ALGEBRAIC SURFACES

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Abstract. This document includes notes for a TCC course I taught in the autumn of 2024 on algebraic surfaces. None of this content is original to me. Almost all the facts here can be found in the excellent texts by Beauville and Reid. Of course, any errors here should be marked down to me.

Contents

1. Some motivation

Standing assumption. $k = \bar{k}$ is an algebraically closed field of characteristic 0. You will lose absolutely nothing by assuming $k = \mathbb{C}$.

This course is ostensibly about birational geometry (though probably we'll spend more time on other stuff). In particular, we are really want to understand the function field $k(X)$ where X is an irreducible variety.

Definition 1.1. If X/k, Y/k are irreducible varieties, we say that X and Y are *birational* if $k(X) \cong k(Y)$.

Question 1.2. *Given X, Y irreducible varieties over k, can we tell if X and Y are birational?*

We want to associate invariants (i.e., numbers) to X and Y which allow us to tell them apart. The first one you probably already know.

Definition 1.3. If X/k is irreducible, the *dimension of* X is the transcendence degree of $k(X)/k$.

It is clear that the dimension is a birational invariant. We break down by dimension.

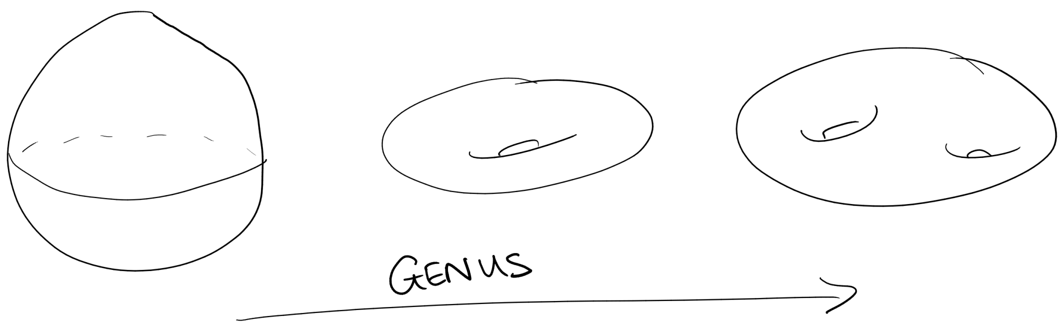
1.1. **Dimension 0.** Over an algebraically closed field there is nothing to say about points, they're points.

1.2. **Dimension 1.** We will need to say a lot about curves to study surfaces. Let X/k be an irreducible variety of dimension 1. There is a very useful fact about curves

Lemma 1.4. *There exists a unique smooth projective curve* $\widetilde{X}$ *birational to* X.

From the lemma, to study curves up to birational equivalence it suffices to study smooth projective curves *up to isomorphism*! This is very convinient. In particular we have a very nice isomorphism invariant (if $k = \mathbb{C}$).

Definition 1.5. Define the *geometric genus* of $\widetilde{X}$ to be the genus of the associated Riemann surface $\widetilde{X}(\mathbb{C})$.



Remark 1.6. Actually, it really is enough to define the genus over $\mathbb{C}$. The field of definition of $\widetilde{X}$ has finite transcendence degree over $\mathbb{Q}$ (even if k is huge, $\widetilde{X}$ is cut out by finitely many equations on finitely many affines) and therefore embeds in $\mathbb{C}$. This is an example of the *Lefschetz Principle.*

In any case, we'll later see how to define the genus without reference to the Riemann surface $\widetilde{X}(\mathbb{C})$.

The genus is a really good invariant. One reason is that for each $g \geq 0$ there exists an irreducible *variety of moduli* $\mathcal{M}_g$ whose $\mathbb{C}$-points are in bijective correspondence with $\mathbb{C}$-isomorphism classes of curves of genus g.

1.3. **Dimension 2.** I am claiming that we can do an entire course on this case, so hopefully it's quite a bit harder. Here's a bunch of questions:

Question 1.7. *Let X/k be an irreducible variety of dimension 2:*

(1) Can we tell if $k(X) \cong k(t_1, t_2)$?
(2) Does there exist a "good" choice of model for X?
(3) Can we get a "curvature trichotomy"-esque invariant?

The answer is yes (Castelnovo's rationality criterion), yes (in the non-ruled case we have the minimal model), and yes (the Kodaira dimension).

2. Housework

Let X/k be an irreducible variety (in particular X is also reduced). Let U be any open affine subvariety of X i.e., so that $U \cong \operatorname{Spec} A$ for some k-algebra A. Since X is irreducible (and reduced) the ring A is an integral domain. The *function field* $k(X)$ of X is defined to be the fraction field of A.

Remark 2.1. If you prefer to minimise scheme words, you could find U so that U is a Zariski open subset of $\mathbb{V}(f_1, ..., f_m) \subset \mathbb{A}^n$ and then take

$$k(X) = \operatorname{Frac} k[t_1, ..., t_n]/(f_1, ..., f_m).$$

Definition 2.2. Let X be a smooth irreducible variety. A *Weil divisor* is a finite formal sum

$$D = \sum_Z n_Z \cdot Z$$

where the sum ranges over prime divisors (closed irreducible subvarieties $Z \subsetneq X$ of codimension 1). Write $\operatorname{Div}(X)$ for the free abelian group supported on the prime divisors.
We say D is *effective* (written $D \geq 0$) if $n_Z \geq 0$ for all Z.

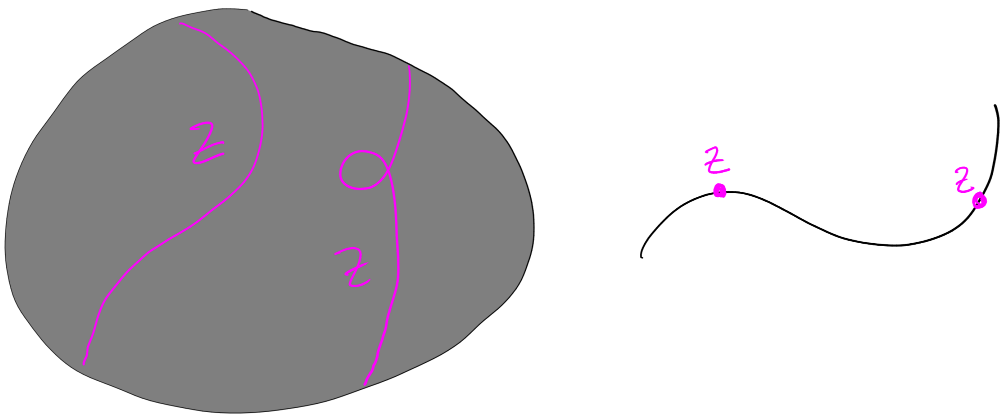
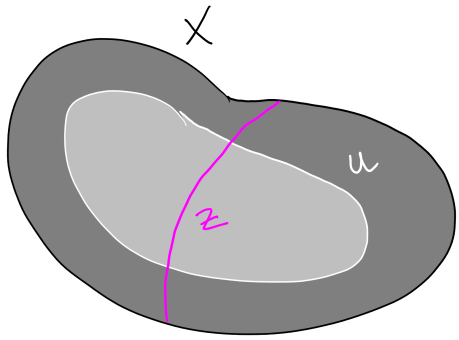


FIGURE 1. Prime divisors on a surface and a curve.

