

# A Bayes Factor Framework for Unified Parameter Estimation and Hypothesis Testing

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Working paper version February 5, 2024

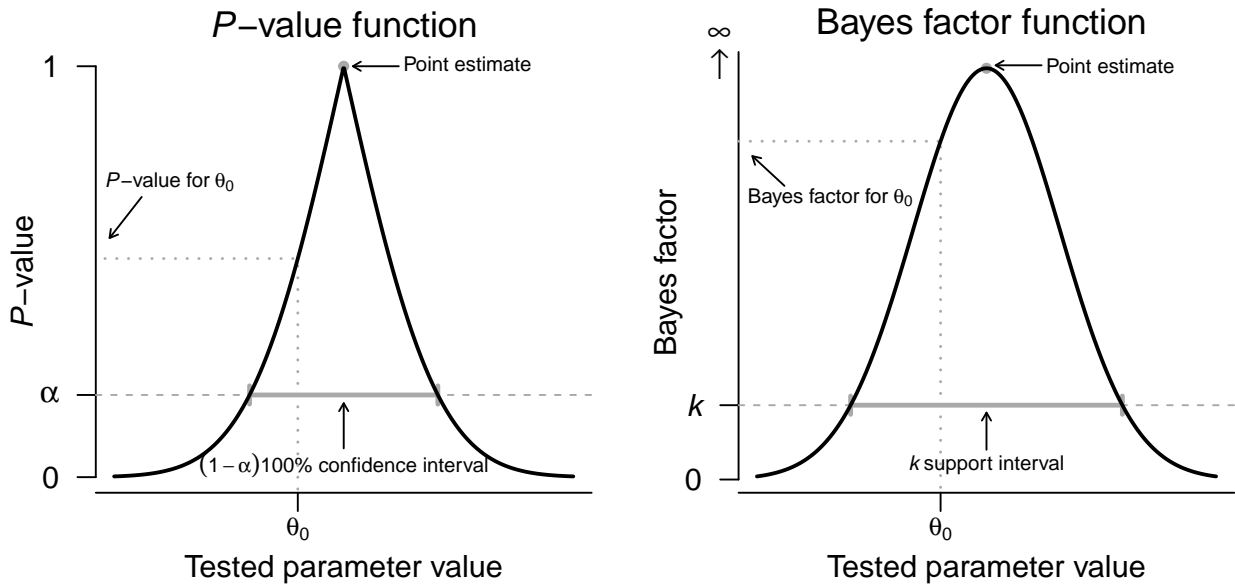
## Abstract

The Bayes factor – the relative likelihood of the data under two competing hypotheses – is a natural measure of statistical evidence or support for one hypothesis over the other. Here we show how Bayes factors can also be used for parameter estimation. The key idea is to consider the Bayes factor as a function of the parameter value under the null hypothesis. This ‘Bayes factor function’ is then inverted to obtain point estimates (‘maximum evidence estimates’) and interval estimates (‘support intervals’), similar to how  $P$ -value functions are inverted to obtain point estimates and confidence intervals. This provides data analysts with a unified inference framework for hypothesis testing and parameter estimation, as Bayes factors (for any tested parameter), support intervals (at any level of interest), and point estimates can be easily read off from a plot of the Bayes factor function. This provides data analysts with a unified framework for statistical inference that is distinct from conventional frequentist and Bayesian approaches: Bayes factor function inference uses the Bayesian evidence calculus, but without synthesizing data and prior. At the same time, Bayes factor function inference is closely related to likelihood-based inference, but also includes a natural way to deal with nuisance parameters. Applications to several real world examples illustrate how our framework is of practical value to data analysts who aim to make quantitative inferences.

*Keywords:* Bayesian inference, integrated likelihood, meta-analysis, nuisance parameters, replication studies, support interval

## 1 Introduction

A universal problem in data analysis is making inferences about unknown parameters of a statistical model based on observed data. In practice, data analysts are often interested in two tasks: (i) estimating the parameters (i.e., finding the most plausible value(s) based on the observed data) and (ii) testing hypotheses related to them (i.e., using the observed data to quantify the evidence that the parameter takes a certain value). While these tasks may seem different at first, there are several statistical concepts that provide a link between the two.



**Figure 1:** Examples of  $P$ -value functions and Bayes factor functions.  $P$ -value are two-sided. Bayes factors are oriented in favor of the tested parameter value over a specified alternative hypothesis (i.e., a higher Bayes factor indicates higher support for the parameter value over the alternative).

In frequentist statistics, there is a duality between parameter estimation and hypothesis testing as  $P$ -values, confidence intervals, and point estimates correspond in the sense that the  $P$ -value for a tested parameter value is less than  $\alpha$  if the  $(1 - \alpha)100\%$  confidence interval excludes that parameter value, and that the  $P$ -value is largest when the tested parameter value is the point estimate. The  $P$ -value function – the  $P$ -value viewed as a function of the tested parameter (for an overview see e.g., Bender et al., 2005; Fraser, 2019) – provides a link between these concepts.<sup>1</sup> A visualization of the  $P$ -value function, such as shown in the left plot in Figure 1, provides the observer with a wealth of information, as  $P$ -values (for any tested parameter), confidence intervals (at any level of interest), and point estimates can be easily read off. As such,  $P$ -value functions and their relatives have been deemed important measures to address common misinterpretations and misuses of  $P$ -values and confidence intervals (see e.g., Greenland et al., 2016; Infanger and Schmidt-Trucksäss, 2019; Rafi and Greenland, 2020; Marschner, 2024, among others).

In Bayesian statistics, the posterior distribution of the unknown parameter plays a similar role to the  $P$ -value function, since point estimates (e.g., posterior modes, medians, or means), credible intervals, and posterior probabilities of hypotheses can all be derived from it. The posterior provides a synthesis of the data and the prior distribution, which can be seen as an advantage but also as a challenge in the absence of prior knowledge. In particular, for testing of hypotheses, it can be difficult to specify prior probabilities such as ' $p$ (the treatment effect is absent)' and ' $p$ (the treatment effect is present)'. One approach to address this, is to report the *Bayes factor* (Jeffreys, 1939; Good, 1958; Kass and Raftery, 1995), i.e., the updating factor of the prior to posterior odds of two hypotheses. As such, Bayes factors allow data analysts to evaluate the relative evidence

<sup>1</sup>One may alternatively look at closely related quantities: One minus the two-sided  $P$ -value function known as *confidence curve* (Cox, 1958; Birnbaum, 1961), one minus the one-sided  $P$ -value function known as *confidence distribution*, or its derivative known as *confidence density* (Xie and Singh, 2013; Schweder and Hjort, 2016).

for two hypotheses without depending on the prior probabilities of the hypotheses themselves; for example, a Bayes factor can quantify the evidence for the presence or absence of a treatment effect without having to assign prior probabilities to these hypotheses (although one still has to specify a prior for the parameter under the alternative, which is challenging in itself). However, the use of Bayes factors comes at the cost of lacking an overarching concept, such as a  $P$ -value function or posterior distribution, that can provide data analyst with a coherent set of point and interval estimates. In practice, data analysts who wish to perform hypothesis testing with Bayes factors but also parameter estimation are therefore faced with a dilemma; they can either supply their Bayes factors with a posterior distribution conditional on one hypothesis being true (e.g., the posterior of a treatment effect, assuming the effect is indeed present), which can lead to contradictory conclusions with the Bayes factor (for examples, see [Aitkin et al., 2005](#); [Wagenmakers et al., 2020](#)), or they can assign prior probabilities to the tested hypotheses and report a posterior averaged over both hypotheses ([Campbell and Gustafson, 2022](#)), but this requires specification of prior probabilities which is highly controversial and the reason why the Bayes factor was reported in the first place rather than the posterior probabilities of the hypotheses.

