

# **On the geometry of the Humbert surface of square discriminant**

**Sam Frengley (University of Bristol)**

# Isogenies of elliptic curves over $\mathbb{Q}$

**Question:** Which integers  $m \geq 2$  can be the degree of a (cyclic) isogeny  $\phi: E \rightarrow E'$  where  $E, E'$ , and  $\phi$  are defined over  $\mathbb{Q}$ ?

**Theorem (Mazur, Kenku):** If  $\phi: E \rightarrow E'$  is a (cyclic) isogeny between elliptic curves defined over  $\mathbb{Q}$ , then the degree of  $\phi$  is  $\leq 19$ , or  $\in \{21, 25, 27, 37, 43, 67, 163\}$ .

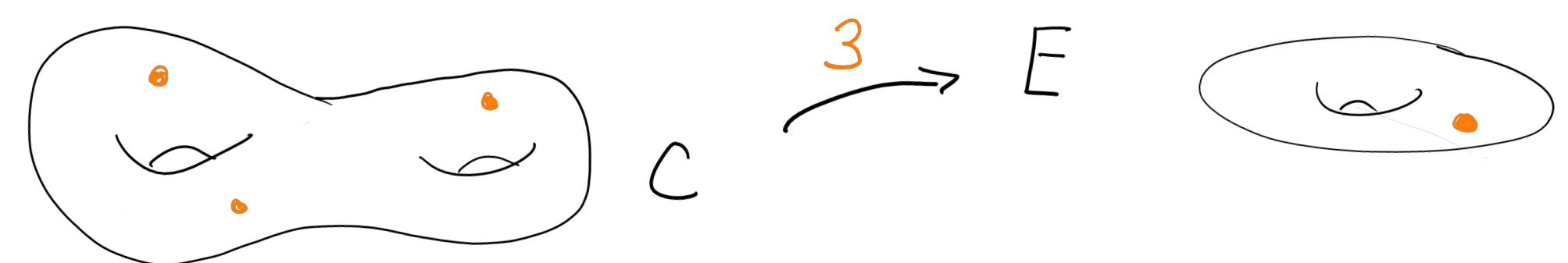
# Upgrade to genus 2

**Question:** For which integers  $N \geq 2$  do there exist some genus 2 curve  $C/\mathbb{Q}$  an elliptic curve  $E/\mathbb{Q}$  and a morphism

$$\psi: C \rightarrow E$$

over  $\mathbb{Q}$  and “minimal” degree  $N$ ?

$\psi$  does not factor through an isogeny

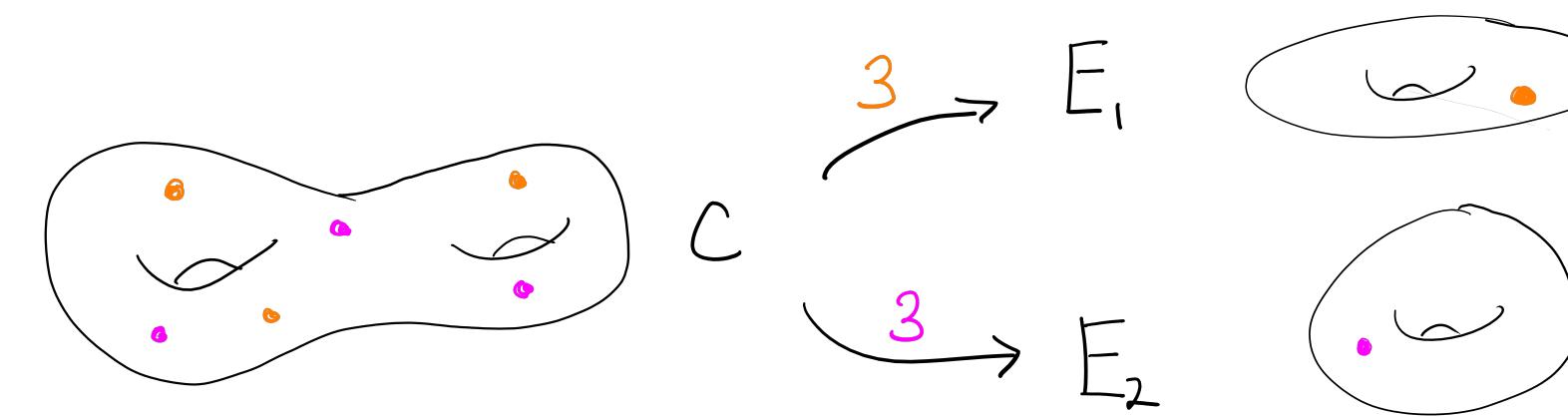


**Example:** Take  $E : y^2 = f(x)$  and  $C : y^2 = f(x^2)$  so that we have  $C \rightarrow E$  induced by  $x \mapsto x^2$  with  $N = 2$ .

For  $N \leq 5$  this was thought about a lot in the context of elliptic integrals by Abel—Legendre—Jacobi + (more 19th century authors)

# In terms of Galois representations

The morphism  $\psi: C \rightarrow E$  induces an isogeny  $E \times E' \rightarrow \text{Jac}(C)$  over  $\mathbb{Q}$  and a different minimal degree  $N$  morphism  $\psi': C \rightarrow E'$ .



