3 Integration of ordinary differential equations

Answers to $\star$ questions must be submitted to webct, as an extensively commented program and any additional text by Monday 18/11/13 4:30pm (usual late penalties will apply).

IMPORTANT The version of GSL installed on phymat is not the same as that covered by the online manual. There are thus several inconsistencies between the description of some of the GSL packages, particularly the ODE solving package. For your convenience we have setup a website with the appropriate version of the manual which you can find below:

http://www.sr.bham.ac.uk/~ddb/gsl-ref/1.13/

The behaviour of a wide range of physical systems can be modelled by differential equations. Even very simple sets of coupled differential equations can produce rich dynamical behavour.

Most systems are not amenable to direct integration. Given a set of initial (or boundary) conditions one can solve these numerically by considering the 'rates of change' in the solution as the dependent variable(s) are changed by small amounts.

We will consider just the class of ordinary differential equations, but partial differential equations, such as those governing the behaviour of viscous fluids or Einstein's equations describing the dynamical response of massive objects to space-time curvature, are another important class of equations.

3.1 Euler's method

Consider a first order differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = f(x, y) \ . \tag{1}$$

By definition,

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} \ . \tag{2}$$

We can rearrange this to find an approximate iterative formula which advances a solution from $y_0 = y(x_0)$ to $y_1 = y(x_0 + h)$ through an interval h:

$$y_1 = y_0 + hf(x_0, y_0) + O(h^2) \ . \tag{3}$$

We can then repeat this for as many steps, n, as is required to develop a solution for $y_n = y(x_n), x_n = x_0 + nh$.

This process of finding successive approximations to y using the slope of the curve at the starting point of each interval is known as *Euler's method* and is depicted in Figure 1.

You should be able to see that because we only used the first derivative $f(x, y)$ of the curve our error in using this method is $O(h^2)$, which is the size of the next correction in the expansion of $y(x)$.

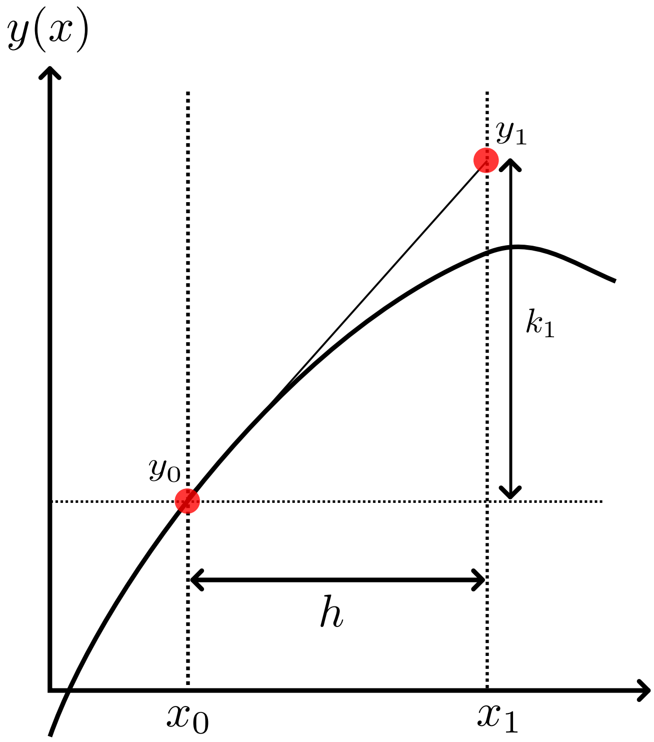

Figure 1: A single step of Euler's method.

1. * Show that for one application of the Euler method the error is $O(h^2)$ by truncating the expansion of $y(x)$ in powers of h.

2. * Using your result from above show that for n applications the error is $O(h)$.

3. * Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + y^2 \,, \tag{4}$$

with the boundary condition $y(0) = 0$.

   (a) Write down the exact solution $y(\pi/4)$.

   (b) Write a program which uses Euler's method find an approximation of $y(\pi/4)$.

   (c) Show that the error in your approximate solution is $O(h)$.

3.2 Runge-Kutta methods

We can improve upon Euler's method by 'averaging' the slope of the curve over the interval h. To do this we take the derivative of the curve at the beginning of the interval, use Euler's method to find an approximate value of y at the midpoint in the interval, and then take the derivative at this point and use Euler's method again to reach the end of the interval. In equations this is

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \\ y_{n+1} &= y_n + k_2 + O(h^3) \end{aligned}$$

and this process is depicted in Figure 2.

Although each application of the Euler method generates errors of $O(h^2)$ you can show that these corrections at $O(h^2)$ cancel out, so that the overall error is $O(h^3)$. This method is known as the *second order Runge-Kutta* method.

