathbb{N}, \quad A \cap B = B.$$

We conclude that,

$$\boxed{A \triangle B = \{\, m \in \mathbb{N} \mid m \neq k^2 - k + 1 \text{ for any } k \in \mathbb{N} \,\}}$$

21.5 Complement

The symbolic notation for Complement is:

$$A^c = \{\, x \in U \mid x \notin A \,\}$$

Example 1:
Let

$$U = \{1, 2, 3, 4, 5, 6\} \quad \text{(the universal set)}, \quad A = \{2, 4, 6\}.$$

Hence,

$$\boxed{A^c = \{1, 3, 5\}}$$

Notice that 2, 4, and 6 are excluded because they are in A.

Example 2:
Consider the universal set and two sets:

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \quad B = \{1, 3, 5, 7, 9\}, \quad C = \{2, 3, 4, 5\}.$$

The complement of B in U is:

$$\boxed{B^c = \{2, 4, 6, 8\}}$$

The complement of C in U is:

$$\boxed{C^c = \{1, 6, 7, 8, 9\}}$$

Example 3:
Let

$$A = \bigcup_{k\in\mathbb{N}} \{\, 2k+1 \,\}, \quad B = \bigcup_{k\in\mathbb{N}} \{\, 2k \,\}.$$

Therefore,

$$\boxed{B^c = \mathbb{N} \setminus B = A}$$

Example 4:
Let

$$A = \{\, p \in \mathbb{P} \mid p+2 \in \mathbb{P} \,\}.$$

Then A is the set of primes p such that $p+2$ is also prime.
Let

$$U = \bigcup_{k\in\mathbb{N}} \{\, 2k+1 \,\}.$$

So

$$U = \{1, 3, 5, 7, 9, 11, 13, \dots\}.$$

Then the complement of A relative to U is

$$A^c = U \setminus A.$$

That is,

$$\boxed{A^c = \{\, 2k+1 \mid k \in \mathbb{N} \,\} \setminus \{\, p \in \mathbb{P} \mid p+2 \in \mathbb{P} \,\}}$$

Example 5:
Let

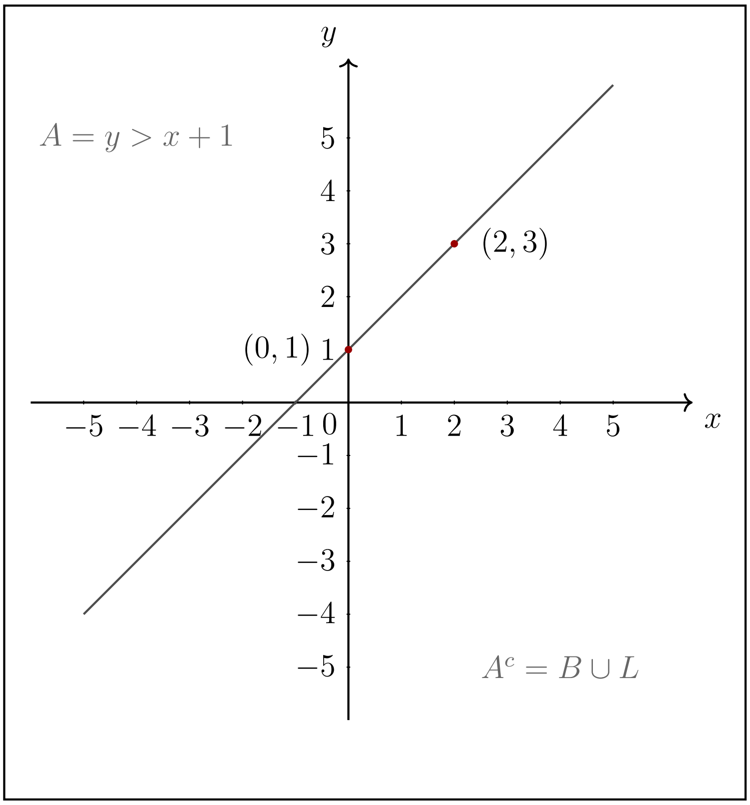
$$L = \{\, (x, y) \in \mathbb{R}^2 \mid y - 3 = 1(x-2) \,\}.$$

That is, L is the line with slope 1 passing through the point $(2, 3)$.
Let

$$A = \{\, (x, y) \in \mathbb{R}^2 \mid y > x + 1 \,\},$$

so A is the set of points above the line $y = x + 1$. Then the complement of A relative to $\mathbb{R}^2$ is

$$A^c = \mathbb{R}^2 \setminus A.$$



That is,

$$\boxed{A^c = \bigcup_{x \in \mathbb{R}} \{\, (x, y) \in \mathbb{R}^2 \mid y \leq x + 1 \,\} \;=\; B \cup L}$$

where

$$B = \{\, (x, y) \in \mathbb{R}^2 \mid y < x + 1 \,\}, \quad L = \{\, (x, y) \in \mathbb{R}^2 \mid y = x + 1 \,\}.$$

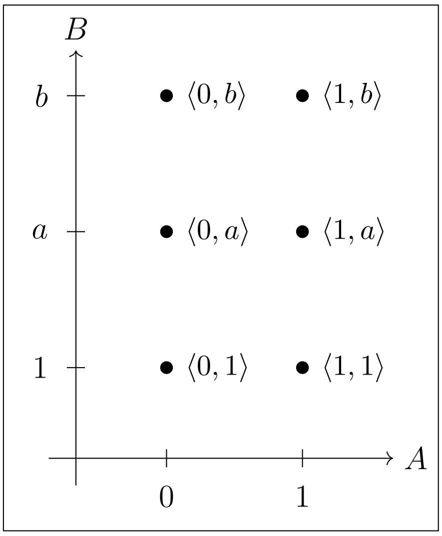
21.6 Cartesian Product

Symbolic notation for Cartesian Product is:

$$A \times B = \{\langle a, b \rangle \mid a \in A \wedge b \in B\}$$

Example 1:

$$A = \{0, 1\}$$
$$B = \{1, a, b\}$$



Then the Cartesian product is

$\langle 0,1\rangle$	$\langle 0,a\rangle$	$\langle 0,b\rangle$
$\langle 1,1\rangle$	$\langle 1,a\rangle$	$\langle 1,b\rangle$

Example 2:
Let

$$A = \{1, 2\}, \quad B = \{3, 4\}.$$

Hence

$\langle 1,3\rangle$	$\langle 1,4\rangle$
$\langle 2,3\rangle$	$\langle 2,4\rangle$

Example 3:
Consider three sets:

$$X = \{1, 2\}, \quad Y = \{a, b\}, \quad Z = \{0, 9\}.$$

First, the Cartesian product of X and Y:

$$X \times Y = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,a\rangle, \langle 2,b\rangle\}$$

Second, the Cartesian product of $(X \times Y)$ with Z gives *triplets*:

$\langle\langle 1,a\rangle,0\rangle$	$\langle\langle 1,a\rangle,9\rangle$
$\langle\langle 1,b\rangle,0\rangle$	$\langle\langle 1,b\rangle,9\rangle$
$\langle\langle 2,a\rangle,0\rangle$	$\langle\langle 2,a\rangle,9\rangle$
$\langle\langle 2,b\rangle,0\rangle$	$\langle\langle 2,b\rangle,9\rangle$

Example 4:
Let

$$A = \{1, 2\}, \quad B = \{3, 4\}, \quad C = \{5, 6\}, \quad D = \{7, 8\}.$$

The Cartesian product of these four sets, denoted

$$A \times B \times C \times D,$$

is the set of all ordered *quadruples* where the first element comes from A, the second from B, the third from C, and the fourth from D.

Explicitly:

$\langle\langle\langle 1,3\rangle,5\rangle,7\rangle$	$\langle\langle\langle 1,3\rangle,5\rangle,8\rangle$	$\langle\langle\langle 1,3\rangle,6\rangle,7\rangle$	$\langle\langle\langle 1,3\rangle,6\rangle,8\rangle$
$\langle\langle\langle 1,4\rangle,5\rangle,7\rangle$	$\langle\langle\langle 1,4\rangle,5\rangle,8\rangle$	$\langle\langle\langle 1,4\rangle,6\rangle,7\rangle$	$\langle\langle\langle 1,4\rangle,6\rangle,8\rangle$
$\langle\langle\langle 2,3\rangle,5\rangle,7\rangle$	$\langle\langle\langle 2,3\rangle,5\rangle,8\rangle$	$\langle\langle\langle 2,3\rangle,6\rangle,7\rangle$	$\langle\langle\langle 2,3\rangle,6\rangle,8\rangle$
$\langle\langle\langle 2,4\rangle,5\rangle,7\rangle$	$\langle\langle\langle 2,4\rangle,5\rangle,8\rangle$	$\langle\langle\langle 2,4\rangle,6\rangle,7\rangle$	$\langle\langle\langle 2,4\rangle,6\rangle,8\rangle$

Example 5:

$$A = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \frac{1}{2} = \frac{y-1}{3x-4}\right\}$$
$$B = \left\{(x,y) \in \mathbb{Z}^2 \;\middle|\; x \geq 6y,\ x \neq 0\right\}$$

Therefore,

$$A = \{\langle 0,-1\rangle, \langle 2/3,0\rangle, \langle 1,0.5\rangle, \dots\}$$
$$B = \{\langle 1,0\rangle, \langle 6,1\rangle, \langle 12,2\rangle, \dots\}$$

Then the Cartesian product $A \times B$ is all possible pairs combining one point from A with one point from B:

$\langle\langle 0,-1\rangle,\langle 1,0\rangle\rangle$	$\langle\langle 0,-1\rangle,\langle 6,1\rangle\rangle$	$\langle\langle 0,-1\rangle,\langle 12,2\rangle\rangle$
$\langle\langle 2/3,0\rangle,\langle 1,0\rangle\rangle$	$\langle\langle 2/3,0\rangle,\langle 6,1\rangle\rangle$	$\langle\langle 2/3,0\rangle,\langle 12,2\rangle\rangle$
$\langle\langle 1,0.5\rangle,\langle 1,0\rangle\rangle$	$\langle\langle 1,0.5\rangle,\langle 6,1\rangle\rangle$	$\langle\langle 1,0.5\rangle,\langle 12,2\rangle\rangle$

Thus, the Cartesian product is:

$$\boxed{A \times B = \{\big((x, \tfrac{3}{2}x - 1), (m,n)\big) \;\mid\; x \in \mathbb{R},\ m,n \in \mathbb{Z},\ m \geq 6n,\ m \neq 0\}}$$

21.7 Subset

The symbolic notation for subset is:

$$A \subseteq B \leftrightarrow \forall x\,(x \in A \rightarrow x \in B)$$

If A is not a subset of B, we write $A \nsubseteq B$. This means that there exists at least one element in A that is not in B:

$$A \nsubseteq B \leftrightarrow \exists x\,(x \in A \wedge x \notin B)$$

Furthermroe, It's crucial to distinguish between an element of a set and a subset of a set:

1. $2 \in \mathbb{Z}$
2. $\mathbb{E} \subseteq \mathbb{Z}$ (the set of even numbers is a subset of the integers)
3. A set can be both an element and a subset of another set.

For example:

$$\{1\} \in \{1, \{1\}\} \quad \text{and} \quad \{1\} \subseteq \{1, \{1\}\}$$

This shows that a set can be both an element and a subset of another set:

1. $\{1\} \in \{1, \{1\}\}$ means the set $\{1\}$ is listed as an element of the larger set.

2. $\{1\} \subseteq \{1, \{1\}\}$ means every element of $\{1\}$ (which is just 1) is also in the larger set.

In contrast, the number 1 itself is only an element, not a subset:

1. $1 \in \{1, \{1\}\}$, because 1 is directly in the set.
2. $1 \nsubseteq \{1, \{1\}\}$ because 1 is not a set and therefore cannot be a subset.

Example 1: Let
$$A = \{1, 2, 3, 4, 5\}, \quad B = \{2, 4\}.$$
We check whether B is a subset of A:

$$B \subseteq A$$

Here, every element of B is present in A, so B is indeed a subset.
Example 2:
Consider three sets:

$$X = \{10, 20, 30, 40\}, \quad Y = \{10, 20, 30, 40\}, \quad Z = \{20, 30\}.$$

Y is a subset of X:

$$Y \subseteq X$$

Z is also a subset of X:

$$Z \subseteq X$$

Example 3:

$$A \subseteq B \ \rightarrow \ A \cup B = B$$

Proof.
By definition of subset:

$$A \subseteq B \ \leftrightarrow \ \forall x \, (x \in A \rightarrow x \in B)$$

By definition of union:

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Thus:

$$x \in A \cup B \ \leftrightarrow \ (x \in A \vee x \in B)$$

Since $A \subseteq B$, we know $\forall x \, (x \in A \rightarrow x \in B)$.
Hence:

$$x \in A \cup B \ \leftrightarrow \ (x \in A \vee x \in B) \ \leftrightarrow \ (x \in B \vee x \in B) \ \leftrightarrow \ x \in B$$

Therefore:

$$\boxed{A \cup B = B. \quad \text{QED}}$$

Example 4:

$$A \subseteq B \;\rightarrow\; A \cap B = A$$

Proof.
By definition of subset:

$$A \subseteq B \;\leftrightarrow\; \forall x\,(x \in A \rightarrow x \in B)$$

By definition of intersection:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Thus:

$$x \in A \cap B \;\leftrightarrow\; (x \in A \wedge x \in B)$$

But since $A \subseteq B$, if $x \in A$, then automatically $x \in B$.
So:

$$(x \in A \wedge x \in B) \;\leftrightarrow\; x \in A$$

Therefore:

$$\boxed{A \cap B = A. \quad \text{QED}}$$

Example 5:
Let

$$A \subseteq \bigcup B$$

Proof.
By definition of subset:

$$A \subseteq \bigcup B \;\leftrightarrow\; \forall x\,(x \in A \rightarrow x \in \bigcup B)$$

By the definition of the union over a set of sets:
If A is a set of sets, then

$$\bigcup A = \{\, x \mid x \text{ belongs to an element of } A \,\} = \{\, x \mid \exists C \in A\ (x \in C) \,\}.$$

Moreover, let $x \in A$.
Since $A \in B$, we may take $C = A$ as one of the sets in B. Thus $x \in C$ with $C \in B$, which means $x \in \bigcup B$.
Therefore:

$$\forall x\,(x \in A \rightarrow x \in \bigcup B),$$

so by the definition of subset:

$$\boxed{A \subseteq \bigcup B. \quad \text{QED}}$$

21.8 Power Set

The symbolic notation for the power set of a set A is:

$$\mathcal{P}(A) = \{x \mid x \subseteq A\}$$

Example 1:
Let

$$S = \{a, b\}.$$

Then the power set of S is

$$\mathcal{P}(S) = \{\ \varnothing,\ \{a\},\ \{b\},\ \{a, b\}\ \}.$$

Since $|S| = 2$, the size of the power set is

$$\boxed{|\mathcal{P}(S)| = 2^{|S|} = 2^2}$$

Example 2:
Let

$$Y = \varnothing.$$

That is, Y is the empty set. Then the power set is

$$\mathcal{P}(Y) = \{\ \varnothing\ \}.$$

Here,

$$|\mathcal{P}(Y)| = 2^{|Y|} = 2^0 = 1.$$

So the power set of the empty set contains exactly one element, the empty set itself.
Example 3:
Let

$$A = \{1, 2, 3\}$$

and we ask for

$$B = \{\mathcal{P}(\mathcal{P}(A))\}.$$

For $\mathcal{P}(A)$,

$$\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

We have

$$|\mathcal{P}(A)| = 2^{|A|} = 2^3.$$

Then for B,

$$\boxed{|\mathcal{P}(\mathcal{P}(A))| = 2^{|\mathcal{P}(A)|} = 256.}$$

Example 4:
Let

$$A = \{1, 2\}, \quad \text{and} \quad B = \{3, 4\}.$$

We ask for

$$C = \mathcal{P}(A) \times \mathcal{P}(B).$$

Since

$$|\mathcal{P}(A)| = 2^{|A|} \quad \text{and} \quad |\mathcal{P}(B)| = 2^{|B|},$$

it follows that

$$\boxed{|C| = 2^{|A|+|B|} = 16.}$$

Explicitly,

$(\varnothing, \varnothing)$	$(\varnothing, \{3\})$	$(\varnothing, \{4\})$	$(\varnothing, \{3,4\})$
$(\{1\}, \varnothing)$	$(\{1\}, \{3\})$	$(\{1\}, \{4\})$	$(\{1\}, \{3,4\})$
$(\{2\}, \varnothing)$	$(\{2\}, \{3\})$	$(\{2\}, \{4\})$	$(\{2\}, \{3,4\})$
$(\{1,2\}, \varnothing)$	$(\{1,2\}, \{3\})$	$(\{1,2\}, \{4\})$	$(\{1,2\}, \{3,4\})$

Example 5:
Let

$$A = \{\, x \in \mathcal{P}(\{1,2,3,4\}) \mid 2 \in x \,\}.$$

From that definition we know that

$$x \subseteq \{1,2,3,4\}, \quad \text{and } 2 \in x.$$

First, the power set of $\{1,2,3,4\}$ is:

$\varnothing$	$\{1\}$	$\{2\}$	$\{3\}$
$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$
$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$
$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$

Now, we only select those subsets that contain 2.
Thus,

$\{2\}$	$\{1,2\}$	$\{2,3\}$	$\{2,4\}$
$\{1,2,3\}$	$\{1,2,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$

Hence the set has

$$\boxed{|A| = 8.}$$

22 Relations Order

22.1 Preorder

A relation $R \subseteq A^2$ is a preorder if

$$\forall x \in A,\ xRx$$

$$\forall x, y, z \in A,\ (xRy \wedge yRz) \to xRz$$

Let $A = \{a, b, c\}$. The Cartesian product is

A^2	a	b	c
a	(a,a)	(a,b)	(a,c)
b	(b,a)	(b,b)	(b,c)
c	(c,a)	(c,b)	(c,c)

Example 1:

$$R = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}$$

Reflexive:

$$\forall x \in A, (x,x) \in R$$

Transitive:

$$\forall x,y,z \in A, (xRy \wedge yRz) \rightarrow xRz$$

But not antisymmetric:

$$aRb \wedge bRa \quad \text{but} \quad a \neq b$$

Example 2:
Universal relation on $L = \{a, b\}$

$$R = L^2$$

Reflexive, transitive, but not antisymmetric if $|L| > 1$.

22.2 Partial Order

A relation $R \subseteq A^2$ is a partial order if

$$\forall x \in A,\ xRx$$

$$\forall x,y,z \in A,\ (xRy \wedge yRz) \rightarrow xRz$$

$$\forall x,y \in A,\ (xRy \wedge yRx) \rightarrow x = y$$

Example 1:
Let $A = \{a, b, c\}$

A^2	a	b	c
a	(a,a)	(a,b)	(a,c)
b	(b,a)	(b,b)	(b,c)
c	(c,a)	(c,b)	(c,c)

$$R = \{(a,a),(b,b),(c,c),(a,b),(a,c)\}$$

Reflexive, transitive, antisymmetric, but not connected:

$$\neg(bRc) \wedge \neg(cRb)$$

Example 2:

$$A = \mathcal{P}(\{a,b\}) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$$

A^2	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\varnothing$	$(\varnothing,\varnothing)$	$(\varnothing,\{a\})$	$(\varnothing,\{b\})$	$(\varnothing,\{a,b\})$
$\{a\}$	$(\{a\},\varnothing)$	$(\{a\},\{a\})$	$(\{a\},\{b\})$	$(\{a\},\{a,b\})$
$\{b\}$	$(\{b\},\varnothing)$	$(\{b\},\{a\})$	$(\{b\},\{b\})$	$(\{b\},\{a,b\})$
$\{a,b\}$	$(\{a,b\},\varnothing)$	$(\{a,b\},\{a\})$	$(\{a,b\},\{b\})$	$(\{a,b\},\{a,b\})$

$$R = \subseteq$$

Reflexive, transitive, antisymmetric, but not connected:

$$\{a\} \nsubseteq \{b\}, \quad \{b\} \nsubseteq \{a\}$$

Example 3:
Divisibility on $\mathbb{N}$

$$n \mid m \leftrightarrow \exists k \in \mathbb{N}, m = kn$$

Partial order, not linear:

$$2 \nmid 3, \quad 3 \nmid 2$$

On $\mathbb{Z}$, not antisymmetric:

$$1 \mid -1 \wedge -1 \mid 1 \quad \text{but} \quad 1 \neq -1$$

22.3 Linear Order

A relation $R \subseteq A^2$ is a linear order if it is a partial order and

$$\forall x, y \in A,\ x \neq y \to (xRy \vee yRx)$$

Example 1:
Let $A = \{a, b, c\}$

A^2	a	b	c
a	(a,a)	(a,b)	(a,c)
b	(b,a)	(b,b)	(b,c)
c	(c,a)	(c,b)	(c,c)

$$R = \{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c)\}$$

Reflexive, transitive, antisymmetric, connected: corresponds to $a < b < c$.
Example 2:

$$x \leq y \leftrightarrow x = y \vee x < y$$

Reflexive, transitive, antisymmetric, connected: $(\mathbb{N}, \leq)$ is a linear order.

22.4 Strict Order

A relation $R \subseteq A^2$ is a strict order if

$$\forall x \in A,\ \neg(xRx)$$

$$\forall x, y \in A,\ xRy \to \neg(yRx)$$

$$\forall x, y, z \in A,\ (xRy \wedge yRz) \to xRz$$

Example 1:
Let $A = \{a, b, c\}$

A^2	a	b	c
a	(a, a)	(a, b)	(a, c)
b	(b, a)	(b, b)	(b, c)
c	(c, a)	(c, b)	(c, c)

$$R = \{(a, b), (a, c), (b, c)\}$$

Irreflexive, asymmetric, transitive.
Example 2:

$$x < y \leftrightarrow \exists k \in \mathbb{N}^+,\ y = x + k$$

Irreflexive, asymmetric, transitive: $(\mathbb{N}, <)$ is a strict order.