The existence of  $E \times E' \rightarrow \text{Jac}(C)$  implies there exists an isomorphism

$$E[N] \cong_{\mathbb{Q}} E'[N]$$

of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -modules.

The converse also holds so long as  $E$  and  $E'$  are not isogenous (and the isomorphism  $E[N] \cong_{\mathbb{Q}} E'[N]$  is antisymplectic wrt the Weil pairing).

Ensures that the image of  
 $E \times E' \rightarrow \text{Jac}(C)$   
is a Jacobian

# Frey–Mazur conjecture

**Conjecture (Frey–Mazur):** There exists an integer  $N_0 > 0$  such that for all  $N \geq N_0$  any pair of elliptic curves  $E/\mathbb{Q}$  and  $E'/\mathbb{Q}$  with  $E[N] \cong_{\mathbb{Q}} E'[N]$  are geometrically isogenous.

**Conjecture (Fisher):** If one restricts to  $N$  is prime, one can take  $N_0 = 19$  in the Frey–Mazur conjecture.

# Frey–Mazur conjecture

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**Theorem (Frey):** If the Frey–Mazur conjecture then the *asymptotic Fermat conjecture* holds.

**Example / Theorem (F.):** There exists a genus 2 curve  $C/\mathbb{Q}$  with minimal degree 15 morphism over  $\mathbb{Q}$  to

$$E : y^2 + xy + y = x^3 - x^2 - 5978298027424617040871x - 177915816685044386506178867920438$$

$$E' : y^2 = x^3 - 2135607437331989841943540710782811x + 37915783123298007085317147066745283477127543370806$$

Which are not (geometrically) isogenous and have conductor

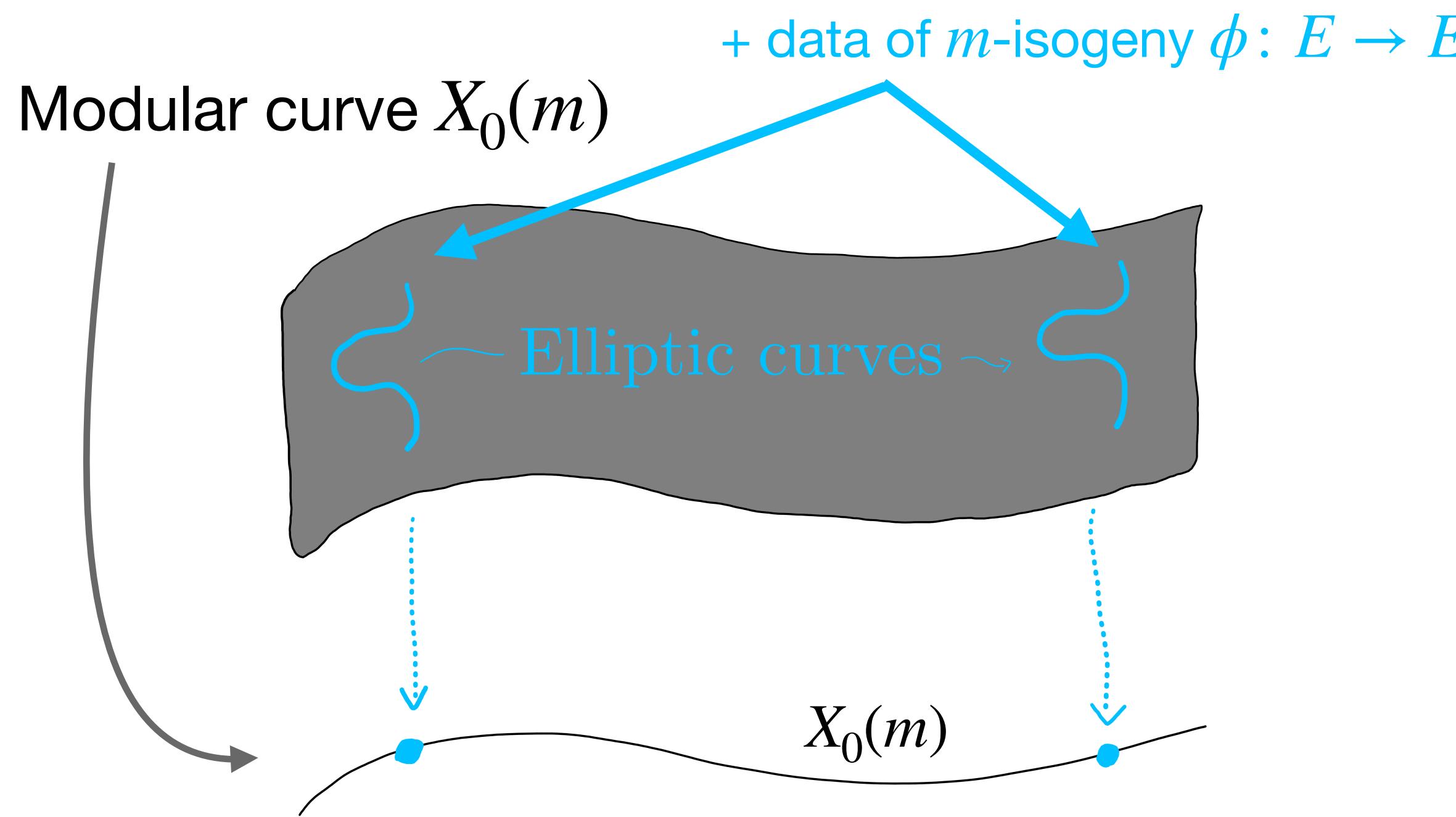
$$3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 61 \cdot 199^2 \cdot 2341^2 \approx 10^{19}.$$