A note on sketches. I am not a capable artist, and this is evidenced by the fact that I cannot accurately draw 4-dimensional manifolds over $\mathbb{R}$. I compromise by drawing the real points of complex surfaces. Similarly, I am bad at drawing open sets in the Zariski topology – you will have to re-imagine my usual-complex-topology open sets.

If $Z \subset X$ is a prime divisor we can choose some open affine $U \subset X$ for which $U \cap Z \neq \emptyset$, say $U \cong \operatorname{Spec} A$. Then $U \cap Z$ is isomorphic to a closed irreducible subvariety of $\operatorname{Spec} A$ and therefore we have $U \cap Z \cong \operatorname{Spec} A/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset A$ (i.e., $U \cap Z$ is cut out by the polynomials in $\mathfrak{p}$). Since Z has codimension 1 the ideal $\mathfrak{p}$ has "height 1" and therefore the local ring



$$\mathcal{O}_{X,Z} = A_{\mathfrak{p}} \subset k(X)$$

is a DVR and comes equipped with a discrete valuation

$$v_Z \colon \mathcal{O}_{X,Z} \twoheadrightarrow \mathbb{Z}_{\geq 0}$$

which then extends to a valuation

$$v_Z \colon k(X) \to \mathbb{Z}$$

which "picks out the order of vanishing of a rational function along Z".

Example 2.3. Take $X = \mathbb{P}^1$ and $f = t_1 = x_1/x_0$. For each $a \in k$ let $P = [1 : a]$. We have $\mathcal{O}_{X,\infty} = k[t_1]_{(t_1-a)}$ and v_P picks out the power of $t_1 - a$ in the numerator of f. Thus

$$v_P(f) = \begin{cases} 1 & \text{if } a = 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

At ∞ we have to use the transition functions. Take $t_0 = x_0/x_1$. Then we have $f = 1/t_0$, so that $v_\infty(f) = -1$.

Exercise 2.4. More generally let $X = \mathbb{P}^n$ and let $F \in k[x_0, ..., x_n]$ be a homogeneous polynomial of degree d. Take $f = F/x_0^d \in k(X)$, let $Z = \mathbb{V}(F)$, and let $H = \mathbb{V}(x_0)$ be the hyperplane at infinity. Show that

$$v_Y(f) = \begin{cases} 1 & \text{if } Y = Z, \\ -d & \text{if } Y = H, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.5. Often people write $\mathcal{O}_{X,\xi}$ where ξ is the generic point of Z and $\mathcal{O}_{X,\xi}$ is the *stalk of the structure sheaf at* ξ.

Exercise 2.6. Check that the preceding definitions do not depend on the choice of open affine.

Definition 2.7. For $f \in k(X)^\times$ define the *divisor of zeroes and poles of f*

$$\operatorname{div}(f) = \sum_Z v_Z(f) \cdot Z.$$

We say that a pair of Weil divisors $D, D' \in \operatorname{Div}(X)$ are *linearly equivalent* (write $D \sim D'$) if $D - D' = \operatorname{div}(f)$ for some $f \in k(X)^\times$.

Definition 2.8. If X/k is a smooth, projective, irreducible variety we define the *Picard group* $\operatorname{Pic}(X) = \operatorname{Div}(X)/\sim$. For a Weil divisor D we write $[D]$ for the class of D in $\operatorname{Pic}(X)$.

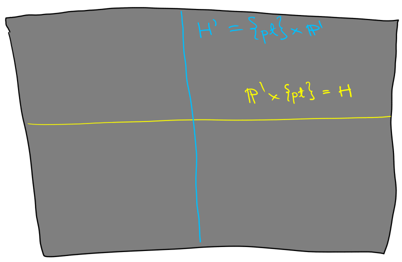
Remark 2.9. The experts will notice that I've imposed enough assumptions in the previous definition to ensure that Cartier divisors talk to linear equivalence classes of Weil divisors, if X is subject to fewer hypotheses (in particular if you need to relax smoothness) you should be more careful.

Example 2.10. Continuing from Exercise 2.4. We have

$$\operatorname{div}(f) = Z - dH.$$

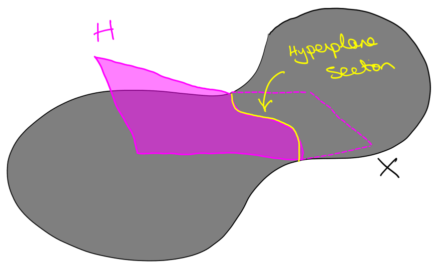
Thus $Z \sim dH$ and therefore $\operatorname{Pic}(X) \cong \mathbb{Z}$ (take $[H] \leftarrow\!\shortmid 1$).

Exercise 2.11. Show that $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \times \mathbb{Z}$ generated by $H \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \times H$.



2.1. **Differentials and canonical divisors.** The goal is to pluck out a "canonical divisor" of a projective variety X/k. Here there is an incomplete treatment but please see Shafarevich's book [4, Chapter 3.5] for something much better.

Try 1: Try the divisor $\operatorname{div}(f)$ for some $f \in k(X)^\times$. This is no good, because we've already used this to define linear equivalence.



Try 2: Consider some projective embedding $X \hookrightarrow \mathbb{P}^n$, and take a hyperplane section. This is ok, but it depends on the extrinsic data of an embedding.

Try 3: Differentials.

Definition 2.12. A *rational* 1*-form* is an expression

$$g\, df \qquad f, g \in k(X)$$

subject to the Leibniz rules

- $da = 0$ for all $a \in k$,
- $d(f+g) = df + dg$ for all $f, g \in k(X)$,
- $d(fg) = f\,dg + g\,df$ for all $f, g \in k(X)$.

We write $\Omega_{k(X)/k}$ for the $k(X)$-module of rational 1-forms.

Lemma 2.13. *If $f_1, \dots, f_n \in k(X)$ is a transcendence basis for $k(X)/k$, then $\Omega_{k(X)/k}$ is generated as a $k(X)$-module by $df_1, \dots, df_n$.*

Proof. Exercise. □

Example 2.14.

(1) $X = \mathbb{P}^1$ so that $k(X) = k(t_1)$. Then $\Omega_{k(X)/k} = \{g\,dt_1 : g \in k(X)\}$.
(2) $X = \mathbb{V}(x_0x_2^2 + x_1^3 + x_0^3) \subset \mathbb{P}^2$. Then $\Omega_{k(X)/k} = \{g\,dt_1 : g \in k(X)\}$.
(3) $X = \mathbb{P}^n$. Then $\Omega_{k(X)/k} = \{g_1\,dt_1 + \dots + g_n\,dt_n : g \in k(X)\}$.

2.1.1. *Canonical divisor on a curve.* Start with curves (irreducible dimension 1 varieties).

Definition 2.15 (Divisor of a 1-form on a curve)**.** Let X/k be a smooth projective curve and non-zero $s \in \Omega_{k(X)/k}$. For each $P \in X(k)$ choose non-constant $f \in k(X)$ so that $v_P(f) = 1$ (a "uniformiser"), then $s = g\,df$ for some $g \in k(X)$. Define $v_P(s) = v_P(g)$ and

$$\operatorname{div}(s) = \sum_P v_P(s) \cdot P.$$

Example 2.16. Let $X = \mathbb{P}^1$ and $s = dt_1$. For all $a \in k$ we have $s = dt_1 = d(t_1 - a)$ so that $v_P(s) = v_P(1) = 0$ for all $P = (1 : a)$. At $\infty = (0 : 1)$ we have $s = dt_1 = d(1/t_0) = -t_0^{-2}\,dt_0$. In particular $v_\infty(dt_0) = -2$ and $\operatorname{div}(s) = -2(\infty)$.

