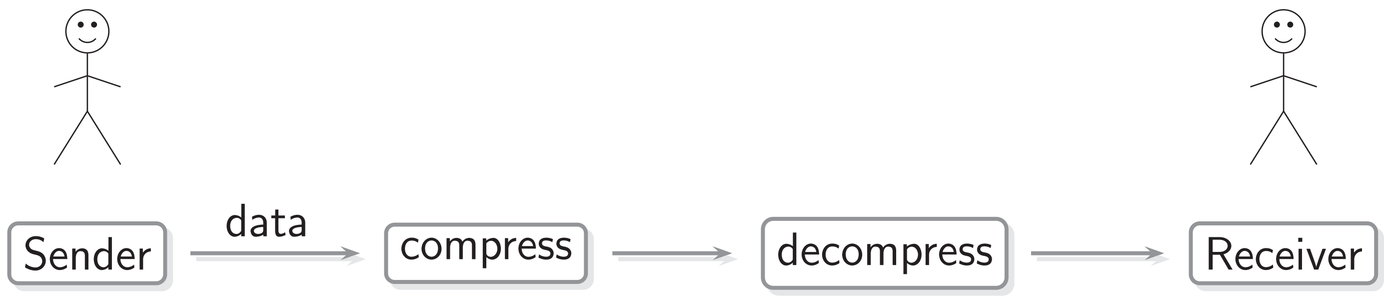
Chapter 3: Compression Coding

Lectures 10-11


Bla Bla Bla Bla
Bla Bla Bla
Bla Bla Bla
Bla Bla
25 words or less!



Compression coding

- Variable length codes
- Assume that there is no channel noise: source coding

Compression coding

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- Assume that there is no channel noise: source coding

Define

source S	with	symbols	$s_1, \dots, s_q$
	with	probabilities	$p_1, \dots, p_q$
code C	with	codewords	$\mathbf{c}_1, \dots, \mathbf{c}_q$
		of lengths	$\ell_1, \dots, \ell_q$
		and radix r	

Morse code

Morse code is a variable-length ternary code (radix $r = 3$).
Its code alphabet is

$\bullet$	called dot (or dit or di)
$-$	called dash (or dah)
$\mathtt{p}$	a pause

The codewords are strings of $\bullet$ and $-$ terminated (separated) by $\mathtt{p}$.

Morse code

A	●—p	N	—●p	1	●————p
B	—●●●p	O	———p	2	●●———p
C	—●—●p	P	●——●p	3	●●●——p
D	—●●p	Q	——●—p	4	●●●●—p
E	●p	R	●—●p	5	●●●●●p
F	●●—●p	S	●●●p	6	—●●●●p
G	——●p	T	—p	7	——●●●p
H	●●●●p	U	●●—p	8	———●●p
I	●●p	V	●●●—p	9	————●p
J	●———p	W	●——p	0	—————p
K	—●—p	X	—●●—p		
L	●—●●p	Y	—●——p		
M	——p	Z	——●●p		

(See Appendix 1 for full list)

Example

The Morse code encodes the word “sky” into

●●●p—●—p—●——p

Morse code

	A	●−p		N	−●p	1	●−−−−p
	B	−●●●p		O	−−−p	2	●●−−−p
	C	−●−●p		P	●−−●p	3	●●●−−p
	D	−●●p	0.11%	Q	−−●−p	4	●●●●−p
12.5%	E	●p		R	●−●p	5	●●●●●p
	F	●●−●p		S	●●●p	6	−●●●●p
	G	−−●p	9.25%	T	−p	7	−−●●●p
	H	●●●●p		U	●●−p	8	−−−●●p
	I	●●p		V	●●●−p	9	−−−−●p
	J	●−−−p		W	●−−p	0	−−−−−p
	K	−●−p		X	−●●−p		
	L	●−●●p		Y	−●−−p		
	M	−−p		Z	−−●●p		

(See Appendix 1 for full list)

Common letters have short codewords, and rarer ones longer:

E is ●p
T is −p
Q is −−●−p

Morse code is thus a compression code.

A code C is

- uniquely decodeable (UD) if it can always be decoded unambiguously
- instantaneous if no codeword is the prefix of another
  Such a code is an I-code.