Also:

- $\infty$  many for  $N \leq 11$  (Kumar + ...)
- $\infty$  many for  $N = 12$  (F.)
- $\infty$  many for  $N = 13$  (Fisher)
- $\sim 10$  for  $N = 14$  (F.)
- One for  $N = 15$  (F.)
- One for  $N = 17$  (Fisher)

# The relevant moduli spaces

Isogeny  $\phi: E \rightarrow E'$



Mazur—Kenku “just” find the rational points on  $X_0(m)$  for every  $m \geq 2$ .

Coverings  $\psi: C \rightarrow E$

Hilbert modular surface  $Y_-(N^2)$  such that  
 $\{k\text{-points on } Y_-(N^2)\} \leftrightarrow \{\psi: C \rightarrow E \text{ over } k\}$

We “just” need to find the rational points on  $Y_-(N^2)$  for every  $N \geq 2$ .

**Question:** Too hard over  $\mathbb{Q}$ , make it easier. For which integers  $N$  does there exist a family of  $\Psi: \mathcal{C} \rightarrow \mathcal{E}$  over  $\mathbb{C}(s, t)$  (or  $\mathbb{Q}(s, t)$ )?

**Theorem (Hermann, Kani–Schanz):** The Hilbert modular surface  $Y_-(N^2)$  is (birational to a) surface which is

- Rational if  $N \leq 5$
- Elliptic K3 if  $N = 6, 7$
- Properly elliptic if  $N = 8, 9, 10$
- General type if  $N \geq 11$ .

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**Thm:**  $\kappa = \min(2, p_g - 1)$  !?

**Theorem (Bakker–Tismerman):** There exists  $p_0$  so that any  $E[p] \cong_{\mathbb{C}(t)} E'[p]$  (symplectic isom.) over  $\mathbb{C}(t)$  are isogenous when  $p \geq p_0$ .

