255 Lecture Notes

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Contents

1 Metric Spaces and Normed Spaces

Definition 1 *A pair (X, d) where X is a set and d is a function $d : X \times X \to R$ is called a metric space if the following holds for all $x, y, z \in X$*
(1) $d(x, y) \geq 0$
(2) $d(x, y) = 0 \iff x = y$
(3) $d(x, y) = d(y, x)$
(4) (Triangle Inequality) $d(x, y) \leq d(x, z) + d(z, y)$

The following is called a metric or a distance function on X.

A metric space (X, d) is a natural abstract "minimal" structure for discussions of the notions like convergence and continuity. To discuss differentiation, one needs more regularity and that is where normed spaces enter. In this course, we will not discuss in much detail thee differentiation theory of normed-spaces- we will only enter the framework to this study.

Example 1 *The most basic one, ande the entire Analayis 1 was devoted to it: $d(x, y) = |x - y|$. This example is the building block fo the mostexamples we will discuss.*

Example 2 $X = R^n; x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$

$$d(x, y) = \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{1/2}$$

(X, d) is sometimes called Euclidean space and d is called Euclidean metric. Another possible choice of metric on R^n is

$$d(x, y) = \max_{1 \leq k \leq n} |x_k - y_k|$$

We will prove later that these functions d are indeed metrics on R^n.

Example 3 *Let $[a, b]$ be a bounded closed interval in R and let $C([a, b])$ be the collection of all continuous functions*

$$f : [a, b] \to R$$

$C([a, b])$ is a vector space over R, with operations of addition and scalar multiplication defined by

$$j(f + g)(f) = f(f) + g(f)$$

$$(\alpha f)(f) = \alpha f(f)$$

The vector space $C[a, b]$ is infinite dimensional.
For $f, y \in C[a, b]$ we set

$$d(f, g) = \sup_{a \leq f \leq b} \mid f(f) - g(f) \mid$$

$$= \sup\{|f(f) - g(f)| : f \in [a, b]\}$$

$$= \max\{|f(f) - g(f)| : f \in [a, b]\}$$

$$= \max_{a \leq f \leq b} |f(f) - g(f)|$$

Let us prove now that d is a metric on $C[a, b]$.

Property 1 $d(f, g) \geq 0$ *obvious*

Property 2

$$d(f, g) = 0$$

$$\Leftrightarrow \sup_{a \leq f \leq b} |f(\xi) - g(f)| = 0$$

$$\Leftrightarrow |f(f) - g(f)| = 0 \quad \forall f \in [a, b] \mid$$

$$\Leftrightarrow f = g$$

Property 3 $d(f, g) = d(g, f)$ *follows from* $|f(f) - g(f) = |(g(f) - f(f)|$ *for all* $f \in [a, b]$

Property 4 *The triangle inequalty. Let $f, g, g \in C([a, b])$.*
Then, for any $t \in [a, b]$,

$$|f(t) - g(t)| \leq |f(t) - g(t)|_h(t) - g(t)|$$

impiles that

$$|f(t) - g(t)| \leq |f(t) - g(t)| \leq \sup_{a \leq s \leq b} |f(s) - f(s)| + \sup_{a \leq s \leq b} |f(s) - g(s)|$$

$$= d(f,h) + d(h,g)$$

So,

$$\sup_a |f(t) - g(t)| \leq d(f,h) + d(h,g)$$

or

$$d(f,g) \leq d(f,h) + d(h,g)$$

Example 4 *Let X be an arbitrary set and let $d : X \times X \to [0, \infty]$ be defined by*

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Then d is a metric on X (sometimes called a discrete metric on X.

To check $(1)-(3)$ of the definition of a metric is trivial. To verify the triangle inequality,

$$d(x, y \leq d(x,z) + d(z,y)$$

Note that this inequality is trivial such that $x = y$. If $x \neq y$, then we must have that either $z \neq x$ or $z \neq y$. (Otherwise, $z = x$ and $z = y$, so $x = y$).

Hence, if x and $d(x,y) = 1$, then either $d(x,z) = 1$ or $d(z,y) = 1$ (of course, they could be also both 1).

so $d(x,y) \leq d(x,z) + d(z,y)$.

We now describe normed spaces.

Definition 2 *Let X be a vector space (over R or over C)*

A function

$$x \in X \to \|x\| \in R$$

is called a norm function, or just a norm on X if the following holds for all $x, x \in X$ and all scalars α.

(1) $\|x\| \geq 0$

(2) $\|x\| = 0 \iff x = 0_x$ *(zero vector in X)*

(3)

$$\|\alpha x\| = |\alpha| \|x\|$$

(4) The triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

A pair $(X, \|\dot{\|})$, where X is a vector space and $\|\dot{\|}$ is a norm on X is called a normed space.