This seems pretty promising (it's at least not linearly equivalent to 0!).

Definition 2.17. Let X/k be a smooth projective curve. A *canonical divisor* K_X for X is any divisor of the form $\operatorname{div}(s)$ for some non-zero $s \in \Omega_{k(X)/k}$.

One should very much hope that this divisor is actually well defined... good news.

Lemma 2.18. *Let X/k be a smooth projective curve. The linear equivalence class of K_X does not depend on the choice of rational 1-form.*

2.1.2. *More general.* An important input in Definition 2.17 is that by Lemma 2.13 there is a 1-dimensional space of 1-forms on a curve. We need to get that back when the dimension is > 1. The idea is to take top wedge powers.

Recall that if V/K is a vector space, for each $p \geq 1$ we have

$$\bigwedge^p V = \left(\bigotimes^p V\right)/R$$

where $R = \operatorname{span}\{v_1 \otimes \dots \otimes v_p : v_i = v_j \text{ for some } i \neq j\}$. The following properties are very useful when you want to compute anything with p-forms.

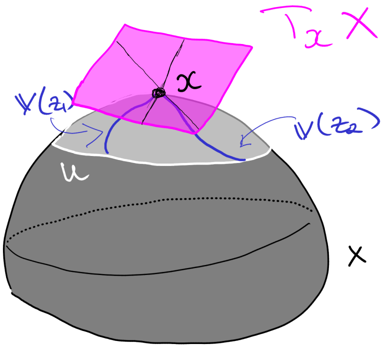
Proposition 2.19. *We have:*

(1) $v_1 \wedge \dots \wedge v_p = 0$ if and only if $v_1, \dots, v_p$ are linearly dependent,

(2) *for a transposition* $\sigma \in S_p$ *we have* $v_1 \wedge ... \wedge v_p = -v_{\sigma(1)} \wedge ... \wedge v_{\sigma(p)}$, *and*
(3) *if* $\dim_K V = n$ *then* $\dim_K \bigwedge^n V = 1$ *and* $\bigwedge^n V$ *is spanned by* $e_1 \wedge ... \wedge e_n$ *for any basis* $\{e_1, ..., e_n\}$ *of* V.

Corollary 2.20. *If* $\dim X = n$ *then the space* $\Omega^n_{k(X)/k} := \bigwedge^n \Omega_{k(X)/k}$ *of* rational n-forms *is a* $k(X)$*-vector space of dimension* 1.

With Corollary 2.20 in our pocket, we are in with a shout of being able to define something reasonable as a canonical divisor.



To get to another n-form we only need to multiply by a rational function – so any good definition of divisor of an n-form will make two divisors-of-n-forms differ by the divisor of a rational function!

Let X/k be a smooth, irreducible, projective variety of dimension n. Recall that at a point $P \in X$ we say $z_1, ..., z_n \in k(X)^\times$ are *local coordinates* at P if $z_1, ..., z_n$ span the *cotangent space* $\mathfrak{m}_P/\mathfrak{m}_P^2$ (where here $\mathfrak{m}_P$ is the maximal ideal of the local ring $\mathcal{O}_{X,P}$).

Example 2.21. If $X = \mathbb{A}^n$ and $P \in X$ is the origin, then $\mathcal{O}_{X,P} = k[t_1, ..., t_n]_{\mathfrak{m}_P}$ where $\mathfrak{m}_P = (t_1, ..., t_n)$. In particular the coordinates $t_1, ..., t_n$ are local coordinates.

Now take $s \in \Omega^n_{k(X)/k}$, a non-zero rational n-form. Let $Z \subset X$ be a prime divisor and choose $z_1, ..., z_n \in k(X)^\times$ to be local coordinates for at (any point in) an open $U \subset X$ for which $Z \cap U \neq \emptyset$. Further suppose that $z_1, ..., z_n$ are regular on U (we can always achieve this by shrinking U if necessary). Then by Corollary 2.20 $s = g\, dz_1 \wedge ... \wedge dz_n$ for some rational function $g \in k(X)^\times$. We define

$$v_Z(s) = v_Z(g).$$

Exercise 2.22. The definition of $v_Z(s)$ above does not depend on the choice of open set U, nor the choice of local coordinates $z_1, ..., z_n$.
Hint: cover X in open affines (one for each point, and small enough so that your favourite local coordinates are regular) and compare the volume forms $dz_1 \wedge ... \wedge dz_n$ by showing that the Jacobian determinant $J = \left|\frac{\partial f_i}{\partial z_j}\right|$ is non-vanishing and regular on overlaps.

Definition 2.23. Let X/k be a smooth, irreducible, projective variety and let $s \in \Omega^n_{k(X)/k}$ be a non-zero rational n-form. We define the *divisor of* s to be

$$\operatorname{div}(s) = \sum_Z v_Z(s) Z$$

where the sum ranges over the prime divisors of X.

Of course, the first thing we should show is the important lemma.

Lemma 2.24. *Let X/k be a smooth, irreducible, projective variety. Then linear equivalence class of the divisor $K_X = \operatorname{div}(s)$ of a rational n-form $s \in \Omega^n_{k(X)/k}$ does not depend on the choice of s. We call K_X a* canonical divisor *on X.*

Example 2.25. Let $X = \mathbb{P}^2$, write $t_1 = x_1/x_0$ and $t_2 = x_2/x_0$. Take $s = dt_1 \wedge dt_2 \in \Omega^2_{k(X)/k}$. Now clearly s has no zeroes or poles on the patch with $x_0 \neq 0$. Then swapping patches by setting $u_0 = x_0/x_2$ and $u_1 = x_1/x_2$ so that $t_1 = u_1/u_0$ and $t_2 = 1/u_0$ we have

$$\begin{aligned} dt_1 \wedge dt_2 &= d(u_1/u_0) \wedge d(1/u_0) \\ &= \frac{u_0\, du_1 - u_1\, du_0}{u_0^2} \wedge \frac{-\, du_0}{u_0^2} \\ &= \frac{-\, du_1 \wedge du_0}{u_0^3} - \frac{u_1\, du_0 \wedge du_0}{u_0^4} \\ &= \frac{-\, du_1 \wedge du_0}{u_0^3} \\ &= \frac{1}{u_0^3}\, du_0 \wedge du_1 \,. \end{aligned}$$

Therefore writing $H = \mathbb{V}(x_0)$ we have $v_H(s) = -3$ and therefore $\operatorname{div}(s) = -3H$. In particular $K_{\mathbb{P}^2} \sim -3H$.

Actually this is more general.

Lemma 2.26. *We have $K_{\mathbb{P}^n} \sim -(n+1)H$ where $H = \mathbb{V}(x_0)$ (or any hyperplane, for that matter).*

Proof. Exercise, follow your nose as above. □

Exercise 2.27. Show that if $X = \mathbb{P}^1 \times \mathbb{P}^1$ then $K_X \sim -2H - 2H'$ where $H = \{\text{pt}\} \times \mathbb{P}^1$ and $H' = \mathbb{P}^1 \times \{\text{pt}\}$.

Again, there is a more general form of Exercise 2.27.