Our goal is therefore to propose a unifying framework for estimation and hypothesis testing based on Bayes factors, building on ideas already hinted at in earlier work ([Pawel et al., 2023b](#)) and also closely related to approaches proposed in the physics community ([Afzal et al., 2023](#); [Fowlie, 2024](#)). The idea is the same as for the  $P$ -value function; we consider the Bayes factor as a function of the tested parameter. We then use this *Bayes factor function*<sup>2</sup> to derive point estimates, interval estimates, and Bayes factors (as shown in the right plot in Figure 1). Our framework builds on the recently proposed Bayesian support interval ([Wagenmakers et al., 2020](#); [Pawel et al., 2023b](#)) and extends it with the novel concept of point estimation based on Bayes factors. We call the resulting estimate the *maximum evidence estimate* (MEE) – the parameter value that receives the most evidential support from the data over a specified alternative hypothesis. Our framework provides data analysts with a unified framework for statistical inference that is in some ways related to Bayesian, likelihood, and frequentist inference, but is also distinct in other ways.

This paper is structured as follows. In the following Section 2, we introduce the general theory of Bayes factors, support sets, and maximum evidence estimates. We then discuss their connection other types of inference in Section 3. Various real data examples in Section 4 illustrate similarities and differences of the proposal to conventional frequentist and Bayesian approaches. We conclude with a discussion of the advantages and limitations of the approach, and opportunities for future research (Section 5).

## 2 Bayes factor function inference

Suppose we observe data  $y$  with an assumed distribution with density/probability mass function  $p(y | \theta, \psi)$  that depends on parameters  $\theta \in \Theta$  and  $\psi \in \Psi$ , with  $\theta$  being the focus parameters and  $\psi$

<sup>2</sup>Our conception of Bayes factor functions differs from the Bayes factor functions introduced by [Johnson et al. \(2023\)](#). The latter are Bayes factors viewed as a function of a hyperparameter of the prior *under the alternative hypothesis* but for a fixed null hypothesis. We acknowledge that introducing another concept with the same name may be confusing, but we think that Bayes factor function is the most appropriate name, in analogy to  $P$ -value function. An alternative might be to call our concept ‘support curve’ in analogy to the confidence curve, but we think Bayes factor function is more appropriate.

being possible nuisance parameters. Consider the Bayes factor (Jeffreys, 1939; Good, 1958; Kass and Raftery, 1995) for quantifying the evidence that the observed data  $y$  provide for the null hypothesis  $H_0: \theta = \theta_0$  relative to the alternative hypothesis  $H_1: \theta \neq \theta_0$ , with prior  $p(\theta, \psi | H_1)$  assigned to the parameters under  $H_1$  and prior  $p(\psi | H_0)$  assigned to the nuisance parameters under  $H_0$ . A natural measure of relative evidence for the two hypotheses is the Bayes factor, the data-based updating factor of the prior odds of the hypotheses to their posterior odds

$$\text{BF}_{01}(y; \theta_0) = \frac{p(H_0 | y)}{p(H_1 | y)} \bigg/ \frac{p(H_0)}{p(H_1)} \quad (1a)$$

$$= \frac{p(y | H_0)}{p(y | H_1)} \quad (1b)$$

$$= \frac{\int_{\Psi} p(y | \theta_0, \psi) p(\psi | H_0) d\psi}{\int_{\Theta} \int_{\Psi} p(y | \theta, \psi) p(\theta, \psi | H_1) d\psi d\theta}. \quad (1c)$$

The Bayes factor hence represents the data-based core of the Bayesian belief calculus, and it remains useful even if one rejects the idea of assigning probabilities to  $H_0$  and  $H_1$ , since this is not necessary (Goodman, 1999). The alternative expression of the Bayes factor in equation (1b) shows that this update is dictated by the relative predictive accuracy of the two hypotheses – how well they predicted the observed data  $y$  (Gneiting and Raftery, 2007; Fong and Holmes, 2020). Finally, the last equation (1c) shows how the Bayes factor is typically calculated, i.e., by dividing the likelihood of  $y$  under the null value  $\theta_0$  (possibly marginalized over the prior of  $\psi$  under  $H_0$ ) by the likelihood of  $y$  marginalized over the prior of  $\theta$  (and possibly  $\psi$ ) under  $H_1$ . The priors for  $\theta$  and  $\psi$  may also be point priors, in which case the Bayes factor reduces to a likelihood ratio.

The idea now is to consider the Bayes factor (1) as a function of  $\theta_0$ , that is, to vary the tested parameter value (the point null hypothesis  $H_0: \theta = \theta_0$ ) in order to assess the support for different parameter values over the alternative, see the right plot in Figure 1 for an example. Like the  $P$ -value function, this *Bayes factor function* (BFF) helps to address cognitive challenges with inferential statistics (Greenland, 2017). For example, it shifts the focus of inference from testing a single privileged null hypothesis (e.g., the hypothesis that there is no treatment effect) to an entire continuum of hypotheses. By looking at the BFF, data analysts can then identify hypotheses that receive equal or even less support from the data than the privileged one; for example, a parameter value indicating a very large treatment effect may receive equal support as the value of no treatment effect (sometimes called ‘counternull’, see Rosenthal and Rubin, 1994). For one- or two-dimensional focus parameters  $\theta$ , the BFF can be graphed so that the relative support for parameter values can be visually assessed. For higher dimensional focus parameters, this becomes more difficult and the BFF may need to be summarized in some form.

## 2.1 Support sets

The BFF can be used to obtain *support sets* (Wagenmakers et al., 2020; Pawel et al., 2023b) which are set-valued estimates for  $\theta$  based on inverting the Bayes factor (1) similar to how  $P$ -value functions

are inverted to obtain confidence sets. Specifically, a support set at level  $k$  is defined by

$$S_k = \{\theta_0 : \text{BF}_{01}(y; \theta_0) \geq k\}$$

that is, the parameter values for which the Bayes factor indicates at least evidence of level  $k$  over the specified alternative. In practice, a  $k$  support set (typically an interval) is obtained from ‘cutting’ the BFF at  $k$  and taking the parameter values above as part of the support set (see the right plot in Figure 1 for illustration). It may happen that for certain choices of  $k$  the support set is empty because the data are not informative enough to constitute relative evidence at that level.

A support set at level  $k = 1$  contains the parameter values that are equally or better supported by the data than the alternative  $H_1$ . The choice of the support level is arbitrary, just as the choice of the confidence level from a confidence set is. One may, for example, report the support level  $k = 1$  as it represents the tipping point at which the parameter values begin to be supported over the alternative. Moreover, conventions for Bayes factor evidence levels can be used. For example, based on the conventions from [Jeffreys \(1961\)](#), a support set at level  $k = 10$  includes the parameter values that receive ‘strong’ relative support from the data, while a  $k = 1/10$  support set includes the parameter values that are at least not strongly contradicted.

## 2.2 The maximum evidence estimate

A natural point estimate for the unknown parameter  $\theta$  based on the BFF is given by

$$\hat{\theta}_{\text{ME}} = \arg \max_{\theta_0 \in \Theta} \text{BF}_{01}(y; \theta_0),$$

and we call it the *maximum evidence estimate* (MEE), since it is the parameter value for which the Bayes factor indicates the most evidence over the alternative. The associated *evidence level*

$$k_{\text{ME}} = \text{BF}_{01}(y; \hat{\theta}_{\text{ME}}),$$

that is, the Bayes factor evaluated at the MEE, quantifies the evidential value of the estimate  $\hat{\theta}_{\text{ME}}$  over the alternative. Evidence levels close to  $k_{\text{ME}} = 1$  indicate that the MEE receives little support over the alternative hypothesis  $H_1$ , whereas large evidence levels  $k_{\text{ME}}$  indicate that the MEE receives substantial support over the alternative hypothesis  $H_1$ . A useful summary of a BFF could hence be to report the MEE, its evidence level, and a support set, similar to how a  $P$ -value function may be summarized with a point estimate and confidence set.

## 2.3 Example: Normal mean

Assume we observe a single observation  $y$  assumed to be sampled (at least approximately) from a normal distribution  $Y \mid \theta \sim N(\theta, \sigma^2)$ . Assume that  $\sigma^2$  is known and we want to conduct inferences regarding  $\theta$ . This is a simple but frequently encountered scenario, for example,  $y$  could be an estimated regression coefficient from a generalized linear model or from a Cox proportional hazard model, and  $\sigma$  its estimated standard error. In the following we will consider an example from [RECOVERY col-](#)

laborative group (2022). This randomised controlled trial found a reduction in mortality of patients hospitalised with COVID-19 when treated with baricitinib compared to usual care (age-adjusted log hazard ratio  $y = -0.14$  with standard error  $\sigma = 0.064$  estimated with Cox regression). To derive a Bayes factor for testing  $H_0: \theta = \theta_0$  against  $H_1: \theta \neq \theta_0$  we need to formulate a prior for  $\theta$  under the alternative  $H_1$ . We will now discuss three choices with different characteristics shown in Table 1.

