Course Notes Summary

Focus Area: Summary of main formulas please

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Prepared for quick revision and reference

Use this sheet as a step-by-step guide when solving problems.

1 Key definitions and concepts

- Conditional probability: $P(B \mid A)$ is the probability B occurs given A has occurred.
- Prior probability: initial probability before new information.
- Posterior probability: revised probability after incorporating new information.
- Likelihood: $P(\text{new data} \mid \text{hypothesis})$, used to update the prior.
- Joint probability: $P(A \cap B)$ is the probability both A and B occur.
- Law of total probability: expresses $P(A)$ via a partition of the sample space.

2 Fundamental formulas

- Conditional probability (definition)
- $$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$
- Symmetric expression for joint probability
- $$P(A \cap B) = P(A)\,P(B \mid A) = P(B)\,P(A \mid B)$$
- Bayes' theorem (basic two-event form)
- $$P(B \mid A) = \frac{P(A \mid B)\,P(B)}{P(A)}$$
- Law of total probability (for a partition $B_1, \ldots, B_n$)
- $$P(A) = \sum_{i=1}^{n} P(A \mid B_i)\,P(B_i)$$
- Bayes' theorem (generalized for a partition $B_1, \ldots, B_n$)
- $$P(B_j \mid A) = \frac{P(A \mid B_j)\,P(B_j)}{\sum_{i=1}^{n} P(A \mid B_i)\,P(B_i)}$$

3 Important relationships and notes

- Prior $\to$ Likelihood $\to$ Posterior sequence:
  1. Start with prior $P(B_j)$.
  2. Use likelihood $P(A \mid B_j)$.
  3. Compute posterior $P(B_j \mid A)$ via Bayes' theorem.
- Normalizing constant in Bayes' rule is $P(A)$ given by the law of total probability.
- $P(B \mid A)$ increases when $P(A \mid B)$ is relatively large compared with $P(A \mid \text{other events})$ weighted by priors.
- Bayes can be applied with discrete counts or probabilities; both formulations are equivalent.

4 Intuitive (frequency/table) method

- Method:
  1. Assume a convenient total N (e.g., 1000 or 100000).
  2. Convert probabilities to counts: $\text{count}(B_i) = N \cdot P(B_i)$.
  3. Compute cells: $\text{count}(A \cap B_i) = \text{count}(B_i) \cdot P(A \mid B_i)$.
  4. Use counts to compute desired conditional:

$$P(B_j \mid A) = \frac{\text{count}(A \cap B_j)}{\sum_i \text{count}(A \cap B_i)}.$$

- Advantage: reduces algebra errors, makes denominators explicit.

5 Worked examples

- Example A: Orange County cigar smokers
  - Given:
    * $P(\text{male}) = 0.51$
    * $P(\text{female}) = 0.49$
    * $P(\text{cigar} \mid \text{male}) = 0.095$
    * $P(\text{cigar} \mid \text{female}) = 0.017$
  - Bayes calculation:

$$P(\text{male} \mid \text{cigar}) = \frac{P(\text{cigar} \mid \text{male})\, P(\text{male})}{P(\text{cigar} \mid \text{male})\, P(\text{male}) + P(\text{cigar} \mid \text{female})\, P(\text{female})} = \frac{0.095 \cdot 0.51}{0.095 \cdot 0.51 + 0.017 \cdot 0.4}$$

$$= \frac{0.04845}{0.04845 + 0.00833} \approx \frac{0.04845}{0.05678} \approx 0.853$$

  - Frequency/table check ($N = 100{,}000$):
    * males $= 51{,}000$; male cigar $= 0.095 \times 51{,}000 = 4{,}845$
    * females $= 49{,}000$; female cigar $= 0.017 \times 49{,}000 = 833$
    * total cigar $= 4{,}845 + 833 = 5{,}678$
    * $P(\text{male} \mid \text{cigar}) = \frac{4{,}845}{5{,}678} \approx 0.853$

- Example B: ELT manufacturers and defectives
  - Given:
    * $P(A) = 0.80, \quad P(B) = 0.15, \quad P(C) = 0.05$
    * $P(\text{defective} \mid A) = 0.04, \quad P(\text{defective} \mid B) = 0.06, \quad P(\text{defective} \mid C) = 0.09$
  - Bayes calculation:

$$P(A \mid \text{defective}) = \frac{P(\text{defective} \mid A)\, P(A)}{P(\text{defective} \mid A)P(A) + P(\text{defective} \mid B)P(B) + P(\text{defective} \mid C)P(C)} = \frac{}{0.04 \cdot 0}$$

$$= \frac{0.032}{0.032 + 0.009 + 0.0045} = \frac{0.032}{0.0455} \approx 0.703$$

  - Frequency/table check ($N = 10{,}000$):
    * A total $= 8{,}000$; defective from $A = 0.04 \times 8{,}000 = 320$
    * B total $= 1{,}500$; defective from $B = 0.06 \times 1{,}500 = 90$
    * C total $= 500$; defective from $C = 0.09 \times 500 = 45$
    * total defective $= 320 + 90 + 45 = 455$
    * $P(A \mid \text{defective}) = \frac{320}{455} \approx 0.703$

6 Quick solution checklist (ordered steps)

1. Identify hypothesis events $B_1, \dots, B_n$ and the observed event A.
2. Record priors $P(B_i)$ and likelihoods $P(A \mid B_i)$.
3. Compute $P(A) = \sum_i P(A \mid B_i)\, P(B_i)$.
4. Compute posterior for target B_j:

$$P(B_j \mid A) = \frac{P(A \mid B_j)\, P(B_j)}{P(A)}.$$

5. Optionally verify with frequency/table method using a convenient N.

7 Compact reference formulas (for quick revision)

- $$P(B \mid A) = \frac{P(A \mid B)\, P(B)}{P(A)}$$
- $$P(A) = \sum_i P(A \mid B_i)\, P(B_i)$$
- $$P(A \cap B) = P(A)\, P(B \mid A) = P(B)\, P(A \mid B)$$