Example

The standard comma code of length 5 is

$$\begin{array}{lclcl} s_1 & \to & \mathbf{c}_1 & = & 0 \\ s_2 & \to & \mathbf{c}_2 & = & 10 \\ s_3 & \to & \mathbf{c}_3 & = & 110 \\ s_4 & \to & \mathbf{c}_4 & = & 1110 \\ s_5 & \to & \mathbf{c}_5 & = & 11110 \\ s_6 & \to & \mathbf{c}_6 & = & 11111 \end{array}$$

Example

The standard comma code of length 5 is

$$\begin{array}{rcrcl} s_1 & \to & \mathbf{c}_1 & = & 0 \\ s_2 & \to & \mathbf{c}_2 & = & 10 \\ s_3 & \to & \mathbf{c}_3 & = & 110 \\ s_4 & \to & \mathbf{c}_4 & = & 1110 \\ s_5 & \to & \mathbf{c}_5 & = & 11110 \\ s_6 & \to & \mathbf{c}_6 & = & 11111 \end{array}$$

This code is an I-code.

Decode

$$1\,1\,0|0|1\,1\,1\,1\,0|1\,1\,0|1\,0|1\,1\,1\,1\,1|1\,1\,0$$

as

$$s_3 s_1 s_5 s_3 s_2 s_6 s_3$$

Example
Consider the code C:

$$\begin{array}{lclcl} s_1 & \to & \mathbf{c}_1 & = & 0 \\ s_2 & \to & \mathbf{c}_2 & = & 01 \\ s_3 & \to & \mathbf{c}_3 & = & 11 \\ s_4 & \to & \mathbf{c}_4 & = & 00 \end{array}$$

This code is not uniquely decodable since, for example,

$$0011$$

can be decoded as $s_1 s_1 s_3$ and $s_4 s_3$.

Example

Consider the code C:

$$\begin{array}{lclcl} s_1 & \rightarrow & \mathbf{c}_1 & = & 0 \\ s_2 & \rightarrow & \mathbf{c}_2 & = & 01 \\ s_3 & \rightarrow & \mathbf{c}_3 & = & 011 \\ s_4 & \rightarrow & \mathbf{c}_4 & = & 0111 \\ s_5 & \rightarrow & \mathbf{c}_5 & = & 1111 \end{array}$$

This code is uniquely decodable but is not instantaneous.

Decode

$$0\,1\,1\,1 \,|\, 1\,1\,1\,1 \,|\, 0\,1\,1 \,|\, 0\,1 \,|\, 0$$

as

$$s_4 s_5 s_3 s_2 s_1$$

Example

Consider the code C:

$$\begin{array}{lclcl} s_1 & \to & \mathbf{c}_1 & = & 0 \\ s_2 & \to & \mathbf{c}_2 & = & 100 \\ s_3 & \to & \mathbf{c}_3 & = & 1011 \\ s_4 & \to & \mathbf{c}_4 & = & 110 \\ s_5 & \to & \mathbf{c}_5 & = & 111 \end{array}$$

This code is an I-code.