Lemma 2.28. *Let $X = Y_1 \times Y_2$ and let $\pi_i \colon X \to Y_i$ be the projection onto the i^{th} factor, then $K_X \sim \pi_1^* K_{Y_1} + \pi_2^* K_{Y_2}$.*

Proof. Exercise. □

Remark 2.29. Later we will prove the *adjunction formula* (Theorem 5.9) which will allow us to get a canonical divisor on a complete intersection by making adjustments, then intersecting with subvarieties.

3. Curves

By a *smooth curve* I mean a smooth irreducible (reduced) projective variety X/k of dimension 1. Remember, the question we're asking is:

Question 3.1. *How do we distinguish curves?*

Try 1: Consider the space $H^0(X, \mathcal{O}_X)$ of *everywhere regular* rational functions $f \in k(X)$. But there is the well known lemma.

Lemma 3.2. *Let X/k be an irreducible proper (e.g., projective) variety. Then $H^0(X, \mathcal{O}_X) \cong k$.*

So unfortunately this can't give us a good number.

Remark 3.3. When $k = \mathbb{C}$ and $X = \mathbb{P}^1$ one should compare this to Liouville's theorem (every bounded holomorphic function $\mathbb{C} \to \mathbb{C}$ is constant – bounded means there is no pole at infinity). More generally when $k = \mathbb{C}$ one can use compactness and the maximum modulus principle.

Try 2: Let D be a Weil divisor on X. Consider the space

$$H^0(X, \mathcal{O}_X(D)) = \{f \in k(X)^\times : \operatorname{div}(f) + D \geq 0\} \cup \{0\}$$

of rational functions with "poles allowed by D and zeroes forced by D". From this we can get a number, namely $h^0(X, \mathcal{O}_X(D)) = \dim_k H^0(X, \mathcal{O}_X(D))$ (in general little h is the dimension of big H).

Example 3.4. Let $X = \mathbb{P}^1$ and take $D = n(\infty)$. Then

$$H^0(X, \mathcal{O}_X(D)) = \{f \in k[t_1] : \deg(f) \leq n\}.$$

In particular $h^0(X, \mathcal{O}_X(D)) = n + 1$ (the number of monomials).

Try 3: The problem with try 2 is of course that we have to name a divisor. But we have a way to do that!

Definition 3.5. Define the *geometric genus* of X to be $p_g(X) = h^0(X, \mathcal{O}_X(K_X))$.

Example 3.6. Take $X = \mathbb{P}^1$. Then we know $K_X \sim -2(\infty)$. Then $h^0(X, \mathcal{O}_X(K_X)) = 0$ because we're asking for those everywhere regular functions (i.e., constants) which have (at least) a double zero at infinity. That's enough to make anyone zero.

Example 3.7. Take X to be a smooth cubic curve in $\mathbb{P}^2$. By the adjunction formula Theorem 5.9 we will see that $K_X \sim \mathcal{O}_X$. Thus by Lemma 3.2 we have $p_g(X) = 1$.

3.1. Riemann–Roch.

We now state the Riemann–Roch theorem, which is a powerful tool for computing the dimensions $h^0(\mathcal{O}_X(D)) := h^0(X, \mathcal{O}_X(D))$ for divisors D on a smooth projective curve X.

Theorem 3.8 (Riemann–Roch)**.** *Let D be a divisor on a smooth irreducible projective curve X/k. Then*

$$\underbrace{h^0(\mathcal{O}_X(D))}_{\text{want this}} - \underbrace{h^0(\mathcal{O}_X(K_X - D))}_{\text{error term}} = \underbrace{\deg D - p_g(X) + 1}_{\text{simple constant}}.$$

Example 3.9. Continuing from Example 3.6 we know that $h^0(\mathcal{O}_X(n(\infty))) = n + 1 = \deg(n(\infty)) + 1$ whenever $n \geq 0$. But Riemann–Roch tells us that this should continue to happen when $n < 0$ so long as we correct by the term $h^0(\mathcal{O}_X(K_X - n(\infty)))$. But we have $K_X = -2(\infty)$ so this dimension is $h^0(\mathcal{O}_X(E))$ where $E = (-n-2)(\infty)))$. Whenever $n \leq -2$ we get $h^0(\mathcal{O}_X(E)) = (-n-2) + 1 = -n - 1$ (as we want). When $n = -1$ we get

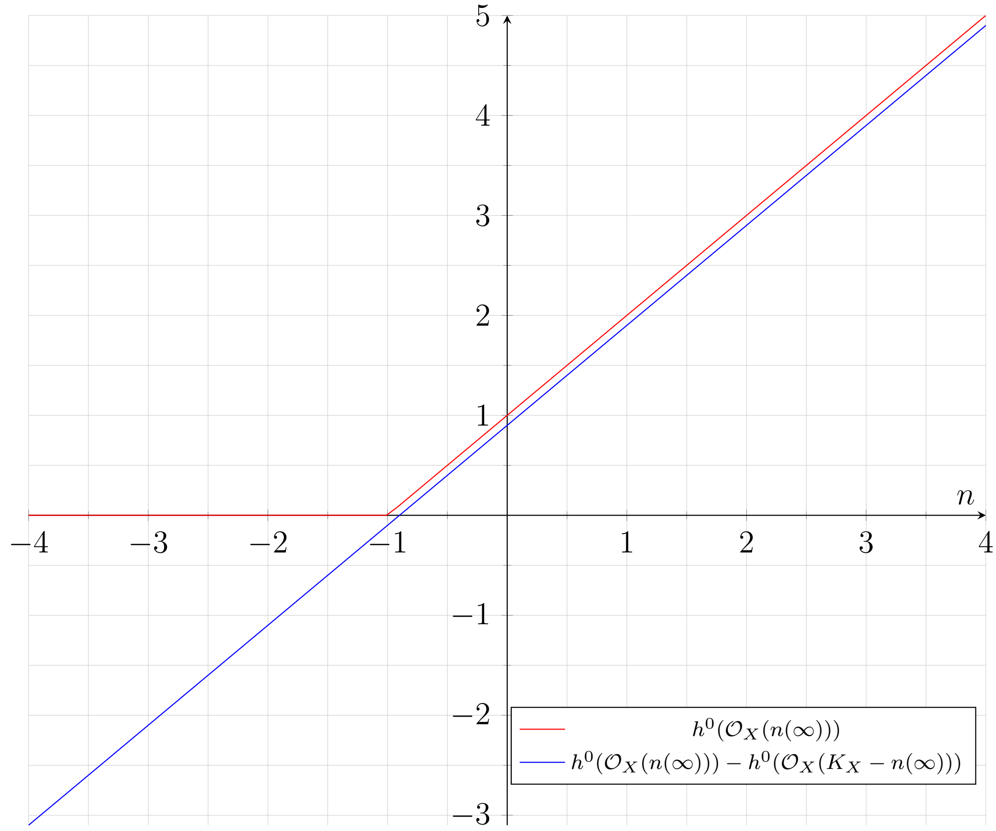


FIGURE 2. Riemann–Roch for $\mathbb{P}^1$ and $D = n(\infty)$.

that both $h^0(\mathcal{O}_X(n(\infty)))$ and $h^0(\mathcal{O}_X(E))$ are zero (as required, again). This example is illustrated in Figure 2.

Remark 3.10. The point is that the correction term makes the dimensions "behave like linear functions in the degree". Later we will see that this is really some manifestation of the Euler characteristic being the "right" thing to use in exact sequences when we want to find dimensions.

Often Riemann–Roch is best used in the form of the following corollary.

