ON DISPERSABLE BOOK EMBEDDINGS OF MOORE GRAPHS

BY

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1 Introduction

In the early 1700's, Königsberg was a Prussian city located on the Pregel River, near modern-day Kaliningrad, Russia. The Pregel river divided the city into four landmasses, which were connected by a total of seven bridges, diagrammed in Figure 1. In time, the children of Königsberg designed a game based on the geography of their city. The goal of the game was to pick a starting location, then cross each of the seven bridges exactly once, and return to the starting location [7]. Years passed, and the children grew into adults, yet no one was ever able to win the game. News of this reached Swiss mathematician Leonhard Euler (1707-1783), who struggled to analyze the problem using existing techniques. Eventually, Euler developed a new type of structure, called a *graph*, to properly convey the necessary information of the problem. Using vertices to represent landmasses and edges to represent bridges, Euler now had a mathematically rigorous structure capable of proving that the children's game was impossible to win [19].

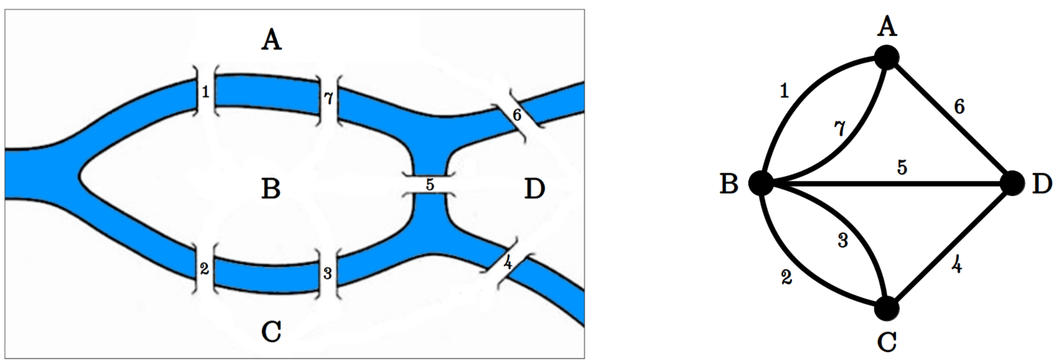


Figure 1: Simplified map of Königsberg and Euler's graphical representation.

In 1736, Euler formed the basis for what would later be known as *graph theory* with his solution to the famous *Seven Bridges of Königsberg* problem. Since then, the field has flourished in every direction. Due to its wide applicability, graph theory has grown to become one of the most powerful and influential branches of modern mathematics [8].

In the field of discrete mathematics, a *graph* is a versatile structure capable of modeling almost anything, including circuit design in electronics, molecular networks in biology, traffic design in civil engineering and much more [13]. Practically any network, relationship or connection, between anything, can be expressed with a graph. A graph is an abstract structure, meaning it can have many different representations, each able to offer unique insight into the fundamental structure of the graph. Graphs are usually expressed in diagrammatic form, using vertices and edges, but they can also be expressed equivalently using sets or matrices. The proper utilization of each type of structure is critical to the exploration of graph theory.

An important concept in graph theory is planarity. A graph is called *planar* if it can be drawn on a plane with no crossing edges. Typically, planarity is considered with respect to the plane $\mathbb{R}^2$, but this definition can be expanded to include graph drawings on other surfaces, such as the Möbius strip or torus. A graph drawn on a surface with no edge crossings is said to be *embedded* on that surface. This thesis examines graph embeddings on a different type of surface, the *book*. First considered by Loyal Taylor Ollman in 1973, *book embeddings* have become increasingly important as computing power has grown exponentially in the last 50 years [34, 14]. Today, book embeddings find high use in the fields of parallel processing optimization and circuit board design [15]. In recent years, there has been increased interest in a variation known as *dispersable* book embeddings, defined in Section 4. Specifically, this paper focuses on dispersable book embeddings of several known families of *Moore graphs*, defined in Section 3. Moore graphs are highly symmetrical, making optimal book embeddings difficult to calculate [23]. This paper focuses on Moore graphs because their highly symmetrical nature makes them the 'worst-case' scenarios for finding book embeddings. Previous scholarship has proved the dispersability of several Moore graph families [35], and for our main result, we add to this by proving the dispersability of a new family, the incidence graphs of the finite projective planes.