**Table 1:** Bayes factor function, maximum evidence estimate, evidence value, and  $k$  support interval for a normal mean based on one observation  $y$  from  $Y \mid \theta \sim N(\theta, \sigma^2)$  with known variance  $\sigma^2$  and for three prior distributions under the alternative  $H_1$ : A normal prior (left), a normal prior centered around the parameter value of the null hypothesis  $\theta_0$  (middle), a point prior shifted away from the parameter value of the null hypothesis  $\theta_0$  by  $d > 0$  (right).

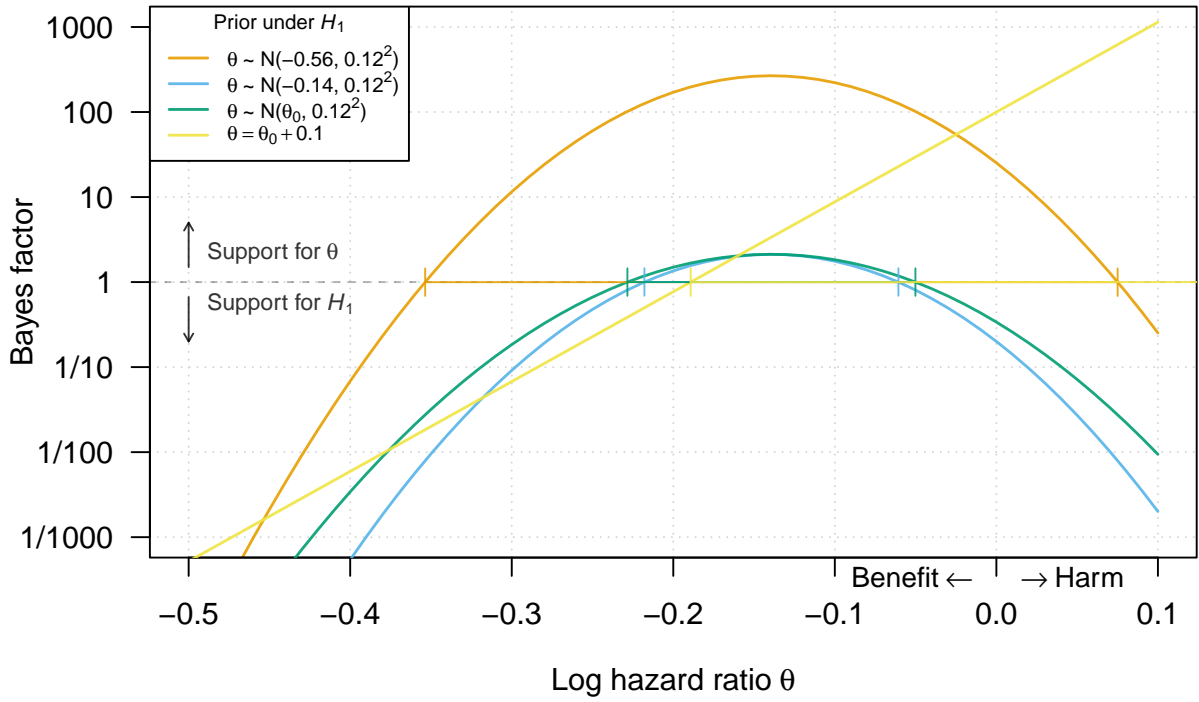
Prior for $\theta$ under $H_1$			
	$\theta \sim N(m, v)$	$\theta \sim N(\theta_0, v)$	$\theta = \theta_0 + d$
BF <sub>01</sub>	$\exp \left[ -\frac{1}{2} \left\{ \frac{(y-\theta_0)^2}{\sigma^2} - \frac{(y-m)^2}{\sigma^2+v} \right\} \right] \sqrt{1 + \frac{v}{\sigma^2}}$	$\exp \left[ -\frac{1}{2} \left\{ \frac{(y-\theta_0)^2}{\sigma^2(1+\sigma^2/v)} \right\} \right] \sqrt{1 + \frac{v}{\sigma^2}}$	$\exp \left\{ \frac{2d(\theta_0-y)+d^2}{2\sigma^2} \right\}$
$\hat{\theta}_{ME}$	$y$	$y$	non-existent
$k_{ME}$	$\exp \left\{ \frac{(y-m)^2}{2(\sigma^2+v)} \right\} \sqrt{1 + \frac{v}{\sigma^2}}$	$\sqrt{1 + \frac{v}{\sigma^2}}$	non-existent
$k$ SI	$y \pm \sigma \sqrt{\log(1 + \frac{v}{\sigma^2}) + \frac{(y-m)^2}{\sigma^2+v} - \log k^2}$	$y \pm \sigma \sqrt{\{\log(1 + \frac{v}{\sigma^2}) - \log k^2\} (1 + \frac{\sigma^2}{v})}$	$\left[ y + \frac{\sigma^2 \log k}{d} - \frac{d}{2}, \infty \right]$

Perhaps the simplest choice is a prior that does not depend on the parameter value  $\theta_0$  of the null hypothesis, for example, a normal prior with mean  $m$  and variance  $v$  (left column in Table 1). The hyperparameters  $m$  and  $v$  may be specified based on external data or based on an alternative hypothesis of interest (e.g., the prior mean  $m$  could be set to a minimum clinically important treatment effect and  $v$  could be set to zero to obtain a point prior as typically used in a power analysis). For example, RECOVERY collaborative group (2022) reported a meta-analytic log hazard ratio and standard error based on eight previous trials, which could be used to specify the mean  $m = -0.56$  and variance  $v = 0.12^2$  of the prior, see Figure 2 for the resulting BFF (orange). In this case, the MEE is given by  $\hat{\theta}_{ME} = y = -0.14$  with the support interval centered around it. Due to the apparent conflict between the observed data and the prior under the alternative, the  $k = 1$  support interval spans an wide range, indicating that large ( $\theta \approx -0.35$ ) up to even slightly harmful ( $\theta \approx 0.08$ ) treatment effects are supported by the data over the alternative.

The formulae in Table 1 (left column) show that as the prior mean  $m$  becomes closer to the observed data  $y$ , the evidence level  $k_{ME}$  decreases and the support interval becomes narrower. This is because an alternative closer to the data clearly has better predictive accuracy of the data than an alternative further away. Figure 2 illustrates this phenomenon with another prior distributions (one with mean at the observed log hazard ratio  $y = -0.14$ , the blue BFF), which is still centered at the observed log hazard ratio estimate but with far narrower  $k = 1$  support interval than the orange one with the mean  $m$  based on the eight previous trials.

Another approach to formulating a prior distribution for  $\theta$  under the alternative commonly sug-





**Figure 2:** Bayes factor function and  $k = 1$  support intervals for a normal mean  $\theta$  based on data  $y = -0.14$  with standard error  $\sigma = 0.064$  for different prior distributions for the mean under the alternative  $H_1$ .

gested in ‘objective’ Bayes theories is to center the prior around the tested parameter value  $\theta_0$  (Jeffreys, 1961; Berger and Delampady, 1987; Kass and Wasserman, 1995). For example, one can specify a normal prior with mean at the null value  $\theta_0$  (middle column in Table 1). Thus, unlike the ‘global’ normal prior with fixed mean  $m$ , the resulting BFF varies both the null and the alternative. As a result, the interpretation of the BFF is different: For such a ‘local’ normal prior, the BFF quantifies the support of parameter values over alternative parameter values in a neighborhood around them. As for the global normal prior, the MEE based on the local normal prior is given by  $\hat{\theta}_{ME} = y$  and support intervals are centered around it, but the associated, Bayes factor, evidence level and support interval are different. Figure 2 illustrates with the data from the RECOVERY trial that when the mean  $m$  of a global normal prior is too different from the observed data  $y$  (as in the case of the orange BFF, where the prior was specified based on the eight previous trials), the  $k = 1$  support interval based on the local prior with the same variance is narrower. On the other hand, when the mean  $m$  of the global prior is equal to the data (blue BFF), the support interval based on the local prior is wider.