Corollary 3.11. *We have:*

(1) $h^0(\mathcal{O}_X(D)) \leq \deg D - p_g(X) + 1$,
(2) $\deg K_X = 2p_g(X) - 2$,
(3) if $\deg D \geq 2p_g(X) - 1$ *then* $h^0(\mathcal{O}_X(K_X - D)) = 0$.

Proof. (1) is clear. For (2) take $D = K_X$ then Riemann–Roch implies that $p_g(X) - 1 = \deg K_X - p_g(X) + 1$ and the claim follows. For (3) combine (1) and (2) to obtain the bound $h^0(\mathcal{O}_X(K_X - D)) \leq \deg(K_X - D) - p_g(X) + 1 \leq \deg(K_X - D) + 1 \leq 0$. □

I'll conclude with a standard example of how to use Riemann–Roch.

Lemma 3.12. *Every genus* 1 *curve* X/k *(over an algebraically closed field) is isomorphic to a smooth cubic curve in* $\mathbb{P}^2$.

Proof. You can find something like this in [5, III]. Take a point $P \in X(k)$ (this is the only place we use the algebraic closure, actually). Now by Corollary 3.11 we have $\deg K_X = 0$ and therefore we can compute

$$h^0(\mathcal{O}_X(3P)) = 3 \qquad \text{and} \qquad h^0(\mathcal{O}_X(9P)) = 9.$$

But if we choose a basis $1, x, y \in H^0(\mathcal{O}_X(3P))$ then each of the 10 homogeneous degree 3 monomials in $1, x, y$ lives in $H^0(\mathcal{O}(9P))$ (count the poles). In particular, there is a relation of degree 3 between $1, x, y$.

Thus the rational map $\phi\colon X \dashrightarrow \mathbb{P}^2$ given by $[1 : x : y]$ lands on a cubic curve X'.

Exercise 3.13. Check that ϕ is in fact an isomorphism.

□

Exercise 3.14. Use a similar trick to show that every genus 1 curve is isomorphic to an intersection of two quadrics in $\mathbb{P}^3$. Hint: consider $H^0(\mathcal{O}_X(4P))$ and $H^0(\mathcal{O}_X(8P))$.

4. Sheaves and stuff

This is not a course about sheaves, it's more about using them. As such, I'm going to give a bunch of examples and if this section makes no sense at all, that's ok. Anything I state you're welcome to take as an axiom until you want to read Hartshorne. I quite like the short treatment in Reid's notes [3, Chapter B], and I follow it relatively closely.

Example 4.1. Let X/k be an irreducible variety and take $U \subset X$ an open subset.

(1) The *structure sheaf* $\mathcal{O}_X$ so that
$$\Gamma(U, \mathcal{O}_X) = \{f \in k(X)^\times : f \text{ is regular on } U\} \cup \{0\}.$$
(2) If X is smooth the sheaf $\mathcal{O}_X(D)$ for a Weil divisor D so that
$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in k(X)^\times :\}.$$
(3) If X is smooth the *sheaf of regular* 1*-forms* $\Omega_{X/k}$ so that
$$\Gamma(U, \Omega_{X/k}) = \{s \in \Omega_{X/k} : s \neq 0 \text{ is regular on } U\} \cup \{0\},$$
where regular means that there exist regular $f, g \in k(X)$ so that $s = g\, df$.
(4) If X is smooth the *sheaf of regular* p*-forms* $\Omega^p_{X/k} = \bigwedge^p \Omega_{X/k}$ so that
$$\Gamma(U, \Omega^p_{X/k}) = \left\{s \in \bigwedge^p \Omega_{X/k} : s \neq 0 \text{ is regular on } U\right\} \cup \{0\},$$
where regular means that there exist regular $g, f_1, ..., f_p, \in k(X)$ so that $s = g\, df_1 \wedge ... \wedge df_p$.
(5) If X is smooth, the *canonical sheaf* $\omega_X = \Omega^{\dim X}_{X/k}$.

Obviously the $\Gamma(U, \mathcal{O}_X)$ are rings, all the others are merely abelian groups. But actually, (2)–(4) are naturally $\Gamma(U, \mathcal{O}_X)$-modules.

Definition 4.2 (Sketch definition)**.** A *sheaf* $\mathcal{F}$ *of "foos"* on a variety X/k is some assignment which takes in open sets and spits out "foos" i.e., $U \mapsto \Gamma(U, \mathcal{F})$ (here "foos" are say sets, rings, abelian groups). To be a sheaf, there is additional data to make sure our sheaf encapsulate "function-ness".

Restriction: Given open sets $V \subset U \subset X$ there exist a restriction map $\rho_{UV} \colon \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$ which staisfies the condition that if $W \subset V \subset U$ then we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.
Slogan: If you're restricting the domain of a function it doesn't matter if you first restrict a little less.

Gluing: If you have something that looks like a function on an open cover of U, then it glues to a function on U and you know it uniquely.

Exercise 4.3.

(1) Look up the actual definition and check that my bad sketch is correct.
(2) Come up with the definition of a *sheaf of $\mathcal{O}_X$-modules* (you want the restrictions to behave with the structure).
(3) Come up with the definition of *locally adjective*.

Example 4.4 (Pushforward)**.** Let $i \colon Y \hookrightarrow X$ be a subvariety. We can "push-forward" a sheaf $\mathcal{F}$ on Y to a sheaf $i_*\mathcal{F}$ on X by defining

$$\begin{aligned}\Gamma(U, i_*\mathcal{F}) &= \Gamma(U \cap Y, \mathcal{F}) \\ &= \begin{cases} \Gamma(U \cap Y, \mathcal{F}) & \text{if } U \cap Y \neq \emptyset, \text{ and} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

I like to think of this as a kind of "δ-function" supported along Y.
A typical example of this is the skyscraper sheaf. Take a finite set of (closed) points $Y = \{P_1, ..., P_r\} \subset X(k)$ and consider $i_*\mathcal{O}_Y$. Since $\dim_k \Gamma(V, \mathcal{O}_Y)$ just counts the number of points in $V \subset Y$ one can think of $i_*\mathcal{O}_Y$ as some kind of indicator function for containing elements of Y. The name "skyscraper" comes from the vision on stalks – we get a k at the stalks $(i_*\mathcal{O}_Y)_{P_i}$ and trivial everywhere else.

FIGURE 3. A sketch of what (I think) the stalks of the pushforward of the structure sheaf looks like. On the left is the skyscraper sheaf. Note that everything is supported along the "spine".

Exercise 4.5. Figure out what the skyscraper sheaf looks like when some of the points are non-reduced.

Exercise 4.6. Define pushforward for any morphism $Y \to X$ (not just immersions).

We get to stalks. One should imagine this as the "algebraic-geometry-version-of-Taylor-series-expansions-of-holomorphic-functions". That is, some kind of "very local" view of a function.

Definition 4.7. Let $\mathcal{F}$ be a sheaf on a variety X. For a point $x \in X$ (here this could be a scheme-theoretic point i.e., non-closed) then the *stalk of $\mathcal{F}$ at x* is defined as

$$\mathcal{F}_x = \varinjlim \Gamma(U, \mathcal{F}).$$

If this is intimidating, not to worry, you can take the following lemma as a definition.

Lemma 4.8. *Let X be a variety and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then for any affine open* $\operatorname{Spec} A \cong U \ni x$*. Identify x with a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, then*

$$\mathcal{F}_x = \Gamma(U, \mathcal{F})_{\mathfrak{p}}$$

the localisation of $\Gamma(U, \mathcal{F})$ *at* $\mathfrak{p}$.