The rest of this paper is structured as follows: Section 2 gives background about sets and graph theory, Section 3 describes Moore graphs and cages, Section 4 details book embeddings, Section 5 considers stacks, Section 6 tackles difference sets, and Section 7 combines it all together and offers the proof of a new theorem on the dispersability of Moore graph families. Also included is an Appendix containing drawings of the various Moore graphs and their book embeddings that we will discuss. Each section begins with a list of key words that reappear throughout the paper. We also define words in **bold**, and *italicize* other words that are of importance. Lastly, many of the included figures require color, so be sure this is not printed in grey-scale.

2 Background on Sets and Graphs

Key words: set, element, order, graph, vertex, edge, drawing, incident, adjacent, planar, embedding.

One of the most fundamental objects in mathematics is the *set*. Formally, a **set** is a well-defined collection of distinct objects, known as **elements**. The elements in a set can be anything including numbers, letters, physical objects, graphs, or other sets. Typically, sets are denoted with capital letters and their elements in brackets. For example, the set of all months M is a twelve element set, written

$M = \{$January, February, March, April, May, June, July, August, September, October, November, December$\}$.

We can also define a set using a rule instead of naming each element explicitly. In this case, we use the vertical bar symbol, $\mid$, which is read as "given that". Then, the set Z of all *integers* (whole numbers) between 0 and 6 (inclusive), can be written equivalently as

$$Z = \{0, 1, 2, 3, 4, 5, 6\} = \{x \mid x \text{ is an integer } \textit{and } 0 \leq x \leq 6\}.$$

Often it is important to know the size of a set, so we define the **order** or **cardinality** of a set S to be the number of elements in S. Thus, as defined above, the order of M is 12 and the order of Z is 7. Sets with a *finite order*, such as M and Z, are said to be *finite sets*. We can also define *infinite sets*, such as the set of all integers, which are said to have *infinite order*.

Remember that sets contain only distinct objects; there are no duplicates allowed. For example, the set of all Super Bowl champions contains only 20 teams, despite the Super Bowl being won 53 times. Also, a set is a *collection* of objects, meaning the order of the objects is irrelevant. Therefore, the set M' defined as

$M' = \{$May, April, July, February, December, March, September, January, November, August, October, June$\}$.

is **equal** to M, because they contain exactly the same elements.

All of the definitions in this section are borrowed and abbreviated from [13]. For a more comprehensive review of these definitions, refer back to that reference. We now shift our focus from the study of sets, or **set theory**, to the study of graphs, known as **graph theory**. Formally, a **graph** is a pair $G = (V, E)$, where V is a finite, non-empty set of objects, called **vertices**, and E is a set of pairs of vertices, called **edges**. In this paper, only simple and undirected graphs will be considered. A graph is **simple** if each edge in E is unique (i.e. no multi-edges) and no edge starts and ends at the same vertex (no loops). Also, a graph is **undirected** if the set E of edges consists only of *unordered* pairs, that is, edges in G have no orientation. Figure 2(a) depicts a simple, undirected graph, like the ones discussed in this paper. By contrast, Figure 2(b-d) shows graph types that will be forbidden.

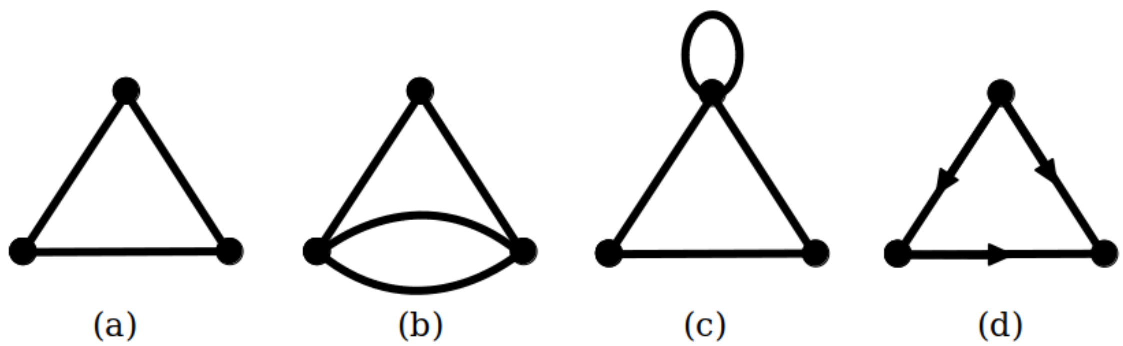


Figure 2: (a) a simple, undirected graph. (b) a non-simple graph with a multi-edge. (c) a non-simple graph with a loop. (d) a directed graph, or digraph.