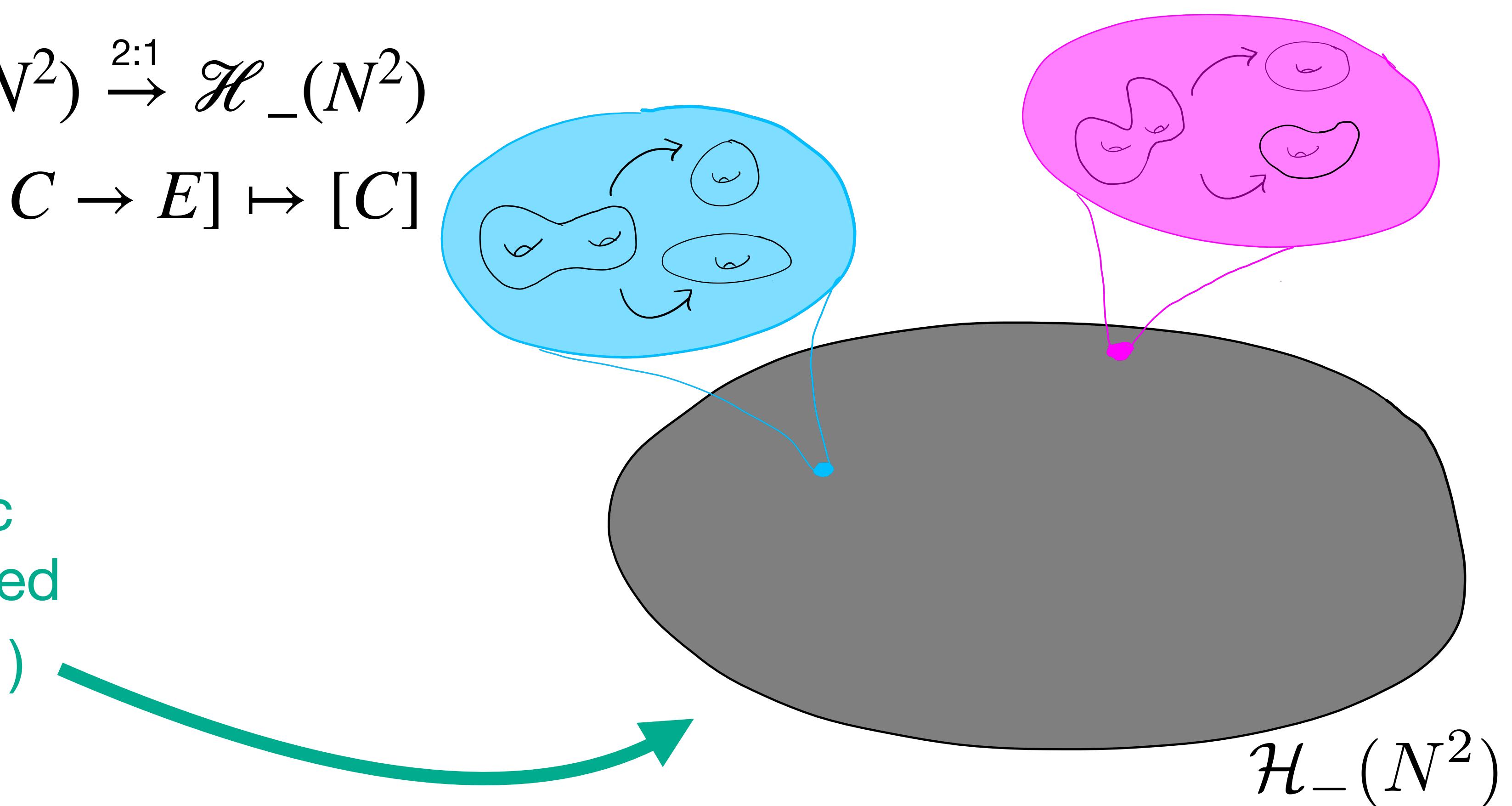
# Humbert surfaces

But: We had an example with  $N = 15, 17$ ? Should this be unexpected since  $Y_-(N^2)$  is general type?

$$Y_-(N^2) \xrightarrow{2:1} \mathcal{H}_-(N^2)$$

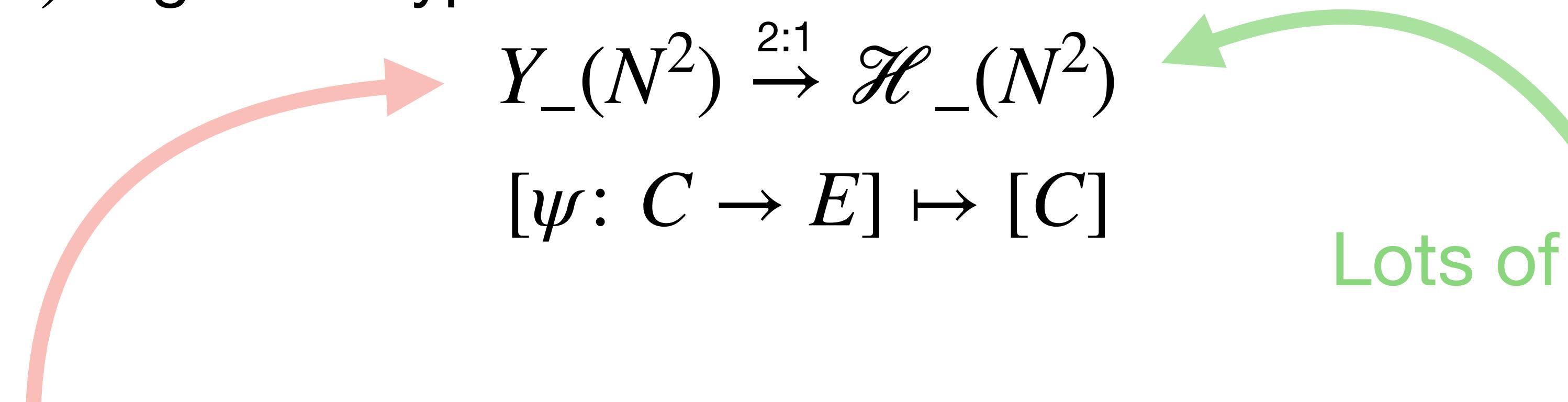
$$[\psi: C \rightarrow E] \mapsto [C]$$

Remembers only that  $C$  admits a map to an elliptic curve (equivalently, unordered pairs  $C \rightarrow E$  and  $C \rightarrow E'$ )



# Humbert surfaces

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$$[\psi: C \rightarrow E] \mapsto [C]$$


Lots of points  $N = 15, 17$

Probably not many points (Bombieri—Lang conjecture)

# Humbert surfaces

**Another reason:** The surface  $\mathcal{H}_-(N^2)$  is very natural from the point of view of moduli. We have a natural way of viewing

$$\mathcal{H}_-(N^2) \subset \mathcal{M}_2$$

As defined it is  
only birational to  
its image



The union

$$\bigcup_N \mathcal{H}_-(N^2)$$

are the points in  $\mathcal{M}_2$  with Jacobians isogenous to a product of elliptic curves.