Exercise 4.9. Define $\otimes$ and quotients for sheaves. If you haven't seen this before you'll probably get it wrong, not to worry – this is a feature not a bug. The point is that we want these things to "look correct" on stalks, but then we may have to add in some "extra" sections to get the gluing to work.

Anyway, even if you don't do the exercise the following should hopefully provide some orientation.

Proposition 4.10. *Let X/k be an irreducible, smooth, projective variety and let D, D' be Weil divisors on X:*

(1) $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ *if and only if* $D \sim D'$,
(2) $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D') \cong \mathcal{O}_X(D + D')$,
(3) *if K_X is a canonical divisor on X then* $\omega_X \cong \mathcal{O}_X(K_X)$.

4.1. **Exactness.** The point here is that exactness is "hyper-local" in the sense that we want to be able to check it on stalks. Why not define it that way then!

Definition 4.11. A sequence

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

of sheaves of abelian groups on X is *exact* if for every $x \in X$ the induced sequence

$$\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$$

of abelian groups is exact.

Example 4.12. If we include a point $i \colon P \hookrightarrow X$ in a curve X we have an exact sequence

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to i_*\mathcal{O}_P \to 0.$$

It suffices to check this stalk-locally. If $x \neq P$ (closed) this is clear, since $\mathcal{O}_X(-P)_x = \mathcal{O}_{X,x}$ and $i_*(\mathcal{O}_P)_x = 0$. If $x = P$ then $\mathcal{O}(-P)_P$ is the unique maximal ideal in $\mathcal{O}_{X,P}$ and $(i_*\mathcal{O}_P)_P = k$.

You will not lose much in this course if you take the following lemma only with closed subvarieties $Y \subset X$ (i.e., no reduced structure).

Lemma 4.13. *If $\iota\colon Y \hookrightarrow X$ is a closed subscheme of a variety X/k then we have a short exact sequence*

$$0 \to \mathcal{I}_{Y|X} \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0.$$

In particular, X is smooth, projective, and irreducible then if D is an effective divisor we have a short exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$$

where here we are abusively identifying D and the corresponding pure codimension 1 subscheme in X (which is non-reduced if and only if $n_Z > 1$ for some Z).

Proof. Exercise, just check it on stalks. $\square$

4.2. **The failure of surjectivity.** It turns out that when we take global sections, right exactness is not preserved. Let's see a couple of examples.

Example 4.14. Let $X = \mathbb{P}^1$ and consider $Y = \{P, Q\}$ and let $D = P + Q$. Then $i_*\mathcal{O}_Y$ is the skyscraper sheaf supported on P and Q. We have a short exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0.$$

But if we take global sections we have

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X, \mathcal{O}_X(-D)) \arrow[r] \arrow[d, equal] & \Gamma(X, \mathcal{O}_X) \arrow[r] \arrow[d, equal] & \Gamma(Y, \mathcal{O}_Y) \arrow[d, equal] \\ 0 \arrow[r] & 0 \arrow[r] & k \arrow[r] & k \oplus k \end{tikzcd}$$

This clearly cannot be surjective! The failure on the right is coming from the "error term" appearing in Riemann–Roch (the term $h^0(\mathcal{O}_X(K_X - D))$).

Exercise 4.15. Generalise Example 4.14 to more general curves X. Give a necessary and sufficient condition on $\deg D$ for the failure to occur.

Example 4.16. I've borrowed this out of Reid's notes. Take $X = \mathbb{P}^n$ with $n \geq 2$ and take distinct points $P, Q, R \in X(k)$. Suppose for simplicity that $P, Q, R \notin H = \mathbb{V}(x_0)$ (for this example, we'll be happy to move our hyperplane in its linear equivalence class anyway). Take $Y \subset X$ to be the three point variety $Y = \{P, Q, R\}$. We have the ideal sheaf exact sequence

$$0 \to \mathcal{I}_{Y|X} \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0.$$

The setup of this example is to tensor this exact sequence with $\mathcal{O}_X(H)$ – this tensoring preserves exactness by the global version of "free modules are flat". So we have:

$$(*) \qquad 0 \to \mathcal{I}_{Y|X}(H) \to \mathcal{O}_X(H) \to (i_*\mathcal{O}_Y)(H) \to 0.$$

I want to try describe the terms and the maps.

<u>$\mathcal{O}_X(H)$:</u> A section $f \in \Gamma(X, \mathcal{O}_X(H))$ is a "linear form" – a rational function of the sort $f = g/x_0$ where $g \in k[x_0, ..., x_n]$ is linear.

<u>$\mathcal{I}_{Y|X}(H)$:</u> This is $\mathcal{I}_{Y|X}(H) = \mathcal{I}_{Y|X} \otimes_{\mathcal{O}_X} \mathcal{O}_X(H)$. Then $\Gamma(X, \mathcal{I}_{Y|X}(H))$ is exactly the set of linear forms which vanish on Y. The map $\mathcal{I}_{Y|X}(H) \to \mathcal{O}_X(H)$ is just inclusion.

<u>$(i_*\mathcal{O}_Y)(H)$:</u> This is isomorphic to $i_*\mathcal{O}_Y(H|_Y) = i_*\mathcal{O}_Y$ because H does not meet Y.

<u>$\mathcal{O}_X(H) \to i_*\mathcal{O}_Y$:</u> We have $\Gamma(Y, \mathcal{O}_Y) \cong k^3$. The map says to take some function $f \in \Gamma(X, \mathcal{O}_X(H))$ and evaluate it on P, Q, and R.

One can also see exactness of $(*)$ visually (without Lemma 4.13). Just go to a small enough affine open.
But now we want to take global section in $(*)$. Then we get

$$0 \to \{\text{linear forms vanishing on } P, Q, R\} \to \operatorname{span}\{x_0, ..., x_n\} \to k^3.$$

Now this *is* right exact so long as there is not an *unexpected linear dependency* – i.e., if they lie on a line! In the language of cohomology $H^1(X, \mathcal{I}_{Y|X}(H)) \neq 0$ if our three points are colinear. All of this is controlled by an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X, \mathcal{I}_{Y|X}(H)) \arrow[r] & \Gamma(X, \mathcal{O}_X(H)) \arrow[r] & \Gamma(Y, \mathcal{O}_Y) \arrow[dll, rounded corners] \\ & H^1(X, \mathcal{I}_{Y|X}(H)) \arrow[r] & H^1(X, \mathcal{O}_X(H)) = 0. & \end{tikzcd}$$

4.3. **Cohomology of coherent sheaves.** The idea for "fixing" these right exactness issues is to consider exact sequences of cohomology groups instead. We'll probably only use these tools in the setting of *coherent sheaves* which is a certain "finite presentation" condition on $\mathcal{O}_X$-modules which allows a lot of theorems to work. At least in our setting of X/k a variety $\mathcal{F}$ being coherent is the same as X admitting a cover by open sets U for which

$$\mathcal{O}_X^{\oplus m}|_U \to \mathcal{O}_X^{\oplus n}|_U \to \mathcal{F}|_U \to 0.$$

To be quasi-coherent is a weakening of this where we don't just allow finite direct sums, but direct sums over any horrible index set. Anyway, if you like, you can take the following theorem as an axiom.

