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Problem 1. It is promised that a given coin is either fair ($Pr(Head) = 1/2$) or biased with $Pr(Head) = 1/2 + \epsilon$ where $0 < \epsilon < 1/2$. Show that $100/\epsilon^2$ coin tosses are sufficient to correctly determine the type of coin (fair or biased) with at least 4/5 probability, i.e., give an algorithm that will need at most $100/\epsilon^2$ coin tosses, and should have the following guarantee: if the coin is fair the algorithm will return 'fair' with probability at least 4/5, and if the coin is biased then algorithm will return 'biased' with probability at least 4/5.

Solution. We need to design an algorithm that can distinguish between a fair coin ($\Pr(\text{Head}) = \frac{1}{2}$) and a biased coin ($\Pr(\text{Head}) = \frac{1}{2} + \epsilon$) with $\epsilon > 0$, using at most $\frac{100}{\epsilon^2}$ tosses, such that the algorithm achieves the following:

- If the coin is fair, the algorithm returns *fair* with probability at least $\frac{4}{5}$.
- If the coin is biased, the algorithm returns *biased* with probability at least $\frac{4}{5}$.

The algorithm is given as follows:

1. Toss the coin $n = \frac{100}{\epsilon^2}$ times.
2. Let S be the number of heads observed in these n tosses.
3. Compute the empirical probability of heads:
$$\hat{p} = \frac{S}{n}.$$
4. Use the following decision rule:
    - If $|\hat{p} - \frac{1}{2}| \geq \frac{\epsilon}{2}$, classify the coin as *biased*.
    - Otherwise, classify the coin as *fair*.

Proof:

Let $X_1, X_2, \ldots, X_n$ be the outcomes of the n tosses, where $X_i = 1$ if the i-th toss is a head, and $X_i = 0$ otherwise. Each X_i is an independent Bernoulli random variable with:

$$\mathbb{E}[X_i] = \Pr(\text{Head}) = p,$$

where $p = \frac{1}{2}$ if the coin is fair, and $p = \frac{1}{2} + \epsilon$ if the coin is biased.

The total number of heads is:

$$S = \sum_{i=1}^{n} X_i,$$

and the empirical probability is:

$$\hat{p} = \frac{S}{n}.$$

Using Hoffding's inequality, for any $t > 0$, we have:

$$\Pr\left(|\hat{p} - p| \geq t\right) \leq 2\exp(-2nt^2).$$

Thus, assuming the coin is fair,

$$\Pr\left(|\hat{p} - \frac{1}{2}| \geq \frac{\epsilon}{2}\right) \leq 2\exp(-2n\frac{\epsilon^2}{2}).$$

For $n = \frac{100}{\epsilon^2}$,

$$\Pr\left(|\hat{p} - \frac{1}{2}| \geq \frac{\epsilon}{2}\right) \leq 2\exp(-2\frac{100}{\epsilon^2}\frac{\epsilon^2}{2}) = 2\exp(-50).$$

Similarly, assuming that the coin is biased,

$$\Pr\left(|\hat{p} - (\frac{1}{2} + \epsilon)| \geq \frac{\epsilon}{2}\right) \leq 2\exp(-2n\frac{\epsilon^2}{2}).$$

For $n = \frac{100}{\epsilon^2}$,

$$\Pr\left(|\hat{p} - (\frac{1}{2} + \epsilon)| \geq \frac{\epsilon}{2}\right) \leq 2\exp(-2\frac{100}{\epsilon^2}\frac{\epsilon^2}{2}) = 2\exp(-50).$$

As the decision boundary is at $1/2 + \epsilon/2$, after $n = \frac{100}{\epsilon^2}$ tosses,

$$\Pr\left(\hat{p} \geq \frac{1}{2} + \frac{\epsilon}{2} \mid Fair\right) \leq \exp(-50)$$

$$\Pr\left(\hat{p} \leq \frac{1}{2} + \frac{\epsilon}{2} \mid Biased\right) \leq \exp(-50)$$

Thus, we end up getting a much better probability of detection than 4/5, for both fair and biased coins, with the probability of being incorrect very small $\approx e^{-50}$ $\square$