22.5 Strict Linear Order

A strict linear order is a strict order that is connected:

$$\forall x, y \in A,\ x \neq y \to (xRy \vee yRx)$$

Example 1:
Let $A = \{a, b, c\}$

A^2	a	b	c
a	(a, a)	(a, b)	(a, c)
b	(b, a)	(b, b)	(b, c)
c	(c, a)	(c, b)	(c, c)

$$R = \{(a, b), (a, c), (b, c)\}$$

Irreflexive, asymmetric, transitive, connected: corresponds to $a < b < c$.
Example 2:

$$x < y \leftrightarrow \exists k \in \mathbb{N}^+,\ y = x + k$$

Irreflexive, asymmetric, transitive, connected: $(\mathbb{N}, <)$ is a strict linear order.

1. $3 \not< 3$

2. $2 < 5$, then $5 \not< 2$

3. $1 < 4 \wedge 4 < 7 \rightarrow 1 < 7$

4. $6 \neq 10 \rightarrow (6 < 10 \vee 10 < 6)$

Example 3:

$$X \subsetneq Y \leftrightarrow X \subseteq Y \wedge X \neq Y$$

1. $\{1\} \subsetneq \{1, 2\}$

2. $\{1, 2\} \subsetneq \{1, 2, 3\}$

3. $\{1, 2\} \not\subsetneq \{1, 2\}$

If $<$ is a strict linear order on A, then

$$\forall a, b \in A, \ \big[(\forall x \in A, \ x < a \leftrightarrow x < b) \rightarrow a = b\big]$$

Example :
For $A = \{a, b, c\}$ with $<= \{(a, b), (a, c), (b, c)\}$:

$$\{x \in A \mid x < a\} = \varnothing, \quad \{x \in A \mid x < b\} = \{a\}, \quad \{x \in A \mid x < c\} = \{a, b\}$$

Property:

1. If R is a strict order on A, then

$$R^+ = R \cup \mathrm{Id}_A$$

is a partial order. Moreover, if R is a strict linear order, then R^+ is a linear order.

2. If R is a partial order on A, define

$$R^- = R \setminus \mathrm{Id}_A$$

R^- is a strict order (irreflexive, asymmetric, transitive). Moreover, if R is a linear order, then R^- is a strict linear order.

Example:
Let $A = \{a, b, c\}$ and

$$R = \{(a, b), (a, c), (b, c)\} \quad \text{(strict linear order)}$$

The Cartesian product A^2:

A^2	a	b	c
a	(a, a)	(a, b)	(a, c)
b	(b, a)	(b, b)	(b, c)
c	(c, a)	(c, b)	(c, c)

Identity relation:

$$\mathrm{Id}_A = \{(a,a),(b,b),(c,c)\}$$

Then

$$R^+ = R \cup \mathrm{Id}_A = \{(a,a),(b,b),(c,c),(a,b),(a,c),(b,c)\}$$

$$R^- = R^+ \setminus \mathrm{Id}_A = \{(a,b),(a,c),(b,c)\}$$

R^+ is a linear order, and R^- is a strict linear order.

23 Operations on Relations

Let $R, S \subseteq A^2$ be binary relations on a set A. The following are standard operations on relations:

23.1 Inverse of a Relation

Definition:

$$R^{-1} = \{\langle y, x\rangle \mid \langle x, y\rangle \in R\}$$

In general, if $R \subseteq \mathbb{N}^2$, the inverse relation R^{-1} contains all pairs of R with their coordinates swapped:

$$R^{-1} = \{(y,x) \in \mathbb{N}^2 \mid (x,y) \in R\}$$

Example:
Let

$$S = \{\langle x, y\rangle \in \mathbb{Z}^2 \mid x + 1 = y\}$$

That is,

S	$\cdots$	-1	0	1	2	3	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-1	$\cdots$		$(-1,0)$				$\cdots$
0	$\cdots$			$(0,1)$			$\cdots$
1	$\cdots$				$(1,2)$		$\cdots$
2	$\cdots$					$(2,3)$	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then:

S^{-1}	$\cdots$	-2	-1	0	1	2	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	$\cdots$		$(0,-1)$				$\cdots$
1	$\cdots$			$(1,0)$			$\cdots$
2	$\cdots$				$(2,1)$		$\cdots$
3	$\cdots$					$(3,2)$	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

23.2 Relative Product of Relations

Definition:

$$(R \mid S) = \{\langle x, z\rangle \mid \exists y\,(xRy \wedge ySz)\}$$

Example:
Suppose we have:

$$R = \{(1,2),(2,3),(3,4)\}, \quad S = \{(2,5),(3,6),(4,7)\}$$

Then the composition of relations is:

$$R \mid S = \{(x,z) \mid \exists y\,((x,y) \in R \wedge (y,z) \in S)\}$$

Thus:

1. $((1,2) \in R)$ and $((2,5) \in S) \to ((1,5) \in R \mid S)$
2. $((2,3) \in R)$ and $((3,6) \in S) \to ((2,6) \in R \mid S)$
3. $((3,4) \in R)$ and $((4,7) \in S) \to ((3,7) \in R \mid S)$

Consequently:

$$R \mid S = \{(1,5),(2,6),(3,7)\}$$

23.3 Restriction of a Relation

Definition:

$$R \restriction A = R \cap A^2$$

In general, if $R \subseteq \mathbb{N}^2$ and $A \subseteq \mathbb{N}$, the restriction of R to A keeps only those pairs in R where both elements are in A:

$$R \restriction A = \{(x,y) \in R \mid (x,y) \in A\}$$

Hence:

A^2	a_1	a_2	$\ldots$	a_n
a_1	(a_1,a_1)	(a_1,a_2)	$\ldots$	(a_1,a_n)
a_2	(a_2,a_1)	(a_2,a_2)	$\ldots$	(a_2,a_n)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_n	(a_n,a_1)	(a_n,a_2)	$\ldots$	(a_n,a_n)

Example 1:
Given:

$$R = \{(1,2),(2,4),(3,6),(4,8),(5,10),(6,12)\} \subseteq \mathbb{Z}^2$$

Restriction to even numbers $E = \{2,4,6,8,10,12,\ldots\}$:

$$R \restriction_E = \{(2,4),(4,8),(6,12)\}$$

Example 2:
Given:

$$f = \{(x, x^2) \mid x \in \mathbb{R}\}$$

Restriction to $\mathbb{Z} \cap [0, 3]$:

$f \restriction_{\mathbb{Z}\cap[0,3]}$	0	1	2	3
0	$(0, 0)$			
1		$(1, 1)$		
2			$(2, 4)$	
3				$(3, 9)$

Example 3:
Let

$\mathbb{R}^2$	$\cdots$	a_{ijk}	$\cdots$	b_{lmn}	$\cdots$	c_{pqr}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{ijk}	$\cdots$	(a_{ijk}, a_{ijk})	$\cdots$	(a_{ijk}, b_{lmn})	$\cdots$	(a_{ijk}, c_{pqr})	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b_{lmn}	$\cdots$	(b_{lmn}, a_{ijk})	$\cdots$	(b_{lmn}, b_{lmn})	$\cdots$	(b_{lmn}, c_{pqr})	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_{pqr}	$\cdots$	(c_{pqr}, a_{ijk})	$\cdots$	(c_{pqr}, b_{lmn})	$\cdots$	(c_{pqr}, c_{pqr})	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Restriction to $\mathbb{N}^2$:

$\mathbb{N}^2$	1	2	3	$\cdots$
1	$(1, 1)$	$(1, 2)$	$(1, 3)$	$\cdots$
2	$(2, 1)$	$(2, 2)$	$(2, 3)$	$\cdots$
3	$(3, 1)$	$(3, 2)$	$(3, 3)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Given relation:

$$R = \{(a_{ijk}, c_{pqr}), (b_{lmn}, 2), (2, 3), (\pi, e), (3, 5)\} \subseteq \mathbb{R}^2$$

Restriction to $\mathbb{N}^2$:

$$R \restriction_{\mathbb{N}^2} = \{(2, 3), (3, 5)\}$$

23.4 Application of a Relation to a Set

Definition:

$$R[A] = \{y \in A \mid \exists x \in A, \langle x, y \rangle \in R\}$$

Example:
Let

$$S = \{(a, b), (b, c), (c, d)\}$$

and consider the set $P = \{a, b, c\}$.
Where:

$$
\begin{aligned}
(a,b) &\to b \in S[\cdot] \\
(b,c) &\to c \in S[\cdot] \\
(c,d) &\to d \notin S[P] \text{ since } d \notin P
\end{aligned}
$$

Then:

$$S[P] = \{b,c\}$$

23.5 Transitive Closure

Definition:

$$R^1 = R, \quad R^{n+1} = R^n \mid R$$

$$R^+ = \bigcup_{n=1}^{\infty} R^n$$

Example:
Let

$$R = \{(1,2),(2,3),(3,4)\}$$

Since:

$$
\begin{aligned}
R^1 =& \{(1,2),(2,3),(3,4)\} \\
R^2 =& R \mid R = \{(1,3),(2,4)\} \\
R^3 =& R^2 \mid R = \{(1,4)\}
\end{aligned}
$$

Therefore:

R^+	1	2	3	4
1		$(1,2)$	$(1,3)$	$(1,4)$
2			$(2,3)$	$(2,4)$
3				$(3,4)$
4				

23.6 Reflexive Transitive Closure

Definition:
Let

$$\mathrm{Id}_A = \{\langle x,x\rangle \mid x \in A\}$$

then

$$R^* = R^+ \cup \mathrm{Id}_A$$

Example:
Since Id_A:

Id_A	1	2	3	4
1	$(1,1)$			
2		$(2,2)$		
3			$(3,3)$	
4				$(4,4)$

We conclude that, $R^* = R^+ \cup \mathrm{Id}_A$:

R^*	1	2	3	4
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$
2		$(2,2)$	$(2,3)$	$(2,4)$
3			$(3,3)$	$(3,4)$
4				$(4,4)$

24 Set Properties and Laws

24.1 Commutative Laws

1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$

Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.

Operation	Result
$A \cup B$	$\{1, 2, 3, 4, 5\}$
$B \cup A$	$\{1, 2, 3, 4, 5\}$
$A \cap B$	$\{3\}$
$B \cap A$	$\{3\}$

24.2 Associative Laws

1. $(A \cup B) \cup C = A \cup (B \cup C)$
2. $(A \cap B) \cap C = A \cap (B \cap C)$

Let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{3, 4\}$.

Operation	Result
$(A \cup B) \cup C$	$\{1, 2, 3, 4\}$
$A \cup (B \cup C)$	$\{1, 2, 3, 4\}$
$(A \cap B) \cap C$	$\varnothing$
$A \cap (B \cap C)$	$\varnothing$

24.3 Distributive Laws

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{3, 4\}$.

Operation	Result
$A \cup (B \cap C)$	$\{1, 2, 3\}$
$(A \cup B) \cap (A \cup C)$	$\{1, 2, 3\}$
$A \cap (B \cup C)$	$\{2\}$
$(A \cap B) \cup (A \cap C)$	$\{2\}$

24.4 De Morgan's Laws

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

Let $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2\}$, and $B = \{2, 3\}$.
First, find the complements: $A^c = \{3, 4, 5\}$ and $B^c = \{1, 4, 5\}$.

Operation	Result
$(A \cup B)^c$	$\{4, 5\}$
$A^c \cap B^c$	$\{4, 5\}$
$(A \cap B)^c$	$\{1, 3, 4, 5\}$
$A^c \cup B^c$	$\{1, 3, 4, 5\}$

24.5 Identity Laws

1. $A \cup \varnothing = A$
2. $A \cap U = A$

Let $U = \{1, 2, 3, 4, 5\}$ and $A = \{2, 3, 4\}$.

Operation	Result
$A \cup \varnothing$	$\{2, 3, 4\}$
$A \cap U$	$\{2, 3, 4\}$

24.6 Idempotent Laws

1. $A \cup A = A$
2. $A \cap A = A$

Let $A = \{1, 3, 5\}$.

Operation	Result
$A \cup A$	$\{1, 3, 5\}$
$A \cap A$	$\{1, 3, 5\}$

24.7 Complement Laws

1. $A \cup A^c = U$
2. $A \cap A^c = \varnothing$
3. $(A^c)^c = A$

Let $U = \{1, 2, 3, 4, 5\}$ and $A = \{1, 3, 5\}$. First, find the complement: $A^c = \{2, 4\}$.

Operation	Result
$A \cup A^c$	$\{1, 2, 3, 4, 5\} = U$
$A \cap A^c$	$\varnothing$
$(A^c)^c$	$\{1, 3, 5\} = A$

24.8 Absorption Laws

1. $A \cup (A \cap B) = A$
2. $A \cap (A \cup B) = A$

Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$.

Operation	Result
$A \cup (A \cap B)$	$\{1, 2, 3\}$
$A \cap (A \cup B)$	$\{1, 2, 3\}$

25 Function

A *function* f from set A to set B is a relation that assigns to each element of A exactly one element of B.

Let A and B be sets. A function $f : A \to B$ is a subset $f \subseteq A \times B$ such that:

$$f = \{(x, y) \in A \times B \mid f(x) = y\}$$

This is a subset of $A \times B$ such that for every $x \in A$, there exists a unique $y \in B$ with $(x, y) \in f$.

Example:

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $x = 3$.

$$f(x) = x + 2$$

Then:

$$f(3) = 3 + 2 = 5$$

We conclude that:

$$f(3) = 5$$

This means the ordered pair $(3, 5) \in f$.

25.1 Components

Domain:

$$\mathrm{dom}(f) = A = \{x \mid \exists y \in B : (x, y) \in f\}$$

Codomain:

$$\mathrm{cod}(f) = B$$

Range (Image):

$$\mathrm{range}(f) = \mathrm{im}(f) = \{y \in B \mid \exists x \in A : f(x) = y\}$$

Image of a subset:
For $S \subseteq A$:

$$f(S) = \{f(x) \mid x \in S\} = \{y \in B \mid \exists x \in S : f(x) = y\}$$

Preimage (Inverse Image):
For $T \subseteq B$:

$$f^{-1}(T) = \{x \in A \mid f(x) \in T\}$$

Example:
Let

$$\begin{aligned} A &= \{1, 2, 3, 4, 5\} \\ B &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \end{aligned}$$

Define

$$f : A \to B,\ f(x) = 2x$$

Let

$$S = \{1, 3, 5\} \subseteq A$$

Then

$$f(S) = \{f(x) \mid x \in S\}$$

$$\begin{aligned} f(1) &= 2 \\ f(3) &= 6 \\ f(5) &= 10 \end{aligned}$$

Therefore

$$\boxed{f(S) = \{2, 6, 10\}}$$

Example 1:
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.

$$\mathrm{range}(f) = \{y \in \mathbb{R} \mid y \geq 0\} = [0, \infty)$$

Image of a set:

$$f(\{-2, 3\}) = \{(-2)^2, 3^2\} = \{4, 9\}$$

Preimage of a set:

$$f^{-1}(\{4\}) = \{x \in \mathbb{R} \mid x^2 = 4\} = \{-2, 2\}$$

Example 2:
Let

$$\times : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

be the usual multiplication function defined by

$$\times(a, b) = a \times b$$

Then:

$$\mathrm{dom}(\times) = \mathbb{N} \times \mathbb{N}, \quad \mathrm{cod}(\times) = \mathbb{N}, \quad \mathrm{range}(\times) = \mathbb{N}$$

For example:

$$(3, 4) \in \mathbb{N} \times \mathbb{N}$$

and

$$3 \times 4 = 12$$

Thus,

$$\times(3, 4) = 12$$

We also have the identity property:

$$\forall n \in \mathbb{N},\ \exists (n, 1) \in \mathbb{N} \times \mathbb{N} : n \times 1 = n$$

Example 3:
Let

$$f(x) = x^2 - 3x + 2$$

We want to find

$$f(2 + x)$$

Substitute $2 + x$ for x:

$$f(2 + x) = (2 + x)^2 - 3(2 + x) + 2$$

Expand and simplify:

$$\begin{aligned} f(2 + x) &= (x^2 + 4x + 4) - 3(2 + x) + 2 \\ &= x^2 + 4x + 4 - 6 - 3x + 2 \\ &= x^2 + x \end{aligned}$$

So:

$$\boxed{f(2 + x) = x^2 + x}$$

Example 4:
Let

$$f(x) = 2x + 3$$

We want to find

$$f^{-1}(x)$$

We define

$$\begin{aligned} y &= 2x + 3 \\ y - 3 &= 2x \\ x &= \frac{y-3}{2} \end{aligned}$$

Now, interchange x and y:

$$\boxed{f^{-1}(x) = \frac{x-3}{2}}$$

Example 5:
Let

$$f : \mathbb{N} \to \mathbb{N}$$

be defined by

$$f(x) = x + 1$$

Then:

$$\text{range}(f) = \mathbb{Z}^{+} = \{1, 2, 3, \ldots\}$$

For example:

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = 3$$

Hence,

$$0 \notin \text{range}(f) \text{ since } \nexists x \in \mathbb{N} : f(x) = 0$$

Example 6:
Let $g : \mathbb{N} \to \mathbb{N}$ be defined by:

$$g(x) = x + 2 - 1$$

Then:

$$\forall x \in \mathbb{N},\ f(x) = x + 1 = x + 2 - 1 = g(x)$$

By the principle of extensionality for functions:

$$\forall x \in \text{dom}(f) : f(x) = g(x) \to f = g$$

provided $\text{dom}(f) = \text{dom}(g)$ and $\text{cod}(f) = \text{cod}(g)$.
Example 7:
Let $h : \mathbb{N} \to \mathbb{N}$ be defined by:

$$h(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ \dfrac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

Since:

$$\forall x \in \mathbb{N},\ (\text{even}(x) \vee \text{odd}(x)) \wedge \neg(\text{even}(x) \wedge \text{odd}(x))$$

the function h is well-defined and $h(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$.

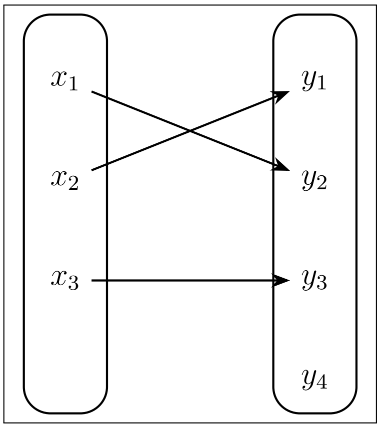
25.2 Types of Functions

25.2.1 Injective

A function $f : A \to B$ is injective or one-to-one if and only if:

$$\forall x_1, x_2 \in A,\ f(x_1) = f(x_2) \implies x_1 = x_2.$$

Equivalently, for each $y \in B$, there is at most one $x \in A$ such that $f(x) = y$. We call such a function an injection from A to B.



Example:

1. $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 2x + 3$

   If $2x_1 + 3 = 2x_2 + 3$, then $x_1 = x_2$.

   Exponential function is strictly increasing.

2. $f : \mathbb{Z} \to \mathbb{N}$, defined by

$$f(x) = \begin{cases} 2x, & \text{if } x \geq 0, \\ -2x - 1, & \text{if } x < 0 \end{cases}$$

3. There exist injective functions from $\mathbb{N}$ into each of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Formally:

$$\exists f_i : \mathbb{N} \to S_i, \quad S_i \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}, \text{ such that } f_i \text{ is injective.}$$

   Example:

$$\begin{aligned} &f_1 : \mathbb{N} \to \mathbb{N}, \quad f_1(x) = x + 1 \\ &f_2 : \mathbb{N} \to \mathbb{Z}, \quad f_2(x) = \begin{cases} \dfrac{x}{2}, & \text{if } x \text{ is even} \\ -\dfrac{x+1}{2}, & \text{if } x \text{ is odd} \end{cases} \\ &f_3 : \mathbb{N} \to \mathbb{Q}, \quad f_3(x) = \frac{x}{x+1} \\ &f_4 : \mathbb{N} \to \mathbb{R}, \quad f_4(x) = x. \end{aligned}$$

   Each f_i is injective, since distinct natural numbers map to distinct elements of the codomain.

Example:

Let

$$f : \{2n+1 \mid n \in \mathbb{N}\} \longrightarrow \mathbb{Q}, \quad f(2n+1) = \frac{2n+1}{2n+2}$$

$$\mathrm{Im}(f) = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; a \in \{2n+1 \mid n \in \mathbb{N}\},\ b = a+1 \right\}$$

Hence,

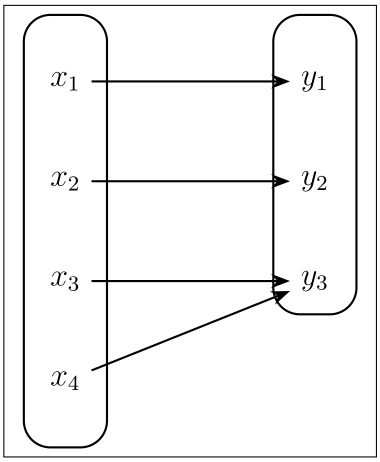
$$\begin{aligned} f(1) &= 1/2 \\ f(3) &= 3/4 \\ f(5) &= 5/6 \\ f(7) &= 7/8 \\ f(9) &= 9/10 \\ &\vdots \end{aligned}$$

25.2.2 Surjective

A function $f : A \to B$ is surjective or onto if and only if B is the range of f, i.e.,

$$\forall y \in B,\ \exists x \in A \mid f(x) = y.$$

We call such a function a surjection from A to B.



