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1. (a) This proof will be shown for the $3 * 3$ kernel size but is the same for any $k > 1$. Consider the following separable 2D filter kernel g of size $2k + 1$ with $k = 1$.

$$\begin{aligned} g_1 &= \begin{bmatrix} w_4 & w_5 & w_6 \end{bmatrix} \\ g_2 &= \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\ g &= g_2 g_1 \\ &= \begin{bmatrix} w_1w_4 & w_1w_5 & w_1w_6 \\ w_2w_4 & w_2w_5 & w_2w_6 \\ w_3w_4 & w_3w_5 & w_3w_6 \end{bmatrix} \end{aligned}$$

Then, consider the following value for our "image" f, where the pixel of interest for this proof is x.

$$f = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & x & a_5 \\ a_6 & a_7 & a_8 \end{bmatrix}$$

Then, the convolution of f with g on the pixel x is:

$$\begin{aligned} h_{2D}(m, n) &= \sum_{k,l} g(k, l) f(m + k, n + l) \\ &= a_1w_1w_4 + a_2w_1w_5 + a_3w_1w_6 + a_4w_2w_4 + \\ & xw_2w_5 + a_5w_2w_6 + a_6w_3w_4 + a_7w_3w_5 + a_8w_3w_6 \end{aligned}$$

Now, we must confirm that applying each of the 1D convolutions separately gives the same result. The convolution of f with g_1 is:

$$\begin{aligned} h_1 &= \sum_{l} g_1(l) f(m, n + l) \\ &= \begin{bmatrix} w_4a_1 + w_5a_2 + w_6a_3 \\ w_4a_4 + w_5x + w_6a_5 \\ w_4a_6 + w_5a_7 + w_6a_8 \end{bmatrix} \end{aligned}$$

Then, the convolution of h_1 (the result from the previous step) with g_2 gives the following.

$$\begin{aligned} h_2 &= \sum_k g_2(k)h_1(m+k) \\ &= a_1w_1w_4 + a_2w_1w_5 + a_3w_1w_6 + a_4w_2w_4 + \\ xw_2w_5 &+ a_5w_2w_6 + a_6w_3w_4 + a_7w_3w_5 + a_8w_3w_6 \end{aligned}$$

Thus, the result is the same in both cases, so convolving an image with a discrete, separable 2D filter kernel is equivalent to convolving with two 1D filter kernels.

(b) For each pixel, a 2D filter kernel would perform $(2k+1)^2$ operations, while each 1D filter kernel would perform $2k+1$ operations. So, the number of operations saved per pixel is:

$$\begin{aligned} (2k+1)^2 - 2*(2k+1) &= (4k^2+4k+1) - (4k+2) \\ &= 4k^2 - 1 \end{aligned}$$

Based on this, the total number of operations saved for an $N * N$ image is $\boxed{N^2(4k^2-1)}$.

Matlab Code