As principally polarised abelian varieties

$$N = p$$

**Theorem (Hermann, F.):** The Humbert surface of square discriminant  $\mathcal{H}_-(N^2)$

is (birational to):

- A rational surface if  $N \leq 16$  or if  $N = 18, 20$ , or  $24$ ,
- An elliptic K3 surface if  $N = 17$ ,
- A properly elliptic surface if  $N = 19$  or  $21$ , and
- A surface of general type if  $N \geq 22$  and  $N \neq 24$ .

$$\mathcal{C}/\mathbb{Q}(u, v)$$

- Kumar ( $N \leq 11$ ) + ...
- Fisher ( $N = 13$ )
- F. ( $N = 12, 14, 15, 18, 20, 24$ )

There exists  $\mathcal{C}/\mathbb{C}(u, v)$  which admits a minimal cover of degree  $N$  to an elliptic curve (after base change to a quadratic extension)

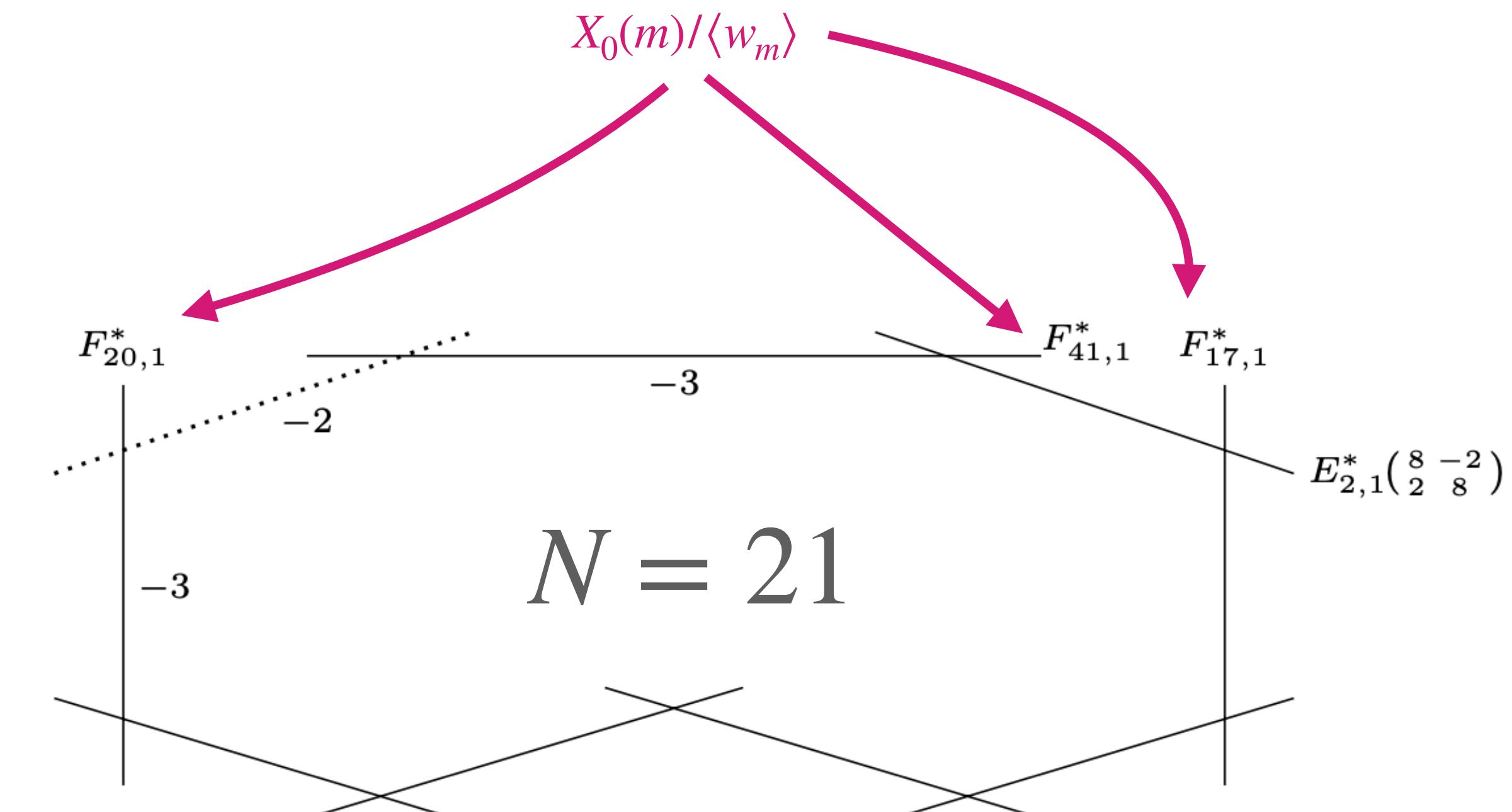
**Thm:**  $\kappa = \min(2, p_g - 1)$  !?

# How?

and use Riemann–Hurwitz++

Use that  $\mathcal{H}_-(N^2)$  is a quotient of  $Y_-(N^2)$  by an involution.