The last prior in the right most column of Table 1 demonstrates potentially pathological behavior of BFF inference. This alternative represents a point prior shifted from the null value  $\theta_0$  by  $d > 0$ . The prior is again ‘local’ in the sense that it is different for each tested parameter value of the null hypothesis  $\theta_0$ , and as such encodes an alternative hypothesis that the log hazard ratio is greater than the tested parameter value. However, this leads to an ever-increasing BFF, see Figure 2 for a numerical illustration. As a result, the MEE and its evidence level do not exist, while the support interval still exists but its right limit extends to infinity.

## 2.4 Choice of the prior

As the previous example showed, the prior assigned to the parameters under the alternative has a substantial impact on BFF inference. The ‘sensitivity’ of Bayes factors to prior distributions enables data analysts to accurately quantify the support of parameter values over informative alternative hypotheses when they are available, but poses a challenge in their absence (Kass and Raftery, 1995). Various approaches have been proposed to deal with this issue, for example, ‘default’ or ‘objective’ prior distributions (Bayarri et al., 2012; Consonni et al., 2018), reverse-Bayes analysis (Held et al., 2022), prior elicitation (O’Hagan, 2019), or sensitivity analysis (Franck and Gramacy, 2019), all with advantages and disadvantages. Here we will not reiterate general considerations on prior specification for Bayes factors (see e.g., Section 5 in Kass and Raftery, 1995) but focus on specific considerations related to BFFs.

A general distinction can be made between *global* priors, which do not depend on the value of  $\theta_0$  under the null hypothesis and *local* priors, which do. In the latter case, the interpretation of the BFF is more intricate, since for each parameter value the BFF quantifies the support over a different alternative. For easier interpretation, global priors may hence be preferred over local priors. At the same time, local priors correspond to the typical use of ‘default’ Bayes factors, which is to center the prior around  $\theta_0$ , and as such may be preferred in the same situations where default Bayes factors would be used.

Finally, it is usually advisable to report sensitivity analyses for plausible ranges of priors, to assess the robustness of the conclusions. A convenient visual sensitivity analysis is, for example, to plot different BFFs resulting from different prior specifications, as shown in Figure 2. One can go a step further and use a ‘reverse-Bayes’ approach (Held et al., 2022), which involves systematically determining the prior that represents the tipping point and changes the conclusions of the analysis. Data analysts can then argue whether or not such a prior is plausible in the light of external knowledge and data.

## 3 Connection to other inference frameworks

We will now explore connections of BFF inference to other inference frameworks.

### 3.1 Bayesian inference

The BFF can, under certain conditions, be transformed into a Bayesian posterior distribution. Specifically, assuming a ‘global’ prior under the alternative, i.e., a prior  $p(\theta \mid H_1)$  which does not depend on the parameter under the null  $\theta_0$ , we have that the posterior distribution based on this prior is

$$p(\theta \mid y, H_1) = \underbrace{\frac{p(y \mid \theta, H_1)}{p(y \mid H_1)}}_{= \text{BFF}_{01}(y; \theta)} \times p(\theta \mid H_1). \quad (2)$$

This means that the posterior can be obtained by multiplying the BFF with the prior. It is, however, important to emphasize that multiplying a BFF based on priors under the alternative that depend on



the null (e.g., commonly used ‘local’ normal or Cauchy priors that are centered around  $\theta_0$ ) cannot be transformed to a genuine posterior distribution in this way, but multiplication with the prior will result in a different posterior for every  $\theta$ .

The link (2) also provides a convenient way to compute BFFs in more general settings: One of the many general Bayesian inference programs, such as Stan (Carpenter et al., 2017), can be used to compute a posterior density, which can then be divided by the prior density to obtain a BFF. The caveats are that this only works for global priors under the alternative and with the same prior assigned to the nuisance parameters under the null and alternative.

The relationship between posterior and BFF also exposes its connection to another Bayesian inference quantity – the relative belief ratio

$$\text{RB}(\theta \mid H_1) = \frac{p(\theta \mid y, H_1)}{\underbrace{p(\theta \mid H_1)}_{=\text{BF}_{01}(y;\theta)}}, \quad (3)$$

see e.g., Evans (2015). This quantity is the updating factor of the prior to the posterior density/probability mass function, and is related to the Bayes factor via the so-called Savage-Dickey density ratio (Dickey, 1971; Verdinelli and Wasserman, 1995; Wagenmakers et al., 2010).

An estimation and testing framework centered around the relative belief ratio was developed by Evans (1997, 2015). This framework is perhaps the closest to the one presented here. However, as with the posterior distribution there is only a correspondence between relative belief ratio inferences and BFF inference if the prior for  $\theta$  is the same for every  $\theta_0$ , and it does not hold anymore when, for example, local priors centered around the value of the null  $\theta_0$  are used.

### 3.2 Frequentist inference

The ‘universal bound’ (Kerridge, 1963; Robbins, 1970; Royall, 1997) bounds the frequentist probability of obtaining misleading Bayesian evidence. That is, for  $k < 1$  the probability of obtaining a Bayes factor against  $H_0$  less than  $k$  is at most  $k$  for any prior under the alternative

$$\Pr\{\text{BF}_{01}(y; H_0) \leq k \mid H_0\} \leq k.$$

If there are nuisance parameters, the bound holds only marginalized over the prior of the nuisance parameters. For the bound to hold in a strict sense (i.e., for every possible value of the nuisance parameter), special priors must be assigned to them (Hendriksen et al., 2021; Grünwald et al., 2024).

The universal bound can thus be used to transform BFFs into conservative  $P$ -values and confidence sets, e.g., a  $k = 1/20$  support set obtained from a BFF corresponds to a 95% conservative confidence set and  $p = \max\{\text{BF}_{01}, 1\}$  corresponds to a conservative  $P$ -value. Remarkably, the bound holds without adjustment even when the data collection is continuously monitored and stopped as soon as evidence against  $H_0$  is found. However, it is important to note that  $P$ -values and confidence sets obtained in this way are usually much more conservative than ordinary ones which are calibrated to have exact type I error rate and coverage, respectively. Finally, if the data model is misspecified, the bound is obviously invalid.

### 3.3 Maximum integrated likelihood

In typical situation where a division of  $p(y | H_0)$  by  $p(y | H_1)$  does not change the maximizer of  $p(y | H_0)$ , the MEE can be obtained by maximizing the marginal likelihood  $p(y | H_0)$  without reference to an alternative  $H_1$ . This is, for instance, the case when a global prior (a prior that does not depend on  $\theta_0$ ) is assigned to  $\theta$  under the alternative, or also in the case of the local normal prior that is centered around  $\theta_0$  from the previous example. The MEE is then equivalent to the maximizer of the *integrated likelihood*

$$\hat{\theta}_{\text{MIL}} = \arg \max_{\theta \in \Theta} \int_{\Psi} p(y | \theta, \psi) p(\psi | H_0) d\psi,$$

based on prior  $p(\psi | H_0)$  assigned to the nuisance parameters (see e.g., [Kalbfleisch and Sprott, 1970](#); [Basu, 1977](#); [Berger et al., 1999](#); [Royall, 1997](#); [Severini, 2007](#)). When there are no nuisance parameters, the MEE reduces to the ordinary maximum likelihood estimate.