Because they are defined in terms of sets, graphs are quite abstract and difficult to conceptualize. In order to visualize a graph, we can construct a **drawing**, where the vertices are points on the plane and edges are lines connecting two points. Graph drawings are highly not unique, as any drawing that contains all of the information of the graph $G = (V, E)$ is a valid drawing of G. In a graph drawing, the position of the vertices is known as the **layout** and, depending on the problem, certain layouts may have benefits over others. In general, it is important to remember the difference between a graph and its drawing: a graph is an abstract mathematical structure consisting of a set of vertices V and a set of edges E, while the drawing of G is just a visualization of this structure, usually on the two dimensional plane. To demonstrate this, Figure 3 gives six different drawings of the same graph.

Before moving on, let's define the graph shown in Figure 3. Within a graph G, a **path** is a sequence of connected edges traveling from one vertex to another, where each edge in G is crossed at most one time and no vertices are repeated. A path that travels over n edges, from start to end, is defined to have length n. If a path C starts and ends at the same vertex, then C is called a **cycle**, and if C has length n, then we

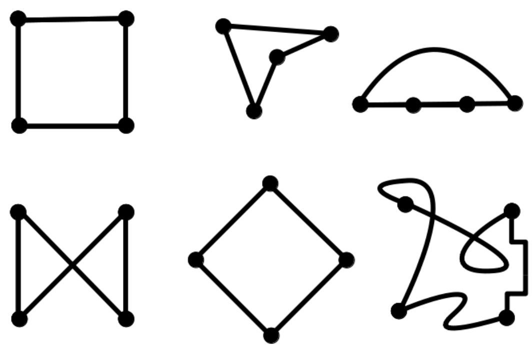


Figure 3: Six drawings of the graph C_4, demonstrating the non-uniqueness of graph drawings.

call C an ***n*-cycle**. Note that each of the drawings in Figure 3 has a 4-cycle. A graph that consists of solely an n-cycle and no additional edges, is called the **cycle on *n* vertices** and is denoted C_n. Thus, Figure 3 depicts six drawings of the graph C_4.

The set of graphs $\{C_n \mid n \geq 3\}$ is known as a **family** of graphs, because each family member exhibits similar characteristics. Drawings of the first eight graphs in the C_n family can be seen in Appendix A.1. Another useful family of graphs to define are the complete graphs. The **complete graph on *n* vertices**, denoted K_n, is a graph with n vertices where each vertex shares an edge with every other vertex. In other words, K_n is a simple, n-vertex graph with every possible edge. The set of graphs $\{K_n \mid n \geq 3\}$ is also a graph family, and the graphs for $n = 4$ through $n = 11$ can be seen in Appendix A.2.

Thus far, the graphs we've seen in Figures 2 and 3 have been *unlabeled*. But, we could choose to name or enumerate each of the vertices and edges to create a **labeled graph**, like in Figure 4. Using labeled graphs, we can distinguish between different vertices and edges, allowing us to define the relationships between them.

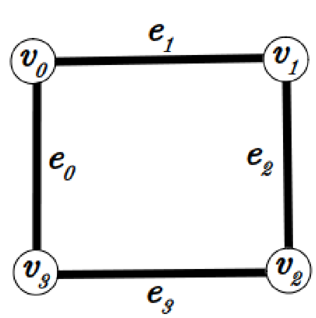


Figure 4: A labeled drawing of C_4

When considering a specific vertex and edge, we say that the two are **incident** if the unordered pair defining the edge contains the vertex. In other words, a vertex and an edge are *incident* if the vertex is an endpoint of the edge. For example, in Figure 4, the edge e_1 is incident to the vertices v_0 and v_1. Additionally, two vertices are **adjacent** if both of them are incident to the same edge. In Figure 4, v_0 and v_1 are adjacent because both are incident to e_1. In a simple graph, each edge must be incident to exactly two vertices. Therefore, two vertices are adjacent if and only if there is an edge whose pair contains both vertices. Recall that in a complete graph K_n, there is an edge between each pair of vertices, thus all vertices are adjacent.