1. Understand fixed point locus on  $Y_-(N^2)$  (genera, singularities, etc)
  - e.g.,  $X_{\text{ns}}^+(p^k)$  or  $X_{\text{s}}^+(p^k)$  and  $N$  odd you get “extended Cartan”
  - $N$  even you get “extended Cartan” and fancier  $X_H$  some  $H \subset \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$
2. Desingularise quotient of  $Y_-(N^2)$ 
  - blow-down Hecke correspondences  
(when  $N$  even, fancier  $X_H$ )
3. Understand (self)-intersections and singularities of Hecke correspondences on quotient to find elliptic fibres etc.



**Challenge:** Below is  $j(\mathcal{E})j(\mathcal{E}') \in \mathbb{Q}(u, v)$  (also have  $j(\mathcal{E}) + j(\mathcal{E}')$ ) for a pair of elliptic curves for which there exists a genus 2 curve  $\mathcal{C}/\mathbb{Q}(u, v)$  with degree  $N = 24$  maps to elliptic curves (over a quadratic extension of  $\mathbb{Q}(u, v)$ ). **Find  $\mathcal{C}$ .**

$$\begin{aligned}
& (u^{32}*v^{16} - 232*u^{32}*v^{15} + 508*u^{32}*v^{14} + 1032*u^{32}*v^{13} + 7814*u^{32}*v^{12} + 17480*u^{32}*v^{11} + 43644*u^{32}*v^{10} + 76312*u^{32}*v^9 + 120769*u^{32}*v^8 + 152864*u^{32}*v^7 + 163968*u^{32}*v^6 + 143584*u^{32}*v^5 + 102656*u^{32}*v^4 + 57984*u^{32}*v^3 + 25024*u^{32}*v^2 + 7168*u^{32}*v^1 - 224*u^{31}*v^{18} \\
& - 1120*u^{31}*v^{17} + 18360*u^{31}*v^{16} + 28520*u^{31}*v^{15} + 1330484*u^{31}*v^{14} + 191544*u^{31}*v^{13} + 413672*u^{31}*v^{12} + 439880*u^{31}*v^{11} + 365256*u^{31}*v^{10} - 214488*u^{31}*v^9 - 1072000*u^{31}*v^8 - 1950672*u^{31}*v^7 - 2375872*u^{31}*v^6 - 2171104*u^{31}*v^5 - 319104*u^{31}*v^4 \\
& - 77568*u^{31}*v^3 - 8192*u^{31}*v^2 - 864*u^{30}*v^{16} + 13568*u^{30}*v^{15} + 409516*u^{30}*v^{14} + 145040*u^{30}*v^{13} - 395680*u^{30}*v^{12} - 381380*u^{30}*v^{11} - 19702684*u^{30}*v^{10} - 17945424*u^{30}*v^9 - 9080920*u^{30}*v^8 + 2367200*u^{30}*v^7 \\
& + 10215408*u^{30}*v^6 - 11509952*u^{30}*v^5 + 8428800*u^{30}*v^4 + 4376320*u^{30}*v^3 + 1530048*u^{30}*v^2 + 307200*u^{30}*v^1 + 25344*u^{29}*v^{22} + 89888*u^{29}*v^{21} + 263584*u^{29}*v^{20} - 1400256*u^{29}*v^{19} + 32200*u^{29}*v^{18} - 5688832*u^{29}*v^{17} - 4287008*u^{29}*v^{16} - 31901208*u^{29}*v^{15} \\
& - 9*v^{14} + 10904432*u^{29}*v^{13} + 10904432*u^{29}*v^{12} + 10904432*u^{29}*v^{11} + 10904432*u^{29}*v^{10} + 217391472*u^{29}*v^{9} + 164065440*u^{29}*v^{8} + 74824880*u^{29}*v^{7} - 6535680*u^{29}*v^{6} - 20910336*u^{29}*v^{5} - 20259456*u^{29}*v^{4} - 10570752*u^{29}*v^{3} - 3270656*u^{29}*v^{2} - 541184*u^{29}*v^{1} - 37376*u^{29} + 9664*u^{28}*v^{1} \\
& + 212000*u^{28}*v^{23} - 482976*u^{28}*v^{22} - 4806464*u^{28}*v^{21} + 85853832*u^{28}*v^{20} - 9140248*u^{28}*v^{19} + 59388456*u^{28}*v^{18} + 11707646*u^{28}*v^{17} + 215968240*u^{28}*v^{16} + 618995064*u^{28}*v^{15} + 659526922*u^{28}*v^{14} - 735691944*u^{28}*v^{10} \\
& - 1082893488*u^{28}*v^{9} - 904578276*u^{28}*v^{8} - 497370336*u^{28}*v^{7} - 174591712*u^{28}*v^{6} - 24186592*u^{28}*v^{5} + 12576160*u^{28}*v^{4} + 313344*u^{28}*v^{3} + 2368128*u^{28}*v^{2} + 18688*u^{28} + 26112*u^{27}*v^{26} + 201152*u^{27}*v^{25} - 2609536*u^{27}*v^{24} - 3213760*u^{27}*v^{23} + 34989728*u^{27}*v^{22} - 7407 \\
