Course Notes Summary

Focus Area: Summary of main formulas please

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Prepared for quick revision and reference

Use this sheet as a step-by-step guide when solving problems.

1 Fundamental definitions

- Prior probability: initial probability before new information.
- Posterior probability: probability revised after new information.
- Conditional probability: probability of B given A has occurred.
- Partition: a set of mutually exclusive, exhaustive events $A_1, \ldots, A_n$.

2 Core formulas

2.1 Conditional probability

- $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$, for $P(A) > 0$.
- Equivalent product form: $P(A \cap B) = P(A)\, P(B \mid A) = P(B)\, P(A \mid B)$.

2.2 Two-event Bayes' theorem

- $P(A \mid B) = \dfrac{P(B \mid A)\, P(A)}{P(B)}$, provided $P(B) > 0$.
- Law of total probability for denominator (two events A and A^c):

$$P(B) = P(B \mid A)\, P(A) + P(B \mid A^c)\, P(A^c).$$

2.3 Generalized (n-event) Bayes' theorem

- Law of total probability (partition $A_1, \ldots, A_n$):

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)\, P(A_i).$$

- General Bayes formula for A_j in partition:

$$P(A_j \mid B) = \frac{P(B \mid A_j)\, P(A_j)}{\displaystyle\sum_{i=1}^{n} P(B \mid A_i)\, P(A_i)}.$$

2.4 Intuitive / frequency (tree-table) method

- Choose a convenient total N (e.g., $N = 100{,}000$).
- Compute cell counts: $\text{count}(A_i \cap B) = N\, P(A_i)\, P(B \mid A_i)$.
- Use counts to get posterior:

$$P(A_j \mid B) = \frac{\text{count}(A_j \cap B)}{\sum_i \text{count}(A_i \cap B)}.$$

3 Short worked examples

3.1 Cigar-smoking example (two-category)

- Given: $P(\text{male}) = 0.51$, $P(\text{cigar} \mid \text{male}) = 0.095$, $P(\text{cigar} \mid \text{female}) = 0.017$.
- Denominator:
$$P(\text{cigar}) = 0.095 \times 0.51 + 0.017 \times 0.49 = 0.04845 + 0.00833 = 0.05678.$$
- Posterior:
$$P(\text{male} \mid \text{cigar}) = \frac{0.095 \times 0.51}{0.05678} \approx \frac{0.04845}{0.05678} \approx 0.854.$$
- Frequency check ($N = 100{,}000$): males smoking $= 100{,}000 \times 0.51 \times 0.095 = 4{,}845$; females smoking $= 100{,}000 \times 0.49 \times 0.017 = 833$; total $= 5{,}678$; ratio $= 4{,}845/5{,}678 \approx 0.854$.

3.2 ELT manufacturers (three-category)

- Given: $P(A) = 0.80$, $P(B) = 0.15$, $P(C) = 0.05$.
- Defective rates: $P(D \mid A) = 0.04$, $P(D \mid B) = 0.06$, $P(D \mid C) = 0.09$.
- Numerator for A: $0.80 \times 0.04 = 0.032$.
- Denominator:
$$0.032 + 0.15 \times 0.06 + 0.05 \times 0.09 = 0.032 + 0.009 + 0.0045 = 0.0455.$$
- Posterior:
$$P(A \mid D) = \frac{0.032}{0.0455} \approx 0.703.$$
- Frequency check ($N = 10{,}000$): defective from $A = 10{,}000 \times 0.80 \times 0.04 = 320$; total defective $= 455$; ratio $= 320/455 \approx 0.703$.

4 Key relationships and connections

- Bayes updates prior to posterior using likelihood $P(B \mid A_i)$.
- Denominator always equals weighted sum of likelihoods across partition (law of total probability).
- Product rule links joint and conditional probabilities: $P(A \cap B) = P(A)\,P(B \mid A)$.
- Using frequencies reduces algebra mistakes and improves intuition.

5 Quick application checklist (ordered steps)

1. Identify hypotheses/partitions $A_1, \ldots, A_n$ and the observed event B.
2. Collect priors $P(A_i)$ and likelihoods $P(B \mid A_i)$.
3. Compute denominator: $\sum_i P(B \mid A_i)\,P(A_i)$.
4. Compute posterior for desired A_j: $P(A_j \mid B) = \dfrac{P(B \mid A_j)\,P(A_j)}{\text{denominator}}$.
5. Optionally verify with a frequency table using a chosen N.

6 Common pitfalls and notes

- Ensure events A_i form a partition (mutually exclusive and exhaustive).
- Check that $P(B)$ (denominator) > 0 before dividing.
- Distinguish $P(B \mid A)$ (likelihood) from $P(A \mid B)$ (posterior).
- Use the frequency method to avoid algebraic substitution errors.

7 Compact formula summary (for quick revision)

- Conditional: $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$.
- Product: $P(A \cap B) = P(A)\, P(B \mid A)$.
- Bayes (two-event): $P(A \mid B) = \dfrac{P(B \mid A)\, P(A)}{P(B)}$.
- Total prob. (partition): $P(B) = \sum_i P(B \mid A_i)\, P(A_i)$.
- Bayes (general): $P(A_j \mid B) = \dfrac{P(B \mid A_j)\, P(A_j)}{\displaystyle\sum_i P(B \mid A_i)\, P(A_i)}$.