Now that we understand graph drawings, we can introduce a core concept in graph theory: *planarity.* A graph G is **planar** if there exists a drawing of G *on the plane* such that no two edges cross. Drawing a graph *on the plane* is analogous to drawing it on a piece of paper, like the figures in this thesis. Referring back to Figure 3, we can clearly see that the graph C_4 is planar. In general, we can use Kuratowski's Theorem to determine whether a finite graph is planar, see [39] for reference and proof.

Theorem 1 (Kuratowski)**.** A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. *Note that $K_{3,3}$ is a complete bipartite graph, which will be defined in Section 3.*

Nearly every graph considered in this paper is non-planar by Kuratowski's Theorem. If a graph *is* planar, we will explicitly state so. Planarity is such a powerful concept that it has sprouted several similar offshoots. For example, we can define a graph G to be **outerplanar** if there is a drawing of G with equidistant, circular layout and no two edges cross [38]. Recall that the *layout* of a graph is the position of the vertices in the plane. Note that outerplanar also fulfills the definition of planar, meaning outerplanarity implies planarity. Again, Figure 3 shows that C_4 is outerplanar as well as planar.

We explore the concept of planarity further by looking at graph embeddings. An **embedding** is a drawing of a graph G on a surface S such that no two edges cross. Consequently, we can say that G can be *embedded* on S. Consider that a planar graph has an embedding on the plane and an outerplanar graph has an embedding on the circle. Since a circle is a subset of the plane, it again follows that outerplanarity implies planarity. Graph embeddings can be extended to more complex surfaces as well. For example, consider the complete graph K_5, which is non-planar by Kuratowski's Theorem. However, Figure 5 shows that K_5 is *torodial*, or embeddable on the torus, since there is a drawing of K_5 on the torus without any edge crossings.

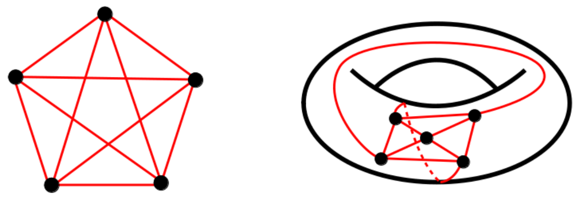


Figure 5: K_5 is non-planar, but has a torodial embedding.

Planarity and embeddings play an important role in real-world graph theory applications. For example, a graph representing a subway system should have as few edge crossings as possible to reduce the risk of accidents and optimize scheduling. Although subway maps are drawn on the plane, a torodial embedding may be more appropriate because tunnels can allow for trains to pass over each other, similar to how K_5 behaved on the torus. There are many more real world uses for graph embeddings, such as circuit board design, which must always exhibit a planar layout or risk short-circuiting. In this section, we reviewed the necessary background on sets and graph theory that we will apply in later sections. It is crucial the reader understands planarity and graph embeddings, especially as they apply to surfaces other than the plane. In the next section, we introduce Moore graphs and cages, which we will eventually embed onto a surface.

3 Moore Graphs and Cages

Key words: degree, regular, matching, girth, Moore, cage, complete, complete bipartite, projective plane.

In this section, we begin by introducing several additional graph properties and definitions that allow us to define two special classes of graphs, known as Moore graphs and cages. It is important to understand the properties and construction of these graphs as these concepts are vital in later sections of the paper.

First, we define the *degree* of a vertex in a graph. For each vertex v in a graph G, there is some number of edges m incident to v. We define this as the **degree** of the vertex v and denote it $\deg(v) = m$. If every vertex has equal degree, then the graph is said to be *regular*. In particular, a graph G is $\boldsymbol{r}$**-regular** if $\deg(v) = r$ for all $v \in V$ [13]. Some classes of regular graphs are given special names. For example, 1-regular graphs are called **matchings** because each vertex is 'matched up' with exactly one other. In addition, 3-regular graphs are called **cubic** graphs [13]. Four examples of cubic graphs are given in Figure 6.

