An element of $\text{Ш}(A/\mathbb{Q})[7]$ for some (absolutely simple) abelian surface

(I promise it uses elliptic curves)

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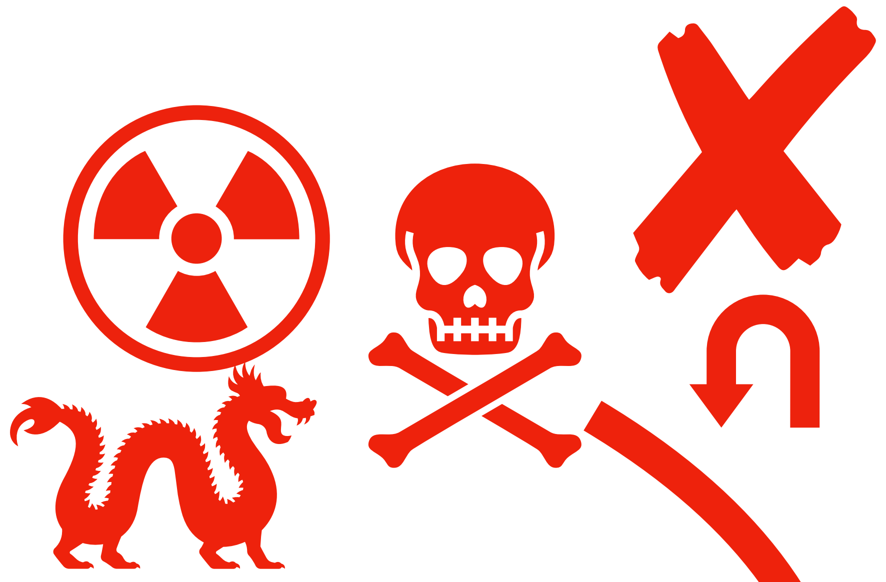
BSD



Conjecture (BSD) Let $A/\mathbb{Q}$ be a (principally polarised) abelian variety of rank r, then

$$\lim_{s\to 1} \frac{L(A/\mathbb{Q}, s)}{(s-1)^r} = \frac{|Ш(A/\mathbb{Q})|\,\Omega_A R_A \prod_p c_p}{|A_{\mathrm{tors}}(\mathbb{Q})|^2}$$

Ш is hard

- Ш$(A/\mathbb{Q})$ is maybe the most mysterious part of the BSD conjecture

- Seemingly very hard to construct elements……. buuuuuuuut

Conjecture (Bhargava—Klagsburn—Lemke Oliver—Shnidman) Fix an abelian variety $A/\mathbb{Q}$ and a prime number p. There exist infinitely many quadratic twists $A^d/\mathbb{Q}$ of $A/\mathbb{Q}$ such that the group Ш$(A^d/\mathbb{Q})$ has a non-trivial p-torsion element.

Theorem (Shnidman—Weiss, Flynn—Shnidman) For fixed p, the group Ш$(A/\mathbb{Q})[p]$ can be arbitrarily large (for absolutely simple $A/\mathbb{Q}$).

But… $\dim A$ is huge!!

An element of $Ш(A/\mathbb{Q})$

Theorem (F.) The Jacobian J_C of the genus 2 curve

$$C : y^2 = -10(x^6 - 4x^5 - 3x^4 + 8x^3 + 25x^2 + 20x + 5)$$

is absolutely simple and has a $(\mathbb{Z}/7\mathbb{Z})^2$ in its Tate—Shafarevich group (unconditionally).

Theorem (Keller—Stoll) $|Ш(J_C/\mathbb{Q})| = 49$ and BSD holds for $J_C/\mathbb{Q}$!!

In this context, weak BSD is already known due to work of Kolyvagin–Logachëv together with Khare—Winterberger ($J_C/\mathbb{Q}$ is of GL_2—type, hence is modular [KW], and has rank $0 \Rightarrow$ weak BSD [KL])

Visibility

Those $T \in H^1(\mathbb{Q}, A[p])$ which are everywhere locally soluble, but not over $\mathbb{Q}$

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Those $T/\mathbb{Q}$ with an isomorphism $T \cong_{\bar{\mathbb{Q}}} A$ (s.t. …)

$$Ш(A/\mathbb{Q})[p] \quad \leftarrow \quad \mathrm{Sel}^{(p)}(A/\mathbb{Q}) \quad \subset \quad H^1(\mathbb{Q}, A[p])$$

Theorem (Selmer) The curve $C: 3x^3 + 4y^3 + 5z^3 = 0$ is an element of order 3 in $Ш(E/\mathbb{Q})[p]$ where $E: y^2 = x^3 - 24300$.