To consider a concrete example, assume a sample of  $n$  normal random variables  $Y_1, \dots, Y_n | \theta, \sigma^2 \stackrel{i.i.d.}{\sim} N(\mu, \sigma^2)$ . Suppose that  $\sigma^2$  is the focus and  $\mu$  the nuisance parameter, and that an improper uniform prior  $p(\mu | H_0) = 1$  is assigned to  $\mu$ . The intergrated likelihood of an observed sample  $y_1, \dots, y_n$  is then

$$p(y_1, \dots, y_n | \sigma^2) = (2\pi\sigma^2)^{-(n-1)/2} n^{-1/2} \exp \left\{ \frac{-\sum_{i=1}^n (y_i - \bar{y})^2}{2\sigma^2} \right\}$$

and maximizing it leads to the sample variance (REML) estimate of the variance

$$\hat{\sigma}_{\text{MIL}}^2 = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n - 1}.$$

The same MEE is obtained when the prior  $p(\mu, \sigma^2 | H_1)$  does not depend on the value of the variance under  $H_0$  as the denominator of the Bayes factor is simply a multiplicative factor that does not change its maximum. It is reassuring that different methods produce the same estimate in these situations. However, the important difference between these methods is the motivation and interpretation of the resulting estimate – the MEE represents a natural estimate for  $\theta$  because it is the parameter value for which the data provide the most evidence over an alternative hypothesis, while the MLE is defined without reference to alternatives.

### 3.4 Likelihoodist inference

The likelihoodist school of statistical inference ([Barnard, 1949](#); [Edwards, 1971](#); [Royall, 1997](#)) rejects the use of prior distributions to formulate alternatives or to eliminate nuisance parameters, but it also shares many features with the BFF paradigm. That is, if point priors are assigned to the parameters, the Bayes factor reduces to a likelihood ratio which is the evidence measure used by likelihoodists. For this reason, BFF inferences correspond to likelihoodist inferences if the Bayesian and likelihoodist agree on the used point priors.

However, there is disagreement when it comes to the use of support sets. Likelihoodists typically

define their support sets based on the relative likelihood

$$L(\theta) = \frac{p(y | \theta)}{p(y | \hat{\theta}_{\text{ML}})}.$$

For example, [Royall \(1997\)](#) recommended reporting the set of parameter values with relative likelihood greater than  $k = 1/8$  (at most ‘strong’ evidence against them) or  $k = 1/32$  (at most ‘quite strong’ evidence against them). From a Bayesian perspective, using the observed MLE as a prior under the alternative seems to hardly represent genuine prior knowledge or an alternative theory, but rather a cherry-picked alternative that gives to the most biased assessment of support for the alternative ([Berger and Sellke, 1987](#)).

## 4 Applications

We will now illustrate application of the BFF inference framework on several real-world examples.

### 4.1 Binomial proportion

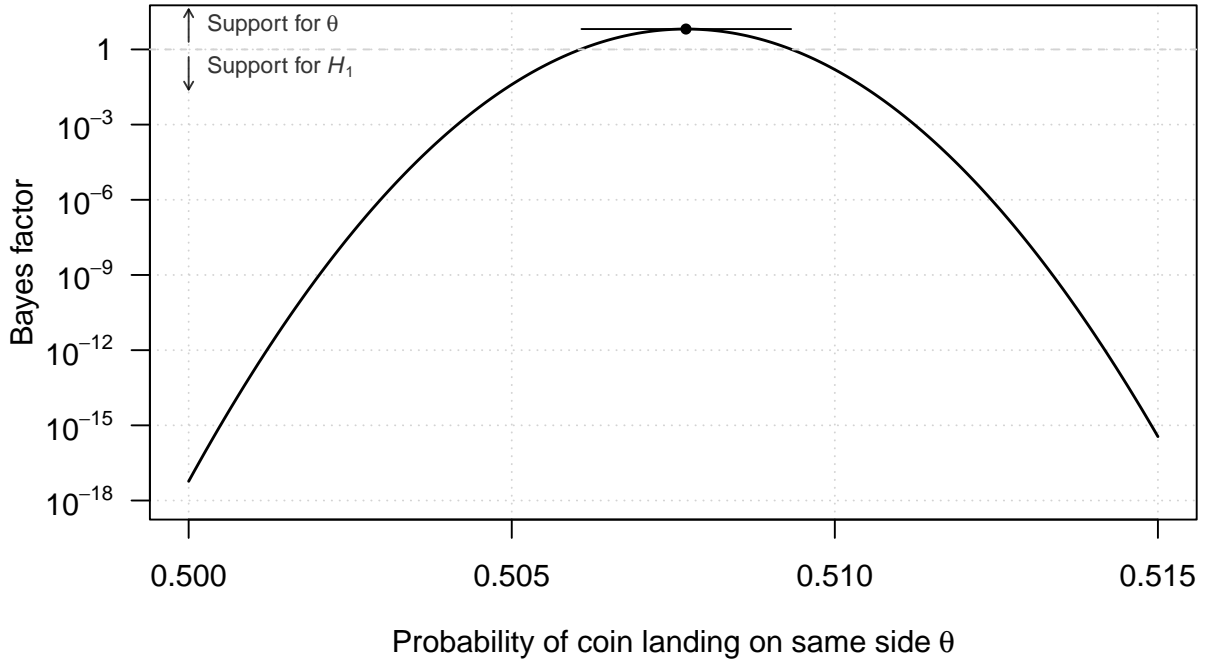
[Bartoš et al. \(2023\)](#) conducted a study to test the hypothesis that fair coins tend to land on the same side as they started slightly more often (with a probability of about 0.51). This hypothesis was formulated by [Diaconis et al. \(2007\)](#) based on a physical model of coin flipping. During the course of the study, 48 participants contributed to the collection of  $n = 350'757$  coin flips among which  $y = 178'078$  landed on the same side as they started.

We will now assume a binomial data model  $Y | \theta \sim \text{Bin}(n, \theta)$  and conduct inferences regarding the unknown probability  $\theta$ . In their pre-registered analysis, [Bartoš et al. \(2023\)](#) specified a truncated beta prior for the probability  $\theta$  under the alternative ( $\theta | H_1 \sim \text{Beta}(a, b)_{[l, u]}$ ). Based on this prior, the Bayes factor for testing  $H_0: \theta = \theta_0$  against  $H_1: \theta \neq \theta_0$  is

$$\text{BF}_{01}(y; \theta_0) = \frac{\theta_0^y (1 - \theta_0)^{n-y}}{B(a + y, b + n - y) / B(a, b)} \times \frac{I_u(a, b) - I_l(a, b)}{I_u(a + y, b + n - y) - I_l(a + y, b + n - y)}$$

with the beta function  $B(a, b) = \int_0^1 t^{a-1} (1 - t)^{b-1} dt$  and the incomplete regularized beta function  $I_x(a, b) = \{\int_0^x t^{a-1} (1 - t)^{b-1} dt\} / B(a, b)$ . Specifically, [Bartoš et al. \(2023\)](#) assigned the hyperparameters  $a = 5100, b = 4900, l = 0.5, u = 1$  to instantiate an alternative hypothesis that closely aligns with the theoretical prediction from [Diaconis et al. \(2007\)](#) of a 0.51 probability with slight uncertainty around it.

Figure 3 shows the resulting BFF for a range of probabilities from 0.5 to 0.515. Looking at the BFF evaluated at  $\theta = 0.5$ , we can see the Bayes factor reported by [Bartoš et al. \(2023\)](#): There is extreme evidence ( $\text{BF}_{01} = 1 / (1.71 \times 10^{17})$ ) against  $\theta = 0.5$  and in favor of the alternative concentrated around  $\theta = 0.51$ . This result hence provides decisive evidence against the hypothesis that coins tend to land on the same side with equal probability. However, the BFF framework permits further insights. For example, we can see that all probability values up to about 0.504 and all values larger than 0.512 are decisively refuted by the data, each having an associated Bayes factor below  $10^{-3}$ . Furthermore, the  $k = 1$  support interval from 0.506 to 0.509 shows the probability values that are better supported by



**Figure 3:** Bayes factor function analysis of data from [Bartoš et al. \(2023\)](#): from  $n = 350'757$  coin flips,  $y = 178'078$  landed on the same side as they started. A beta prior tightly concentrated around the theoretically predicted probability of 51% is assigned to the probability under the alternative ( $\theta \mid H_1 \sim \text{Beta}(5100, 4900)_{[0.5, 1]}$ ).