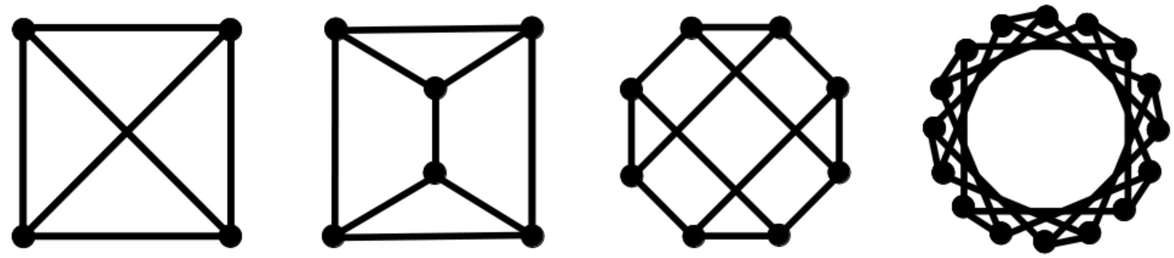


Figure 6: Four examples of simple, 3-regular graphs.

Analogous to sets, we define the **order** of graph G to be the number of vertices in G. Thus, the graphs in Figure 6 have order 4, 6, 8, and 16, respectively. Then, recall the definition of a *cycle* from earlier. A graph that contains a cycle is called **cyclic**, and the length of the shortest cycle is called the **girth** of the graph. Thus, the graphs in Figure 6 are all cyclic and have girth 3, 3, 4, and 4, respectively. Earlier, we defined the complete graphs on n vertices (denoted K_n) for all $n \geq 3$. Since all vertices are adjacent in K_n, every set of 3 vertices forms a 3-cycle. Thus, the girth of any complete graph is 3. In addition to girth, we can also define the **diameter** of a graph to be the maximum distance between any two vertices, or the length of the longest uv-geodesic in the graph [13]. As an exercise, try to determine the diameters of each graph in Figure 6.

In 1960, Hoffman and Singleton [25] proved that a k-regular graph with diameter d, has order n where,

$$n \leq 1 + k + k(k-1) + k(k-1)^2 + \ldots + k(k-1)^{d-1}.$$

The graphs that achieve this bound are known as **Moore graphs**, after Edward F. Moore, and have girth $g = 2d+1$ [20]. Moore graphs can also be defined in terms of their girth. In this case, a Moore graph is an r-regular graph with girth g and order $n = M(r,g)$, where $M(r,g)$ is the **Moore bound** defined by

$$M(r,g) = \begin{cases} \dfrac{r(r-1)^{(g-1)/2} - 2}{r-2} & \text{if } g \text{ is odd} \\[2ex] \dfrac{2(r-1)^{g/2} - 2}{r-2} & \text{if } g \text{ is even} \end{cases}$$

In general, Moore graphs are rare [20]. All known Moore graphs can be defined with the following theorem.

Theorem 2 ([20])**.** There exists a Moore graph of degree r and girth g if and only if

(i) $r = 2$ and $g > 3$. These are the cycles, C_n;

(ii) $g = 3$ and $r > 2$. These are the complete graphs, K_n;

(iii) $g = 4$ and $r > 2$. These are the complete bipartite graphs, $K_{m,m}$;

(iv) $g = 5$ and:

  (a) $r = 3$. The Petersen graph,

  (b) $r = 7$. The Hoffman-Singleton graph,

  (c) and possibly $r = 57$;

(v) $g = 6, 8,$ or 12, and there exists a finite projective plane of order $r - 1$.

For the remainder of this section, we rigorously define many of the Moore graphs and their properties as well as introduce cages. Notice that Moore graphs are only defined for specific values of r and g, but we could still consider the general problem of finding the r-regular graph with girth g and minimum order. In graph theory, this is known as the *cage problem* [20].

First, we define an **(r, g)-graph** to be any r-regular graph with girth g. First proposed in 1947 by Tutte [40], an **(r, g)-cage** is an (r, g)-graph with minimum order. In other words, a *cage* is a cyclic graph with uniform vertex degree r and shortest cycle of length g, and most importantly, there is no graph with fewer vertices satisfying these parameters. Thus, all Moore graphs are cages but not all cages are Moore graphs. Throughout this paper, we focus on Moore graph families, but remember that these are cage families as well.

Consider the possible small values of r, the uniform vertex degree, in a regular graph. A 0-regular graph consists of only disconnected vertices because each vertex is incident to exactly 0 edges. Similarly, a 1-regular graph consists of only disconnected edges. Thus, both 0 and 1-regular graphs are acyclic, so we begin considering cages where $r \geq 2$.

We should also consider the possible values of g, the girth of a graph. Clearly, it is not possible to have a cycle of length 0 or 1 since we need at least two vertices to 'cycle between.' Also, since this paper is restricted to simple graphs without multi-edges, it is impossible to have a cycle of length 2. Therefore, we begin the study of (r, g)-cages by defining that $r \geq 2$ and $g \geq 3$.