Theorem 4.17. *Let X/k be a variety and let $\mathcal{F}$ be a (quasi-)coherent sheaf on X. Then there exist k-vector spaces $H^i(X, \mathcal{F})$ which satisfy:*

(A) <u>Global sections:</u> H^0 interpolates Γ (i.e., $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$).

(B) <u>Functoriality:</u> a morphism $\mathcal{F} \to \mathcal{G}$ induces $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$.

(C) <u>Long exact sequence:</u> If we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

then we can extract a long exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}') \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{F}'') \arrow[dll, rounded corners] \\
 & H^1(X,\mathcal{F}') \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{F}'') \arrow[dll, rounded corners] \\
 & H^2(X,\mathcal{F}') \arrow[r] & \cdots &
\end{tikzcd}
$$

(D) *Affines: If X is affine then $H^i(X,\mathcal{F}) = 0$ for all $i > 0$.*

(E) *Dimension: If X is irreducible and $\dim X = n$ then $H^i(X,\mathcal{F}) = 0$ for all $i > n$.*

(F) *Finite dimensional vector spaces: If X is proper (e.g., projective) and $\mathcal{F}$ is coherent then $\dim_k H^i(X,\mathcal{F}) < \infty$ for all i.*

(G) *Serre vanishing: If $X \subset \mathbb{P}^m$ is a closed subvariety, $\mathcal{F}$ is coherent, and H is a hyperplane section of X (i.e., $\mathcal{O}_X(H)$ is very ample) then there exists $N > 0$ such that for all $r > N$ and $i > 0$ we have $H^i(X,\mathcal{F}(rH)) = 0$ (here $\mathcal{F}(rH) = \mathcal{F} \otimes \mathcal{O}_X(rH)$).*

(H) *Serre duality: If X is smooth, projective, and irreducible of dimension n, then $H^n(X,\omega_X) \cong k$. For any line bundle $\mathcal{L} \cong \mathcal{O}_X(D)$ there exists a perfect pairing*

$$H^i(X,\mathcal{L}) \times H^{n-i}(X,\mathcal{L}^{-1} \otimes \omega_X) \to k$$

or if you prefer

$$H^i(X,\mathcal{O}_X(D)) \times H^{n-i}(X,\mathcal{O}_X(K_X - D)) \to k.$$

In particular $h^i(\mathcal{O}_X(D)) = h^{n-i}(X,\mathcal{O}_X(K_X - D))$.

(I) *Euler characteristic: If X is irreducible, projective of dimension n, and $\mathcal{F}$ is coherent we define the Euler characteristic of $\mathcal{F}$ to be the alternating sum*

$$\chi(\mathcal{F}) = \sum_{i=0}^{\infty}(-1)^i h^i(\mathcal{F}) = \sum_{i=0}^{n}(-1)^i h^i(\mathcal{F}).$$

Then $\chi(\mathcal{F})$ is additive in short exact sequences i.e., if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact, then $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.

Maybe most of this stuff shouldn't be too surprising if you keep the goal in mind. Here's some remarks to convince you that it is so.

Remark 4.18.

(1) Of course, if you were paying attention you should have noticed we have some obsession with attaching a number to varieties – in which case you will be very pleased to see that (F) allows us to get a number!
(2) If you were paying even closer attention to the Riemann–Roch theorem you'll remember some error term of the form $h^0(\mathcal{O}_X(K_X - D))$ – so you'll be very suspicious of Serre duality.
(3) The additivity of χ in exact sequences can be viewed as some "corrected" version of the dimension (which would be exact if the $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ was just an exact sequence of old-fashioned k-vector spaces).

4.4. **Sketch proof of Riemann–Roch using Serre duality.**

Sketch. When $D = 0$ Riemann–Roch is true by definition of the genus and the fact $h^0(\mathcal{O}_X) = 1$. The proof now goes by applying Serre duality to get adding and subtracting points.

We have the ideal sheaf exact sequence

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to i_*\mathcal{O}_P \to 0$$

and tensor with any invertible sheaf $\mathcal{L} = \mathcal{O}_X(D')$. Note that $\mathcal{L} \otimes i_*\mathcal{O}_P \cong \mathcal{O}_P$ (this is clear when the support of D' does not contain P, more generally replace D' with a linearly equivalent divisor whose support does not contain P – more on this in Lemma 5.3).

By additivity of Euler characteristics we get $\chi(\mathcal{L}) - \chi(\mathcal{L}(-P)) = \chi(\mathcal{O}_P) = h^0(\mathcal{O}_P) = 1$ (the second last equality is because the dimension of P is 0). Now Serre duality says that $\chi(\mathcal{L}(-P)) = h^0(\mathcal{L}(-P)) - h^1(\mathcal{L}(-P)) = h^0(\mathcal{L}(-P)) - h^0(\mathcal{L}(K_X + P)$.

Exercise 4.19. Prove $\chi(\mathcal{L}(P)) = h^0(\mathcal{L}(P)) - h^0(\mathcal{L}(K_X - P))$.

The claim follows by induction (adding and subtracting points, as required). □

5. The adjunction formula

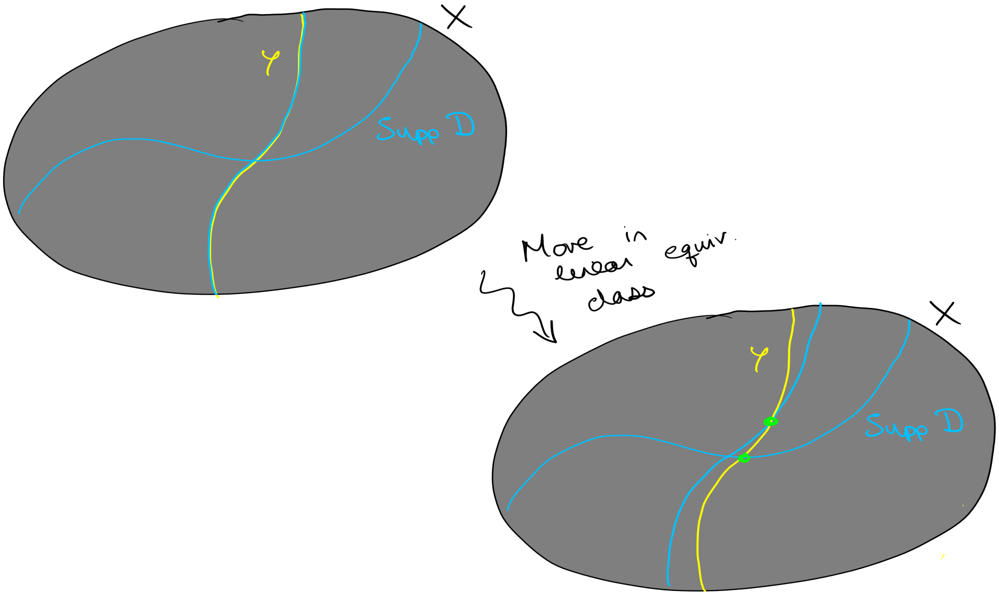
Let X/k be a smooth irreducible variety.

5.1. **The moving lemma and restriction.** If $Z \subset X$ is a prime divisor and $Y \subset X$ is an irreducible subvariety which 1is not contained in Z then $Z \cap Y$ (where here I actually mean scheme theoretic intersection $Z \times_X Y$ which adds appropriate multiplicities to intersections) is an effective divisor on Y which we denote $Z|_Y$ and call the *restriction of Z to Y*. More generally if $D = \sum_Z n_Z \cdot Z$ is a divisor on X whose support does not contain Y we define $D|_Y = \sum_Z n_Z \cdot Z|_Y$.