Example:

1. $f : \mathbb{Z} \to \mathbb{N}$, where $f(x) = |x|$

x	-3	-2	-1	0	1	2	3
$\lvert x\rvert$	3	2	1	0	1	2	3

2. $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^3$

$$\forall y \in \mathbb{R}, \exists x = \sqrt[3]{y} \in \mathbb{R} \mid f(x) = y$$

3. $f : \mathbb{R} \to [0, \infty)$, where $f(x) = x^2$

$$f(\sqrt{x}) = (\sqrt{x})^2 = x$$

4. Floor function: $f : \mathbb{R} \to \mathbb{Z}$, where $f(x) = \lfloor x \rfloor$

x	-2.7	-1.2	0.5	1.0	2.9
$\lfloor x \rfloor$	-3	-2	0	1	2

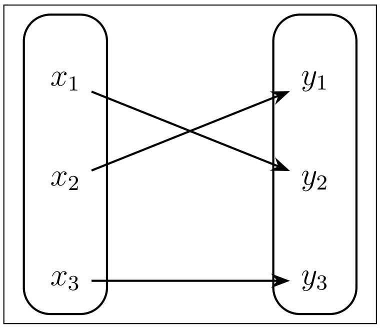
25.2.3 Bijective

A function $f : A \to B$ is bijective if and only if it is both injective and surjective. We call such a function a bijection from A to B, or a one-to-one correspondence.

Equivalently:

$$\forall y \in B,\ \exists! x \in A \mid f(x) = y.$$

(For every element in B, there exists exactly one element in A that maps to it.)



Example:
Let
$$A = \{1, 2, 3\}, \quad B = \{a, b, c\}.$$
Define a function
$$f : A \to B$$
by
$$f(1) = a, \quad f(2) = b, \quad f(3) = c.$$
Each element of A maps to a unique element of B, so f is injective. Every element of B is the image of some element of A, so f is surjective. Therefore, f is bijective.

25.3 Combinations

25.3.1 Bijective (Both injective and surjective)

1. Identity function: $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x$

   Injective and surjective, therefore bijective:
   $$\forall y \in \mathbb{R}, \exists! x \in \mathbb{R} \mid f(x) = y$$

2. $f : [0, \infty) \to [0, \infty)$, where $f(x) = x^2$

   Injective:
   $$\forall x_1, x_2 \in [0, \infty), x_1^2 = x_2^2 \implies x_1 = x_2$$
   Surjective:
   $$\forall y \in [0, \infty), \exists x \in [0, \infty) \mid f(x) = y$$
   Therefore bijective.

25.3.2 Injective but not surjective

Recall the Von Neumann Ordinal,
$$\begin{aligned}
0 &= \varnothing = \{\} \\
1 &= \{\varnothing\} = \varnothing \cup \{\varnothing\} \\
2 &= \{\varnothing, \{\varnothing\}\} = 1 \cup \{1\} \\
&\vdots \\
n &= n - 1 \cup \{n - 1\}
\end{aligned}$$

Based on this notion, we can define a function that is injective but not surjective:
Let $A = \{1, 2, 3\}$ and define $f : A \to \mathcal{P}(A)$ by $f(x) = \{x\}$.
Then:

$$f(1) = \{1\}, \quad f(2) = \{2\}, \quad f(3) = \{3\}$$

Hence:

$$\mathrm{Im}(f) = \{\{1\}, \{2\}, \{3\}\} \subset \mathcal{P}(A)$$

Since $\varnothing \in \mathcal{P}(A)$ but $\varnothing \notin \mathrm{Im}(f)$, we conclude that f is injective but not surjective.

25.3.3 Surjective but not injective

Piecewise function: $f : \mathbb{N} \to \mathbb{N}$, where

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ \dfrac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

Surjective:

$$\forall y \in \mathbb{N}, \exists x \in \mathbb{N} : f(x) = y$$

Not injective:

$$f(1) = f(2) = 1 \text{ but } 1 \neq 2$$

25.3.4 Neither injective nor surjective

1. Constant function: $f : \mathbb{N} \to \mathbb{N}$, where $f(x) = 1$ for all x

   Not injective:

   $$f(1) = f(2) = 1 \text{ but } 1 \neq 2$$

   Not surjective:

   $$\nexists x \in \mathbb{N} \mid f(x) = 2$$

2. $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$

   Not injective:

   $$f(-2) = f(2) = 4 \text{ but } -2 \neq 2$$

   Not surjective:

   $$\nexists x \in \mathbb{R} \mid f(x) = -1$$

25.4 Equivalence Relation with Function Properties

Example 1:

Let $E = \{a, b, c, d, e, f\}$. Define $R \subseteq E^2$:

R	a	b	c	d	e	f
a	(a, a)	(a, b)				
b	(b, a)	(b, b)				
c			(c, c)	(c, d)		
d			(d, c)	(d, d)		
e					(e, e)	(e, f)
f					(f, e)	(f, f)

1. Reflexive:
$$\forall x \in E,\ (x, x) \in R$$
2. Symmetric:
$$\forall x, y \in E,\ (x, y) \in R \to (y, x) \in R$$
3. Transitive:
$$\forall x, y, z \in E,\ (x, y) \in R \wedge (y, z) \in R \to (x, z) \in R$$

Equivalence classes:

$$[a] = \{a, b\}, \quad [c] = \{c, d\}, \quad [e] = \{e, f\}$$

Thus:
$$E/R = \{[a], [c], [e]\}$$

Function $f : E \to E/R$
Defined by:
$$f(x) = [x]$$

1. Not injective:
$$a \neq b \ \wedge \ f(a) = f(b) \to$$
2. Surjective:
$$\forall y \in E/R,\ \exists x \in E : f(x) = y$$
3. Not bijective:
$$\neg\text{Injective} \to \neg\text{Bijective}$$

Therefore:
$$E \ncong E/R$$

Bijective Function Defined on Quotient Classes
Let:
$$S = \{a, c, e\}$$

Define:
$$g : E/R \to S, \quad g([a]) = a,\ g([c]) = c,\ g([e]) = e$$

1. Injective:

$$\forall x_1, x_2 \in E/R,\ g(x_1) = g(x_2) \to x_1 = x_2$$

2. Surjective:

$$\forall y \in S,\ \exists x \in E/R,\ g(x) = y$$

3. Bijective:

$$\text{Injective} \wedge \text{Surjective} \to \text{Bijective}$$

Therefore:

$$E/R \cong S$$

Example 2:
Let $X = \mathbb{N}^2$. Define $R \subseteq X^2$ by:

$$(x_1, y_1)\, R\, (x_2, y_2) \leftrightarrow x_1 + y_2 = y_1 + x_2$$

By definition of equivalence:

1. Reflexive:

$$\forall (x, y) \in X,\ (x, y)\, R\, (x, y)$$

2. Symmetric:

$$\forall (x_1, y_1), (x_2, y_2) \in X,\ ((x_1, y_1)\, R\, (x_2, y_2) \to (x_2, y_2)\, R\, (x_1, y_1))$$

3. Transitive:

$$\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in X,\ ((x_1, y_1)\, R\, (x_2, y_2) \wedge (x_2, y_2)\, R\, (x_3, y_3) \to (x_1, y_1)\, R\, (x_3, y_3))$$

Equivalence Classes:

$$[(x_1, y_1)] = \{(x_2, y_2) : x_2 - y_2 = x_1 - y_1\}$$

Quotient set:

$$X/R = \{[(x, y)] : (x, y) \in \mathbb{N}^2\}$$

Define:

$$\Sigma : X/R \to \mathbb{Z}, \quad \Sigma([(x, y)]) = x - y$$

Values:

$\Sigma(x, y)$	1	2	3	4	5
1	0	-1	-2	-3	-4
2	1	0	-1	-2	-3
3	2	1	0	-1	-2
4	3	2	1	0	-1
5	4	3	2	1	0

1. Injective

$$\Sigma([(x_1, y_1)]) = \Sigma([(x_2, y_2)]) \to x_1 - y_1 = x_2 - y_2 \to (x_1, y_1)\, R\, (x_2, y_2) \to [(x_1, y_1)] = [(x_2, y_2)]$$

$\Sigma([(x, y)])$	-2	-1	0	1	2	$\dots$
Pairs	$(0, 2)$	$(0, 1)$	$(0, 0)$	$(1, 0)$	$(2, 0)$	$\dots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

2. Surjective

$$\forall z \in \mathbb{Z},\ \exists (x,y) \in \mathbb{N}^2,\ \Sigma([(x,y)]) = z$$

$z \in \mathbb{Z}$	-2	-1	0	1	2	$\dots$
Pair	$(0,2)$	$(0,1)$	$(0,0)$	$(1,0)$	$(2,0)$	$\dots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

3. Bijective

$$\text{Injective} \land \text{Surjective} \to \text{Bijective}$$

We conclude that:

$$X/R \cong \mathbb{Z}$$

26 Functions as Relations

A function $f : A \to B$ defines a relation between A and B:

$$x \in A \text{ relates to } y \in B \leftrightarrow f(x) = y$$

We identify f with its set of ordered pairs:

$$f = \{(x,y) \mid x \in A \land f(x) = y\} \subseteq A \times B$$

26.1 Graph of a Function

Let $R \subseteq A \times B$ satisfy:

1. $(xRy \land xRz) \to y = z$, and
2. $\forall x \in A, \exists y \in B \mid \langle x, y\rangle \in R$.

Then R is the graph of a function $f : A \to B$ defined by $f(x) = y$ iff xRy.
Let $f : A \to B$ be a function. The graph of f is the relation $R_f \subseteq A \times B$ defined by:

$$R_f = \{\langle x, y\rangle \mid f(x) = y\}.$$

Example:
Define a function $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = (-1)^n \cdot \left\lfloor \frac{n+1}{2} \right\rfloor$$

Explicitly:

$$
\begin{aligned}
n = 0: &\quad \left\lfloor \frac{0+1}{2} \right\rfloor = \lfloor 0.5 \rfloor = 0 \to f(0) = (-1)^0 \cdot 0 = 0 \\
n = 1: &\quad \left\lfloor \frac{1+1}{2} \right\rfloor = \lfloor 1 \rfloor = 1 \to f(1) = (-1)^1 \cdot 1 = -1 \\
n = 2: &\quad \left\lfloor \frac{2+1}{2} \right\rfloor = \lfloor 1.5 \rfloor = 1 \to f(2) = (-1)^2 \cdot 1 = 1 \\
n = 3: &\quad \left\lfloor \frac{3+1}{2} \right\rfloor = \lfloor 2 \rfloor = 2 \to f(3) = (-1)^3 \cdot 2 = -2 \\
n = 4: &\quad \left\lfloor \frac{4+1}{2} \right\rfloor = \lfloor 2.5 \rfloor = 2 \to f(4) = (-1)^4 \cdot 2 = 2
\end{aligned}
$$

Therefore: $f(0) = 0, \quad f(1) = -1, \quad f(2) = 1, \quad f(3) = -2, \quad f(4) = 2, \ldots$
Then the graph is:

$$R_f = \{(0,0), (1,-1), (2,1), (3,-2), (4,2), \ldots\}$$

Each element in $\mathbb{N}$ maps to exactly one element in $\mathbb{Z}$.
Example 2:
Let

$$A = \{a_i \mid i \geq 1\}, \qquad B = \{b_k \mid k \geq 1\}, \qquad C = \{c_j \mid j \geq 1\}.$$

Define functions

$$
\begin{aligned}
f : A \to B, &\qquad f(a_i) = b_i \quad \forall i \geq 1, \\
g : B \to C, &\qquad g(b_k) = c_k \quad \forall k \geq 1.
\end{aligned}
$$

Their graphs are

$$
\begin{aligned}
R_f &= \{(a_i, b_i) \mid i \geq 1\}, \\
R_g &= \{(b_k, c_k) \mid k \geq 1\}.
\end{aligned}
$$

1. Relational composition

$$R_f \mid R_g$$

Take an arbitrary pair

$$(a_i, c_j) \in R_f \mid R_g.$$

By definition of relational composition, there exists some index k such that

$$(a_i, b_k) \in R_f \ \wedge \ (b_k, c_j) \in R_g.$$

Because

$$R_f = \{(a_i, b_i) \mid i \geq 1\},$$

the only pair whose first component is a_i is (a_i, b_i). Thus

$$b_k = b_i \to k = i.$$

Similarly, since

$$R_g = \{(b_k, c_k) \mid k \geq 1\},$$

then

$$c_j = c_k \to j = k.$$

Therefore,

$$j = k = i,$$

which implies

$$R_f \mid R_g = \{(a_i, c_i) \mid i \geq 1\}.$$

2. Function composition

$$g \circ f$$

For each $i \geq 1$,

$$(g \circ f)(a_i) = g(f(a_i)) = g(b_i) = c_i,$$

so the graph of the composite is

$$R_{g \circ f} = \{(a_i, c_i) \mid i \geq 1\}.$$

Thus,

$$\boxed{R_f \mid R_g = R_{g \circ f} = \{(a_i, c_i) \mid i \geq 1\}.}$$

26.2 Restriction and Image

Let $f : A \to B$ be a function and let $C \subseteq A$. The restriction of f to C, denoted $f \upharpoonright C : C \to B$, is defined by

$$(f \upharpoonright C)(x) = f(x), \forall x \in C.$$

Equivalently, in terms of graphs:

$$R_{f \upharpoonright C} = \{(x, y) \in R_f \mid x \in C\}.$$

The image of C under f is:

$$f[C] = \{f(x) \mid x \in C\}.$$

Example 1:
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2.$$

Let $C = [0, \infty) \subseteq \mathbb{R}$. The restriction of f to C is the function

$$f \upharpoonright C : C \to \mathbb{R},$$

defined by

$$(f \upharpoonright C)(x) = x^2 \quad \forall x \in C.$$

The graph of f is

$$R_f = \{(x, x^2) \mid x \in \mathbb{R}\}.$$

The graph of the restricted function is

$$R_{f \upharpoonright C} = \bigcup_{x \in [0, \infty)} \{(x, x^2)\}.$$

Example 2:
Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(n) = (-1)^n \cdot n.$$

Let $\mathbb{P}$ be the set of prime numbers:

$$\mathbb{P} = \{2, 3, 5, 7, 11, \dots\} \subseteq \mathbb{N}.$$

The restriction of f to $\mathbb{P}$ is the function

$$f \restriction \mathbb{P} : \mathbb{P} \to \mathbb{Z},$$

defined by

$$(f \restriction \mathbb{P})(p) = (-1)^p \cdot p \quad \forall p \in \mathbb{P}.$$

The graph of f is

$$R_f = \{(n, (-1)^n \cdot n) \mid n \in \mathbb{N}\}.$$

The graph of the restricted function is

$$R_{f\restriction\mathbb{P}} = \bigcup_{p \in \mathbb{P}} \{(p, (-1)^p \cdot p)\}.$$

26.3 Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be functions. The *composition* of f and g, denoted $g \circ f$, is the function:

$$(g \circ f) : A \to C$$

defined by:

$$(g \circ f)(x) = g(f(x)), \forall x \in A$$

This operation is only defined when the range of f is a subset of the domain of g.
Example 1:
Consider two functions:

$$f(x) = x + 1, \quad g(x) = 2x$$

To compute $(g \circ f)(x)$:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x + 1) \\ &= 2(x + 1) \\ &= 2x + 2\end{aligned}$$

So the composed function is:

$$(g \circ f)(x) = 2x + 2$$

Example 2:
Consider two functions:

$$f : \mathbb{N} \to \mathbb{Z}, \quad f(n) = 2n - 5$$

$$g : \mathbb{Z} \to \mathbb{Q}, \quad g(z) = \frac{z+1}{3}$$

To compute $(g \circ f)(n)$:

$$\begin{aligned}(g \circ f)(n) &= g(f(n)) \\ &= g(2n-5) \\ &= \frac{(2n-5)+1}{3} \\ &= \frac{2n-4}{3}\end{aligned}$$

So the composed function is:

$$(g \circ f) : \mathbb{N} \to \mathbb{Q}, \quad (g \circ f)(n) = \frac{2n-4}{3}$$

Example 3:
Let

$$f(x) = x + 1, \quad g(x) = 2x$$

Then:

$$(g \circ f)(x) = g(f(x)) = 2(x+1) = 2x + 2$$

$$(f \circ g)(x) = f(g(x)) = (2x) + 1 = 2x + 1$$

26.4 Partial function

A partial function $f : A \rightharpoonup B$ is a mapping that assigns to every element of A *at most one* element of B.

1. If f assigns a value to $x \in A$, we say $f(x)$ is defined, and write $f(x) \downarrow$.
2. Otherwise, $f(x)$ is undefined, and we write $f(x) \uparrow$.
3. The domain of a partial function f is:

$$\mathrm{dom}(f) = \{x \in A : f(x) \downarrow\}$$

Example 1:
Every total function $f : A \to B$ is also a partial function where $\mathrm{dom}(f) = A$.
Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Define a function $f : A \to B$ by:

$$f(1) = a, \quad f(2) = b, \quad f(3) = c$$

This function is total, since every element of A is assigned an element of B. We can also view it as a partial function:

$$f : A \rightharpoonup B \quad \text{with } \mathrm{dom}(f) = A$$

This illustrates that a total function is just a special case of a partial function where the function is defined for all inputs.

Example 2:

Let $f : \mathbb{R} \rightharpoonup \mathbb{R}$ be defined by:

$$f(x) = \frac{1}{x}$$

Then f is undefined at $x = 0$:

$$f(0) \uparrow \quad \text{but} \quad f(x) \downarrow \text{ for } x \in \mathbb{R} \setminus \{0\}$$

The function f is defined on all real numbers except 0. So the domain of f is:

$$\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$$

Example 3:

Define $g : \mathbb{N} \rightharpoonup [0, 1]$ by:

$$g(n) = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is even} \\ \uparrow & \text{if } n \text{ is odd} \end{cases}$$

This is a partial function because $g(n)$ is undefined for all odd natural numbers. Here $\mathrm{dom}(g) = \{0, 2, 4, 6, ...\} \subset \mathbb{N}$.

26.5 Graph of a Partial Function

The graph of a partial function $f : A \rightharpoonup B$ is the set:

$$R_f = \{(x, y) \in A \times B \mid f(x) = y\}$$

Let $R \subseteq A \times B$ satisfy:

$$\forall x \in A, \forall y, y' \in B, \, (x, y) \in R \wedge (x, y') \in R \rightarrow y = y'$$

Then R is the graph of a partial function $f : A \rightharpoonup B$, defined by:

$$f(x) = \begin{cases} y & \text{if } (x, y) \in R, \exists y \in B \\ \uparrow & \text{otherwise} \end{cases}$$

If in addition, R is serial, i.e., $\forall x \in A, \exists y \in B \mid (x, y) \in R$, then f is a total function.

Example:

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b\}$. Define a relation $R \subseteq A \times B$ by:

$$R = \{(1, a), (2, a), (4, b)\}$$

This satisfies:

$$\forall x \in A, \forall y, y' \in B,\ (x,y) \in R \wedge (x,y') \in R \to y = y'$$

So R is the graph of a partial function $f : A \rightharpoonup B$, where:

$$f(1) = a, \quad f(2) = a, \quad f(4) = b, \quad f(3) \uparrow$$

27 Inverse

A function $g : Y \to X$ is an inverse of a function $f : X \to Y$ if

$$f(g(y)) = y \quad \text{and} \quad g(f(x)) = x, \quad \forall x \in X, \forall y \in Y.$$

Example:
Let

$$f(x) = ax + b, \qquad g(y) = \frac{y-b}{a}.$$

Then we can verify that

$$f(g(y)) = a\left(\frac{y-b}{a}\right) + b = y,$$

and

$$g(f(x)) = \frac{(ax+b)-b}{a} = x.$$

Therefore, g is the inverse of f.

27.1 Left and Right Inverse

Let $f : X \to Y$

1. A function $g : Y \to X$ is a *left inverse* of f if $g(f(x)) = x$, $\forall x \in X$. The condition is:

$$g(f(x)) = x, \quad \forall x \in X.$$

Logical implication:

$$(\exists g : Y \to X)\,[g \circ f = \mathrm{id}_X] \iff f \text{ is injective.}$$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

Example:

Let

$$f : [0, \infty) \to \mathbb{R},\ f(x) = a^x,\ a > 0,\, a \neq 1.$$

Define

$$g(y) = \log_a(y).$$

Then

$$g(f(x)) = \log_a(a^x) = x,$$

so g is a left inverse of f.

2. A function $h : Y \to X$ is a *right inverse* of f if $f(h(y)) = y$, $\forall y \in Y$.

   The condition is:

$$f(h(y)) = y, \quad \forall y \in Y.$$

Logical implication:

$$(\exists h : Y \to X)\,[f \circ h = \mathrm{id}_Y] \iff f \text{ is surjective.}$$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, "h"] \end{tikzcd}$$

Example:
Let

$$f : [0, \infty) \to \mathbb{R},\ f(x) = x^2.$$

Define

$$h(y) = \sqrt{y}.$$

Then

$$f(h(y)) = (\sqrt{y})^2 = y,$$

so h is a right inverse of f.