In general, finding some r-regular graph with girth g is not difficult [20]. The cage problem derives its complexity from finding the *minimum order* of a graph with these properties. In [36], Sachs proved that a cage exists for all possible r and g values. However, there are only a small number of cages where the order is actually known. Other than the Moore graphs, only 12 other cages have ever been confirmed. Finding cages is extremely difficult and discoveries are rare, with the last cage being discovered in 2007 by Exoo [20].

In addition, Moore graphs and cages are required to be connected. A graph is connected if there is a path, or sequence of adjacent edges, between every pair of vertices. With the exception of the $(3,5)$-cage, or Petersen Graph, all known cages are Hamiltonian [27]. A graph is **Hamiltonian** if it has a cycle that contains every vertex; known as a **Hamilton cycle** [13]. Try to find a Hamilton cycle in each graph shown in Figure 6. Although the Petersen Graph does not contain a Hamilton cycle, it does contain a *Hamilton path*, which is a path that contains every vertex, but does not start and end at the same vertex. The Petersen Graph is a classic counter-example to many conjectures in graph theory [13]. Additionally, it is also the only non-planar Moore graph and cage [20].

As stated before, the cage problem is not defined for $r < 2$ or $g < 3$, so we begin by considering the $(2, g)$-cages. In the C_n graph, each vertex is adjacent to two other vertices in the cycle, so all graphs in the C_n family are 2-regular. Also, since there is only one cycle in the graph and it contains all of the vertices, the girth must be equal to the order of the graph. Using this, we can classify our first infinite family of cages, which consists of exactly the Moore graphs from Theorem 2(i).

Theorem 3 (Wong [43])**.** The cycle graph C_n is a Moore graph and the *unique* $(2, n)$-cage for all $n \geq 3$.

In graph theory, two graphs G and H are **isomorphic** if there is a bijection $f\colon V(G) \to V(H)$ between the vertex sets of G and H, such that two vertices u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H [13]. If there is only one non-isomorphic cage for a given regularity and girth, then that cage is said to be **unique** [20]. Proving the uniqueness of a cage becomes very difficult for larger parameters, but the reader should reason through why these first few families should all be unique.

Next, we consider the $(r, 3)$-cage, which we denote as G. Since G is r-regular, each vertex is adjacent to r other vertices. Then, G must have a lower bound of $r + 1$ vertices to be r-regular. Now, recall from earlier that K_r, the complete graph on r vertices, is an $(r-1)$-regular graph with girth 3. Then, K_{r+1} is

r-regular with order $r+1$ and girth 3, fulfilling all of the necessary properties of the $(r,3)$-cage. Since K_{r+1} has exactly $r+1$ vertices, the order equals the lower bound, meaning $G = K_{r+1}$ must be a cage. We use this to define another infinite family of graphs.

Theorem 4 ([20])**.** The complete graph K_{r+1} is a Moore graph and the unique $(r,3)$-cage for all $r \geq 3$.

This theorem defines another Moore graph family, which is also the infinite family of cages of girth 3. In a similar way, we can also define the infinite family of cages of girth 4 using bipartite graphs. First, we define an **independent set** in a graph as a set of vertices where no pair of them are adjacent. Then, a graph is **bipartite** if its vertices can be partitioned into two independent sets (called *partite sets*) such that every edge is incident to one vertex in each set. Also, a graph is **complete bipartite** if each vertex is adjacent to *all* of the vertices in the opposite partite set. For notation, we denote complete bipartite graphs $K_{m,n}$, where m and n are the orders of the two partite sets. Thus, the order of $K_{m,n}$ is $m+n$. Two drawings of $K_{4,4}$ are given in Figure 7, and the graphs $K_{n,n}$, $3 \leq n \leq 10$, are depicted in Appendix A.3.

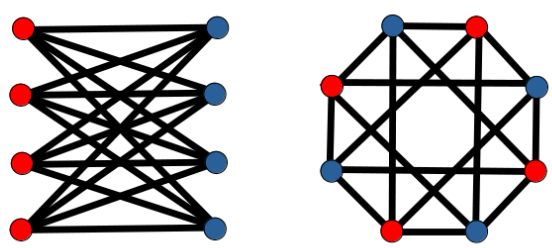


Figure 7: Two drawings of the complete bipartite graph $K_{4,4}$, with partite sets shown in red and blue.