& 1964*u^{27}*v^{21} - 58381416*u^{27}*v^{20} - 154293864*u^{27}*v^{19} + 590355704*u^{27}*v^{18} + 155443742*u^{27}*v^{17} + 1936557072*u^{27}*v^{16} + 29389536*u^{27}*v^{15} - 4019789760*u^{27}*v^{14} - 7490760136*u^{27}*v^{13} - 7552592480*u^{27}*v^{12} - 4033307695*u^{27}*v^{11} + 340367840*u^{27}*v^{10} + 2665148800*u^{27}*v^{9} + 2535517888 \\
& *u^{27}*v^{8} + 1415827632*u^{27}*v^{7} + 532323136*u^{27}*v^{6} + 139553824*u^{27}*v^{5} + 28391424*u^{27}*v^{4} + 6796288*u^{27}*v^{3} + 1807104*u^{27}*v^{2} + 2494912*u^{27}*v^{1} + 36288*u^{26}*v^{27} - 433340*u^{26}*v^{26} + 10323008*u^{26}*v^{25} + 35562144*u^{26}*v^{24} - 261881920*u^{26}*v^{23} + 267584912*u^{26} \\
& *v^{2} + 6*v^{1} + 1117351256*u^{26}*v^{21} + 3753193420*u^{26}*v^{20} + 28275347200*u^{26}*v^{19} - 4849258960*u^{26}*v^{18} - 18428217792*u^{26}*v^{17} - 26562106272*u^{26}*v^{16} - 16673131504*u^{26}*v^{15} + 8962517540*u^{26}*v^{14} + 32778836944*u^{26}*v^{13} + 38081897528*u^{26}*v^{12} + 24722329920*u^{26}*v^{11} + 7032705064*u^{26}*v^{10} - 2834 \\
& *v^{9} + 903120*u^{26}*v^{9} - 4163548360*u^{26}*v^{8} - 2245872416*u^{26}*v^{7} - 721954480*u^{26}*v^{6} - 305097766*u^{26}*v^{5} - 452736*u^{26}*v^{4} - 20016608*u^{26}*v^{3} - 2165 \\
& 90454*u^{25}*v^{25} - 823602332*u^{25}*v^{24} + 2148563928*u^{25}*v^{23} - 22310071216*u^{25}*v^{22} - 141856391488*u^{25}*v^{21} + 18428256314388*u^{25}*v^{20} + 11121672624*u^{25}*v^{19} + 31891637664*u^{25}*v^{18} + 4608*u^{25} + 9216*u^{24}*v^{32} - 49766 \\
& 4*u^{24}*v^{31} + 1036224*u^{24}*v^{30} + 32303880*u^{24}*v^{29} - 69487936*u^{24}*v^{28} + 2044736160*u^{24}*v^{27} + 193669223960*u^{24}*v^{26} + 413741571754*u^{24}*v^{25} - 448901760712*u^{24}*v^{24} + 12742 \\
& 0128076*u^{24}*v^{18} - 4103634395420*u^{24}*v^{17} - 779262138811*u^{24}*v^{16} - 710584027200*u^{24}*v^{15} - 310776184544*u^{24}*v^{14} - 28735947616*u^{24}*v^{13} - 728105920*u^{24}*v^{12} + 91788948*u^{24}*v^{11} + 890952*u^{24}*v^{10} + 20808 \\
& 8035808*u^{24}*v^{9} + 28846459312*u^{24}*v^{8} + 409285388008*u^{24}*v^{7} + 23047287077404*u^{24}*v^{6} - 2326037465032*u^{24}*v^{5} + 1179583215728*u^{24}*v^{4} \\
& + 19321431616*u^{24}*v^{3} - 2057958576776*u^{24}*v^{2} - 251289123488*u^{24}*v^{1} - 123143871792*u^{23}*v^{11} - 356560033762*u^{23}*v^{10} - 4888395520*u^{23}*v^{9} + 612248080*u^{23}*v^{8} + 471675072*u^{23}*v^{7} - 109150560*u^{23}*v^{6} + 187766*u^{23}*v^{5} - 20920372*u^{23}*v^{4} \\
& + 222*v^{34} - 13271040*u^{22}*v^{33} - 131983488*u^{22}*v^{32} + 14129664*u^{22}*v^{31} - 174481536*u^{22}*v^{30} + 2002814848*u^{22}*v^{29} + 69121372320*u^{22}*v^{28} + 156019312896*u^{22}*v^{27} + 49615156112*u^{22}*v^{26} - 723900752104*u^{22}*v^{25} - 2869351847264*u^{22}*v^{24} \\
& + 3849083498296*u^{22}*v^{23} + 8971858630884*u^{22}*v^{22} + 1035933880992*u^{22}*v^{21} + 6349285481632*u^{22}*v^{20} - 43010484748748*u^{22}*v^{19} + 53019588536*u^{22}*v^{18} + 112751278304*u^{22}*v^{17} + 93230923606*u^{22}*v^{16} + 1316564832*u^{22}*v^{15} + 1537984*u^{22}*v^{14} + 163584*u^{22}*v^{13} + 6340032*u^{22}*v^{12} + 1622016*u^{22}*v^{11} + 14675163329*u^{22}*v^{10} + 204404405888*u^{22}*v^{9} + 30652595104*u^{22}*v^{8} \\