3. *Two-Sided* Inverse

   The conditions are:

$$f(f^{-1}(y)) = y, \quad f^{-1}(f(x)) = x.$$

Logical implication:

$$f \text{ is bijective}, \quad f^{-1} \text{ is unique.}$$

$$X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} Y$$

Example:

Let

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}, \quad \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

Define a bijection $f : \mathbb{N} \to \mathbb{Z}$:

$$f(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even} \\ -\dfrac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Define its inverse $f^{-1} : \mathbb{Z} \to \mathbb{N}$:

$$f^{-1}(z) = \begin{cases} 2z, & \text{if } z > 0 \\ 1, & \text{if } z = 0 \\ -2z + 1, & \text{if } z < 0 \end{cases}$$

Left inverse condition:

$$f^{-1}(f(n)) = n, \quad \forall n \in \mathbb{N}.$$

Right inverse condition:

$$f(f^{-1}(z)) = z, \quad \forall z \in \mathbb{Z}.$$

Thus,

$$f \text{ is bijective} \quad \text{and} \quad f^{-1} \text{ is unique.}$$

27.2 Injective Case

If $f : X \to Y$ is injective, then there exists a left inverse $g : Y \to X$ such that $g(f(x)) = x$, $\forall x \in X$.

Example 1:

Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $f : X \to Y$ by

$$f(1) = a, \quad f(2) = b, \quad f(3) = c.$$

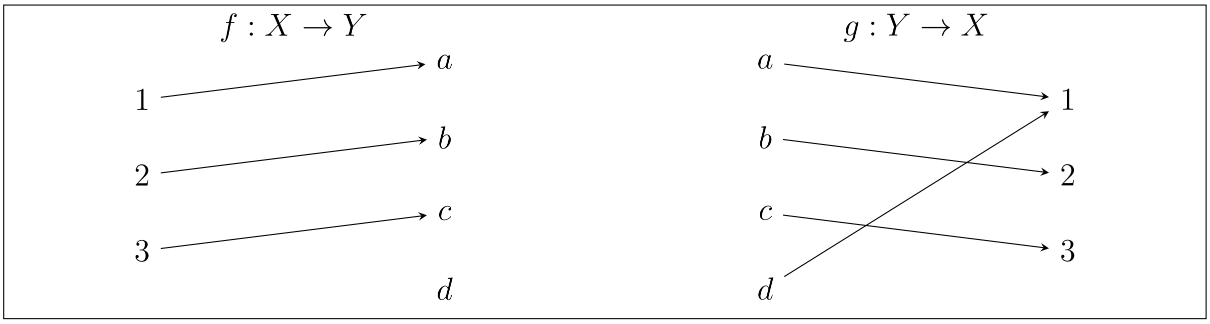
Then f is injective but not surjective (since $d \notin \text{ran}(f)$).

Example 2:

Define a function $g : Y \to X$ by

$$g(a) = 1, \quad g(b) = 2, \quad g(c) = 3, \quad g(d) = 1.$$

Then $\forall x \in X$, $g(f(x)) = x$, so g is a left inverse of f.



27.3 Surjective Case

If $f : X \to Y$ is surjective, then there exists a right inverse $h : Y \to X$ such that $f(h(y)) = y$, $\forall y \in Y$.

Example:

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b\}$. Define $f : X \to Y$ by

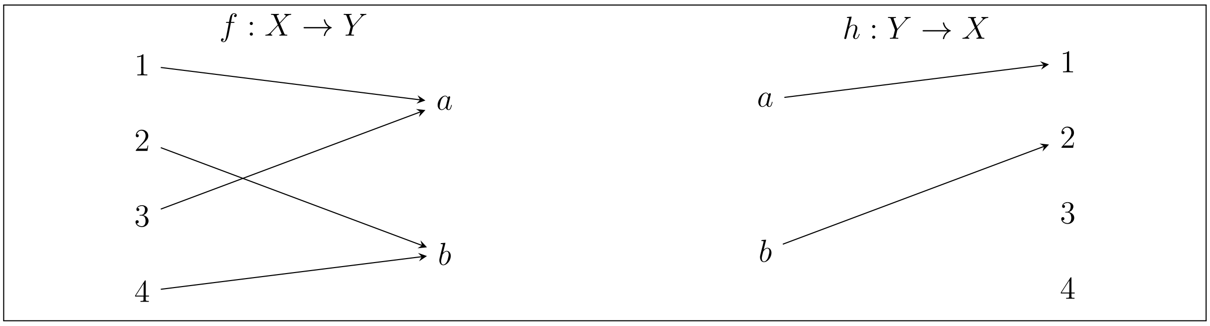
$$f(1) = a, \quad f(2) = b, \quad f(3) = a, \quad f(4) = b.$$

Then f is surjective but not injective.

A right inverse $h : Y \to X$ may be defined as

$$h(a) = 1, \quad h(b) = 2.$$

Then $\forall y \in Y$, $f(h(y)) = y$, so h is a right inverse of f.



.

27.4 Bijective Case

If $f : X \to Y$ is bijective, then there exists a function $f^{-1} : Y \to X$ such that

$$f^{-1}(f(x)) = x, \forall x \in X, \quad \text{and} \quad f(f^{-1}(y)) = y, \forall y \in Y.$$

Example:

Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$ with

$$f(1) = a, \quad f(2) = b, \quad f(3) = c.$$

Then f is bijective. Its inverse f^{-1} satisfies

$$f^{-1}(a) = 1, \quad f^{-1}(b) = 2, \quad f^{-1}(c) = 3.$$

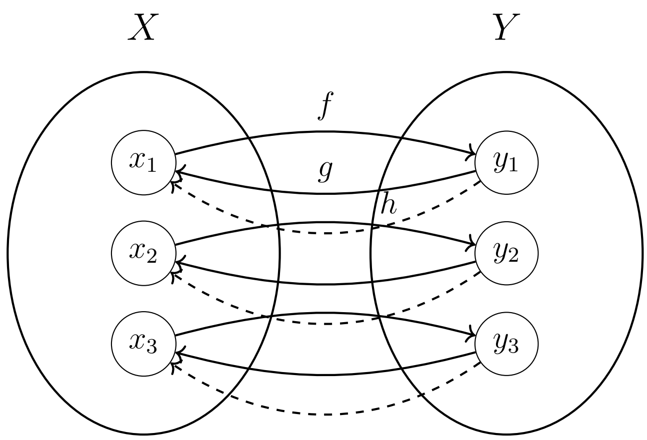
$$
\begin{array}{ccc}
 & f : X \to Y & \\
1 & \longrightarrow & a \\
2 & \longrightarrow & b \\
3 & \longrightarrow & c
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & f^{-1} : Y \to X & \\
a & \longrightarrow & 1 \\
b & \longrightarrow & 2 \\
c & \longrightarrow & 3
\end{array}
$$

27.5 Uniqueness of Inverse

If $f : X \to Y$ has a left inverse g and a right inverse h, then $g = h$. Hence, every function has at most one inverse.

Formally:

$$\forall f : X \to Y, \forall g, h : Y \to X$$
$$[(g \circ f = \mathrm{id}_X) \wedge (f \circ g = \mathrm{id}_Y) \wedge (h \circ f = \mathrm{id}_X) \wedge (f \circ h = \mathrm{id}_Y)] \to g = h$$



$$\boxed{\begin{array}{cc} g \circ f = \mathrm{id}_X & f \circ g = \mathrm{id}_Y \\ h \circ f = \mathrm{id}_X & f \circ h = \mathrm{id}_Y \\ \multicolumn{2}{c}{\therefore g = h} \end{array}}$$

Example 1:

Let

$$X = \{1, 2, 3\}, \quad Y = \{a, b, c\}.$$

Define

$$f : X \to Y, \quad f(1) = a,\ f(2) = b,\ f(3) = c.$$

Then f is bijective.

Let

$$g : Y \to X, \quad g(a) = 1,\ g(b) = 2,\ g(c) = 3,$$

and

$$h : Y \to X, \quad h(a) = 1,\ h(b) = 2,\ h(c) = 3.$$

Then

$$g(f(x)) = x \quad \text{and} \quad f(h(y)) = y.$$

Thus g is a left inverse, h is a right inverse, and

$$g = h = f^{-1}.$$

Hence, the inverse of f is unique.

Example 2:

Let

$$f : \mathbb{N} \to \mathbb{Q}, \quad f(n) = n,$$

where we regard $\mathbb{N} \subset \mathbb{Q}$. Then f is injective but not surjective, since many rational numbers (e.g., $\frac{1}{2}, \frac{3}{4}$) are not images of any natural number.

Define

$$g : \mathbb{Q} \to \mathbb{N}, \quad g(q) = \begin{cases} n, & \text{if } q = n \in \mathbb{N}, \\ 1, & \text{otherwise.} \end{cases}$$

Then $\forall n \in \mathbb{N}$,

$$g(f(n)) = g(n) = n,$$

so g is a left inverse of f.

However, f has no right inverse, since it is not surjective; there is no $h : \mathbb{Q} \to \mathbb{N}$ such that

$$f(h(q)) = q, \forall q \in \mathbb{Q}.$$

Hence, f has a left inverse but no right inverse, and therefore no true inverse. If a true inverse existed, it would be unique.

Let A, B be sets. If there exist injective functions

$$f : A \to B, \qquad g : B \to A,$$

then there exists a bijection

$$h : A \to B.$$

Proof.

Define the starting layers in A:

$$A_0 = A \setminus g(B), \qquad A_{n+1} = g(f(A_n)), \quad n \geq 0.$$

For $a_0 \in A$, consider the alternating chain:

$$\begin{array}{ccccccccc} A_0 & \xrightarrow{f} & B & \xrightarrow{g} & A & \xrightarrow{f} & B & \xrightarrow{g} & \dots \\ a_0 \in A_0 & & f(a_0) & & g(f(a_0)) & & f(g(f(a_0))) & & \dots \end{array}$$

Similarly, define the starting layers in B:

$$B_0 = B \setminus f(A), \qquad B_{n+1} = f(g(B_n)), \quad n \geq 0.$$

$$\begin{array}{ccccccccc} \dots & \xleftarrow{f} & A & \xleftarrow{g} & B & \xleftarrow{f} & A & \xleftarrow{g} & B_0 \\ \dots & & g(f(g(b_0))) & & f(g(b_0)) & & g(b_0) & & b_0 \in B_0 \end{array}$$

If, after finitely many steps, this chain leaves $g(B)$, then $a_0 \in X$.

Formally,

$$X = \{\, a \in A \mid \exists n \geq 0 : (g \circ f)^n(a) \notin g(B) \,\},$$

and equivalently,

$$X = \bigcup_{n=0}^{\infty} A_n.$$

Define

$$Y = A \setminus X = \{\, a \in A \mid \forall n \geq 0 : (g \circ f)^n(a) \in g(B) \,\},$$

the set of elements that always remain within the image $g(B)$ under repeated application of f and g, or equivalently,

$$Y = \bigcap_{n=0}^{\infty} (A \setminus A_n).$$

Similarly, define

$$Z = f(X) = \{\, b \in B \mid \exists a \in X,\, b = f(a) \,\},$$

and

$$W = B \setminus Z = \{\, b \in B \mid b \notin f(X) \,\}.$$

Define

$$h(a) = \begin{cases} f(a), & a \in X, \\ g^{-1}(a), & a \in Y. \end{cases}$$

1. **Injective:**
   (a) If $a_1, a_2 \in X$, then $h(a_1) = h(a_2) \implies f(a_1) = f(a_2) \implies a_1 = a_2$.
   (b) If $a_1, a_2 \in Y$, then $h(a_1) = h(a_2) \implies g^{-1}(a_1) = g^{-1}(a_2) \implies a_1 = a_2$.
   (c) If $a_1 \in X, a_2 \in Y$, then $h(a_1) = h(a_2)$. By definition of h, this gives

   $$f(a_1) = g^{-1}(a_2) \quad \implies \quad a_2 = g(f(a_1)).$$

   But $a_2 \in Y = A \setminus X = \bigcap_{n=0}^{\infty} (A \setminus A_n)$,
   while $a_2 = g(f(a_1)) \in g(f(X)) \subseteq X = \bigcup_{n=0}^{\infty} A_n$.
   Therefore $a_2 \in X \cap Y$, contradiction. Hence h injective.

2. **Surjective:** $\forall b \in B$,
   (a) If $b \in f(X)$ then $\exists a \in X : b = f(a) = h(a)$.
   (b) If $b \notin f(X)$ then $g(b) = a \in Y$, and $h(a) = g^{-1}(g(b)) = b$.

   Hence h surjective.

Therefore h bijective,

$$|A| = |B|. \quad \square$$

Example
Firstly, define injective functions:

$$f : \mathbb{N} \to \mathbb{N}, \quad f(x) = x,$$

$$g : \mathbb{N} \to \mathbb{N}, \quad g(0) = 1, \quad g(y) = y + 1 \text{ for } y \geq 1.$$

Moreover, the forward roots

$$X_0 = \mathbb{N} \setminus g(\mathbb{N}) = \varnothing,$$

while the backward root is

$$Y_0 = \mathbb{N} \setminus f(\mathbb{N}) = \{0\}.$$

Then, consider the alternating chain starting from the backward root Y_0:

$$0 \xrightarrow{g} 1 \xrightarrow{f} 1 \xrightarrow{g} 2 \xrightarrow{f} 2 \xrightarrow{g} 3 \xrightarrow{f} 3 \cdots$$

From this, we identify the sets X and Y. Since $X_0 = \varnothing$, we have

$$X = \bigcup_{n \geq 0} X_n = \varnothing, \qquad Y = \mathbb{N} \setminus X = \mathbb{N}.$$

Consequently, the corresponding subsets under f are

$$Z = f(X) = \varnothing, \qquad W = \mathbb{N} \setminus Z = \mathbb{N}.$$

Finally, the bijection $h : \mathbb{N} \to \mathbb{N}$ is defined by

$$h(x) = \begin{cases} f(x), & x \in X = \varnothing, \\ g^{-1}(x), & x \in Y = \mathbb{N}. \end{cases}$$

Explicitly, for all $x \geq 1$, this gives

$$h(x) = g^{-1}(x) = x - 1,$$

so that the bijection is

$$h : 1 \mapsto 0, \; 2 \mapsto 1, \; 3 \mapsto 2, \; 4 \mapsto 3, \ldots$$

28 Truth Value for FOL

In propositional logic (PL), every atomic sentence is directly assigned a truth value (true or false). On the other hand, in first-order logic (FOL), the truth value of a predicate is determined relative to an interpretation and a variable assignment.

For example:

Let domain $D = \{1, 2, 3\}$, for $E(x) =$ "x is even".

Hence:

1. $E(1) =$ False
2. $E(2) =$ True
3. $E(3) =$ False

28.1 Example 1

Let the domain:

$$D = \{\langle x, y\rangle \in \mathbb{N}^2\}$$

Evaluate:

$$\forall x, y \in D\ (x \cdot y > x + y)$$

However, by antisymmetric schema, we can find a counterexample:

$$\exists x \exists y\ \big((\langle x, y\rangle \in \mathbb{N}^2)\ \wedge\ x = y\ \wedge\ x = 2\ \wedge\ y = 2\ \wedge\ (x \cdot y = x + y)\ \wedge\ \neg(x \cdot y > x + y)\big)$$

Therefore:

$$\boxed{\neg\forall x, y \in D\ (x \cdot y > x + y)}$$

Note that other counterexamples include $\langle 1, 1\rangle$, $\langle 1, 2\rangle$, $\langle 2, 1\rangle$, and $\langle 1, n\rangle$ for any $n \in \mathbb{N}$.

28.2 Example 2

Let the domain:

$$D = \mathbb{Z}$$

Evaluate:

$$\forall x \forall y \forall z\ (x, y, z \in \mathbb{Z} \rightarrow ((x + y = z) \rightarrow z > 0))$$

However, we can find a counterexample:

$$\exists x \exists y \exists z\ (x, y, z \in \mathbb{Z}\ \wedge\ x = -1\ \wedge\ y = -2\ \wedge\ z = -3\ \wedge\ (x + y = z)\ \wedge\ \neg(z > 0))$$

Since $-3 \not> 0$, we have shown that there exist integers where the sum equals z but z is not positive.

Consequently:

$$\boxed{\neg\forall x \forall y \forall z\ ((x + y = z) \rightarrow z > 0)}$$

28.3 Example 3

Let the domain:

$$D = \left\{ x \in \mathbb{Z}\ \middle|\ \frac{3x}{2} + 1 + \frac{4x}{3} = -\frac{31}{8} + \frac{2x}{3} \right\}.$$

Evaluate:

$$\exists x\, (x \in D).$$

Since $-\dfrac{9}{4} \notin \mathbb{Z}$,
Thus:

$$\boxed{D = \varnothing}$$

And the statement is false, because the domain is empty.

28.4 Example 4

Let the domain:

$$D = \{\langle x, y\rangle \in \mathbb{R}^2 \mid x + 2y = 16\}$$

Evaluate:

$$\forall x, y \in D\,((x \in \mathbb{R} \wedge y \in \mathbb{Z}) \rightarrow x > y)$$

Since we can find a counterexample:

$$\exists x \exists y\ (x \in \mathbb{R}\ \wedge\ y \in \mathbb{Z}\ \wedge\ x = 0\ \wedge\ y = 8\ \wedge\ (x + 2y = 16)\ \wedge\ \neg(x > y))$$

Therefore:

$$\boxed{\neg\forall x, y \in D\,((x \in \mathbb{R} \wedge y \in \mathbb{Z}) \rightarrow x > y)}$$

28.5 Example 5

Let the domain:

$$D = \{\langle x, y\rangle \in \mathbb{Z}^2\}$$

Evaluate:

$$\forall\langle x, y\rangle \in D\ (\text{even}(x) \leftrightarrow \text{even}(y))$$

However, we can find a counterexample:

$$\exists x \exists y\ \big((\langle x, y\rangle \in \mathbb{Z}^2)\ \wedge\ x = 4\ \wedge\ y = 3\ \wedge\ \text{even}(x)\ \wedge\ \neg\text{even}(y)\ \wedge\ \neg(\text{even}(x) \leftrightarrow \text{even}(y))\big)$$

Therefore:

$$\boxed{\neg\forall\langle x, y\rangle \in D\ (\text{even}(x) \leftrightarrow \text{even}(y))}$$

Note that other counterexamples include $\langle 2, 5\rangle$, $\langle 7, 8\rangle$, and any pair where one number is even and the other is odd.

29 Natural Deduction for FOL

29.1 Universal Elimination ($\forall$E)

$$\begin{array}{r|ll}
1 & \forall x.P(x) & \\
\hline
2 & P(a) & \forall\text{E, 1} \\
3 & P(b) & \forall\text{E, 1}
\end{array}$$

29.2 Universal Introduction ($\forall$I)

$$\begin{array}{r|ll}
1 & P(a) & \\
\hline
2 & P(a) \vee Q(a) & \forall\text{I, 1} \\
\end{array}$$
$$\begin{array}{rll}
3 & \forall x.(P(x) \vee Q(x)) & \forall\text{I, 1–2}
\end{array}$$

29.3 Existential Introduction (∃I)

1	$P(a)$	
2	$\exists x.P(x)$	∃I, 1
3	$\exists x.(P(x) \land Q(a))$	∃I, 1

29.4 Existential Elimination (∃E)

1	$\exists x.P(x)$	
2	$\forall x.(P(x) \to Q(x))$	
3	a │ $P(a)$	
4	│ $P(a) \to Q(a)$	∀E, 2
5	│ $Q(a)$	⇒E, 3, 4
6	$Q(a)$	∃E, 1, 3–5
7	$\exists x.Q(x)$	∃I, 6

29.5 Identity Introduction (=I)

1	$P(a)$	
2	$Q(a) \to R(a)$	...
3	$S(a)$	...
4	$a = a$	=I

29.6 Identity Elimination (=E)

1	$a = b$	
2	$P(a)$	
3	$P(b)$	=E, 1, 2

30 Boolean, DNF, and CNF

30.1 Boolean

In short, the application of Boolean Algebra is essentially the same as Formal Logic. The main difference lies in the symbols used. For example, in Formal Logic, the symbol $\land$ is used for logical AND, whereas in Boolean Algebra we often use $\cdot$ (dot) for AND. Logical OR, represented by $\lor$ in Formal Logic, corresponds to $+$ in Boolean Algebra.

Negation, $\neg x$ in Formal Logic, is written as $\bar{x}$ in Boolean Algebra. Sometimes, you may also encounter the AND and OR symbols inside circles: $\oplus$ for XOR (exclusive OR) and $\odot$ or $\otimes$ for AND, depending on the text. These are just alternative notations and do not change the underlying logic.

Furthermore, it is important to note that Boolean Algebra is not the same as conventional mathematics. Specifically, Boolean Algebra operates strictly on binary values (1 and 0). General mathematics typically deals with a wider range of numbers, including integers, fractions, and complex numbers.