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$$Ш(A/\mathbb{Q})[p] \quad \leftarrow \quad \mathrm{Sel}^{(p)}(A/\mathbb{Q}) \quad \subset \quad H^1(\mathbb{Q}, A[p])$$

Depends only on $A[p]$ as a module under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

$$B(\mathbb{Q})/pB(\mathbb{Q}) \quad \subset \quad \mathrm{Sel}^{(p)}(B/\mathbb{Q}) \quad \subset \quad H^1(\mathbb{Q}, B[p])$$

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$$\begin{tikzcd}
Ш(A/\mathbb{Q})[p] & \mathrm{Sel}^{(p)}(A/\mathbb{Q}) \arrow[l] \arrow[d, "\wr"'] \arrow[r, phantom, "\subset"] & H^1(\mathbb{Q}, A[p]) \arrow[d, "\wr"'] \\
B(\mathbb{Q})/pB(\mathbb{Q}) \arrow[r, phantom, "\subset"] & \mathrm{Sel}^{(p)}(B/\mathbb{Q}) \arrow[r, phantom, "\subset"] & H^1(\mathbb{Q}, B[p])
\end{tikzcd}$$

$\boxed{\text{If } A[p] \cong_{\mathbb{Q}} B[p]}$

$\cup$

$\{$points$\}$

You can realise Selmer's example this way using $B/\mathbb{Q}$ an elliptic curve.

Modular form associated to $E/\mathbb{Q}$

$$E : y^2 = x^3 - 100x + 400$$

$$f(q) = q - \underbrace{3q^3}_{-3 = 3 + 1 - \#E(\mathbb{F}_3)} - \overbrace{4q^7}^{-2 = 7 + 1 - \#E(\mathbb{F}_7)} + 6q^9 - \underbrace{3q^{11}}_{-3 = 11 + 1 - \#E(\mathbb{F}_{11})} - \overbrace{2q^{13}}^{-2 = 13 + 1 - \#E(\mathbb{F}_{13})} - O(q^{17})$$

Modular form associated to $J_C/\mathbb{Q}$

$$C : y^2 = -10(x^6 - 4x^5 - 3x^4 + 8x^3 + 25x^2 + 20x + 5)$$

$$\Big\updownarrow \ \mathrm{End}(J_C/\mathbb{Q}) = \mathbb{Z}[\sqrt{2}]$$

$$g(q) = q + \underbrace{(\sqrt{2}+1)}_{\mathrm{Tr}(\sqrt{2}+1) = 2 = 3+1-\#C(\mathbb{F}_3)}q^3 + \overbrace{(-2\sqrt{2}+2)}^{\mathrm{Tr}(-2\sqrt{2}+2) = 4 = 7+1-\#C(\mathbb{F}_7)}q^7 + 2\sqrt{2}q^9 + \underbrace{(-3\sqrt{2}-1)}_{\mathrm{Tr}(-3\sqrt{2}-1) = -2 = 11+1-\#C(\mathbb{F}_{11})}q^{11} + 4\sqrt{2}q^{13} + \dots$$

A congruence of modular forms

$$f(q) = q \quad -3q^3 \quad -4q^7 + 6q^9 \quad -3q^{11} \quad -2q^{13} + \ldots$$

Mod 7 $\quad -3 = 4 \in \mathbb{F}_7 \quad -3 = 4 \in \mathbb{F}_7 \quad -2 = 5 \in \mathbb{F}_7$

$$g(q) = q + (\sqrt{2} + 1)q^3 + (-2\sqrt{2} + 2)q^7 + 2\sqrt{2}q^9 + (-3\sqrt{2} - 1)q^{11} + 4\sqrt{2}q^{13} + \ldots$$

Mod $(\sqrt{2} - 3)$ $\quad 4 \in \mathbb{F}_7 \quad -10 = 4 \in \mathbb{F}_7 \quad 12 = 5 \in \mathbb{F}_7$

Upshot

$$\boxed{E[7] \quad \cong_{\mathbb{Q}} \quad J_C[\sqrt{2} - 3]}$$

BUT:

- The rank of $E(\mathbb{Q})$ is 2 (just look for points),
- The rank of $J_C(\mathbb{Q})$ is 0 (by an “easy” $2-$descent), and

$$(\mathbb{Z}/7\mathbb{Z})^2 \subset E(\mathbb{Q})/7E(\mathbb{Q}) \subset \mathrm{Sel}^{(7)}(E/\mathbb{Q}) \cong \mathrm{Sel}^{(\sqrt{2}-3)}(J_C/\mathbb{Q}) \subset \mathrm{Sel}^{(7)}(J_C/\mathbb{Q})$$

Cannot come from $J(\mathbb{Q})$