IVP Analytic vs. Numerical Solution

June 25, 2020

1 Problem

Compare the analytic and numerical solution of η of the following shallow water problem:

$$\begin{aligned} \eta &= e^{-(x-3.5)^2} \\ u &= 0 \\ h &= x \\ m &= \infty \end{aligned}$$

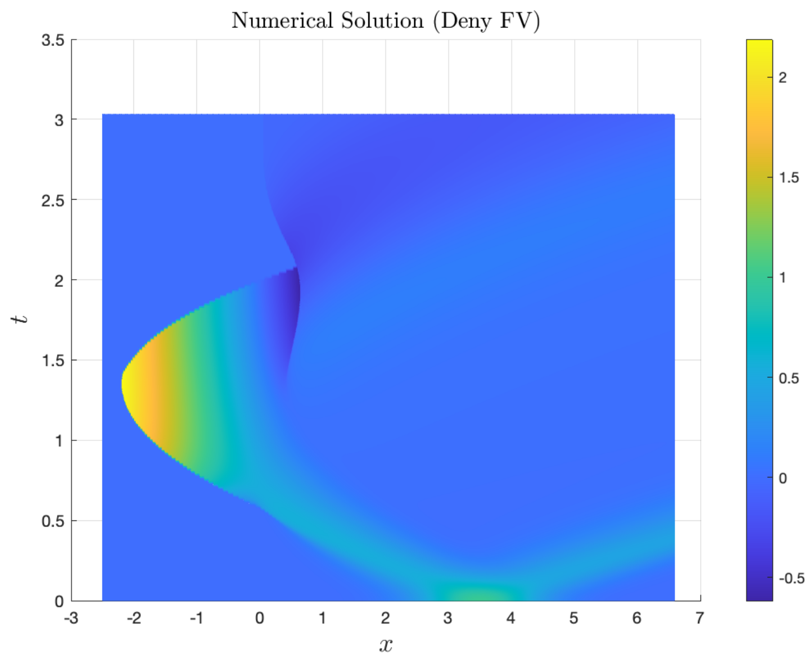
In other words, a Gaussian initial wave with no initial velocity, and a plane-inclined shape (y^∞). This reduces to a 1-1 SWE. We can reproduce this with a different slope and initial conditions easily.

2 Setup

Statistical comparison was done on an equally spaced grid of 1000 points in time on [0,3] and at 1000 points in x on [-2.5, 6.5]

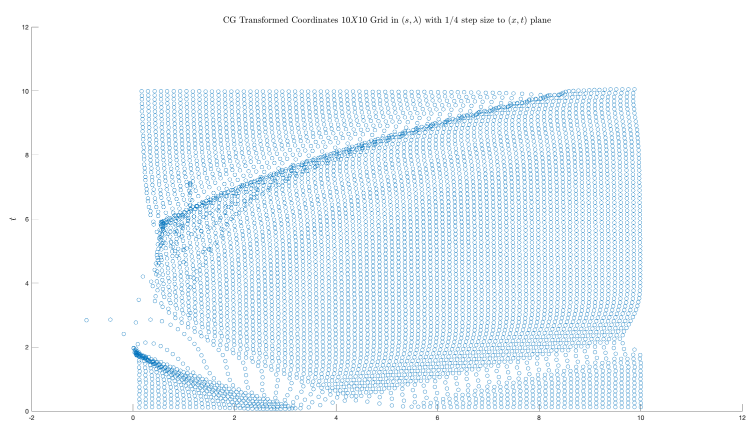
2.1 Numerical

I set Deny's Catalina 1 "runwave.m" with the initial conditions. The following displays eta in the (x, t) plane



2.2 Anaylytic

Chebfun was used to calculate the Hankel transform solution to the CG transform on a grid in (s, λ) then CG transform to (x, t)
The following figure shows the a grid in (s, λ) transformed to (x, t)



Note the distinct non-linear nature caused by the $-u^2$ of η
The analytical solution of η was computed using formulas in Nicolsky (2018)

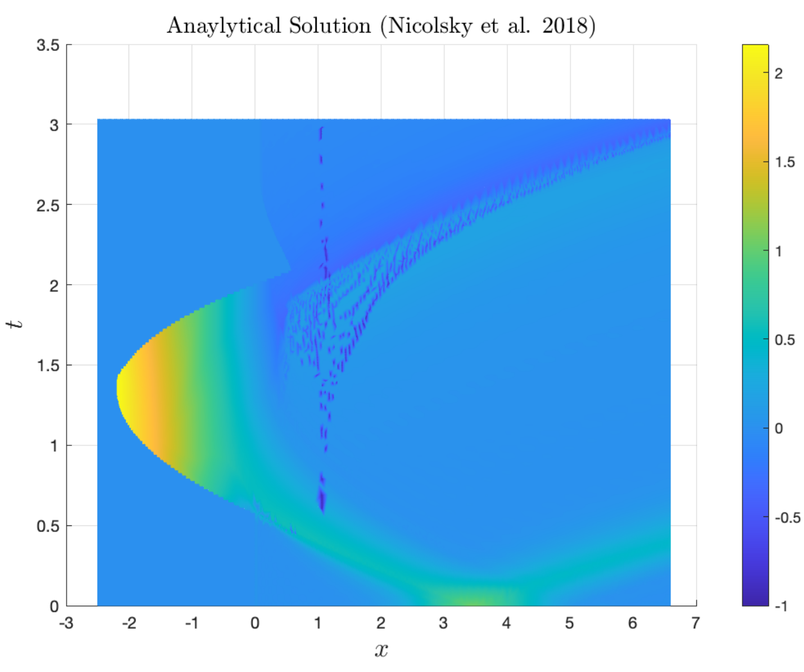
$$\psi(s,\lambda) = \int_0^\infty (a(k)cos(\beta k\lambda) + b(k)sin(\beta k\lambda))J_0(2k\sqrt{s})dk$$
$$\varphi(s,\lambda) = s^{-1/2}\int_0^\infty (a(k)sin(\beta k\lambda) + b(k)cos(\beta k\lambda))J_1(2k\sqrt{s})dk$$

