Point Representation in the Complex Plane

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The Complex Plane

When using geometry to represent complex numbers, we place them on a two-dimensional coordinate system known as the **complex plane**, or the **z-plane**.

- The horizontal axis (real axis) represents real numbers.
- The vertical axis (imaginary axis) represents imaginary numbers.

A complex number $z = a + bi$, where $a, b \in \mathbb{R}$, is associated with the point (a, b) in this plane. This point is often simply denoted by z.

Absolute Value (Modulus)

The **absolute value** or **modulus** of a complex number $z = a+bi$ is its distance from the origin $(0, 0)$ in the complex plane. By the Pythagorean Theorem:

$$|z| = \sqrt{a^2 + b^2} \tag{1}$$

Example:

$$|3 - 4i| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Note: The modulus $|z|$ is always a non-negative real number.

Distance Between Two Complex Numbers

Given two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, the distance between them in the complex plane is:

$$|z_1 - z_2| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2} \tag{2}$$

Geometric Interpretation

The geometric representation of modulus is useful for describing loci (sets of points) in the complex plane.

Circles in the Complex Plane

The set of all complex numbers z satisfying the equation

$$|z - z_0| = r \tag{3}$$

represents a **circle** of radius r centered at $z_0 \in \mathbb{C}$. This includes all points whose distance from z_0 is exactly r.

Example: Consider the equation:

$$|z + 2| = |z - 1|$$

This means the point z is equidistant from -2 and 1 on the real axis. Let $z = x + iy$, then:

$$|x+iy+2| = |x+iy-1| \Rightarrow (x+2)^2+y^2 = (x-1)^2+y^2 \Rightarrow x+2 = -x+1 \Rightarrow x = -\frac{1}{2}$$

This describes a vertical line $x = -\frac{1}{2}$ in the complex plane.

Complex Conjugate

The **complex conjugate** of a complex number $z = a + bi$ is defined as:

$$\overline{z} = a - bi \tag{4}$$

Example:

$$\overline{-1 + 2i} = -1 - 2i$$

Properties:

- If $z \in \mathbb{R}$, then $\overline{z} = z$.
- Conjugation distributes over addition and subtraction:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2} \tag{5}$$

- Conjugation distributes over multiplication and division:

$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \quad (z_2 \neq 0) \tag{6}$$

Extracting Real and Imaginary Parts

For any complex number z, the real and imaginary parts can be written as:

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i} \tag{7}$$

Product with the Conjugate

Multiplying a complex number by its conjugate yields:

$$z \cdot \overline{z} = |z|^2 \tag{8}$$

Rationalizing with the Conjugate

To divide complex numbers, especially when the denominator is complex, we use the conjugate to rationalize:

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z}_2}{z_2 \cdot \overline{z}_2} = \frac{z_1 \cdot \overline{z}_2}{|z_2|^2}, \quad z_2 \neq 0 \tag{9}$$

Special case:

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}, \quad z \neq 0 \tag{10}$$