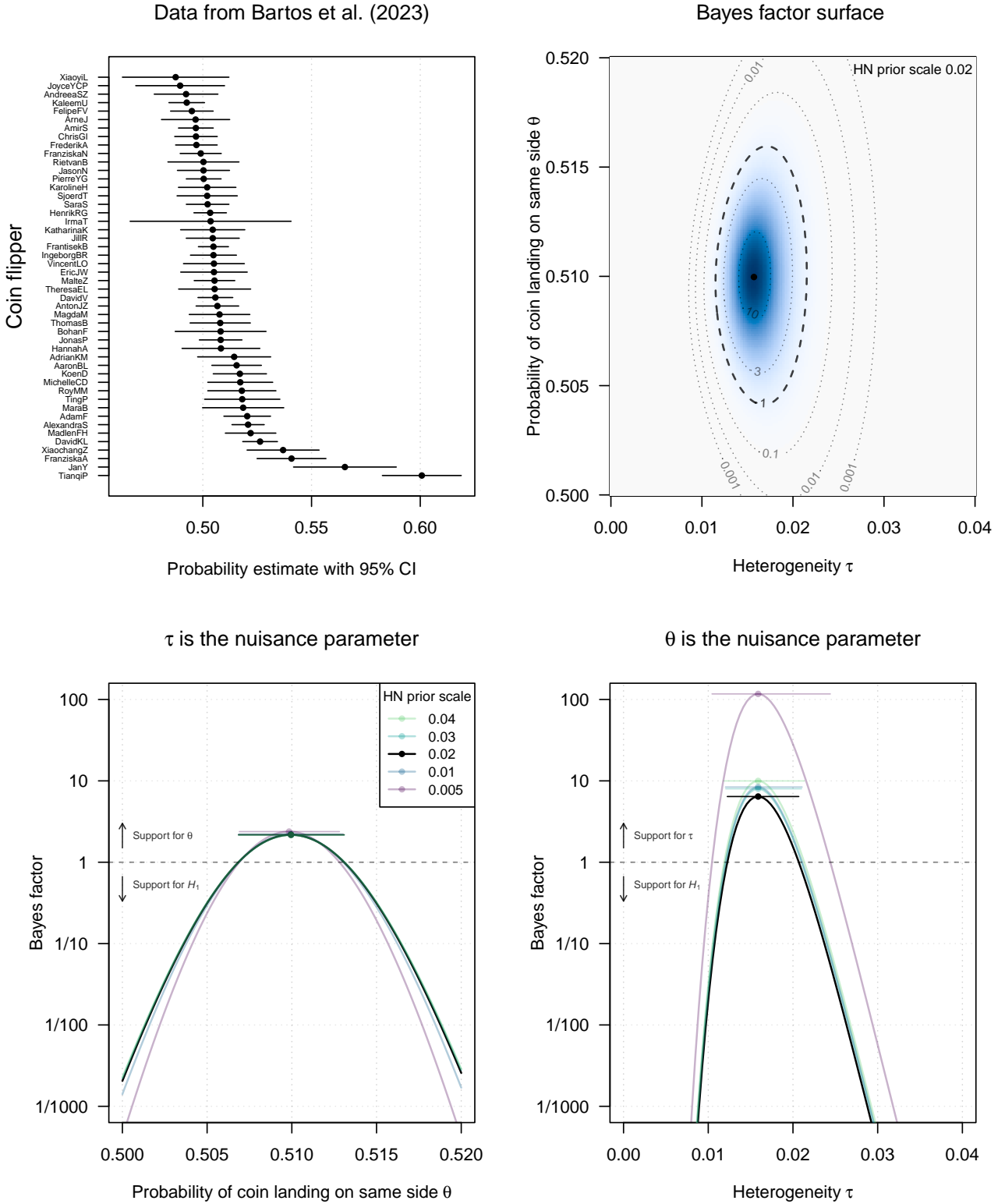
the data than the specified alternative, which excludes the theoretically predicted  $\theta = 0.51$ . The MEE at  $\hat{\mu}_{\text{ME}} = 0.508$  is the best supported value, with  $k_{\text{ME}} = 6.51$  indicating substantial evidence over the alternative concentrated around 0.51.

Both the  $k = 1$  support interval and the MEE coincide with the 95% credible interval and posterior mean based on a uniform prior distribution which were reported by [Bartoš et al. \(2023\)](#) alongside the Bayes factor for  $\theta = 0.5$ . The difference, however, is that MEE, Bayes factor, and support intervals are all linked to the same BFF based on the same prior and data model, whereas this is not the case for their Bayes factor, posterior mean, and credible interval approach.

## 4.2 Meta-analysis

The previous analysis assumed that coin flips were independent among participants and trials. The top left plot in Figure 4 shows that this assumption seems violated as the estimated probabilities that a coin lands on the same side for each of the 48 study participants are clearly heterogeneous. This suggests that the analysis should be modified to account for heterogeneity. In the following, we will therefore synthesize these estimates while accounting for heterogeneity with a meta-analysis, as [Bartoš et al. \(2023\)](#) did.

Suppose we have  $i = 1, \dots, n$  estimates  $y_i$  with (assumed to be known) standard errors  $\sigma_i$ . The



**Figure 4:** Bayes factor analysis of coin flipping experiments from Bartoš et al. (2023), taking into account between-flipper heterogeneity. The product of a truncated beta prior ( $\theta \mid H_1 \sim \text{Beta}(5100, 4900)_{[0.5, 1]}$ ) for  $\theta$  and a half-normal prior with scale 0.02 for  $\tau$  are assigned under the alternative  $H_1$ . The same priors are assumed when the parameters are nuisance parameters under  $H_0$  (bottom plots). The bottom plots also show the BFF for other scale parameters of the half-normal prior.

estimates are assumed to be normally distributed around a subject specific parameter  $\theta_i$ , i.e.,

$$\begin{aligned} y_i \mid \theta_i, \sigma_i^2 &\sim \mathcal{N}(\theta_i, \sigma_i^2) \\ \theta_i \mid \theta, \tau^2 &\sim \mathcal{N}(\theta, \tau^2). \end{aligned}$$

Marginalized over the study-specific parameters, the distribution of an estimate is then

$$y_i \mid \theta, \tau, \sigma_i^2 \sim \mathcal{N}(\theta, \sigma_i^2 + \tau^2).$$

There are two unknown parameters,  $\theta$  and  $\tau$ . The mean  $\theta$  quantifies the average true parameter across units (participants, studies, etc.), while the heterogeneity standard deviation  $\tau$  quantifies the heterogeneity of these true parameters. The Bayes factor for testing  $H_0: \theta = \theta_0, \tau = \tau_0$  against  $H_1: \theta \neq \theta_0, \tau \neq \tau_0$  is then given by

$$\text{BF}_{01}(y_1, \dots, y_n; \theta_0, \tau_0) = \frac{\prod_i^n \mathcal{N}(y_i \mid \theta_0, \tau_0^2)}{\int_0^\infty \int_{-\infty}^{+\infty} \prod_i^n \mathcal{N}(y_i \mid \theta, \tau^2) p(\theta, \tau \mid H_1) d\theta d\tau}$$

with  $\mathcal{N}(x \mid m, v)$  denoting the normal density with mean  $m$  and variance  $v$  evaluated at  $x$ .

As in the previous analysis we assigned a  $\theta \mid H_1 \sim \text{Beta}(5100, 4900)_{[0.5, 1]}$  prior to the average probability  $\theta$  under the alternative  $H_1$ . In addition, we assigned a half-normal prior  $p(\tau \mid H_1) = \sqrt{2/\pi} \exp\{-\tau^2/(2s^2)\}/s$  to the heterogeneity standard deviation  $\tau$ , and assumed it to be independent of  $\theta$ . Half-normal priors are commonly used in meta-analysis due to their simplicity and desirable properties such as nearly uniform behavior around zero  $\tau = 0$  (see e.g., Röver et al., 2021). We choose a scale  $s = 0.02$  because the resulting prior gives a 95.4% probability to  $\tau$  values smaller than 0.04, thus encoding the possibility of no heterogeneity (all probabilities are the same when  $\tau = 0$ ) up to small amounts of heterogeneity (the true participant probabilities differ by a few percentage points). Several BFFs for priors with smaller or larger scale parameters are also shown in Figure 4 as sensitivity analyses.