& + 21*v^{30} - 150027159392*u^{21}*v^{29} - 711727075296*u^{21}*v^{28} - 1352443802880*u^{21}*v^{27} + 405318668384*u^{21}*v^{26} + 790106853429352*u^{21}*v^{25} - 2853307373136*u^{21}*v^{24} - 23842866137888*u^{21}*v^{23} + 1567745536*u^{21}*v^{22} + 8008 \\
& 8035808*u^{21}*v^{21} + 17684644*u^{21}*v^{20} + 193536*u^{21}*v^{19} + 144292224*u^{21}*v^{18} - 1567745536*u^{21}*v^{17} + 4393896928*u^{21}*v^{16} + 200808 \\
& + 17353045*u^{21}*v^{15} + 2089552*u^{21}*v^{14} + 1737416200*u^{21}*v^{13} + 1501521581344*u^{21}*v^{12} + 102752752*u^{21}*v^{11} + 14515200*u^{21}*v^{10} + 1179583215728*u^{21}*v^{9} \\
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& + 222*v^{34} - 13271040*u^{22}*v^{33} - 131983488*u^{22}*v^{32} + 14129664*u^{22}*v^{31} - 174481536*u^{22}*v^{30} + 2002814848*u^{22}*v^{29} + 69121372320*u^{22}*v^{28} + 156019312896*u^{22}*v^{27} + 49615156112*u^{22}*v^{26} - 723900752104*u^{22}*v^{25} - 1094959340524*u^{22}*v^{24} \\
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& 10750654616*u^{21}*v^{21} + 1723514593088*u^{21}*v^{20} - 5847882189152*u^{21}*v^{19} + 78059585337620*u^{21}*v^{18} + 387373562383384*u^{21}*v^{17} + 19149136500856*u^{21}*v^{16} + 1422057000608*u^{21}*v^{15} - 72921096926213488*u^{21}*v^{14} - 4428700644*u^{21}*v^{13} + 1787766*u^{21}*v^{12} + 205271000608*u^{21}*v^{11} + 1079583215728*u^{21}*v^{10} \\
& + 19321431616*u^{21}*v^{9} - 2057958576776*u^{21}*v^{8} - 251289123488*u^{21}*v^{7} - 123143871792*u^{21}*v^{6} - 356560033762*u^{21}*v^{5} - 4888395520*u^{21}*v^{4} + 612248080*u^{21}*v^{3} + 147675072*u^{21}*v^{2} - 109150560*u^{21}*v^{1} + 187766*u^{21}*v^{0} \\
& + 222*v^{34} - 13271040*u^{22}*v^{33} - 131983488*u^{22}*v^{32} + 14129664*u^{22}*v^{31} - 174481536*u^{22}*v^{30} + 2002814848*u^{22}*v^{29} + 69121372320*u^{22}*v^{28} + 156019312896*u^{22}*v^{27} + 49615156112*u^{22}*v^{26} - 723900752104*u^{22}*v^{25} - 1094959340524*u^{22}*v^{24} \\
& + 3849083498296*u^{22}*v^{23} + 8971858630884*u^{22}*v^{22} + 1035933880992*u^{22}*v^{21} + 6349285481632*u^{22}*v^{20} - 43010484748748*u^{22}*v^{19} + 53019588536*u^{22}*v^{18} + 112751278304*u^{22}*v^{17} + 93230923606*u^{22}*v^{16} + 1316564832*u^{22}*v^{15} + 1537984*u^{22}*v^{14} + 163584*u^{22}*v^{13} + 6340032*u^{22}*v^{12} + 1622016*u^{22}*v^{11} + 14675163329*u^{22}*v^{10} + 204404405888*u^{22}*v^{9} + 30652595104*u^{22}*v^{8} \\
& + 21*v^{30} - 150027159392*u^{21}*v^{29} - 711727075296*u^{21}*v^{28} - 1352443802880*u^{21}*v^{27} + 405318668384*u^{21}*v^{26} + 790106853429352*u^{21}*v^{25} - 2853307373136*u^{21}*v^{24} - 23842866137888*u^{21}*v^{23} + 1567745536*u^{21}*v^{22} + 8008 \\
& 8035808*u^{21}*v^{21} + 17684644*u^{21}*v^{20} + 193536*u^{21}*v^{19} + 144292224*u^{21}*v^{18} - 1567745536*u^{21}*v^{17} + 4393896928*u^{21}*v^{16} + 200808 \\
& + 17353045*u^{21}*v^{15} + 2089552*u^{21}*v^{14} + 1737416200*u^{21}*v^{13} + 102752752*u^{21}*v^{12} + 14515200*u^{21}*v^{11} + 1079583215728*u^{21}*v^{10} - 72921096926213488*u^{21}*v^{9} - 1567745536*u^{21}*v^{8} + 200808 \\
& 7- 78529536*u^{21}*v^{7} - 54074488*u^{21}*v^{6} - 300672*u^{21}*v^{5} + 1628*u^{21}*v^{4} + 2494912418*u^{2$$