Proposition 1 *Let $(X, \|\dot{\|})$ be a normed space and let $d : X \times X \to R$ be defined by $d(x,y) = \|x - y\|$*

Then d is a metric on X.

proof:

Property 1: $d(x,y) \geq 0$ *is obvious.*

Property 2: $d(x, y) = 0 \Leftrightarrow \|x - y\| = 0 \Leftrightarrow x - y = 0_x \Leftrightarrow x = y$

Property 3: $d(x, y) = \|x - y\| = \|(-1)(y - x)\| = |(-1)|\|y - x\| = \|y - x\|$ $= d(y, x)$

Property 4: $d(x, y) = \|x - y\| \leq \|(x - z) + (z - y)\| = \|x - z\| + \|z - y\| =$ $d(x, z), d(z, y)$

Proposition 2 *Let $x = R^n$ and $x = (x_1, \ldots, x_n) \in R^n$ For $1 \leq p < \infty$ we set*

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}$$

and for $p = \infty$, one sets $\|x\|_\infty = \max_{1 \leq k \leq n} |x_k|$.

Then, for any $1 \leq p \leq \infty$, $\| \cdot \|_p$ is a norm on X.

proof.

For any $1 \leq p \leq \infty$, the properties (1)-(3) in the definition of the norm

$$(\|x\|_p \geq 0, \|x\|_p = 0 \Leftrightarrow x = 0_x, \|\alpha x\|_p = |\alpha| \cdot \|x\|_p)$$

are obvious. So one needs to verify only the triangle inquality.

In the case of $p = 1$,

$$\|x + y\|_1 = \sum_{k=1}^{n} |x_k + y_k| \leq \sum_{k=1}^{n} (|x_k| + |y_k|) = \sum_{k=1}^{n} |x_k| + \sum_{k=1}^{n} |y_k| = \|x\|_1 + \|x\|_1$$

and so indeed, $\| \cdot \|_1$ is a norm.

In the case $p = \infty$, one verifies triangle inequality as follows:

For any $1 \leq k \leq n$,

$$\begin{aligned} |x_k + y_k| &\leq |x_k| + |y_k| \\ &\leq \max_{1 \leq k \leq n} |x_k| + \max_{1 \leq k \leq n} |y_k| \\ &= \|x\|_\infty + \|x\|_\infty \end{aligned}$$

So

$$max_{1 \leq k \leq n} |x_k + y_k| \, \|x\|_\infty + \|y\|_\infty$$

or

$$\|1 \leq k \leq n\|_\infty \leq \|x\|_\infty + \|y\|_\infty$$

So $\| \cdot \|_\infty$ is indeed a norm on $X = R^n$.

To verify the triangle inequality in the case $1 < p < \infty$ is more tricky. Before doing so, we will make a detour and discuss first the so called Holder and Minkowsky inequalities.

2 Holder and Minkowski Inequalities

Definition 3 *Let $1 < p < \infty$ which we will call exponent. The number q is called the exponent conjugate to p if*

$$\frac{1}{p} + \frac{1}{q} = 1$$

Obviously, $q = \frac{p}{p-1}$ and $1 < q < \infty$. The exponent conjugate to q is p; Note that if $p = 2$ then $q = 2$.

Lemma 1 *Let $a, b \geq 0$ and let $1 < p, q < \infty$ be conjugate exponents. Then*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

.

Remark: Note that in the case p=q=2, the inequality follows from $(a-b)^2 \geq 0$.

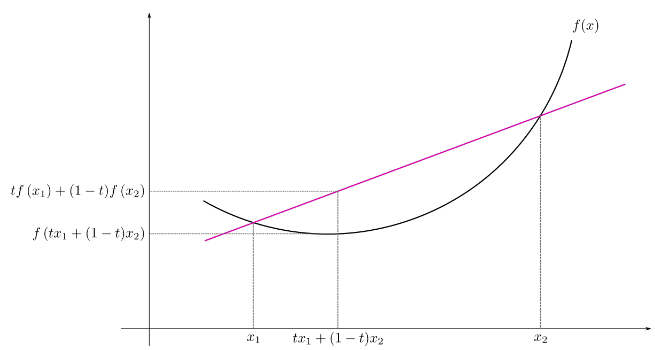
proof: The inequality is obvious if $a = 0$ or $b = 0$. So we will assume that $a > 0$ and $b > 0$.

Recall that a function $f : R \to R$ is called convex if for all $\alpha, \beta \in R$ and all $Q \in [a, b]$,

$$f(Q\alpha + (1-Q)\beta) \leq Qf(\alpha) + (1-Q)f(\beta)$$

If f is twice differentiable, then f is convex if and only if $f''(x) \geq 0$ for all $x \in R$

The geometric interpretation of convexity can be seen in this picture.



f is convex if and only if its graph on the segment $[\alpha, \beta]$ is below the straight line connecting the points $(\alpha, f\alpha))$ and $\beta, f(\beta)$.

The function $f(x) = e^x$ is convex since $f''(x) = e^x$ is convex since $f''(x) = e^x > 0$ for all x.