The top-right plot in Figure 4 shows the BFF in a two-dimensional surface when both parameters are considered as focus parameters. In contrast to the analysis that ignored between-participant heterogeneity, we see that the MEE for the average probability ( $\hat{\theta}_{\text{ME}} = 0.51$ ) is now consistent with the theoretical prediction of Diaconis et al. (2007). In addition, the MEE for the heterogeneity standard deviation ( $\hat{\tau}_{\text{ME}} = 0.016$ ) suggests small but non-negligible heterogeneity. This MEE receives strong support over the alternative ( $k_{\text{ME}} = 14$ ). The relatively concentrated  $k = 1$  support region indicates that probabilities from around 0.505 to 0.515 along with heterogeneity standard deviations from 0.012 to 0.021 are supported by the data over the alternative. Finally, the BFF shows that probabilities of  $\theta = 0.5$  and no heterogeneity  $\tau = 0$  are clearly refuted by the data over the alternative ( $\log \text{BF}_{01} = -1.81 \times 10^5$ ).

The two bottom plots in Figure 4 show BFFs when either  $\tau$  or  $\theta$  is considered as nuisance parameter. In both cases, the same prior as for the alternative  $H_1$  was assigned to the corresponding nuisance parameter under  $H_0$ . In addition, BFFs for other choices of the scale parameter of the half-normal prior were computed to assess the sensitivity of the results to this choice. We see that the two MEEs ( $\hat{\theta}_{\text{ME}} = 0.51$  and  $\hat{\tau}_{\text{ME}} = 0.016$ ) align with the joint MEEs, but their evidence values ( $k_{\text{ME}} = 2.2$



and  $k_{ME} = 6.4$ , respectively) indicate less support over the alternative than for the joint one. Finally, looking at the colored BFFs obtained by changing the scale parameter of the half-normal prior assigned to  $\tau$ , we see that the scale has little effect on inferences about the probability  $\theta$ , but a more pronounced effect on inferences about  $\tau$ . Increasing the scale of the prior does not seem to change the BFF too much, while decreasing the scale to a value of  $s = 0.005$  dramatically increases the height of the BFF, increasing the support of the MEE and surrounding values over the alternative. This seems reasonable, since the data show clear signs of heterogeneity, while a prior with such a small scale would predict almost none.

### 4.3 Replication studies

In a replication study, researchers repeat an original study as closely as possible in order to assess whether consistent results can be obtained (National Academies of Sciences, Engineering, and Medicine, 2019). Various types of Bayes factor approaches have been proposed to quantify the degree to which a replication study has replicated an original study (Verhagen and Wagenmakers, 2014; Ly et al., 2018; Harms, 2019; Pawel and Held, 2022; Pawel et al., 2023a). A common idea of all approaches is that the posterior distribution of the unknown parameters based on the data from the original study is used as the prior distribution in the analysis of the replication data. If the replication data support this prior distribution, this suggests replication success. We will now show how this idea translates to analyzing replication studies with BFFs.

Suppose that original and replication study provide an effect estimate  $y_o$  and  $y_r$  with standard error  $\sigma_o$  and  $\sigma_r$ , respectively. Each is supposed to be normally distributed around the underlying effect size  $\theta$  with (assumed to be known) variance equal to its squared standard error  $\sigma^2$ , i.e.,  $y_i | \theta \sim N(\theta, \sigma_i^2)$  for  $i \in \{o, r\}$ . A ‘replication BFF’ may then be obtained by contrasting the null hypothesis  $H_0: \theta = \theta_0$  to the alternative  $H_1: \theta \sim N(y_o, \sigma_o^2)$ , where the prior under the alternative is the posterior distribution of  $\theta$  based on the original data and a flat prior for  $\theta$  (Verhagen and Wagenmakers, 2014). This leads to the following BFF

$$BF_{01}(y_r; \theta_0) = \sqrt{1 + \frac{\sigma_o^2}{\sigma_r^2}} \exp \left[ -\frac{1}{2} \left\{ \frac{(y_r - \theta_0)^2}{\sigma_r^2} - \frac{(y_r - y_o)^2}{\sigma_r^2 + \sigma_o^2} \right\} \right]$$

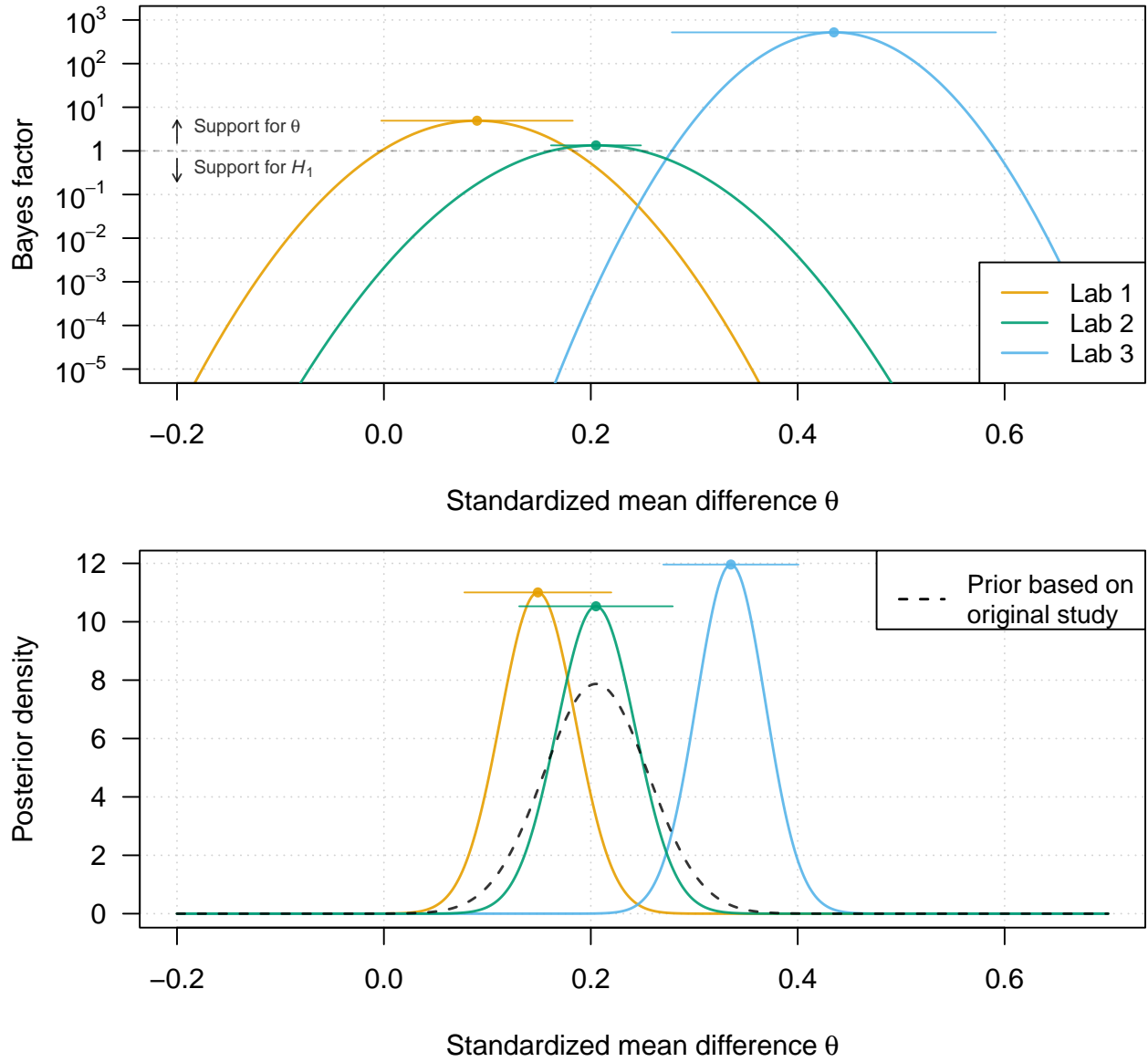
with MEE at the replication effect estimate  $\hat{\theta}_{ME} = y_r$ , evidence value

$$k_{ME} = \sqrt{1 + \frac{\sigma_o^2}{\sigma_r^2}} \exp \left\{ \frac{-(y_r - y_o)^2}{2(\sigma_o^2 + \sigma_r^2)} \right\},$$

and  $k$  support interval

$$y_r \pm \sigma_r \sqrt{\log \left( 1 + \frac{\sigma_o^2}{\sigma_r^2} \right) + \frac{(y_r - y_o)^2}{\sigma_r^2 + \sigma_o^2} - \log k^2}.$$