Example 5.1. If $Y \subset \mathbb{P}^2$ is a cubic curve and H is a hyperplane which meets Y with multiplicity 3 at an inflection point P, then $H|_Y = 3P$.

Remark 5.2. This is a typical example of the abuse of notation which trades effective Weil divisors and closed subschemes of pure codimension 1 (add multiplicity along components by taking a power of the ideal sheaf).

The moving lemma is a nice tool which allows us to restrict divisors (even ones which contain Y!) so long as we are willing to work up to linear equivalence (which we are of course).



Lemma 5.3 (Moving lemma). *Let X/k be a smooth irreducible variety, let $D \in \mathrm{Div}(X)$, and let $Y \subset X$ be a closed subvariety. Then there exists a linearly equivalent divisor $D' \sim D$ such that Y is not contained in the support of D'.*

Proof. This is borrowed from [2, Prop. 9.1.11]. It suffices to prove the claim when $Y = \{x\}$ is a closed point, and therefore we may assume X is affine. By writing $D = A - B$ with A and B effective, we are also free to assume D is effective. But because X is affine there is some $\pi \in H^0(X, \mathcal{O}_X(-D))$ which generates $\mathcal{O}_X(-D)_x$ as an $\mathcal{O}_{X,x}$-module (it has rank 1). But π is exactly the rational function we need to adjust by, let $D' = D + \mathrm{div}(\pi)$. By construction $\mathcal{O}_X(D')_x = \mathcal{O}_{X,x}$ as subsets of $k(X)$, and this is exactly what it means to be disjoint from the support of D'. □

Remark 5.4. The moving lemma is true in more generality [2, Prop. 9.1.11] (allowing Y to be reducible and knowing that D does not contain any of the components of Y). The more general result of Chow proves something like this for "algebraic cycles" up to an appropriate equivalence notion [6, Tag 0B0D].

Let X be a smooth projective variety and let $Y \subset X$ be a smooth closed subvariety. Let $[D] \in \mathrm{Pic}(X)$ be a linear equivalence class of divisors on X. Let $D' \in [D]$ be a divisor whose support does not contain Y, then the restriction $[D'|_Y]$ is a well defined linear equivalence class of divisors on Y.

Upshot. When you're given a divisor to restrict: move it, then restrict it.

Exercise 5.5. Prove that $\mathcal{O}_X(D) \otimes i_*\mathcal{O}_Y \cong i_*\mathcal{O}_Y(D|_Y)$.

Remark 5.6. In light of this I may write $\mathcal{F}|_Y = \mathcal{F} \otimes i_*\mathcal{O}_Y$ (actually, this is a bit of an abuse of notation because this is a sheaf on X but I don't want to define the terms in $i^*\mathcal{F} = i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_Y$ which *is* a sheaf on Y, but anyway just pushforward if you landed on the wrong topological space).

Example 5.7. Previously we had $Y = \{P, Q, R\} \subset \mathbb{P}^2$ and H a hyperplane. After choosing our favourite hyperplane, we are free to assume that H does not meet Y. In particular $\mathcal{O}_{\mathbb{P}^2}(H) \otimes i_* \mathcal{O}_Y \cong \mathcal{O}_Y(H|_Y) \cong \mathcal{O}_Y$ as expected.

Example 5.8. Let $Y \subset \mathbb{P}^2$ be a cubic curve. We have $\mathcal{O}_{\mathbb{P}^2}(Y)|_Y \cong \mathcal{O}_{\mathbb{P}^2}(3)|_Y \cong \mathcal{O}_Y(9P)$ (where P is an inflection point on Y – choose a hyperplane which passes through P with muliplicity 3).

Theorem 5.9 (Adjunction formula)**.** *Let X/k be a smooth projective irreducible variety and let $Y \subset X$ be a smooth irreducible subvariety of codimension 1 then we have*

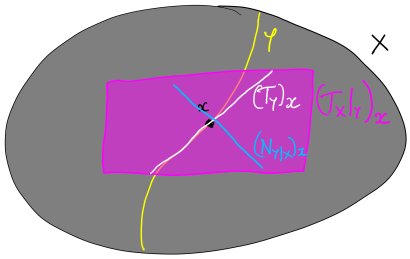
$$K_Y = (K_X + Y)|_Y$$

is an canonical divisor for Y. Equivalently,

$$\omega_Y \cong \omega_X(Y)|_Y.$$

Sketch proof. The idea is actually quite simple, what takes more effort is convincing yourself that the definitions are correct. We start with the tangent–normal exact sequence

$$0 \to T_Y \to T_X|_Y \to N_{X|Y} \to 0.$$



In the setting of vector bundles, it is quite clear that this sequence is exact (and of course, we're only checking exactness stalk-locally). Anyway, the dual of this exact sequence is really what we're after, and we get

$$0 \to \Omega_Y \to \Omega_X|_Y \to \mathcal{O}_X(-Y) \to 0.$$

Now taking top wedge powers we get an isomorphism $\omega_X|_Y \cong \omega_Y \otimes \mathcal{O}_X(-Y) = \omega_Y(-Y)$. Now tensor both sides with $\mathcal{O}_X(Y)$ so that $\omega_X(Y)|_Y \cong \omega_Y$. $\square$

Let's conclude this section with some nice applications.

Example 5.10.

(1) Let Y be a smooth cubic curve in $X = \mathbb{P}^2$. Then $\omega_X \cong \mathcal{O}_X(-3)$ and in particular $\omega_Y \cong \mathcal{O}_X(-3H + Y)|_Y \cong \mathcal{O}_X|_Y \cong \mathcal{O}_Y$ is trivial.

(2) Let Y be a smooth quadric intersection $Q_1 \cap Q_2$ in $\mathbb{P}^3$. Then $\omega_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4)$ and
$$\omega_{Q_1} \cong \mathcal{O}_{\mathbb{P}^3}(-4H + Q_1)|_{Q_1} \cong \mathcal{O}_{\mathbb{P}^3}(-2)|_{Q_1}$$
and thus
$$\omega_Y \cong \mathcal{O}_{\mathbb{P}^3}(-2H + Q_2)|_Y \cong \mathcal{O}_Y$$
is again trivial.

(3) Let Y be a smooth quartic curve in $X = \mathbb{P}^2$. Then $\omega_Y \cong \mathcal{O}_{\mathbb{P}^2}(1)|_Y$.

(4) Let Y be a smooth intersection of a quadric and cubic surface in $X = \mathbb{P}^3$. Then $\omega_Y \cong \mathcal{O}_{\mathbb{P}^3}(1)|_Y$.

(5) Let Y be a smooth quartic surface in $X = \mathbb{P}^3$. Then $\omega_Y \cong \mathcal{O}_Y$.

The examples in (1)–(2) are "elliptic normal curves". A curve $Y \subset \mathbb{P}^n$ which satisfies the condition $\omega_Y \cong \mathcal{O}_{\mathbb{P}^n}(1)|_Y$ as in (3)–(4) are known as "canonical curves". The example in (5) is a K3 surface – more on this later.

Exercise 5.11. Use the adjunction formula to prove the *genus degree formula*. If $X \subset \mathbb{P}^2$ is a smooth curve of degree d, then

$$p_g(X) = \frac{(d-1)(d-2)}{2}.$$

We'll see a different proof of this later.

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