Decode

$$0|0|1\,1\,0|0|1\,0\,1\,1|1\,1\,1$$

as

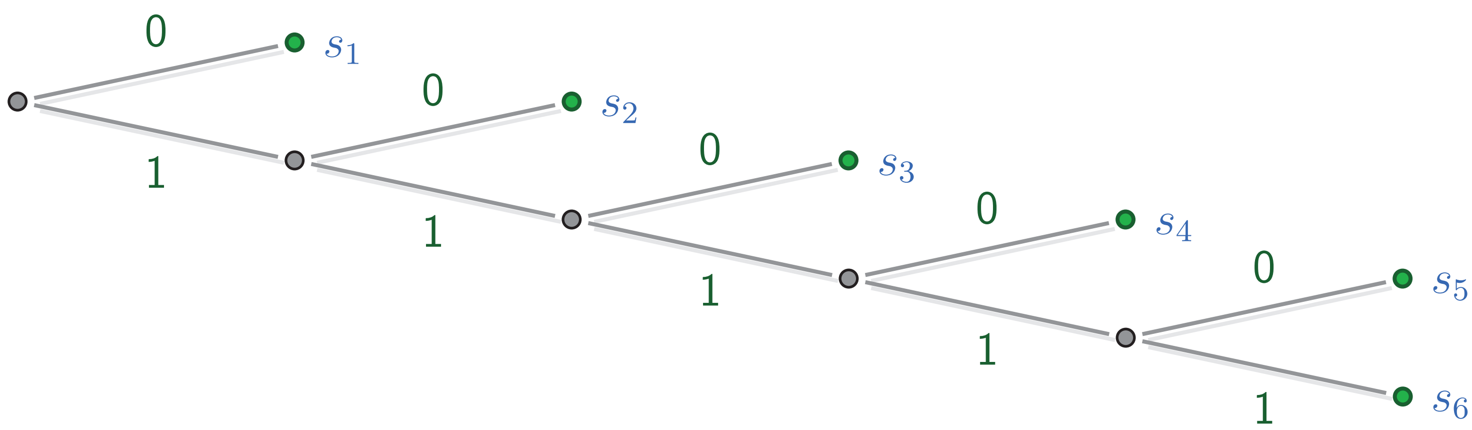
$$s_1 s_1 s_4 s_1 s_3 s_5$$

Decision trees can represent I-codes.

Example

The standard comma code of length 5 is

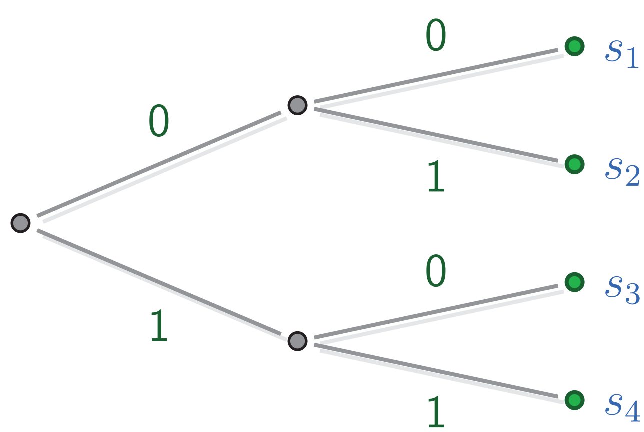
$$\begin{array}{llll} s_1 & \rightarrow & \mathbf{c}_1 & = & 0 \\ s_2 & \rightarrow & \mathbf{c}_2 & = & 10 \\ s_3 & \rightarrow & \mathbf{c}_3 & = & 110 \\ s_4 & \rightarrow & \mathbf{c}_4 & = & 1110 \\ s_5 & \rightarrow & \mathbf{c}_5 & = & 11110 \\ s_6 & \rightarrow & \mathbf{c}_6 & = & 11111 \end{array}$$



Example

Consider the block code C:

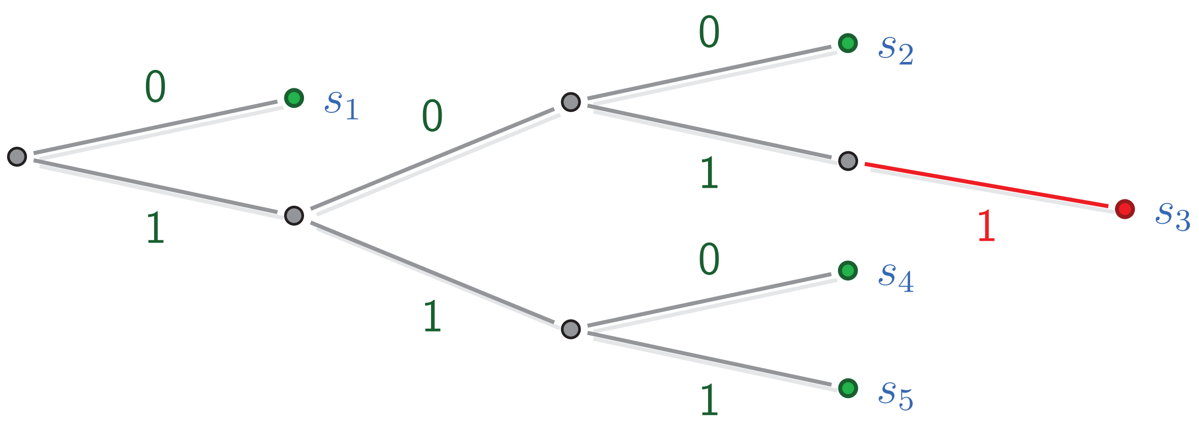
$$\begin{array}{llll} s_1 & \rightarrow & \mathbf{c}_1 & = & 00 \\ s_2 & \rightarrow & \mathbf{c}_2 & = & 01 \\ s_3 & \rightarrow & \mathbf{c}_3 & = & 10 \\ s_4 & \rightarrow & \mathbf{c}_4 & = & 11 \end{array}$$