Property	**Expression**
Binary Operation +	$1 + 0 = 1$
Commutative Property of +	$1 + 0 = 0 + 1$
Associative Property of +	$(1 + 0) + 1 = 1 + (0 + 1)$
Binary Operation $\cdot$	$1 \cdot 1 = 1$
Commutative Property of $\cdot$	$1 \cdot 0 = 0 \cdot 1$
Associative Property of $\cdot$	$(1 \cdot 0) \cdot 1 = 1 \cdot (0 \cdot 1)$
Distributive Properties	$1 \cdot (0 + 1) = (1 \cdot 0) + (1 \cdot 1)$
Special Combinations	$a = a \cdot (a + b)$
Unary Operation $^-$(Negation)	$\bar{1} = 0, \bar{0} = 1$
Identity with +	$a + 0 = a$
Annihilation with $\cdot$	$a \cdot 0 = 0$
Proof using Commutativity of +	$a + b = b + a$
Proof using Commutativity of $\cdot$	$a \cdot b = b \cdot a$
Identity Property of +	$a + 0 = a$
Zero for +	$0 + a = a$
Commutative Property for Zero	$0 + a = a + 0$
Identity with $\cdot$	$a \cdot 1 = a$
Idempotent Law for +	$a + a = a$
Idempotent Law for $\cdot$	$a \cdot a = a$
Annihilation with +	$a + 1 = 1$
Annihilation with $\cdot$	$a \cdot 0 = 0$
Double Negation	$\bar{\bar{a}} = a$
De Morgan's Law for +	$\overline{a \cdot b} = \bar{a} + \bar{b}$
De Morgan's Law for $\cdot$	$\overline{a + b} = \bar{a} \cdot \bar{b}$

Let $k \geq 1$. A k-ary Boolean function f takes as input k True/False values and outputs a True/False value; namely, it is a mapping.

$$f : \{1, 0\}^k \to \{1, 0\}$$

Example:

$$f_{p_1 \cdot p_2}(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \text{ and } x_2 \text{ both equal 1} \\ 0 & \text{otherwise} \end{cases}$$

Let $k \geq 1$ and let A be a propositional formula that uses (at most) the variables $p_1, \ldots, p_k$. The k-ary Boolean function $f_A(x_1, x_2, \ldots, x_k)$ is defined by

$$f_A(x_1, \dots, x_k) = \varphi(A), \text{ where } \varphi(p_i) = x_i \text{ for } i = 1, 2, \dots, k$$

Construction:

$$f(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = 1 \text{ or } x_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

x_1	x_2	$f(x_1, x_2)$
1	1	1
1	0	1
0	1	0
0	0	1

$$(p_1 \cdot p_2) + (p_1 \cdot \bar{p}_2) + (\bar{p}_1 \cdot \bar{p}_2)$$

30.1.1 General Construction

For $1 \leq i \leq 2^k$ and $1 \leq j \leq k$:
Form Disjunctions (Clauses):

$$\psi_i = \bigvee_{j=1}^{k} L_{i,j} := L_{i,1} + L_{i,2} + \dots + L_{i,k}$$

Form Conjunctions (Clauses):

$$\Sigma_i = \bigwedge_{j=1}^{k} L_{i,j} := L_{i,1} \cdot L_{i,2} \cdot \dots \cdot L_{i,k}$$

Example:
Let's take two propositional variables:

$$p_1, p_2$$

Possible truth assignments:

i	p_1	p_2
1	1	1
2	1	0
3	0	1
4	0	0

For each row i, we will form literals $L_{i,1}$ and $L_{i,2}$:

1. If $\varphi_i(p_j) = 1$, then $L_{i,j} = p_j$
2. If $\varphi_i(p_j) = 0$, then $L_{i,j} = \bar{p}_j$

Conjunction:

$$C_i = L_{i,1} \cdot L_{i,2}$$

i	(p_1, p_2)	C_i
1	(1, 1)	$p_1 \cdot p_2$
2	(1, 0)	$p_1 \cdot \bar{p}_2$
3	(0, 1)	$\bar{p}_1 \cdot p_2$
4	(0, 0)	$\bar{p}_1 \cdot \bar{p}_2$

Each C_i corresponds to *one minterm*, True for exactly one assignment.
Disjunction:

$$D_i = L_{i,1} + L_{i,2}$$

i	(p_1, p_2)	D_i
1	(1, 1)	$p_1 + p_2$
2	(1, 0)	$p_1 + \bar{p}_2$
3	(0, 1)	$\bar{p}_1 + p_2$
4	(0, 0)	$\bar{p}_1 + \bar{p}_2$

Each D_i corresponds to *one maxterm*, False for exactly one assignment.

30.2 Disjunctive Normal Form (DNF)

The idea of DNF is that we can reduce all connectives in propositional logic (PL) to only + (Disjunction), . (Conjunction), and $\bar{x}$ (negation) in order to determine the truth value of a sentence.

Example:

$$(s \cdot a) + (\bar{m} \cdot b) + (\bar{h} \cdot \bar{l}) + (r \cdot \bar{i})$$

Moreover, for convenience we use $(\pm L)$ to indicate that L is an atomic sentence which may or may not be prefaced with an occurrence of negation. DNF example:

$$(\pm L_1 \cdot \cdots \cdot \pm L_{i+1}) + (\pm L_j \cdot \cdots \cdot \pm L_{k+1})$$

In detail:
Let L be a sentence containing three atomic sentences a, b, c.

a	b	c	L
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	1

From the truth table, we observe that L is true on lines $1, 3, 7, 8$. For each of these lines, we construct a conjunction that captures the exact truth value assignment on that line:

1. Line 1: $(a \cdot b \cdot c)$
2. Line 3: $(a \cdot \bar{b} \cdot c)$
3. Line 7: $(\bar{a} \cdot \bar{b} \cdot c)$
4. Line 8: $(\bar{a} \cdot \bar{b} \cdot \bar{c})$

Therefore, a DNF representation logically equivalent to L is:

$$(a \cdot b \cdot c) + (a \cdot \bar{b} \cdot c) + (\bar{a} \cdot \bar{b} \cdot c) + (\bar{a} \cdot \bar{b} \cdot \bar{c})$$

This gives us a sentence in DNF which is true on exactly those lines where one of the disjuncts is true, namely, lines 1, 3, 7, and 8. Hence, the DNF sentence has exactly the same truth table as L. In other words, we have produced a sentence in DNF that is logically equivalent to L, which is precisely the goal of normalization.

30.2.1 General Construction

Let L be a sentence containing atomic sentences $x_1, \ldots, x_n$. We aim to construct a new sentence, G, that is *logically equivalent* to L but written in DNF. Intuitively, G will serve as a *truth-functional reconstruction* of L: it will reproduce the truth value of L on every possible assignment of truth values to $x_1, \ldots, x_n$. That is, for each possible line of the truth table for L, G will be true if and only if L is true on that line. Hence, G acts as the DNF counterpart of L, the formula that expresses L's truth behavior in canonical disjunctive form.

Line	x_1	x_2	$\ldots$	x_n	L
1	1	1	$\ldots$	1	L_1
2	1	1	$\ldots$	0	L_2
3	1	0	$\ldots$	1	L_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^n	0	0	$\ldots$	0	L_{2^n}

Case 1:

If L is false on every line of its truth table, then L is a contradiction. In that case, a contradiction of the form is in DNF and is logically equivalent to L.

Case 2:

If L is true on at least one line of its truth table, we proceed as follows. For each line i where $L = 1$, let ψ_i be a conjunction of the form:

$$\psi_i = \bigwedge_{m=1}^{n} (\pm x_m)$$

where

$$(\pm x_m) = \begin{cases} x_m, & \text{if } x_m = 1 \text{ on line } i, \\ \bar{x}_m, & \text{if } x_m = 0 \text{ on line } i. \end{cases}$$

That is, x_m appears unnegated in ψ_i if it is true on line i, and negated if it is false on line i. Thus, ψ_i is true on, and only on, line i of the truth table.

Now, let $i_1, i_2, \ldots, i_j$ be the indices of all lines where $L = 1$. Then we define G as the disjunction of all such ψ_i:

$$G = \psi_{i_1} + \psi_{i_2} + \cdots + \psi_{i_j} = \bigvee_{k=1}^{j} \left[\bigwedge_{m=1}^{n} (\pm x_m) \right]$$

Since G is a disjunction of conjunctions of literals, G is in DNF. By construction, L and G are true on exactly the same lines of the truth table, and therefore

$$L \equiv G$$

Example:
Let S contain three atomic sentences x_1, x_2, x_3.

x_1	x_2	x_3	S
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	1

$S = 1$ on lines 1, 3, 7, and 8.

1. For line 1: $f_1 = (x_1 \cdot x_2 \cdot x_3)$
2. For line 3: $f_3 = (x_1 \cdot \bar{x}_2 \cdot x_3)$
3. For line 7: $f_7 = (\bar{x}_1 \cdot \bar{x}_2 \cdot x_3)$
4. For line 8: $f_8 = (\bar{x}_1 \cdot \bar{x}_2 \cdot \bar{x}_3)$

Form the disjunction:

$$B = f_1 + f_3 + f_7 + f_8$$

Hence,

$$\left[(x_1 \cdot x_2 \cdot x_3) + (x_1 \cdot \bar{x}_2 \cdot x_3) + (\bar{x}_1 \cdot \bar{x}_2 \cdot x_3) + (\bar{x}_1 \cdot \bar{x}_2 \cdot \bar{x}_3)\right]$$

We conclude that,

$$S \equiv B$$

30.3 Conjunctive Normal Form (CNF)

Shortly, the formal definition of CNF is also analogous to the definition of DNF.
Example:

$$(s + a) \cdot (\bar{m} + b) \cdot (\bar{h} + \bar{l}) \cdot (r + \bar{i})$$

and

$$(\pm T_1 + \cdots + \pm P_m) \cdot (\pm T_i + \cdots + \pm T_k)$$

In detail:
Consider a sentence T with atomic sentences p, q, r.

p	q	r	B
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	1

The truth table shows T is false on lines $2, 4, 6$. We build a clause for each false row by taking the disjunction of literals that would make that row true:

1. Line 2: $(\bar{p} + \bar{q} + r)$
2. Line 4: $(\bar{p} + q + r)$
3. Line 6: $(p + \bar{q} + r)$

A CNF formula logically equivalent to T:

$$(\bar{p} + \bar{q} + r) \cdot (\bar{p} + q + r) \cdot (p + \bar{q} + r)$$

Each clause eliminates exactly one false row. The conjunction ensures all three clauses hold simultaneously, making the formula false precisely where M is false and true elsewhere.

30.3.1 General Construction

Given sentence L with atomic sentences $y_1, \ldots, y_n$, we construct an equivalent sentence H in CNF. The strategy: identify where L fails and build clauses that fail at exactly those points.

Line	y_1	y_2	$\ldots$	y_n	L
1	1	1	$\ldots$	1	L_1
2	1	1	$\ldots$	0	L_2
3	1	0	$\ldots$	1	L_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^n	0	0	$\ldots$	0	L_{2^n}

Case 1:
When L is true everywhere, it is a tautology. Any tautological formula in CNF serves as an equivalent, such as $(p + \bar{p})$ or any logically equivalent tautology.
Case 2:
When L is false on some lines, for each line j where $L = 0$, construct clause ϕ_j:

$$\phi_j = \bigvee_{t=1}^{n} (\pm y_t)$$

where

$$(\pm y_t) = \begin{cases} \bar{y}_t, & \text{if } y_t = 1 \text{ on line } j, \\ y_t, & \text{if } y_t = 0 \text{ on line } j. \end{cases}$$

Each literal is the opposite of its truth value on line j, making ϕ_j false only on line j. Let $j_1, j_2, \ldots, j_s$ index all lines where $L = 0$. Define:

$$H = \phi_{j_1} \cdot \phi_{j_2} \cdot \cdots \cdot \phi_{j_s} = \bigwedge_{w=1}^{s} \left[\bigvee_{t=1}^{n} (\pm y_t) \right]$$

H is in CNF. Since H is false exactly where L is false:

$$L \equiv H$$

Example:
Let R have atomic sentences y_1, y_2, y_3.

y_1	y_2	y_3	R
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

$R = 0$ on lines 2, 5, and 7.

1. For line 2: $g_2 = (\bar{y}_1 + \bar{y}_2 + y_3)$
2. For line 5: $g_5 = (y_1 + \bar{y}_2 + \bar{y}_3)$
3. For line 7: $g_7 = (y_1 + y_2 + \bar{y}_3)$

Form the conjunction:

$$C = g_2 \cdot g_5 \cdot g_7$$

Hence,

$$\left[(\bar{y}_1 + \bar{y}_2 + y_3) \cdot (y_1 + \bar{y}_2 + \bar{y}_3) \cdot (y_1 + y_2 + \bar{y}_3) \right]$$

We conclude that,

$$R \equiv C$$

Example 1:
Let

$$\Sigma = a \to \bar{b}$$

First, we rewrite using the definition of implication:

$$\Sigma = (a \to \bar{b} \equiv \bar{a} + \bar{b})$$

Truth Table:

a	b	$\bar{a}$	$\bar{b}$	Σ
1	1	0	0	0
1	0	0	1	1
0	1	1	0	1
0	0	1	1	1

Hence,

1. DNF Construction:

   (a) Line 2: $(a \cdot \bar{b})$
   (b) Line 3: $(\bar{a} \cdot b)$
   (c) Line 4: $(\bar{a} \cdot \bar{b})$

   Therefore,

$$\Sigma = (a \cdot \bar{b}) + (\bar{a} \cdot b) + (\bar{a} \cdot \bar{b})$$

2. CNF Construction:

   (a) Line 1: $(\bar{a} + \bar{b})$

   Thus,

$$\Sigma = (\bar{a} + \bar{b})$$

3. Both forms are logically equivalent:

$$\Sigma \equiv \bar{a} + \bar{b}$$

Example 2:
Let

$$\gamma = \neg(p \equiv q)$$

First, we rewrite using the definition of biconditional:

$$p \leftrightarrow q \equiv (p \cdot q) + (\bar{p} \cdot \bar{q})$$

Then, applying negation:

$$\neg(p \leftrightarrow q) \equiv \neg[(p \cdot q) + (\bar{p} \cdot \bar{q})]$$

Using De Morgan's law:

$$\gamma = (\bar{p} + \bar{q}) \cdot (p + q)$$

Truth Table:

p	q	$\bar{p}$	$\bar{q}$	$p \equiv q$	γ
1	1	0	0	1	0
1	0	0	1	0	1
0	1	1	0	0	1
0	0	1	1	1	0

1. DNF Construction

   $\gamma = 1$ on lines 2 and 3.

   (a) Line 2: $(p \cdot \bar{q})$
   (b) Line 3: $(\bar{p} \cdot q)$

   Therefore,

$$\gamma = (p \cdot \bar{q}) + (\bar{p} \cdot q)$$

2. CNF Construction

   $\gamma = 0$ on lines 1 and 4.

   (a) Line 1: $(\bar{p} + \bar{q})$
   (b) Line 4: $(p + q)$

   Thus,

$$\gamma = (\bar{p} + \bar{q}) \cdot (p + q)$$

3. Both forms are logically equivalent

$$\gamma \equiv (p \cdot \bar{q}) + (\bar{p} \cdot q) \equiv (\bar{p} + \bar{q}) \cdot (p + q)$$

Example 3:
Let

$$\delta = \neg x \vee \neg(y \wedge z)$$

First, we apply De Morgan's law to $\neg(y \wedge z)$:

$$\neg(y \wedge z) \equiv \bar{y} + \bar{z}$$

Then, the expression becomes:

$$\delta = \bar{x} + \bar{y} + \bar{z}$$

This is already in sum-of-products form (disjunction of literals).
Truth Table:

x	y	z	$\bar{x}$	$y \wedge z$	$\neg(y \wedge z)$	δ
1	1	1	0	1	0	0
1	1	0	0	0	1	1
1	0	1	0	0	1	1
1	0	0	0	0	1	1
0	1	1	1	1	0	1
0	1	0	1	0	1	1
0	0	1	1	0	1	1
0	0	0	1	0	1	1

1. DNF Construction

   $\delta = 1$ on lines 2, 3, 4, 5, 6, 7, and 8.

   (a) Line 2: $(x \cdot y \cdot \bar{z})$
   (b) Line 3: $(x \cdot \bar{y} \cdot z)$
   (c) Line 4: $(x \cdot \bar{y} \cdot \bar{z})$
   (d) Line 5: $(\bar{x} \cdot y \cdot z)$
   (e) Line 6: $(\bar{x} \cdot y \cdot \bar{z})$
   (f) Line 7: $(\bar{x} \cdot \bar{y} \cdot z)$
   (g) Line 8: $(\bar{x} \cdot \bar{y} \cdot \bar{z})$

   Therefore,

   $$\delta = (x \cdot y \cdot \bar{z}) + (x \cdot \bar{y} \cdot z) + (x \cdot \bar{y} \cdot \bar{z}) + (\bar{x} \cdot y \cdot z) + (\bar{x} \cdot y \cdot \bar{z}) + (\bar{x} \cdot \bar{y} \cdot z) + (\bar{x} \cdot \bar{y} \cdot \bar{z})$$

   Simplified form:

   $$\delta = \bar{x} + \bar{y} + \bar{z}$$

2. CNF Construction

   $\delta = 0$ on line 1 only.

   (a) Line 1: $(\bar{x} + \bar{y} + \bar{z})$

   Thus,

   $$\delta = \bar{x} + \bar{y} + \bar{z}$$

3. Both forms are logically equivalent

   $$\delta \equiv \bar{x} + \bar{y} + \bar{z} \equiv \neg x \vee \neg(y \wedge z)$$

Example 4:
Let

$$\psi = \neg(a \to b) \land (c \to d)$$

First, we rewrite using the definition of implication:

$$a \to b \equiv \bar{a} + b$$
$$c \to d \equiv \bar{c} + d$$

Then, applying negation to the first part:

$$\neg(a \to b) \equiv \neg(\bar{a} + b)$$

Using De Morgan's law:

$$\neg(\bar{a} + b) \equiv a \cdot \bar{b}$$

Therefore:

$$\psi = (a \cdot \bar{b}) \cdot (\bar{c} + d)$$

Expanding using distributive law:

$$\psi = (a \cdot \bar{b} \cdot \bar{c}) + (a \cdot \bar{b} \cdot d)$$

Truth Table:

a	b	c	d	$a \to b$	$\neg(a \to b)$	$c \to d$	ψ
1	1	1	1	1	0	1	0
1	1	1	0	1	0	0	0
1	1	0	1	1	0	1	0
1	1	0	0	1	0	1	0
1	0	1	1	0	1	1	1
1	0	1	0	0	1	0	0
1	0	0	1	0	1	1	1
1	0	0	0	0	1	1	1
0	1	1	1	1	0	1	0
0	1	1	0	1	0	0	0
0	1	0	1	1	0	1	0
0	1	0	0	1	0	1	0
0	0	1	1	1	0	1	0
0	0	1	0	1	0	0	0
0	0	0	1	1	0	1	0
0	0	0	0	1	0	1	0

1. DNF Construction

   $\psi = 1$ on lines 5, 7, and 8.

   (a) Line 5: $(a \cdot \bar{b} \cdot c \cdot d)$
   (b) Line 7: $(a \cdot \bar{b} \cdot \bar{c} \cdot d)$

(c) Line 8: $(a \cdot \bar{b} \cdot \bar{c} \cdot \bar{d})$

Therefore,

$$\psi = (a \cdot \bar{b} \cdot c \cdot d) + (a \cdot \bar{b} \cdot \bar{c} \cdot d) + (a \cdot \bar{b} \cdot \bar{c} \cdot \bar{d})$$

Simplified form:

$$\psi = (a \cdot \bar{b} \cdot \bar{c}) + (a \cdot \bar{b} \cdot d)$$

2. CNF Construction

   $\psi = 0$ on lines 1, 2, 3, 4, 6, 9, 10, 11, 12, 13, 14, 15, and 16.

   (a) Line 1: $(\bar{a} + b + \bar{c} + \bar{d})$
   (b) Line 2: $(\bar{a} + b + \bar{c} + d)$
   (c) Line 3: $(\bar{a} + b + c + \bar{d})$
   (d) Line 4: $(\bar{a} + b + c + d)$
   (e) Line 6: $(\bar{a} + b + \bar{c} + d)$
   (f) Line 9: $(a + \bar{b} + \bar{c} + \bar{d})$
   (g) Line 10: $(a + \bar{b} + \bar{c} + d)$
   (h) Line 11: $(a + \bar{b} + c + \bar{d})$
   (i) Line 12: $(a + \bar{b} + c + d)$
   (j) Line 13: $(a + b + \bar{c} + \bar{d})$
   (k) Line 14: $(a + b + \bar{c} + d)$
   (l) Line 15: $(a + b + c + \bar{d})$
   (m) Line 16: $(a + b + c + d)$

   Thus (product of all clauses),

   Simplified form:

$$\psi = (a) \cdot (\bar{b}) \cdot (\bar{c} + d)$$

3. Both forms are logically equivalent

$$\psi \equiv (a \cdot \bar{b} \cdot \bar{c}) + (a \cdot \bar{b} \cdot d) \equiv (a) \cdot (\bar{b}) \cdot (\bar{c} + d)$$

Example 5:
Let

$$\varepsilon = \neg(a \vee b) \leftrightarrow ((\neg c \wedge \neg a) \rightarrow \neg b)$$

First, we rewrite using the definition of implication and biconditional:

$$\neg(a \vee b) \equiv \bar{a} \cdot \bar{b}$$

$$(\neg c \wedge \neg a) \rightarrow \neg b \equiv \neg(\bar{c} \cdot \bar{a}) + \bar{b} \equiv (c + a) + \bar{b} \equiv c + a + \bar{b}$$

For the biconditional:

$$p \leftrightarrow q \equiv (p \cdot q) + (\bar{p} \cdot \bar{q})$$

Let $p = \bar{a} \cdot \bar{b}$ and $q = c + a + \bar{b}$
Then:

$$\varepsilon = [(\bar{a} \cdot \bar{b}) \cdot (c + a + \bar{b})] + [\overline{(\bar{a} \cdot \bar{b})} \cdot \overline{(c + a + \bar{b})}]$$

Simplifying:

$$\varepsilon = [(\bar{a} \cdot \bar{b}) \cdot (c + a + \bar{b})] + [(a + b) \cdot (\bar{c} \cdot \bar{a} \cdot b)]$$

$$\varepsilon = (\bar{a} \cdot \bar{b} \cdot c) + (\bar{a} \cdot \bar{b} \cdot a) + (\bar{a} \cdot \bar{b} \cdot \bar{b}) + (\bar{a} \cdot \bar{c} \cdot b) + (\bar{c} \cdot b \cdot b)$$

$$\varepsilon = (\bar{a} \cdot \bar{b} \cdot c) + (\bar{a} \cdot \bar{c} \cdot b) + (\bar{c} \cdot b)$$

Truth Table:

a	b	c	$\neg(a \vee b)$	$(\neg c \wedge \neg a) \rightarrow \neg b$	ε
1	1	1	0	1	0
1	1	0	0	1	0
1	0	1	0	1	0
1	0	0	0	1	0
0	1	1	0	1	0
0	1	0	0	0	1
0	0	1	1	1	1
0	0	0	1	1	1

1. DNF Construction

   $\varepsilon = 1$ on lines 6, 7, and 8.

