Notes on Chapter 1.2: Some Preliminaries

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1 Sets

- A **set** is any collection of objects (elements)
- Notation:
  - $x \in A$: x is an element of A
  - $x \notin A$: x is not an element of A
- Operations:
  - Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
  - Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$
  - Complement: $A^c = \{x \in \mathbb{R} : x \notin A\}$
- **Example 1.2.1**:
  - Let $A = \{1, 2, 3\}, B = \{2, 4, 6\}$
  - $A \cup B = \{1, 2, 3, 4, 6\}$
  - $A \cap B = \{2\}$
  - If universal set is $\mathbb{N}$, then $A^c = \{4, 5, 6, \ldots\}$

2 Functions

- A **function** $f : A \to B$ is a rule associating each $x \in A$ to a single element of B
- Key properties:
  - **One-to-one (1-1)**: $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$
  - **Onto**: For every $b \in B$, there exists $a \in A$ with $f(a) = b$

- **Example 1.2.4** (Dirichlet's function):

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

  This function is:

  - Not one-to-one (both 1 and 2 map to 1)
  - Not onto if codomain is $\mathbb{R}$ (no x maps to 2)

3 Absolute Value and Triangle Inequality

- Absolute value definition:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- Properties:
  - $|ab| = |a||b|$
  - $|a + b| \leq |a| + |b|$ (Triangle Inequality)
  - $|a - b| \leq |a - c| + |c - b|$
- **Example**:
  - $|-3| = 3$
  - $|2 \cdot (-5)| = |2| \cdot |-5| = 10$
  - For $a = 1, b = -2$: $|1 + (-2)| = 1 \leq |1| + |-2| = 3$

4 Logic and Proofs

- **Proof by contradiction**: Assume the opposite of what you want to prove and find a contradiction
- **Example 1.2.6** (Uniqueness of limits):
  - Assume $\lim a_n = a$ and $\lim a_n = b$ with $a \neq b$
  - Take $\epsilon = |a - b|/2$
  - By definition, $\exists N_1$ such that $n \geq N_1 \implies |a_n - a| < \epsilon$
  - And $\exists N_2$ such that $n \geq N_2 \implies |a_n - b| < \epsilon$
  - For $n \geq \max\{N_1, N_2\}$:

$$|a - b| \leq |a - a_n| + |a_n - b| < 2\epsilon = |a - b|$$

  - Contradiction: $|a - b| < |a - b|$

5 Induction

- **Principle of Induction**:
  1. Show statement holds for $n = 1$ (base case)
  2. Assume holds for n, prove for $n + 1$ (inductive step)
- **Example 1.2.7**:
  - Define $x_1 = 1$, $x_{n+1} = \frac{1}{2}x_n + 1$
  - **Claim**: $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$
  - **Base case**: $x_1 = 1 \leq 1.5 = x_2$
  - **Inductive step**: Assume $x_n \leq x_{n+1}$

$$x_{n+2} = \frac{1}{2}x_{n+1} + 1 \geq \frac{1}{2}x_n + 1 = x_{n+1}$$