Example

Consider the code C:

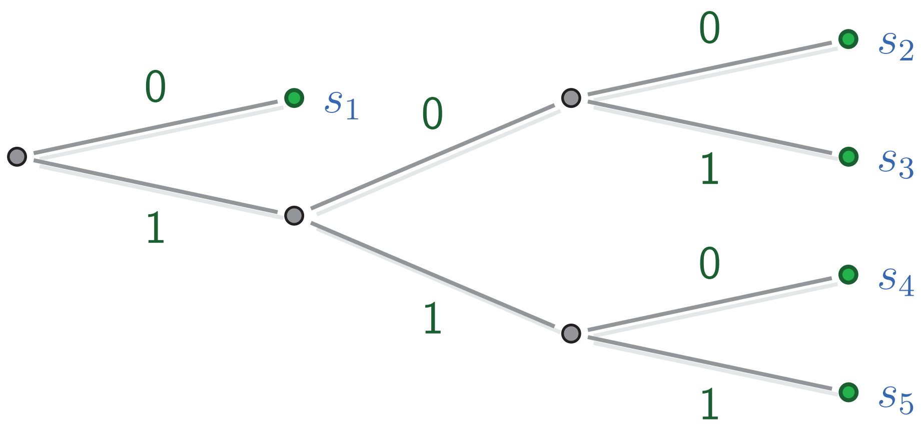
$$\begin{array}{lcccl} s_1 & \rightarrow & \mathbf{c}_1 & = & 0 \\ s_2 & \rightarrow & \mathbf{c}_2 & = & 100 \\ s_3 & \rightarrow & \mathbf{c}_3 & = & 1011 \\ s_4 & \rightarrow & \mathbf{c}_4 & = & 110 \\ s_5 & \rightarrow & \mathbf{c}_5 & = & 111 \end{array}$$



Example

Consider the code C:

$$
\begin{aligned}
s_1 &\rightarrow & \mathbf{c}_1 &= 0 \\
s_2 &\rightarrow & \mathbf{c}_2 &= 100 \\
s_3 &\rightarrow & \mathbf{c}_3 &= 1011 \\
s_4 &\rightarrow & \mathbf{c}_4 &= 110 \\
s_5 &\rightarrow & \mathbf{c}_5 &= 111
\end{aligned}
$$



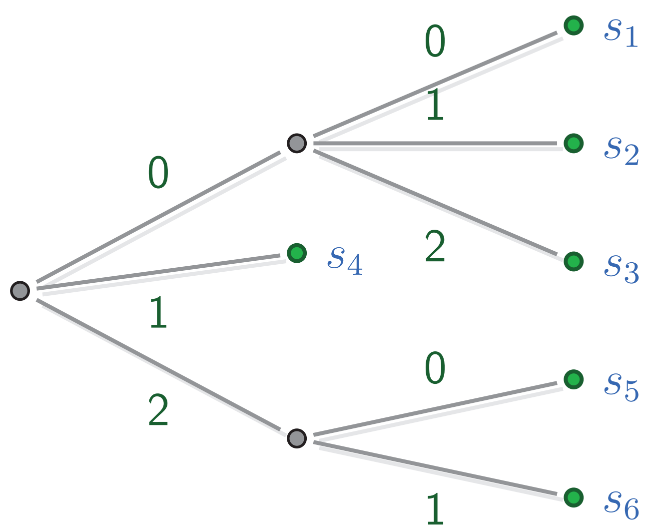
Decision trees can represent I-codes.

- Branches are numbered from the top down.
- Any radix r is allowed.
- Two codes are equivalent if their decision trees are isomorphic.