   (a) Line 6: $(\bar{a} \cdot b \cdot \bar{c})$
   (b) Line 7: $(\bar{a} \cdot \bar{b} \cdot c)$
   (c) Line 8: $(\bar{a} \cdot \bar{b} \cdot \bar{c})$

   Therefore,

   $$\varepsilon = (\bar{a} \cdot b \cdot \bar{c}) + (\bar{a} \cdot \bar{b} \cdot c) + (\bar{a} \cdot \bar{b} \cdot \bar{c})$$

   Simplified form:

   $$\varepsilon = (\bar{a} \cdot \bar{c} \cdot b) + (\bar{a} \cdot \bar{b})$$

2. CNF Construction

   $\varepsilon = 0$ on lines 1, 2, 3, 4, and 5.

(a) Line 1: $(\bar{a} + \bar{b} + \bar{c})$

(b) Line 2: $(\bar{a} + \bar{b} + c)$

(c) Line 3: $(\bar{a} + b + \bar{c})$

(d) Line 4: $(\bar{a} + b + c)$

(e) Line 5: $(a + \bar{b} + \bar{c})$

Thus,

$$\varepsilon = (\bar{a} + \bar{b} + \bar{c}) \cdot (\bar{a} + \bar{b} + c) \cdot (\bar{a} + b + \bar{c}) \cdot (\bar{a} + b + c) \cdot (a + \bar{b} + \bar{c})$$

Simplified form:

$$\varepsilon = (\bar{a}) \cdot (\bar{b} + \bar{c})$$

3. Both forms are logically equivalent

$$\varepsilon \equiv (\bar{a} \cdot \bar{c} \cdot b) + (\bar{a} \cdot \bar{b}) \equiv (\bar{a}) \cdot (\bar{b} + \bar{c})$$

31 Modal Logic

In short, Modal logic is a logical system that extends classical logic by incorporating modalities such as possibility and necessity. Modal logic operators are $\Box$ and $\Diamond$, where $\Box$ may be read as "necessarily" and $\Diamond$ as "possibly." So, $\Box p$ means "p is necessarily true" and $\Diamond p$ means "p is possibly true." In modal logic, the core consideration is context, meaning that a statement may be true or valid in a particular context (or possible world) but not necessarily true or valid in all contexts. In other words, a statement might hold in some scenarios without being universally valid. For example, consider the following modus ponens reasoning in clasical logic:

Let:

P: It is raining

Q: The ground is wet

According to the rule of modus ponens, if P is true, then Q must also be true. However, intuitively and based on empirical experience, we know that even if it is raining (P), the ground might not be wet (Q), for instance, if the ground is covered by a tent. In such a context, the conclusion Q does not hold, even though the premise P is true. In cases like this, modal logic becomes relevant because it accommodates statements that are evaluated not only in terms of absolute truth, but also in terms of modality, such as: Necessarily ($\Box$): something that must be true in all possible worlds ("$\Box Q$" means Q is true in every possible context). Possibly ($\Diamond$): something that may be true in at least one possible world ("$\Diamond Q$" means Q is true in at least one possible context).

The idea of modal logic was first discussed by Aristotle in *De Interpretatione*. Aristotle noticed that: Necessity implies possibility, but not vice versa:

$$\Box P \rightarrow \Diamond P$$

but not necessarily

$$\Diamond P \to \Box P$$

Possibility and necessity are interdefinable.
If $A \wedge B$ is possibly true, then both A and B are possibly true, but not conversely:

$$\Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$$

but not necessarily

$$(\Diamond A \wedge \Diamond B) \to \Diamond(A \wedge B)$$

If $A \to B$ is necessary, then if A is necessary, so is B:

$$\Box(A \to B) \to (\Box A \to \Box B)$$

In modern times, particularly in the twentieth century, the field of modal logic advanced significantly through the work of Clarence Irving Lewis. He began investigating the foundations of material implication and its limitations. Lewis and Langford argued that the truth-functional connective $A \to B$ is a poor substitute for the philosophical statement "*A implies B*." Instead, they proposed defining implication in modal terms: "Necessarily, if A then B." This can be symbolized as:

$$A \to B$$

Following this, Lewis introduced five axiom systems for modal logic, known as *S1-S5*.

1. S1: The First Modal System

   $S1$ includes all tautologies in classical logic, statements that are always true regardless of the truth value of their variables. Examples of tautologies:

   (a) $P \to P$
   (b) $P \vee \neg P$ (Law of the Excluded Middle)
   (c) $\neg\neg P \leftrightarrow P$ (Double Negation Elimination)
   (d) $\neg(P \wedge \neg P)$ (Law of Non-Contradiction)
   (e) $(P \wedge Q) \to P$ (Conjunction Elimination)
   (f) $\neg(P \vee Q) \leftrightarrow (\neg P \wedge \neg Q)$ (De Morgan's Law)
   (g) $(P \vee (Q \vee R)) \leftrightarrow ((P \vee Q) \vee R)$ (Disjunction Associativity)

   Although $S1$ is not yet strong in adequately expressing modal implication, it remains an important starting point in the exploration of modal logic.

2. The Second Modal System

   $S2$ extends $S1$ by adding formulas that govern the interaction between the modal operators $\Box$ and $\Diamond$. Common forms in S2 include:

   (a) $\Diamond(P \wedge Q) \to \Diamond P$
   (b) $\Box A \to A$
   (c) $\Box(A \to B) \to \Box(\neg\Diamond A \vee B)$

This system begins to introduce the idea that something necessary ($\Box A$) must be true, and it strengthens the relationship between modality and implication.

3. S3: The Third Modal System

   $S3$ gives greater control over the relationship between modality and implication. Representative formulas in $S3$ include:

   (a) $(P \to Q) \vDash (\neg\Diamond Q \to \neg\Diamond P)$
   (b) $(\Box A \land \Box(A \to B)) \to \Box B$
   (c) $\Box(A \to B) \to \Box(\neg\Diamond A \lor B)$

   This system enhances the internal structure of modal operators, although it is still developed within an axiomatic framework.

4. S4: The Fourth Modal System

   $S4$ strengthens $S3$ by adding the principle that something necessary remains necessary in all subsequent contexts. $S4$ is often associated with the idea that necessity is "stable" or persists throughout chains of reasoning. For example:

   (a) $\Box A \to A$
   (b) $\Box A \to \Box\Box A$

5. S5: The Fifth Modal System

   $S5$ is the most powerful system in the axiomatic tradition of Lewis. Here, necessity and possibility are fully connected. For example:

   (a) $\Diamond A \to \Box\Diamond A$

   S5 models modality in the most ideal sense, where any possibility is considered universally applicable across the system. In modern day, only S4 and S5 are still widely used in modal logic systems. Why are $S1$–$S3$ rarely used? Simply put, $S1$–$S3$ are systems that are only syntactically valid, but cannot accommodate strong semantics within a realistic possible-worlds framework. This limitation became evident as modal logic developed and gave rise to other branches of logic, such as temporal logic, epistemic logic, deontic logic, and various dynamic logic systems. These systems require frameworks that are not only syntactically valid but also flexible and sound.

31.1 The Role of Semantics

In relation to semantic issues, Rudolf Carnap also made an important contribution in clarifying how a modal proposition should be evaluated. He introduced the concept of a "state description", a complete representation of a world's state via a list of propositions that are true in it. Through this approach, even though a proposition like $\Box A$ states that A is necessary, in practice, the truth value of A still depends on the context of a particular possible world. That is, $\Box A$ can be true in one world but not in another that has a different structure or information. In other words, $\Box A$ does not always reflect universal truth but rather contextual truth, valid in one state description, but not necessarily valid across all possible worlds. Carnap's ideas eventually influenced the development of modal logic by Saul Kripke.

31.2 Kripke Semantics and Accessibility Relations

Building upon Carnap's early work, Saul Kripke developed the semantic approach that is now the standard in modal logic, using accessibility relations between possible worlds. The relation between worlds is denoted by $w\,R\,w'$, meaning "world w' is accessible from world w." This relation is crucial in determining how modal operators like $\Box$ and $\Diamond$ are evaluated. For example, $\Box A$ is true in world w if and only if A is true in all worlds w' that are accessible from w.

The original development of $S1$ through $S5$ by C. I. Lewis was purely syntactic, relying on the addition and combination of axioms. It was only decades later that modal logic was further advanced by Saul Kripke, who introduced accessibility relations between possible worlds. This led to the mapping of axioms such as K, T, B, 4, and 5 onto semantic properties like *reflexivity*, *symmetry*, and *transitivity*.

1. Axiom K (Distribution + Necessitation): $\Box(A \to B) \to (\Box A \to \Box B)$
2. Axiom T / M (*Reflexive*): Axiom K + $\Box A \to A$
3. Axiom B (*Reflexive* + *Symmetric*): Axiom T + Brouwer's Axiom: $A \to \Box\Diamond A$
4. Axiom 4 (*Transitive*): $\Box A \to \Box\Box A$
5. Axiom 5 (Euclidean; *Reflexive* + *Transitive* + *Symmetric*): $\Diamond A \to \Box\Diamond A$

31.3 The Significance of Possibility and Necessity

Why are possibility and necessity important? A compelling answer can be found in Vern Sheridan Poythress book, *Logic: A God-Centered Approach to the Foundation of Western Thought*.

What are the extra symbols $\Diamond$ and $\Diamond Q$ intended to represent? They represent possibility and necessity, we say. But what kind of possibility and necessity? We must be careful, because the words possible and necessary in English have a range of usages. As usual, natural language is flexible. A logical formalism will capture only one dimension of this richness.

We can talk of something being "necessary" in the context of moral obligation. "It is necessary for me to carry out the terms of the contract that I signed." Moral necessity has been studied by philosophers, and the term "deontic logic" has been used. Deontic logic has been treated as one kind of modal logic. But in this case the operation denoting necessity is customarily written with the symbol O ("It is obligatory that . . .") rather than the symbol $\Diamond Q$. We can also talk about possible or impossible events in the context of discussing the limitations of our knowledge. We might say, for example, that "It is possible that Don is guilty of the crime of which he is accused." In most situations of this type, the crime has already taken place. Either in fact Don is guilty or he is not (though we are ignoring the possibility that he is an accessory to some other person who directly did the deed). But we are in a situation where we do not know whether Don is guilty. This kind of possibility has been called epistemic possibility, that is, a kind of possibility having to do with capabilities in knowledge. It asks what follows from what we know or do not know. The operation denoting epistemic necessity is sometimes denoted K (for "known").

Epistemic possibility may also crop up in situations of future predictions. We might say, "It is possible that it will rain tomorrow." Or "It is not possible that it might rain tomorrow" (because we have received what we consider to be a definitive weather report on tonight's news). This situation might be regarded as essentially the same as the situation involving Don's guilt. But some philosophers have considered that future events are innately indeterminate rather than simply unknown to us. This kind of situation gives rise to what are known as tense logics (since truth value can vary with the tense of the verb, which indicates whether the proposition in question has to do with the past, the present, or the future). As usual, ordinary conversation contains fuzzy boundaries. Just how unlikely or absurd does something have to be before we will say, at least loosely speaking, that it is "impossible"? What we say depends, as usual, on context, which enables hearers to discern whether we are speaking precisely, and whether we are including very unlikely "possibilities."

More often philosophers are concerned with what has been called logical or alethic possibility and necessity. They are not asking whether it is possible for it to rain tomorrow, given the weather report, but whether it is possible in general. Are unicorns possible? Maybe not within this world. But are they possible in some world? Such very general possibility is to be distinguished both from moral possibility and from the kind of "possibility" deriving from limited knowledge.

Ultimately, because modal logic takes into account context or possible worlds, rather than relying solely on formal rules, it serves as the foundation for various other branches of logic. Moreover, in this course, for the most part, we will follow Saul Kripke's innovation in modal logic.

32 Modal Logic Language

1. Propositional variables: $p, q, r, x, y, z \ldots$ also infinite set of propositional variables $x_1, \ldots x_n$
2. Connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
3. Modal operators: $\Box$ and $\Diamond$
4. Falsity: $\bot$
5. Truth: $\top$

Note: We use $\Vdash$ for satisfaction (truth in a model or at a world), and $\models$ for validity (truth in all models)

32.1 Construction of Formulas

Let U be a set of propositional variables. We define by induction the set of formulas based on P as the smallest set $L(U)$ satisfying the following conditions:

1. Atomic propositions. Every propositional variable belongs to the set of formulas:

$$x \in L(U), \quad \text{for every } x \in U$$

2. Falsity.

   The constant falsity is a formula:

$$\bot \in L(U)$$

3. Negation.

   If $\phi \in L(U)$, then its negation is also a formula:

$$\neg\phi \in L(U)$$

4. Implication.

   If $\phi, \psi \in L(U)$, then the conditional formula is also in the set:

$$(\phi \to \psi) \in L(U)$$

5. Modal operators.

   If $\phi \in L(U)$, then its necessity is a formula:

$$\Box\phi \in L(U)$$

6. Modal-free formulas.

   If a formula A does not contain $\Box$ and $\Diamond$, we say it is modal-free.

Furthermore, in classical logic, if x is a well-formed formula (wff), then so is $\neg x$. This follows directly from the syntax and semantics of the system: if x represents a true proposition, then $\neg x$ is its negation, and the truth of each can be verified with a truth table. Such relationships capture the semantics of propositions in the classical setting.

Modal logic, however, adds an extra layer of complexity. Here, we cannot negate propositions as straightforwardly as in classical logic because modal operators introduce the notions of necessity and possibility. For example, if x is necessarily true, written $\Box x$, then negating it leads not simply to $\neg\Box x$, but to an expression equivalent to $\Diamond\neg x$. Likewise, the negation of $\Diamond x$ is $\Box\neg x$. These shifts occur because modal semantics depend on relationships between possible worlds, not just truth assignments in a single world.

This complexity reflects deeper epistemological limitations, we may believe a proposition to be true, only to later discover it false, and vice versa. Constructing a proposition that perfectly matches reality is therefore not merely a logical challenge, but also a linguistic and epistemological one. Despite this, modal logic provides a way to represent and reason about such uncertainty. It does so by evaluating propositions relative to a set of possible worlds, each representing a different way reality might be.

33 Relational Models

In order to evaluate the *semantics* of every proposition in modal logic, we need to define a structure called a *relational model*. This model provides a framework in which the truth of formulas can be interpreted relative to different "possible worlds." Yet, Before we define relational formulas, we must understand how to assign each propositional variable to a set of worlds where it is considered true.

A relational model (or Kripke model) for the basic modal language is a triple:

$$M = \langle W, R, V \rangle$$

where:

1. W is a non-empty set of possible worlds. Each element of W represents a distinct "state of affairs" or scenario.

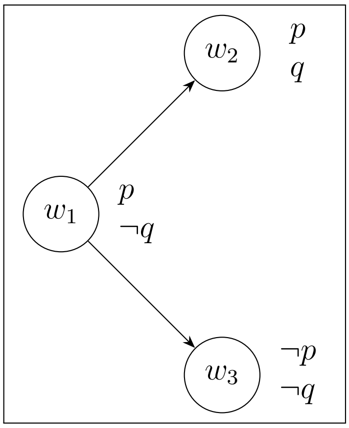
2. $R \subseteq W \times W$ is a binary accessibility relation. If $(w, w') \in R$, we say that the world w' is accessible from the world w. The accessibility relation encodes the modal structure of necessity and possibility. Sometimes, this relation is also written as wRw'.

3. V is a valuation function that assigns to each propositional variable $p \in U$ a subset of worlds in which p is true:

$$V : U \to \mathcal{P}(W)$$

   where $\mathcal{P}(W)$ denotes the power set of W. That is, for each $p \in U$, $V(p) \subseteq W$.

Consider the following simple model:



In this diagram, each circle represents a world, labeled w_1, w_2, w_3, and the propositions that are true in each world are listed next to them. The arrows indicate the accessibility relation R between worlds. Formally, when we write wRw', we mean that world w' is accessible from world w.

33.1 Truth at a world

Every modal model specifies which formulas are true at which worlds.

Let

$$M = \langle W, R, V \rangle$$

be a Kripke model over propositional variables U. The satisfaction relation

$$M, w \Vdash \varphi$$

means "φ is true at world w in M." It is defined inductively:

1. Atomic propositions:

$$M, w \Vdash p \leftrightarrow w \in V(p), \quad p \in U$$

2. Falsity:

$$M, w \nVdash \bot$$

3. Negation:

$$M, w \Vdash \neg\varphi \leftrightarrow M, w \nVdash \varphi$$

4. Conjunction:

$$M, w \Vdash \varphi \wedge \psi \leftrightarrow M, w \Vdash \varphi \wedge M, w \Vdash \psi$$

5. Disjunction:

$$M, w \Vdash \varphi \vee \psi \leftrightarrow M, w \Vdash \varphi \vee M, w \Vdash \psi$$

6. Implication:

$$M, w \Vdash \varphi \to \psi \leftrightarrow M, w \nVdash \varphi \vee M, w \Vdash \psi$$

7. Necessity:

$$M, w \Vdash \Box\varphi \leftrightarrow \forall v \in W, (w, w') \in R \to M, w' \Vdash \varphi$$

8. Possibility:

$$M, w \Vdash \Diamond\varphi \leftrightarrow \exists w' \in W, (w, w') \in R \wedge M, w' \Vdash \varphi$$

33.2 Truth in a Model

While

$$M, w \Vdash \varphi$$

represents truth at a specific world w (local truth), we sometimes want global truth. Formulas that are true at all worlds in a given model.

$$M \Vdash \varphi \;\leftrightarrow\; \forall w \in W,\; M, w \Vdash \varphi$$

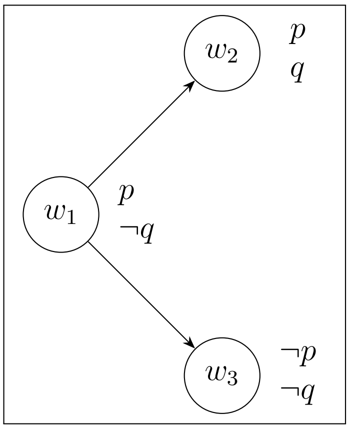
That is, φ holds at every world in M.

1. If $M \Vdash \varphi$, then φ is globally valid in that model.

2. If $M, w \Vdash \varphi$ for some but not all w, then φ is only locally true in M.

Example 1:

Recall the simple model, then we can check which of p, $\Box p$, $\Diamond p$, $\Diamond q$ hold and which do not:



1. For p:

$$M, w_1 \Vdash p \leftrightarrow w_1 \in V(p)$$

From the model,

$$w_1 \in V(p)$$

Therefore,

$$M, w_1 \Vdash p$$

2. For $\Box p$:

By definition of $\Box$:

$$M, w_1 \Vdash \Box p \leftrightarrow \forall w' \,(w_1 R w' \to M, w' \Vdash p)$$

However:

$$\exists w_3 \,(\, w_1 R w_3 \wedge M, w_3 \nVdash p\,)$$

Consequently, the universal condition fails. Thus,

$$M, w_1 \nVdash \Box p$$

3. For $\Diamond p$:

By definition of $\Diamond$:

$$M, w_1 \Vdash \Diamond p \leftrightarrow \exists w' \,(w_1 R w' \wedge M, w' \Vdash p)$$

From the model,

$$\exists w_2(w_1 R w_2 \land M, w_2 \Vdash p)$$

So the existential condition is satisfied. It follows that,

$$M, w_1 \Vdash \Diamond p$$

4. For $\Diamond q$:

   similarly, by definition of $\Diamond$:

$$M, w_1 \Vdash \Diamond q \leftrightarrow \exists w' \, (w_1 R w' \land M, w' \Vdash q)$$

   From the model,

$$w_1 R w_2 \land M, w_2 \Vdash q$$

   Thus, the existential condition is satisfied. We conclude that:

$$M, w_1 \Vdash \Diamond q$$

34 Validity and Tautology

34.1 Validity

Validity in modal logic is always a property of formulas. A formula is called valid if it holds in every model and at every world of that model. However, which formulas count as valid depends on the semantics, in particular, on the accessibility relation R that structures the models.

Example 1:

$$\models \Box(p \land q) \to \Box p$$

is valid in all Kripke models, since if every accessible world satisfies both p and q, then certainly every accessible world satisfies p.