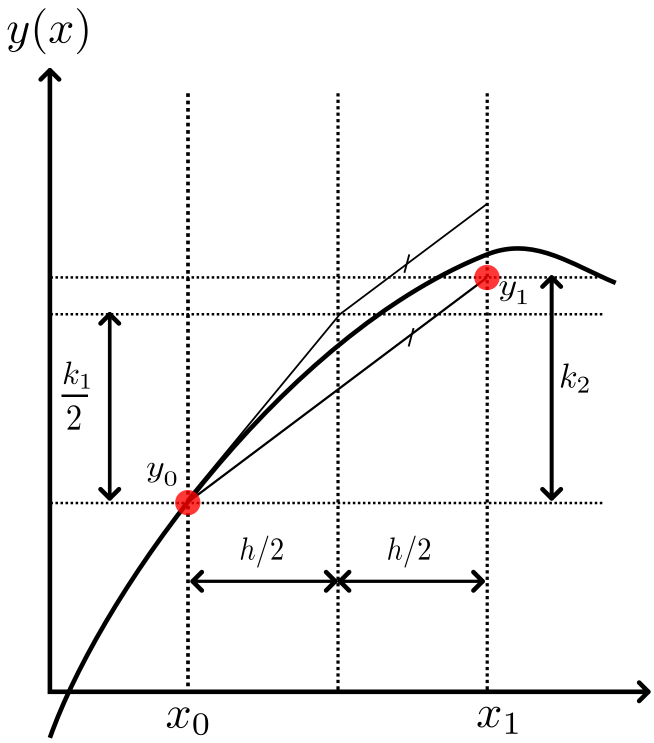

Figure 2: A single step of second order Runge-Kutta.

Higher order Runge-Kutta methods rely on the same idea as this but use more sub-intervals. There are many different ways of expanding $f(x, y)$ so that we can continue to use Euler's method for each sub-interval but successively higher order corrections in the different expansions from each sub-interval cancel each other out.

For more about Runge-Kutta and other methods for solving ODEs see [1].

4. Repeat Question 3 using the second order Runge-Kutta method and demonstrate that the error behaviour is now $O(h^3)$.

3.3 Coupled differential equations

Many physical systems are described by a second-order differential equation, or by a set of coupled differential equations. To numerically solve such a system, we can first split any second (or higher) order equations into two (or more) first order equations, then write the resulting set of first order equations as a vector differential equation:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}(x) = \mathbf{f}(x, \mathbf{y}) \; , \tag{5}$$

where $\mathbf{y}$ is the vector of dependent variables and $\mathbf{f}$ the corresponding vector of right hand sides. (The independent variable x is still a scalar.)

The preceding numerical methods can then be applied to this vector equation, e.g. 2nd order Runge-Kutta becomes:

$$\begin{aligned}
\mathbf{k}_1 &= h\mathbf{f}(x_n, \mathbf{y}_n) \\
\mathbf{k}_2 &= h\mathbf{f}(x_n + \frac{h}{2}, \mathbf{y}_n + \frac{\mathbf{k}_1}{2}) \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + \mathbf{k}_2 + O(h^3) \; .
\end{aligned}$$

5. * Consider the *Simple Harmonic Oscillator*, with unit mass and potential

$$V(x) = \frac{1}{2}x^2 \tag{6}$$

   (a) Write down the second order differential force equation and decompose into a system of coupled first order equations.
   (b) Write down the solution of the equation.
   (c) Solve the equations for the simple harmonic oscillator potential using
      i. Second order Runge-Kutta (not using GSL)
      ii. A Runge-Kutta 4$^{\text{th}}$ order GSL routine (HINT: Look at the examples in the GSL manual, Ordinary Differential Equations).
      iii. A GSL routine with adaptive step-size control.
   (d) Provide a summary of the error behaviour for each method. You may find it useful to remember that energy $E = \frac{p^2}{2} + \frac{x^2}{2}$, should be a conserved quantity.
   (e) Produce a phase space plot for each solution ($p = \frac{\mathrm{d}x}{\mathrm{d}t}$ vs. x).

6. Consider the *Anharmonic Oscillator* potential equation

$$V(x) = \frac{1}{4}x^4 \tag{7}$$

(a) Write down the second order differential force equation and decompose it into a system of coupled first order equations.

(b) Solve the equations for the anharmonic oscillator potential using a GSL routine.

(c) find the period of oscillation if the amplitude is 0.1, 1 and 10.

References

[1] Press, W.H. and Flannery, B.P. and Teukolsky, S.A. and Vetterling, W.T. and others *Numerical Recipes* . Cambridge Univ Press, 3rd Edition, 2007.