Example

Code

$$\begin{array}{llll} s_1 & \to & \mathbf{c}_1 & = & 00 \\ s_2 & \to & \mathbf{c}_2 & = & 01 \\ s_3 & \to & \mathbf{c}_3 & = & 02 \\ s_4 & \to & \mathbf{c}_4 & = & 1 \\ s_5 & \to & \mathbf{c}_5 & = & 20 \\ s_6 & \to & \mathbf{c}_6 & = & 21 \end{array}$$



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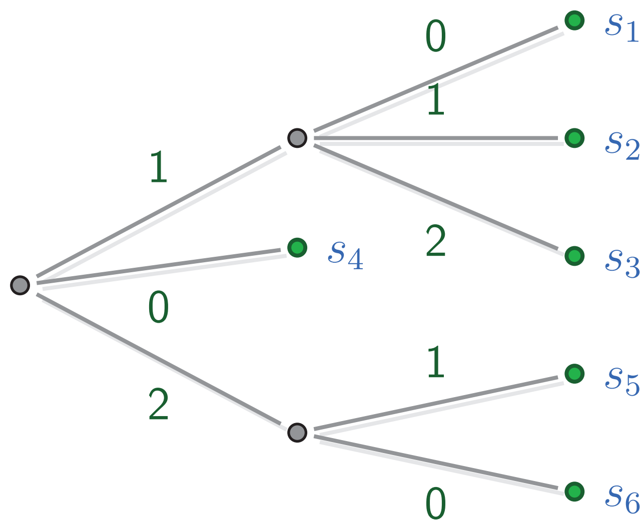
- Two codes are equivalent if their decision trees are isomorphic.

Example

				Code	Equivalent code
s_1	$\rightarrow$	$\mathbf{c}_1$	$=$	00	10
s_2	$\rightarrow$	$\mathbf{c}_2$	$=$	01	11
s_3	$\rightarrow$	$\mathbf{c}_3$	$=$	02	12
s_4	$\rightarrow$	$\mathbf{c}_4$	$=$	1	0
s_5	$\rightarrow$	$\mathbf{c}_5$	$=$	20	21
s_6	$\rightarrow$	$\mathbf{c}_6$	$=$	21	20



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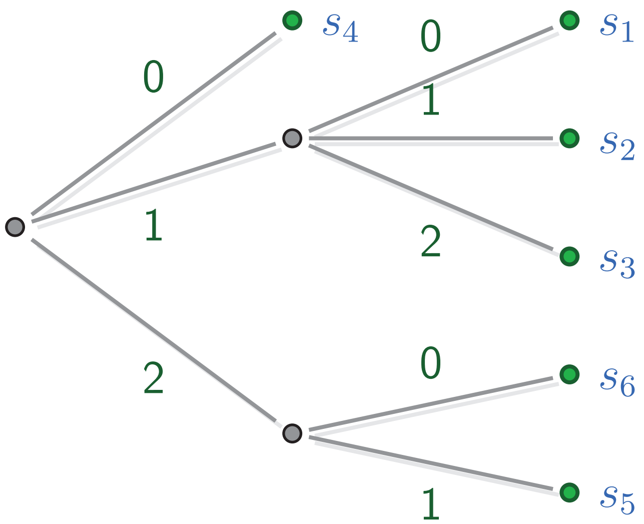

Decision trees can represent I-codes.

- Branches are numbered from the top down.
- Any radix r is allowed.
- Two codes are equivalent if their decision trees are isomorphic.
- By shuffling source symbols, we may assume that $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_q$.

The Kraft-McMillan Theorem

The following are equivalent:

1. There is a radix r UD-code with codeword lengths $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_q$
2. There is a radix r I-code with codeword lengths $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_q$
3. $K = \sum_{i=1}^{q} \left(\frac{1}{r}\right)^{\ell_i} \leq 1$

Example

Is there a radix 2 UD-code with codeword lengths $1, 2, 2, 3$?

No, by the Kraft-McMillan Theorem:

$$\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 = \frac{9}{8} \not\leq 1$$

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2. There is a radix r I-code with codeword lengths $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_q$
3. $K = \sum_{i=1}^{q} \left(\frac{1}{r}\right)^{\ell_i} \leq 1$

Example

Is there a radix 3 I-code with codeword lengths $1, 2, 2, 2, 2, 3$?
Yes, by the Kraft-McMillan Theorem:

$$\left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 = \frac{22}{27} \leq 1$$