Proof

Let

$$M = (W, R, V)$$

be an arbitrary Kripke model and let

$$w \in W$$

be an arbitrary world. We need to show:

$$M, w \Vdash \Box(p \land q) \to \Box p$$

First, assume the antecedent

$$M, w \Vdash \Box(p \land q)$$

By the semantics of $\Box$, this means:

$$\forall w' \in W, \text{ if } wRw' \text{ then } M, w' \Vdash p \wedge q$$

Second, for each world w' such that wRw':

$$M, w' \Vdash p \wedge q \to M, w' \Vdash p$$

moreover, generalize over accessible worlds.
Since the above holds for all w' accessible from w, we have:

$$\forall w' \in W, \text{ if } wRw' \text{ then } M, w' \Vdash p$$

By the semantics of $\Box$, this is equivalent to:

$$M, w \Vdash \Box p$$

finally, we conclude the implication

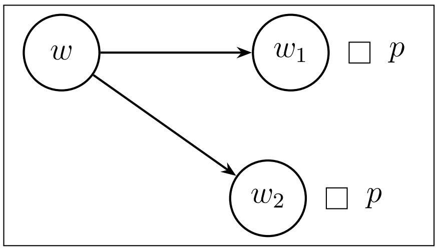
$$M, w \Vdash \Box(p \wedge q) \to M, w \Vdash \Box p$$

Ultimatelly,

$$M, w \Vdash \Box(p \wedge q) \to \Box p$$

Since M and w were arbitrary, the formula holds in every world of every Kripke model.
Therefore,

$$\models \Box(p \wedge q) \to \Box p. \quad QED$$



Example 2:

$$\Box(p \to q) \not\models p \to \Box q$$

and

$$p \to \Box q \not\models \Box(p \to q)$$

Proof (by counterexample)
Let

$$M = (W, R, V), \quad W = \{w, v\}, \quad R = \{(w, v)\},$$

with valuation

$$V(p) = \{w\}, \quad V(q) = \varnothing.$$

At world w:

1. Antecedent:

$$M, w \Vdash \Box(p \to q)$$

because the only accessible world is v, and there p is false, so $p \to q$ is true.

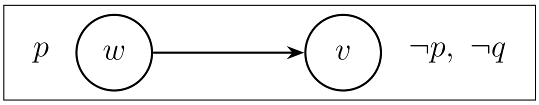
2. Consequent:

$$M, w \nVdash p \to \Box q$$

since $M, w \Vdash p$ but $M, w \nVdash \Box q$ (because q is false at v).

Therefore,

$$\Box(p \to q) \not\models p \to \Box q.$$



This model makes $\Box(p \to q)$ true at w but $p \to \Box q$ false.
Let

$$M = (W, R, V), \quad W = \{w, v\}, \quad R = \{(w, v)\},$$

with valuation

$$V(p) = \{v\}, \quad V(q) = \varnothing.$$

At world w:

1. Antecedent:

$$M, w \Vdash p \to \Box q$$

because $M, w \nVdash p$, so the implication holds.

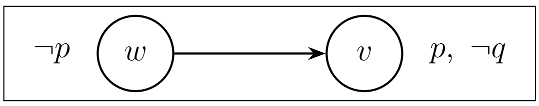
2. consequent:

$$M, w \nVdash \Box(p \to q)$$

since at v, $M, v \Vdash p$ but $M, v \nVdash q$, so $p \to q$ is false at v.

Therefore,

$$p \to \Box q \not\models \Box(p \to q). \quad QED$$



This model makes $p \to \Box q$ true at w but $\Box(p \to q)$ false.
Moreover:

1. If A is valid in U, then A is valid in every subclass $U' \subseteq U$.
2. If A is valid, then $\Box A$ is also valid.

Example:

$$\models \Box X$$

Proof
Assume

$$\models X$$

i.e. X is valid in every Kripke model.
Let

$$M = \langle W, R, V \rangle$$

be an arbitrary Kripke model and let $w \in W$. Since X is valid, for every world $w' \in W$ we have

$$M, w' \Vdash X$$

Now suppose wRw'. By validity of X, it follows again that

$$M, w' \Vdash X$$

Therefore, by the semantics of $\Box$, we obtain

$$M, w \Vdash \Box X$$

Since both M and w were arbitrary, we conclude

$$\models \Box X$$

34.2 Tautology

In classical logic, it is easy to check whether a formula is a tautology or not. For example,

$$P \vee \neg P$$

is a tautology, and we can also prove this using a truth table. However, in modal logic the term tautology is usually not used. Instead, we talk about valid formulas, since the semantics of modal logic are more complex than in classical logic. In this context, the concept of simultaneous substitution becomes helpful.

For example, recall

$$\varphi \equiv (\psi \leftrightarrow \chi)$$

Then under simultaneous substitution we have

$$\varphi[\delta_1/p_1, \ldots, \delta_n/p_n] \;=\; \psi[\delta_1/p_1, \ldots, \delta_n/p_n] \;\leftrightarrow\; \chi[\delta_1/p_1, \ldots, \delta_n/p_n].$$

In other words, substitution distributes structurally through the connectives.

35 Schema

A schema is a collection of formulas consisting precisely of all substitution instances of a given modal formula φ. Formally:

$$\{\psi : \exists \delta_1, \ldots, \delta_n \; (\psi = \varphi[\delta_1/x_1, \ldots, \delta_n/x_n])\}$$

The formula φ is called the characteristic formula of the schema, and it is unique up to renaming of propositional variables. A formula ψ is said to be an instance of a schema if ψ belongs to this set.

For convenience, a schema is usually denoted by a meta-linguistic expression obtained by substituting symbols such as $A, B, \ldots$ for propositional variables. For example:

1. The schema A corresponds to the characteristic formula p.
2. The schema $A \to \Box A$ corresponds to $p \to \Box p$.
3. The schema $A \to (B \to A)$ corresponds to $p \to (q \to p)$.

In particular, the schema A denotes the set of all formulas, since every formula is a substitution instance of p.

35.1 Truth and Validity

A schema is said to be true in a model if and only if all of its instances are true in that model. Similarly, a schema is valid if and only if it is true in every model. Valid schemas have a particularly important connection with the axiom schema K:

$$K = \Box(A \to B) \to (\Box A \to \Box B).$$

It can be shown that K is valid in every model. Because of this, schema K serves as the cornerstone of the modal system K, the weakest normal modal logic.

35.2 Valid Schemas

The following schemas are valid in every Kripke model:

1. $\Box(A \to B) \to (\Diamond A \to \Diamond B)$
2. $\Diamond(A \to B) \to (\Box A \to \Diamond B)$
3. $\Box(A \land B) \leftrightarrow (\Box A \land \Box B)$
4. $\Box A \to \Box(B \to A)$
5. $\neg\Diamond A \to \Box(A \to B)$
6. $\Diamond(A \lor B) \leftrightarrow (\Diamond A \lor \Diamond B)$
7. $\Diamond A \leftrightarrow \neg\Box\neg A$

Example:

Let $M = (W, R, V)$ be an arbitrary Kripke model and $w \in W$ an arbitrary world. We show the duality of $\Diamond$ and $\Box$:

1. By definition:

$$M, w \Vdash \neg\Box\neg A \leftrightarrow M, w \nVdash \Box\neg A.$$

2. By the semantics of $\Box$:

$$M, w \Vdash \Box\neg A \leftrightarrow \forall w' \in W,\ (wRw' \to M, w' \Vdash \neg A).$$

Therefore:

$$M, w \nVdash \Box\neg A \leftrightarrow \neg(\forall w' \in W,\ (wRw' \to M, w' \Vdash \neg A)).$$

3. Negating the universal gives an existential:

$$\neg(\forall w' \in W,\ (wRw' \to M, w' \Vdash \neg A)) \leftrightarrow \exists w' \in W,\ (wRw' \land M, w' \nVdash \neg A).$$

4. By classical negation:

$$M, w' \nVdash \neg A \leftrightarrow M, w' \Vdash A.$$

5. By definition of $\Diamond$:

$$\exists w' \in W,\ (wRw' \land M, w' \Vdash A) \leftrightarrow M, w \Vdash \Diamond A.$$

Thus, for an arbitrary model M and world w:

$$M, w \Vdash \Diamond A \leftrightarrow M, w \Vdash \neg\Box\neg A.$$

Since M and w were arbitrary, the formula is valid in all Kripke models:

$$\models \Diamond A \leftrightarrow \neg\Box\neg A.$$

35.3 Invalid Schemas

To show that a formula φ is invalid, we construct a model M and a world w such that

$$M, w \nVdash \varphi$$

Such models are called falsifying models. To show that a schema X is invalid, it suffices to construct a falsifying model for one of its instances.

Example 1:

Schema D is

$$\Box p \to \Diamond p.$$

Counterexample:
Let

$$M = (W, R, V)$$

with

$$W = \{w\}, \quad R = \varnothing, \quad V(p) = \varnothing.$$

1. Antecedent:

$$M, w \Vdash \Box p \leftrightarrow \forall v \in W\ (wRv \Rightarrow M, v \Vdash p).$$

   Since $R = \varnothing$,

$$M, w \Vdash \Box p.$$

2. Consequent:

$$M, w \Vdash \Diamond p \leftrightarrow \exists v \in W\ (wRv \wedge M, v \Vdash p).$$

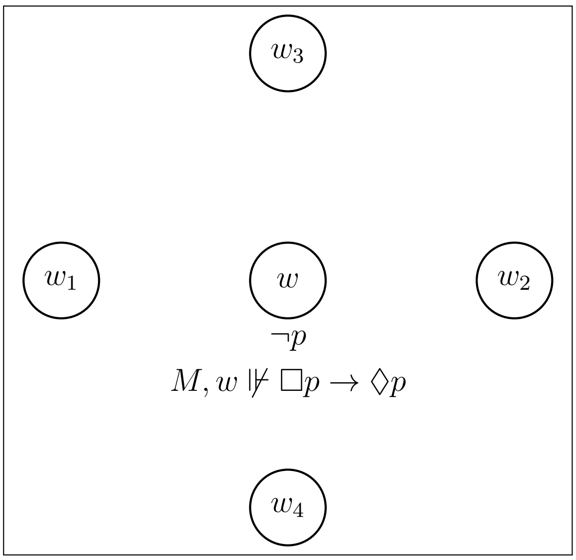
   Since $R = \varnothing$,

$$M, w \nVdash \Diamond p.$$

3. Therefore:

$$M, w \nVdash \Box p \to \Diamond p.$$

Hence $\Box p \to \Diamond p$ is not valid. Q.E.D.



Example 2:
Schema T is

$$\Box p \to p.$$

Counterexample:
Let

$$M = (W, R, V)$$

with

$$W = \{w\}, \quad R = \varnothing, \quad V(p) = \varnothing.$$

1. Antecedent:

$$M, w \Vdash \Box p \leftrightarrow \forall v \in W \; (wRv \Rightarrow M, v \Vdash p).$$

   Since $R = \varnothing$,

$$M, w \Vdash \Box p.$$

2. Consequent:

$$M, w \Vdash p$$

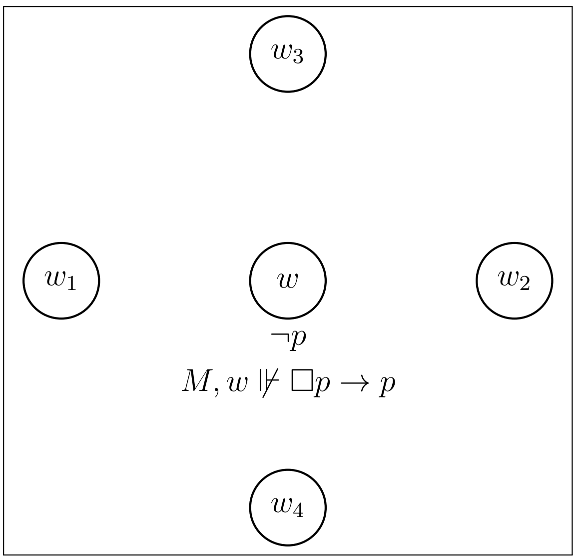
   is false because $V(p) = \varnothing$. Hence,

$$M, w \nVdash p.$$

3. Therefore:

$$M, w \nVdash \Box p \to p.$$

Hence $\Box p \to p$ is not valid.   Q.E.D.



Example 3:
Schema B is:

$$p \to \Box\Diamond p.$$

Counterexample:
Let

$$M = (W, R, V)$$

with

$$W = \{w, v\}, \quad R = \{(w, v)\}, \quad V(p) = \{w\}.$$

1. Antecedent:

   $$M, w \Vdash p$$

   since $w \in V(p)$.

2. Consequent:

   $$M, w \Vdash \Box\Diamond p \leftrightarrow \forall x \in W\,(wRx \Rightarrow M, x \Vdash \Diamond p).$$

   There is $x = v$ with wRv. But

   $$M, v \Vdash \Diamond p \leftrightarrow \exists u \in W\,(vRu \wedge M, u \Vdash p),$$

   and there is no u with vRu. Hence,

   $$M, v \nVdash \Diamond p,$$

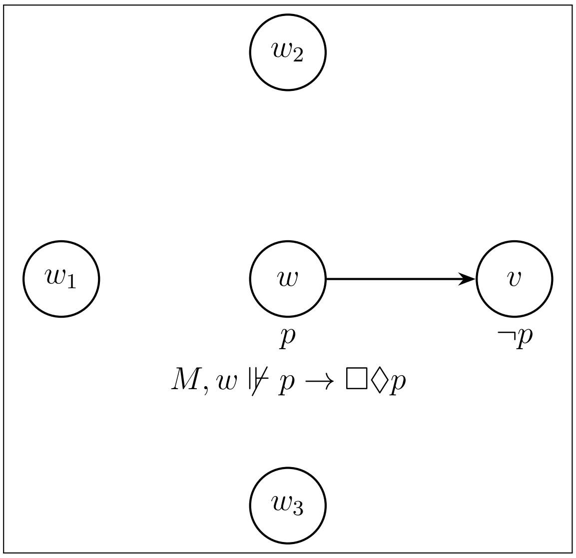
   so

   $$M, w \nVdash \Box\Diamond p.$$

3. Therefore:

   $$M, w \nVdash p \to \Box\Diamond p.$$

Hence $p \to \Box\Diamond p$ is not valid. Q.E.D.



Example 4:
Schema **4** is

$$\Box p \to \Box\Box p.$$

Counterexample:
Let

$$M = (W, R, V)$$

with

$$W = \{w, v, u\}, \quad R = \{(w, v), (v, u)\}, \quad V(p) = \{v\}.$$

1. Antecedent:

$$M, w \Vdash \Box p \leftrightarrow \forall x \in W\ (wRx \Rightarrow M, x \Vdash p).$$

The only x with wRx is v, and $v \in V(p)$, so

$$M, w \Vdash \Box p.$$

2. Consequent:

$$M, w \Vdash \Box\Box p \leftrightarrow \forall x \in W\ (wRx \Rightarrow M, x \Vdash \Box p).$$

For $x = v$ we have

$$M, v \Vdash \Box p \leftrightarrow \forall y \in W\ (vRy \Rightarrow M, y \Vdash p).$$

But vRu and $u \notin V(p)$, hence

$$M, v \nVdash \Box p,$$

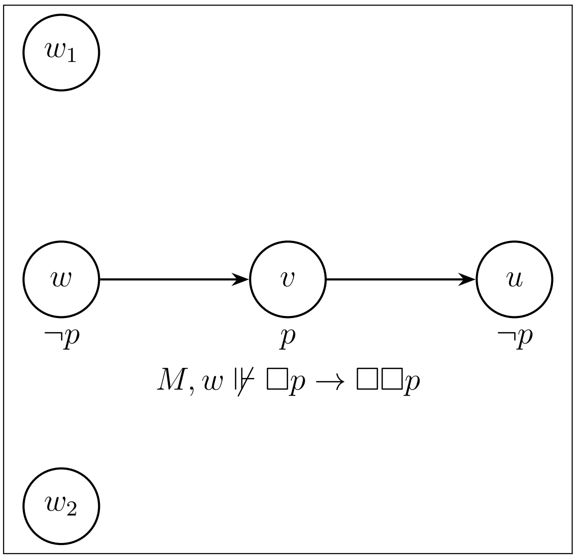
so

$$M, w \nVdash \Box\Box p.$$

3. Therefore:

$$M, w \nVdash \Box p \to \Box\Box p.$$

Hence $\Box p \to \Box\Box p$ is not valid. Q.E.D.



Example 5:
Schema **5** is

$$\Diamond p \to \Box\Diamond p.$$

Counterexample:
Let

$$M = (W, R, V)$$

with

$$W = \{w, v\}, \quad R = \{(w, v)\}, \quad V(p) = \{v\}.$$

1. Antecedent:

$$M, w \Vdash \Diamond p \leftrightarrow \exists x \in W\ (wRx \wedge M, x \Vdash p).$$

Since wRv and $v \in V(p)$,

$$M, w \Vdash \Diamond p.$$

2. Consequent:

$$M, w \Vdash \Box\Diamond p \leftrightarrow \forall x \in W\ (wRx \Rightarrow M, x \Vdash \Diamond p).$$

The only x with wRx is v, and

$$M, v \Vdash \Diamond p \leftrightarrow \exists y \in W\ (vRy \wedge M, y \Vdash p).$$

There is no y with vRy, hence

$$M, v \nVdash \Diamond p,$$

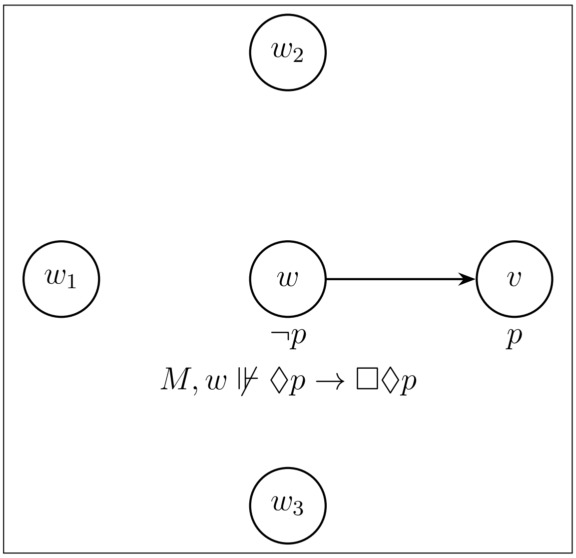
so

$$M, w \nVdash \Box\Diamond p.$$

3. Therefore:

$$M, w \nVdash \Diamond p \to \Box\Diamond p.$$

Hence $\Diamond p \to \Box\Diamond p$ is not valid. Q.E.D.



35.4 Properties of Schemas

If A and $A \to B$ are true at a world w in a model M, then B must also be true at w. This follows immediately from the semantics of implication. Hence, the class of valid formulas is closed under modus ponens. A formula A is valid if and only if all its substitution instances are valid. Equivalently, a schema is valid if and only if its characteristic formula is valid. The "if" direction is straightforward: A is trivially a substitution instance of itself, so if A is valid, then one substitution instance (namely A itself) is valid. For the "only if" direction, suppose $M = \langle W, R, V \rangle$ is a modal model, and let

$$B \equiv A[D_1/p_1, \ldots, D_n/p_n]$$

be a substitution instance of A. Construct a new model $M' = \langle W, R, V' \rangle$ where

$$V'(p_i) = \{w \in W : M, w \Vdash D_i\}.$$

One proves by induction on the structure of A that, for all $w \in W$,

$$M, w \Vdash B \quad \text{iff} \quad M', w \Vdash A.$$

If A were valid but some substitution instance B were not valid, then there would exist M, w such that $M, w \nVdash B$. By the equivalence above, this would imply $M', w \nVdash A$, contradicting the assumption that A is valid. Thus, every substitution instance of A must also be valid. By contrast, truth in a particular model does not behave in the same way.

For example, let $A = p$ in a model with a single world w and valuation $V(p) = \{w\}$. Then p is true at w. However, $\bot$ is also a substitution instance of p, yet $\bot$ is not true at w. Thus, while validity is preserved under substitution, truth in a specific model is not.

36 Frame Definability

A frame is a pair $F = \langle W, R \rangle$ where W is a non-empty set of worlds and R is a binary relation on W. A model M is based on a frame $F = \langle W, R \rangle$ iff $M = \langle W, R, V \rangle$ for some valuation V. We say that a formula A is valid in a frame F, written $F \vDash A$, if $M \vDash A$ for every model M based on F.

More generally, if $\mathcal{F}$ is a class of frames, then A is valid in $\mathcal{F}$, written $\mathcal{F} \vDash A$, iff $F \vDash A$ for every frame $F \in \mathcal{F}$. If $\mathcal{F}$ is a class of frames, we say that A *defines* $\mathcal{F}$ iff $\mathcal{F} \vDash A$ for all and only frames $F \in \mathcal{F}$.

While a model M may guarantee the truth of a formula, this can occur only because of a particular valuation V. In such cases, the truth is merely an *accidental truth*: it holds due to how truth values were assigned, not because of the frame's structure. A frame, on the other hand, concerns only the behavior and relationships of worlds, not the propositional variables themselves. Thus, if a formula is valid on a frame, it reflects a genuine structural property of that frame rather than a coincidental choice of valuation.

Example:

1. Reflexive frame:

   $$\forall w \in W,\ wRw$$

   Therefore:

   $$M, w \Vdash \Box A \to A \quad \text{for all } M \text{ on } F$$

2. Accidental truth in a non-reflexive model:

   $$F = \langle \{w\}, \varnothing \rangle$$

   $$M = \langle \{w\}, \varnothing, V \rangle \quad \text{with } V(p) = \{w\}$$

   Then:

$$M, w \Vdash \Box p \to p$$

3. Non-reflexive frame:

$$\exists w \in W,\ \neg wRw$$

where:

$$V(p) = W \setminus \{w\}$$

Then:

$$M, w \nVdash \Box p \to p$$

As we can see from the example, a frame F is simply a pair

$$F = \langle W, R \rangle$$

consisting of a set of worlds W with an accessibility relation R.
Every model

$$M = \langle W, R, V \rangle$$

is then, as we say, based on the frame $\langle W, R \rangle$.
We can now define $F \Vdash A$, the notion of a formula being *valid in a frame*, as:

$$M \Vdash A \quad \text{for all } M \text{ based on } F$$

With this notation, we can establish correspondence relations between formulas and classes of frames.
For example:

$$F \Vdash \Box p \to p \quad \text{iff} \quad F \text{ is reflexive.}$$

36.1 Properties of Accessibility Relations

Recall that in Invalid Schemas we constructed counterexamples showing that the schemas $(D, T, B, 4, 5)$ are not valid in general Kripke frames. In this part, we will prove that these schemas are indeed valid whenever the accessibility relation R has the corresponding property.
Let

$$M = \langle W, R, V \rangle$$

be a Kripke model. If R has one of the properties (serial, reflexive, symmetric, transitive, euclidean), then the corresponding modal axiom

$$(D, T, B, 4, 5) \text{ is valid in } M.$$

1. Serial (D)

   Let

$$M = (W, R, V)$$

   be an arbitrary Kripke model with R serial.