We will now reanalyze data from three replication studies that were part of the large-scale replication project in the social-behavioral sciences (Protzko et al., 2023). The original experiment termed “Label” found the following central result:



**Figure 5:** Bayes factor functions with maximum evidence estimates and  $k = 1$  support intervals (top) and posterior distribution with posterior modes and 95% highest posterior density credible intervals (bottom) for the three replication studies from the “Labels” experiment (Protzko et al., 2023). The original study found an estimated standardized mean difference of  $y_o = 0.205$  with standard error  $\sigma_o = 0.051$  which is used to formulate the prior distribution under the alternative  $\theta \mid H_1 \sim N(y_o, \sigma_o^2)$ . The replication effect estimates were  $y_{ri} \in \{0.09, 0.205, 0.435\}$  with standard errors  $\sigma_{ri} \in \{0.052, 0.057, 0.044\}$ , respectively, and are assumed to be normally distributed  $y_{ri} \mid \theta \sim N(\theta, \sigma_{ri}^2)$ . The posterior is obtained by multiplying the BFF with the prior density.

*“When a researcher uses a label to describe people who hold a certain opinion, he or she is interpreted as disagreeing with that opinion when a negative label is used and agreeing with that opinion when a positive label is used.” Protzko et al. (2023, p. 2)*

which was based on an estimated standardized mean difference  $y_o = 0.205$  with standard error  $\sigma_o = 0.051$ . The replication studies conducted in three other labs found a smaller, a similar, and a much larger effect estimate ( $y_{ri} \in \{0.09, 0.205, 0.435\}$  with standard errors  $\sigma_{ri} \in \{0.052, 0.057, 0.044\}$ ,

respectively). The top plot in Figure 5 shows the associated BFFs, MEEs, and  $k = 1$  support intervals. We see that, the BFFs peak at the corresponding replication estimate, but the height of these peaks differs between replications. The replication from lab 2 produced an effect estimate identical to the original one, so there is little support of its MEE over the alternative based on the original study as the estimates from both studies are in close agreement. In contrast, the MEEs from lab 1 and lab 3 receive substantial and very strong support over the alternative because they are either smaller or larger. In turn, their  $k = 1$  support intervals are much wider compared to the narrow support interval from lab 2. Finally, we can also see that the BFF from lab 1 indicates absence of evidence for or against no effect ( $\text{BF}_{01} \approx 1$  at  $\theta = 0$ ), whereas the BFFs from labs 2 and 3 indicate strong and decisive evidence against no effect up to very small effects of around  $\theta = 0.1$ .

The bottom plot in Figure 5 illustrates the posterior distributions, conveniently obtained by multiplying the BFF by the prior distribution based on the original data. We can see that these posteriors represent a synthesis of the original and replication studies. For example, the posterior based on the replication from lab 3 is centered around 0.34 with 95% credible interval from 0.27 to 0.4. Clearly, this interval excludes both the original ( $y_o = 0.205$ ) and the replication effect estimates ( $y_{r3} = 0.435$ ), leading to an opposite conclusion from the BFF, which indicates decisive evidence for the MEE at  $y_r$ . This phenomenon is perhaps best described in the words of Stephen Senn

*“Bayesian: One who, vaguely expecting a horse and catching a glimpse of a donkey, strongly concludes he has seen a mule.”* (Senn, 2007, p. 46)

It raises the question of whether a synthesis of prior and likelihood is appropriate in this situation. The BFF, on the other hand, seems to be more robust to such prior-data conflicts because it relies only on the data-based core of Bayesian inference.

## 5 Discussion

We showed how Bayes factors can be used for parameter estimation, extending their traditional use cases of hypothesis testing and model comparison. We also linked these ideas to the overarching concept of Bayes factor functions (BFFs), which are Bayes factor analogues of  $P$ -value functions, and are likewise particularly useful for reporting of analysis results. This provides data analysts with a unified framework for statistical inference that is distinct from conventional frequentist and Bayesian approaches: BFF inference uses the Bayesian evidence calculus, but without synthesizing data and prior. At the same time, BFF inference is closely related to likelihood-based inference, but also includes a natural way to deal with nuisance parameters.

Like the likelihoodist and Neyman-Pearson paradigms of statistical inference, BFF inference requires the formulation of alternative hypotheses. For this reason, BFFs are particularly valuable in contexts where there are strong theories or prior data available to formulate alternative hypotheses. In cases where there are no clear alternative hypotheses, data analysts may use BFFs based on ‘default’ prior distribution (e.g., unit-information priors, see Kass and Wasserman, 1995) but should acknowledge this limitation and report sensitivity analyses (e.g., BFFs for different prior distributions). Another possibility is to base BFF inferences on Bayes factor bounds, which give a bound on the maximum evidence against parameter values, but at the cost of losing the ability to quantify

evidence *in favour* of parameter values (Pawel et al., 2023b). Finally, where under their control, data analysts should design experiments and studies so that conclusive inferences can be drawn from the data collected, and this also applies to BFF inferences. While some research has been done on study design with support intervals (Pawel et al., 2023b), future research needs to investigate how experiments need to be designed to enable conclusive inference with BFFs.

Bayesian, likelihoodist, or predictive reasoning may all motivate the Bayes factor as a natural tool for quantifying the relative evidence or support of competing hypotheses. Nevertheless, neither the Bayes factor nor any other measure of statistical evidence is infallible or suitable for all purposes. Any type of statistical inference can lead to distorted scientific inferences when used in a bright-line fashion without consideration of contextual factors (Goodman, 2016; Greenland, 2023). We believe that BFFs are useful in this regard because they shift the focus from finding evidence against a single null hypothesis to making gradual and quantitative inferences.

## Acknowledgments

I thank František Bartoš, Andrew Fowlie, Leonhard Held, and Eric-Jan Wagenmakers for valuable comments on drafts of the manuscript. The acknowledgment of these individuals does not imply their endorsement of the paper.

## Conflict of interest

We declare no conflict of interest.

## Software and data

The code and data to reproduce our analyses is openly available at XXXX. A snapshot of the repository at the time of writing is available at XXX. We used the statistical programming language R version XX (R Core Team, 2022) for analyses.

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## Computational details

```
cat(paste(Sys.time(), Sys.timezone(), "\n"))

## 2024-02-05 10:07:38.343612 Europe/Zurich

sessionInfo()

## R version 4.3.2 (2023-10-31)
## Platform: x86_64-pc-linux-gnu (64-bit)
## Running under: Ubuntu 22.04.3 LTS
##
## Matrix products: default
## BLAS: /usr/lib/x86_64-linux-gnu/blas/libblas.so.3.10.0
## LAPACK: /usr/lib/x86_64-linux-gnu/lapack/liblapack.so.3.10.0
##
## locale:
##  [1] LC_CTYPE=en_US.UTF-8      LC_NUMERIC=C
##  [3] LC_TIME=de_CH.UTF-8      LC_COLLATE=en_US.UTF-8
##  [5] LC_MONETARY=de_CH.UTF-8  LC_MESSAGES=en_US.UTF-8
##  [7] LC_PAPER=de_CH.UTF-8     LC_NAME=C
##  [9] LC_ADDRESS=C             LC_TELEPHONE=C
## [11] LC_MEASUREMENT=de_CH.UTF-8 LC_IDENTIFICATION=C
##
## time zone: Europe/Zurich
## tzcode source: system (glibc)
##
## attached base packages:
## [1] stats      graphics  grDevices  utils      datasets  methods   base
##
## other attached packages:
## [1] metabf_0.1      ReplicationSuccess_1.3.1 metadat_1.3-0
## [4] knitr_1.44
##
## loaded via a namespace (and not attached):
## [1] cubature_2.1.0  compiler_4.3.2  tools_4.3.2     Rcpp_1.0.12
## [5] mathjaxr_1.6-0  codetools_0.2-19 highr_0.10       digest_0.6.33
## [9] xfun_0.40       evaluate_0.21
```