Example

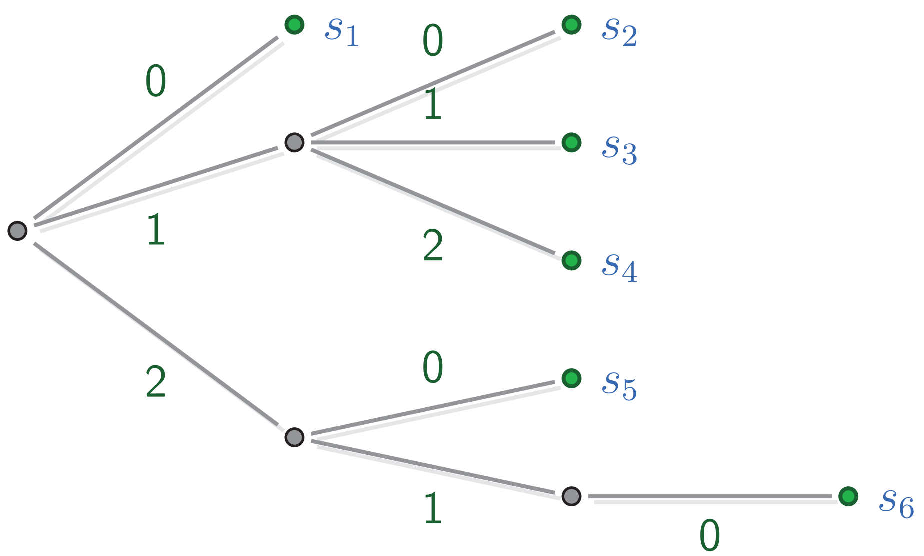
Is there a radix 3 I-code with codeword lengths $1, 2, 2, 2, 2, 3$?

Yes, by the Kraft-McMillan Theorem:

$$\left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 = \frac{22}{27} \leq 1$$

For instance,

$$\begin{array}{lcccl} s_1 & \to & \mathbf{c}_1 & = & 0 \\ s_2 & \to & \mathbf{c}_2 & = & 10 \\ s_3 & \to & \mathbf{c}_3 & = & 11 \\ s_4 & \to & \mathbf{c}_4 & = & 12 \\ s_5 & \to & \mathbf{c}_5 & = & 20 \\ s_6 & \to & \mathbf{c}_6 & = & 210 \end{array}$$



This is a standard I-code.

The Kraft-McMillan Theorem

The following are equivalent:

① There is a radix r UD-code with codeword lengths $\ell_1 \le \ell_2 \le \cdots \le \ell_q$

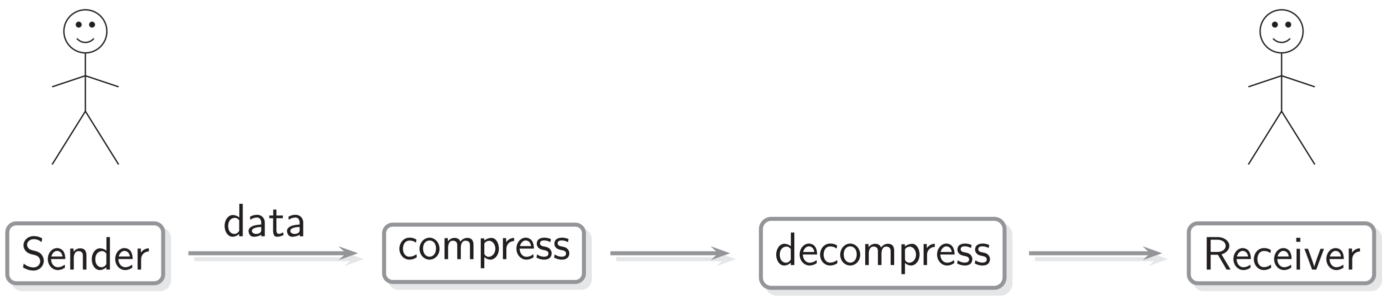
② There is a radix r I-code with codeword lengths $\ell_1 \le \ell_2 \le \cdots \le \ell_q$

③ $K = \displaystyle\sum_{i=1}^{q} \left(\frac{1}{r}\right)^{\ell_i} \le 1$

Proof ② $\Rightarrow$ ① is trivial. We now prove that ① $\Rightarrow$ ③ .
Suppose that a radix r UD-code has codeword lengths $\ell_1 \le \ell_2 \le \cdots \le \ell_q$.
Note that

$$K^n = \left(\sum_{i=1}^{q} \left(\frac{1}{r}\right)^{\ell_i}\right)^n = \sum_{j=1}^{\infty} \frac{N_j}{r^j}$$

where $N_j = |\{(i_1, \ldots, i_n) \in \{\ell_1, \ldots, \ell_q\}^n \,:\, i_1 + \cdots + i_n = j\}|$.



Compression coding

- Variable length codes
- Assume that there is no channel noise: source coding