   Let

$$w \in W$$

   be an arbitrary world. We need to show

$$M, w \Vdash \Box p \to \Diamond p$$

   First, assume

$$M, w \Vdash \Box p$$

   by definition of $\Box$,

$$\forall v \in W \ (wRv \to M, v \Vdash p)$$

   Second, since R is serial

$$\exists v \in W \ (wRv)$$

   From the assumption

$$M, v \Vdash p$$

   Third

$$\exists v \in W \ (wRv \wedge M, v \Vdash p)$$

   Thus

$$M, w \Vdash \Diamond p$$

   Finally

$$M, w \Vdash \Box p \to \Diamond p$$

   We conclude that

$$\models \Box p \to \Diamond p \quad Q.E.D.$$

2. Reflexive (T)
   Let

$$M = (W, R, V)$$

be an arbitrary Kripke model with R reflexive.
Let

$$w \in W$$

be an arbitrary world. We need to show

$$M, w \Vdash \Box p \to p$$

First, assume

$$M, w \Vdash \Box p$$

By definition of $\Box$,

$$\forall v \in W \ (wRv \to M, v \Vdash p)$$

Second, since R is reflexive

$$wRw$$

From the assumption

$$M, w \Vdash p$$

Finally

$$M, w \Vdash \Box p \to p$$

As a consequence

$$\models \Box p \to p \quad Q.E.D.$$

3. Symmetric (B)
   Let

$$M = (W, R, V)$$

be an arbitrary Kripke model with R symmetric.
Let

$$w \in W$$

be an arbitrary world. We need to show

$$M, w \Vdash p \to \Box\Diamond p$$

First, assume

$$M, w \Vdash p$$

Second, let $v \in W$ with

$$wRv$$

Since R is symmetric

$$vRw$$

Third, from the assumption

$$M, w \Vdash p$$

we obtain

$$M, v \Vdash \Diamond p$$

Finally

$$M, w \Vdash \Box\Diamond p$$

As a result

$$\models p \to \Box\Diamond p \quad Q.E.D.$$

4. Transitive (4)
   Let

   $$M = (W, R, V)$$

   be an arbitrary Kripke model with R transitive.
   Let

   $$w \in W$$

   be an arbitrary world. We need to show

   $$M, w \Vdash \Box p \to \Box\Box p$$

   First, assume

$$M, w \Vdash \Box p$$

By definition of $\Box$,

$$\forall v \in W \ (wRv \to M, v \Vdash p)$$

Second, take an arbitrary $v \in W$ such that

$$wRv$$

From the assumption,

$$M, v \Vdash p$$

And for all $u \in W$ with

$$vRu$$

Since R is transitive,

$$wRu$$

Thus from the assumption,

$$M, u \Vdash p$$

Therefore

$$M, v \Vdash \Box p$$

Finally

$$M, w \Vdash \Box\Box p$$

Consequently

$$\models \Box p \to \Box\Box p \quad Q.E.D.$$

5. Euclidean (5)
   Let

$$M = (W, R, V)$$

be an arbitrary Kripke model with R euclidean.
Let

$$w \in W$$

be an arbitrary world. We need to show

$$M, w \Vdash \Diamond p \to \Box \Diamond p$$

First, assume

$$M, w \Vdash \Diamond p$$

By definition of $\Diamond$,

$$\exists v \in W \ (wRv \land M, v \Vdash p)$$

Second, take an arbitrary $u \in W$ such that

$$wRu$$

Since R is euclidean and wRv,

$$uRv$$

Third, from

$$M, v \Vdash p$$

we obtain

$$M, u \Vdash \Diamond p$$

Finally

$$M, w \Vdash \Box \Diamond p$$

It follows that

$$\models \Diamond p \to \Box \Diamond p \quad Q.E.D.$$

36.2 Partially Functional Relation

A relation R on a set of worlds W is partially functional if

$$\forall w \in W,\ \forall u, v \in W,\ (wRu \wedge wRv \to u = v)$$

This means each world has at most one accessible world.
Example:
Let

$$M = (W, R, V), \quad W = \{w, v\}, \quad R = \{(w, v)\}.$$

Since from w there is at most one successor, we have

$$wRv \text{ and no other } R\text{-successor of } w.$$

Therefore, the partial functionality condition holds:

$$\forall x \in W,\ \forall y, z \in W,\ (xRy \wedge xRz)\ \to\ y = z.$$

Therefore,

$$R \text{ is partially functional.}$$

36.3 Functional Relation

A relation R on a set of worlds W is functional if

$$\forall w \in W,\ \exists v \in W\ \big(wRv\ \wedge\ \forall u \in W, (wRu \to u = v)\big)$$

This ensures exactly one accessible world per world.
Example:
Let

$$M = (W, R, V), \quad W = \{w, v, u\}, \quad R = \{(w, v), (v, u), (u, w)\}.$$

Since from each world there is exactly one successor, we have

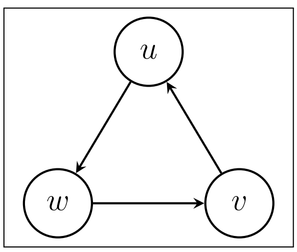
$$(wRv) \wedge (vRu) \wedge (uRw).$$

Therefore, the functionality condition holds:

$$\forall x \in W,\ \exists! y \in W\ \ (xRy).$$

Thus,

$$R \text{ is functional.}$$



36.4 Weakly Dense Relation

A relation R is weakly dense if whenever uRv, there is a w "between" u and v.

$$\forall u, v \in W,\ (uRv \to \exists w \in W\ (uRw \wedge wRv))$$

Example:
Let

$$M = (W, R, V), \quad W = \{w, u, v\}, \quad R = \{(w, u), (u, v), (w, v)\}.$$

Since there is a direct relation wRv, and also an intermediate path through u, we have

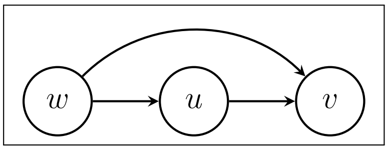
$$\exists w, u, v \in W\ \ (wRv \wedge wRu \wedge uRv).$$

Therefore, the weak density condition holds:

$$\forall w, v \in W,\ (wRv)\ \to\ \exists u \in W\ (wRu \wedge uRv).$$

Accordingly,

$$R \text{ is weakly dense.}$$



36.5 Confluence (Diamond Property)

A relation R on W is confluent if

$$\forall w, u, v \in W,\ (wRu \wedge wRv) \to \exists t \in W\ (uRt \wedge vRt)$$

Example:
Let

$$M = (W, R, V), \quad W = \{w, u, v, t\}, \quad R = \{(w, u), (w, v), (u, t), (v, t)\}.$$

Since from w we can reach both u and v, we have

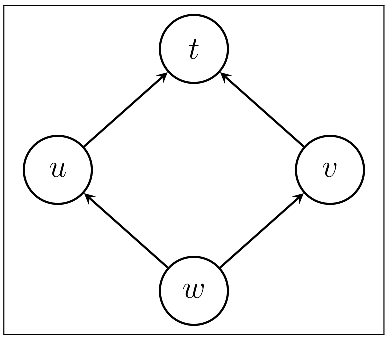
$$\exists w, u, v, t \in W\ \ (wRu \wedge wRv \wedge uRt \wedge vRt).$$

Therefore, the confluence condition holds:

$$\forall w, u, v \in W,\ (wRu \wedge wRv)\ \to\ \exists t \in W\ (uRt \wedge vRt).$$

Because of this,

$$R \text{ is confluent.}$$



37 Combining Frame Properties

1. $T + 4 = S4$

   Suppose

   $$M = (W, R, V), \quad W = \{w, v, u\}$$

   with the relation

   $$R = \{(w, w), (v, v), (u, u), (w, v), (v, u), (w, u)\}.$$

   Reflexivity (T):

   $$\forall x \in W \ (xRx)$$

   Holds, since $wRw, vRv, uRu \in R$.

   Transitivity (4):

   $$\forall x, y, z \in W \ (xRy \land yRz \to xRz)$$

   Holds, since $wRv \land vRu \to wRu \in R$.

   Modal validity at w:

   $$M, w \Vdash \Box p \to p$$

   $$M, w \Vdash \Box p \to \Box\Box p$$

   It follows that R is reflexive and transitive, validating $S4$.

2. $T + B$

   Suppose

   $$M = (W, R, V), \quad W = \{w, v, u\}$$

   with the relation

$$R = \{(w,w),(v,v),(u,u),(w,v),(v,w)\}.$$

Reflexivity (T):

$$\forall x \in W\ (xRx)$$

Symmetry (B):

$$\forall x, y \in W\ (xRy \to yRx)$$

Modal validity at w:

$$M, w \Vdash \Box p \to p$$

$$M, w \Vdash p \to \Box\Diamond p$$

As a result R is reflexive and symmetric.

3. $T + 5 = S5$

   Suppose

$$M = (W, R, V), \quad W = \{w, v, u\}$$

   with

$$R = W \times W.$$

   Reflexivity (T):

$$\forall x \in W\ (xRx)$$

   Euclidean (5):

$$\forall x, y, z \in W\ (xRy \wedge xRz \to yRz)$$

   Modal validity at w:

$$M, w \Vdash \Box p \to p$$

$$M, w \Vdash \Diamond p \to \Box\Diamond p$$

   Thus, R is reflexive and Euclidean, validating $S5$.

4. $B + 5$

Suppose

$$M = (W, R, V), \quad W = \{w, v, u\}$$

with the relation

$$R = \{(w, w), (v, v), (u, u), (w, v), (v, w), (v, u), (u, v)\}.$$

Symmetry (B):

$$\forall x, y \in W \ (xRy \to yRx)$$

Euclidean (5):

$$\forall x, y, z \in W \ (xRy \land xRz \to yRz)$$

Modal validity at w:

$$M, w \Vdash p \to \Box\Diamond p$$

$$M, w \Vdash \Diamond p \to \Box\Diamond p$$

Accordingly R is symmetric and Euclidean.

5. $4 + 5$

Suppose

$$M = (W, R, V), \quad W = \{w, v, u\}$$

with the relation

$$R = \{(w, w), (v, v), (u, u), (w, v), (v, u), (w, u)\}.$$

Transitivity (4):

$$\forall x, y, z \in W \ (xRy \land yRz \to xRz)$$

Euclidean (5):

$$\forall x, y, z \in W \ (xRy \land xRz \to yRz)$$

Modal validity at w:

$$M, w \Vdash \Box p \to \Box\Box p$$

$$M, w \Vdash \Diamond p \to \Box\Diamond p$$

We conclude that R is transitive and Euclidean.

38 Logical Relationships Between Properties

1. $T \to D$

   Proof.

   Assume R is reflexive:

   $$\forall w \in W : wRw$$

   We show R is serial:

   $$\forall w \in W \mid \exists v \in W : wRv$$

   Let $w \in W$. By reflexivity, wRw, so choose $v = w$.

   As a consequence R is serial. QED

2. $B + 4 \to 5$

   Proof.

   Assume symmetry and transitivity:

   $$\forall x, y : xRy \to yRx$$

   $$\forall x, y, z \mid (xRy \wedge yRz) \to xRz$$

   Let

   $$xRy \wedge xRz$$

   By symmetry,

   $$yRx$$

   By transitivity,

   $$yRx \wedge xRz \to yRz$$

   Thererfore R is Euclidean. QED

3. $T + 5 \to B$

   Proof.

   Assume reflexivity and Euclideanness.

   Let

   $$xRy$$

   By reflexivity,

   $$xRx$$

   By Euclidean property:

   $$(xRy \wedge xRx) \to yRx$$

   Thus R is symmetric. QED

4. $B + 5 \to 4$

   Proof.

   Assume symmetry and Euclideanness.

   Let

$$xRy \land yRz$$

   By symmetry,

$$yRx$$

   By Euclidean property:

$$(yRx \land yRz) \to xRz$$

   Hence R is transitive. QED

5. $D + B + 4 \to T$

   Proof

   Assume seriality, symmetry, and transitivity.

   Let

$$w \in W$$

   By seriality,

$$\exists v \mid wRv$$

   By symmetry,

$$vRw$$

   By transitivity:

$$(wRv \land vRw) \to wRw$$

   Thus R is reflexive. QED

39 Equivalence Relations

39.1 Basic Duality

1. $\Diamond X \equiv \neg\Box\neg X$ (Definition of possibility)
2. $\Box X \equiv \neg\Diamond\neg X$ (Definition of necessity)
3. $\neg\Diamond X \equiv \Box\neg X$ (Negation of possibility)
4. $\neg\Box X \equiv \Diamond\neg X$ (Negation of necessity)

39.2 Double Negation

5. $\Box\neg\neg X \equiv \Box X$ (Double negation in necessity)
6. $\Diamond\neg\neg X \equiv \Diamond X$ (Double negation in possibility)
7. $\neg\neg\Box X \equiv \Box X$ (Double negation outside)
8. $\neg\neg\Diamond X \equiv \Diamond X$ (Double negation outside)

39.3 De Morgan's Laws for Modalities

9. $\neg(\Box X \wedge \Box Y) \equiv \Diamond\neg X \vee \Diamond\neg Y$ (De Morgan's)
10. $\neg(\Diamond X \vee \Diamond Y) \equiv \Box\neg X \wedge \Box\neg Y$ (De Morgan's)
11. $\neg(\Box X \vee \Box Y) \equiv \Diamond\neg X \wedge \Diamond\neg Y$ (De Morgan's)
12. $\neg(\Diamond X \wedge \Diamond Y) \equiv \Box\neg X \vee \Box\neg Y$ (De Morgan's)

39.4 Implication Equivalences

13. $X \to Y \equiv \neg X \vee Y$ (Material implication)
14. $\neg(X \to Y) \equiv X \wedge \neg Y$ (Negation of implication)
15. $X \to Y \equiv \neg Y \to \neg X$ (Contrapositive)
16. $\Box X \to \Box Y \equiv \neg\Box Y \to \neg\Box X$ (Modal contrapositive)
17. $\Diamond X \to \Diamond Y \equiv \neg\Diamond Y \to \neg\Diamond X$ (Modal contrapositive)

39.5 Conjunction and Disjunction

18. $\Box(X \wedge Y) \equiv \Box X \wedge \Box Y$ (Distribution over conjunction)
19. $\Diamond(X \vee Y) \equiv \Diamond X \vee \Diamond Y$ (Distribution over disjunction)

39.6 Substitution Lemma

To justify this notion (e.g. $\varphi \equiv (\psi \leftrightarrow \chi)$), we prove the Substitution Lemma, which shows that the truth of propositional formulas under an assignment aligns with the truth of their tautological instances in Kripke models.

Suppose X is a modal-free formula whose propositional variables are $x_1, \ldots, x_n$, and let $\delta_1, \ldots, \delta_n$ be modal formulas. Then for any assignment v, any model $M = \langle W, R, V\rangle$, and any $w \in W$ such that

$$v(x_i) = T \quad \text{iff} \quad M, w \Vdash \delta_i,$$

we have

$$v \Vdash X \quad \text{iff} \quad M, w \Vdash X[\delta_1/x_1, \ldots, \delta_n/x_n].$$

Proof. By induction on the structure of X.

1. **Atomic variable:**

$$X \equiv x_i$$
$$v \Vdash x_i \leftrightarrow v(x_i) = T \leftrightarrow M, w \Vdash \delta_i \leftrightarrow M, w \Vdash x_i[\delta_1/x_1, \ldots, \delta_n/x_n].$$

2. **Falsity:**

$$X \equiv \bot$$
$$v \nVdash \bot \quad \text{and} \quad M, w \nVdash \bot$$

3. **Negation:**

$$X \equiv \neg\alpha$$
$$v \Vdash \neg\alpha \leftrightarrow v \nVdash \alpha \leftrightarrow M, w \nVdash \alpha[\delta_1/x_1, \ldots, \delta_n/x_n] \leftrightarrow M, w \Vdash \neg\alpha[\delta_1/x_1, \ldots, \delta_n/x_n].$$

4. **Conjunction:**

$$X \equiv (\alpha \wedge \beta)$$
$$v \Vdash \alpha \wedge \beta \leftrightarrow (v \Vdash \alpha \wedge v \Vdash \beta)$$
$$\leftrightarrow (M, w \Vdash \alpha[\delta_1/x_1, \ldots, \delta_n/x_n] \wedge M, w \Vdash \beta[\delta_1/x_1, \ldots, \delta_n/x_n])$$
$$\leftrightarrow M, w \Vdash (\alpha \wedge \beta)[\delta_1/x_1, \ldots, \delta_n/x_n].$$

5. **Disjunction:**

$$X \equiv (\alpha \vee \beta)$$
$$v \Vdash \alpha \vee \beta \leftrightarrow (v \Vdash \alpha \vee v \Vdash \beta)$$
$$\leftrightarrow (M, w \Vdash \alpha[\delta_1/x_1, \ldots, \delta_n/x_n] \vee M, w \Vdash \beta[\delta_1/x_1, \ldots, \delta_n/x_n])$$
$$\leftrightarrow M, w \Vdash (\alpha \vee \beta)[\delta_1/x_1, \ldots, \delta_n/x_n].$$

6. **Implication:**

$$X \equiv (\alpha \rightarrow \beta)$$
$$v \Vdash \alpha \rightarrow \beta \leftrightarrow (v \nVdash \alpha \vee v \Vdash \beta)$$
$$\leftrightarrow (M, w \nVdash \alpha[\delta_1/x_1, \ldots, \delta_n/x_n] \vee M, w \Vdash \beta[\delta_1/x_1, \ldots, \delta_n/x_n])$$
$$\leftrightarrow M, w \Vdash (\alpha \rightarrow \beta)[\delta_1/x_1, \ldots, \delta_n/x_n].$$

From this lemma we can show that all tautological instances are valid.

Proof.

Contrapositively, suppose U is such that

$$M, w \nVdash U[\delta_1/x_1, \ldots, \delta_n/x_n]$$

for some model M and world w. Define an assignment v such that

$$v(x_i) = T \quad \leftrightarrow \quad M, w \Vdash \delta_i,$$

and let v assign arbitrary values to $q \notin \{x_1, \ldots, x_n\}$. Then by the lemma,

$$v \Vdash U,$$

so U is not a tautology.

40 Limit of Logic

40.1 Contextual Meaning

On one hand, formal logic is precise in systematically expressing sentences. On the other hand, it tends to flatten nuance into rigid structures, and as a result, it lacks the richness and flexibility inherent in natural language.

Consider the following syllogistic reasoning:

Premise 1: All foxes are carnivores

Premise 2: King Herod is a fox

Conclusion: Therefore, King Herod is a carnivore

In syllogistic reasoning, the form is clearly valid, but the content is not. Here, formal reasoning, especially when translated into symbols, offers precision in structure, but it does not capture the semantic or contextual dimension. Specifically, in this case, Jesus metaphorically referred to King Herod as a "fox". The statement was meant as a figurative critique rather than a literal claim, which formal logic cannot fully represent. In this case, formal logic operates in only one dimension, it captures structural validity but fails to account for meaning, context, or figurative nuance.

Moreover, consider the formula:

$$\forall x\, (P(x) \to Q(x))$$

This is valid in first-order logic, but the interpretation of x and the predicates P and Q is context-dependent.

40.2 Vagueness

Natural language is full of vague terms, such as "some," "soon," and "enough." Classical logic, however, cannot naturally capture this vagueness because it requires precision. When we make these terms precise, we lose part of the expressive subtlety that makes language rich. For example, if we try to formalize "enough" in classical logic, we might define it as a quantity greater than or equal to a specific threshold x. But in natural language, "enough" is context-dependent, enough food for one person might not be enough for others. Forcing precision in this way risks losing the flexibility and adaptability that give language its nuance.

40.3 Presuppositions

Consider definite descriptions, although formal logic can represent the underlying intuitions, it may produce interpretations that differ from human intentions. This double-layered meaning is difficult to capture fully within a formal system. Yet at the same time, we also claim that, for our formal reasoning to remain pure, the logic must not depend on agents or subjectivity.

40.4 Idiomatic Layer

Similarly, formal logic is precise but cannot capture the idiomatic expressions, which often carry non-literal meanings.

Example:

"Rain cat and dog"

A formal translation can interpret this syntactically, but it cannot capture the real-world meaning, that it is raining heavily, without extra context.

40.5 Naturalness

Natural language is rich, flexible, and expressive, but also messy. Formal logic, by contrast, is precise but brittle. Yet, the key point is that natural language, despite being abstract, conveys subtleties and nuances that formal logic often cannot.

For example, consider the sentence:

"Every student who studies hard will likely succeed."

A formal translation might be:

$$\forall x\,(\mathrm{Student}(x) \wedge \mathrm{StudiesHard}(x) \to \mathrm{Succeeds}(x))$$

This captures the syntactic structure and a strict logical relation. However, the original sentence conveys probabilistic nuance and contextual subtleties that the formal statement ignores, showing that natural language expresses meaning beyond strict logical forms.