where

$$a(k) = 2k\int_0^\infty \psi(s*,0)J_0(2k\sqrt{s*})ds*$$
$$b(k) = 2k\int_0^\infty \varphi(s*,0)s*^{1/2}\,J_1(2k\sqrt{s*})ds*$$

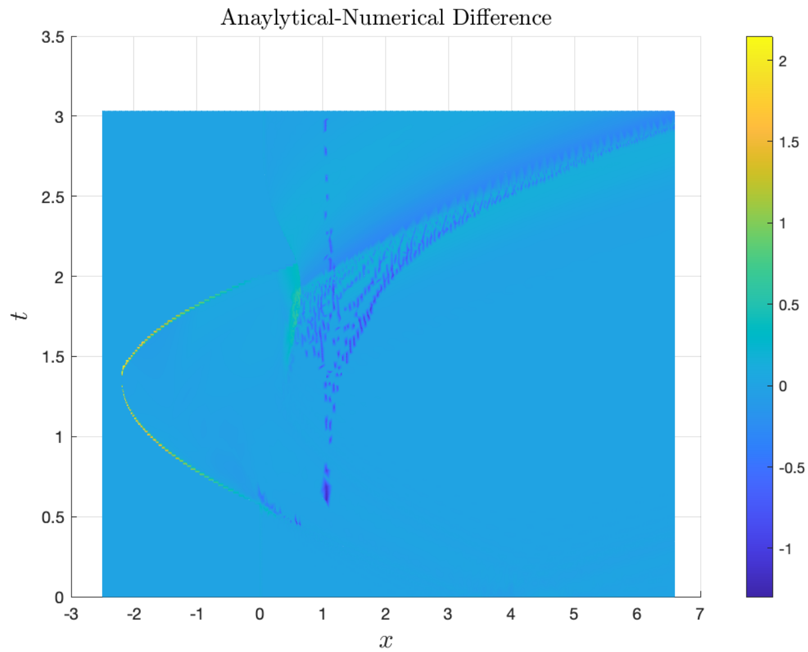
and the projection of φ and ψ onto $\lambda = 0$ were computed via a first order taylor expansion. Note that these eqautions require $\eta_0'(x) > -1$. See Nicolsky for derivation.

$$\Phi(s,\lambda) = \begin{pmatrix}\varphi(s,\lambda)\\ \psi(s,\lambda)\end{pmatrix}$$
$$\Phi_0(x) = \begin{pmatrix}u_0(x)\\ \eta_0(x) + u_0^2(x)/2\end{pmatrix}$$
$$\Phi_1 = \Phi_0 + u_0(u_0' AD^{-1}B\Phi_0 - B\Phi_0 - AD^{-1}\Phi_0')$$

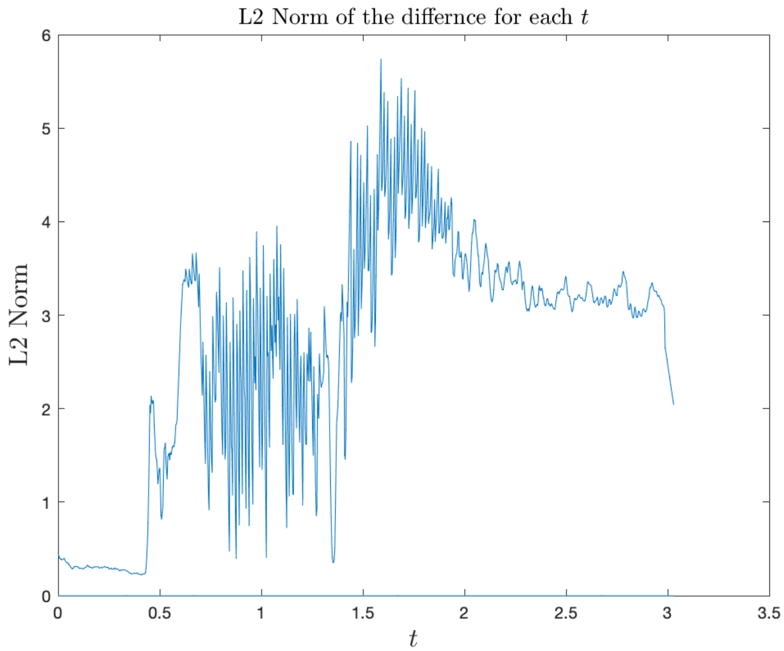


3 Statistical Analysis

This is the difference between the two ie. numerical - anaytical



The following is the L2 norm at each value of t. The difference increases in a sporadic fashion at the beginning and end of run-up. The primary explaination for this is problems with the computation of the analytic solution.



4 Further Problems

1. Analytic solution stability.
2. Comparison of the speed wasn't completed.
3. Different initial conditions.
4. NOAA anaytic solution.