Irrationality of $\sqrt{2}$

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Abstract

A short and classical proof of the irrationality of $\sqrt{2}$ by contradiction.

Proof

Assume, for the sake of contradiction, that $\sqrt{2}$ is *rational*. Then, there exist integers p and q such that

$$\frac{p}{q} = \sqrt{2},$$

where $\frac{p}{q}$ is in lowest terms (irreducible), i.e., $\gcd(p, q) = 1$.

Squaring both sides:

$$\left(\frac{p}{q}\right)^2 = 2 \quad \Rightarrow \quad \frac{p^2}{q^2} = 2 \quad \Rightarrow \quad p^2 = 2q^2.$$

Since p^2 is even, p must also be an even number. If it were odd, its square would also be odd. It can't be odd, then it is even. Thus, we can write $p = 2k$ for some integer k. Substituting:

$$(2k)^2 = 2q^2 \quad \Rightarrow \quad 4k^2 = 2q^2 \quad \Rightarrow \quad q^2 = 2k^2.$$

This implies q^2 is also even, and therefore q is even. Hence, both p and q are even, contradicting our assumption that $\frac{p}{q}$ is in lowest terms.

Therefore, our assumption must be false. We conclude that $\sqrt{2}$ is irrational.