Let now α and β be real numbers such that

$$a = e^{\frac{\alpha}{p}} \quad \text{and} \quad b = e^{\frac{beta}{q}}$$

The numbers α *and* β *are given by*
$\alpha = pln(a), \beta = qln(b)$ *(ln for* $\log_e$ *in the base)*
Recall that

$$\frac{1}{p} + \frac{1}{q} = 1$$

and. set $Q = \frac{1}{p}$ *so that* $\frac{1}{q} = 1 - Q$
The convexity of $f(x) = e^x$*, gives*

$$f(\frac{1}{p}\alpha + \frac{1}{q}\beta) \leq \frac{1}{p}f(\alpha) + \frac{1}{q}f(\beta)$$

or

$$e^{\frac{1}{p}\alpha + \frac{1}{q}\beta} \leq \frac{1}{p}e^{\alpha} + \frac{1}{q}e^{\beta}$$

or

$$e^{\frac{1}{\rho}\alpha} \cdot e^{\frac{1}{q}\beta} \leq \frac{1}{p}\left(e^{\frac{\alpha}{p}}\right)^p + \frac{1}{q}\left(e^{\frac{B}{q}}\right)^q$$

Since

$$a = e^{\frac{\alpha}{p}} \text{ and } b = e^{\frac{b}{a}}$$

we get

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

Theorem 1 *Hölder Inequality Let* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$ *be two vectors in* R^n *Let* $1 < p, q < \infty$ *be cojugate expoents. Then*
Remark: Recalling our p-functions

$$\sum_{k=1}^{n} ||x_k||y_k| \leq (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{n} |y_k|^q)^{\frac{1}{q}}$$

The Hölder Inequality as

$$\sum_{k=1}^{n} |x_k||y_k| \leq \|x_k\|\|y_k\|$$

proof:

$$A = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}, B = \left(\sum_{k=1}^{n} |\ x_k|^q\right)^{\frac{1}{q}}$$

If $A = 0$*, then* $x_k = 0$ *for all* $1 \leq k \leq n$ *and the Holder Inequality holds trivially. Similarly, if* $B = 0$*, the inequality again holds trivially. So we can assume that* $A > 0, B > 0$.
For fixed k $1 \leq k \leq n$ *Let*

$$a = \frac{|x_k|}{A}, \quad b = \frac{|x_k|}{B}$$

Applying our lemma,

$$ab \leq \frac{[}{1}]p^p + \frac{1}{q}b^q$$

we get

$$\frac{|x_k|}{A} \cdot \frac{|y_k|}{B} \leq \frac{1}{P}\frac{|x_k|^p}{A^p} + \frac{1}{q}\frac{|y_k|^2}{B^q}$$

Summarizing the last inequality over k we get

$$\sum_{k=1}^{n} \frac{|x_k|\,|y_k|}{A \cdot B} \frac{1}{p}\frac{1}{A^p}\left(\sum_{k=1}^{n} |x_k|^p\right) + \frac{1}{q}\frac{1}{B^q}\left(\sum_{k=1}^{n} |y_k|^2\right)$$

But

$$\frac{1}{A^p}\sum_{k=1}^{n} |x_k|^p = 1, \quad \frac{1}{B^q}\sum_{k=1}^{n} |y_k|^q = 1$$

and

$$\frac{1}{p} + \frac{1}{q} = 1$$

So

$$\sum_{k=1}^{n} \frac{|x_k||y_k|}{AB} \leq 1$$

or

$$\sum_{k=1}^{n} |x_k||y_k| \leq AB$$

and that ispreciesely the statment of the Holder Inequality.

Theorem 2 *(Minkowski Inequality) Let $X = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two vectors in R^n and $1 \leq p < \infty$. Then*

$$\left(\sum_{k=1}^{n} (|x_k| + |y_k|)^p\right) \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}$$

proof: The statement is obvious if $p = 1$ so we may assume that $1 < p < \infty$. Let q be the exponent conjugate to p. We write

$$\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^p = \sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^{p-1}\left(|x_k| + |y_k|\right)$$

$$= \underbrace{\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^{p-1}|x_k|}_{I} + \underbrace{\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^{p-1}|y_k|}_{II}$$

We apply Holder's inequality to I:

$$I \leq \left(\sum_{k=1} \left(\left(|x_k| + |y_k|^{p-1} \right)^q \right)^{\frac{1}{q}} \cdot \left(\sum_{k=1}^{n} |x_k|^p \right) \right)^{\frac{1}{p}}$$

$$= \left(\sum_{k=1}^{n} (|x_k| + |y_k|)^{(p-1)q} \right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}$$

Since p and q are conjugate exponenents, we have that $p = q(p-1)$

$$I \leq \left(\sum_{k=1}^{n} (|x_k| + |y_k|)^p \right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}$$

In identical way, one derives that

$$II \leq \left(\sum_{k=1}^{n} (|x_k| + |y_k|)^p \right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} |y_k|^p \right)^{\frac{1}{p}}$$

So

$$I + II = \sum_{k=1}^{n} (|x_k| + |y_k|)^p \leq (\sum_{k=1}^{n} (|x_k| + |y_k|)^p)^{\frac{1}{q}} \left((\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}} \right)$$

So

$$\left(\sum_{k=1}^{n} (|x_k| + |y_k|)^p \right)^{1-\frac{1}{q}} \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^q \right)^{\frac{1}{q}}$$

Since $p = 1 - \frac{1}{q}$, *we have proven the Minkowski inequality.*

We now return to the last proposition of the last lecture and complete the proof that the p-function