Notice that the $K_{m,n}$ notation is very similar to that of complete graphs. In graph theory, complete graphs and complete bipartite graphs exhibit many shared and interesting properties [13]. While complete graphs define the cages of girth 3, the complete bipartite graphs define the cages of girth 4. First, we prove that all complete bipartite graphs have girth 4 when $m,n \geq 2$.

Theorem 5. The complete bipartite graph $K_{m,n}$ has girth 4 when $m,n \geq 2$.

Proof. Let the graph $G = K_{m,n}$ be a complete bipartite graph with two bipartite sets, A and B, of order m and n, respectively. We consider the cases when $m,n \geq 2$, since if $m=1$ or $n=1$, then G is a star (see [13]) and non-cyclic. The graph G is shown in Figure 8(a) with partite set A shown in red and partite set

B shown in blue. Without loss of generality, choose an edge e_1 in G, with endpoints v_0 and v_1. Our goal is to find the shortest cycle in G. Since G is bipartite, either v_0 or v_1 is in A while the other is in B. Without loss of generality, assume $v_0 \in A$ and $v_1 \in B$. Choose another edge $e_2 = (v_1, v_2)$. We know e_2 must exist since G is complete bipartite and $m \geq 2$. Then, the path, $P = \{e_1, e_2\} = \{v_0, v_1, v_2\}$ travels from A to B to A, see [Fig. 8(b)]. There is no edge from v_2 to v_0 since both of these vertices are in set A. Thus, there is no cycle of length 3 in a complete bipartite graph. Instead, choose edge $e_3 = (v_2, v_3)$, which goes from A to B. Again, we know v_3 and e_3 both exist since G is complete bipartite and $n \geq 2$. Finally, choose $e_4 = (v_3, v_0)$, which must exist for the same reasons as before. Adding e_4 to the path P, we get a cycle of length 4, where $P = \{e_1, e_2, e_3, e_4\} = \{v_0, v_1, v_2, v_3\}$, see [Fig. 8(c)]. Since all of the chosen edges in G were arbitrary, the shortest cycle in $K_{m,n}$ for $m, n \geq 2$ has length 4. Thus, the girth of $K_{m,n}$ is 4. $\square$

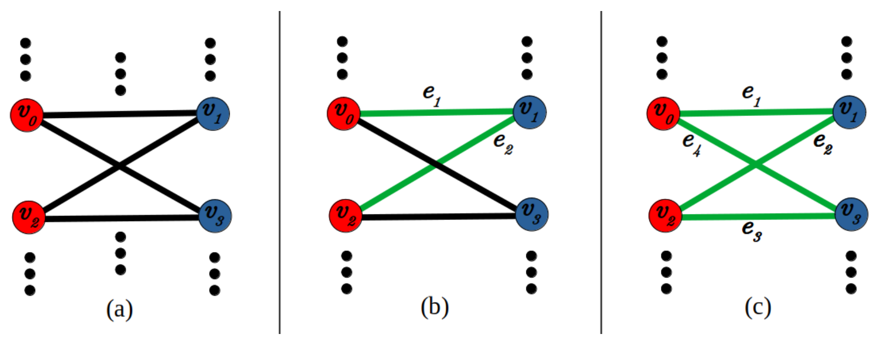


Figure 8

In a complete bipartite graph, each vertex is adjacent to all of the vertices in the opposite partite set. Then $K_{r,r}$ is a graph with two partite sets of order r where each vertex is adjacent to r others, meaning the graph is r-regular. Therefore, the complete bipartite graph, $K_{r,r}$ is an r-regular graph with girth 4.

Theorem 6 ([20])**.** The complete bipartite graph $K_{r,r}$ is a Moore graph and the unique $(r, 4)$-cage for $r \geq 3$.

We now have theorems classifying three infinite families of Moore graphs and cages: those of regularity 2 and those of girths 3 and 4. It is easy to check that all of the graphs in these families achieve the Moore bound, proving they are indeed Moore graphs. Each of these families is based on a basic graph family – the cycles, complete graphs and complete bipartite graphs – but the remaining families are generated from more complicated graphs. Next, we explore the construction of the girth 6 cages and then generalize our strategy to cages of girth 8 and 12.

We proceed by defining a **finite projective plane of order n**, denoted $PG(2,n)$, as n^2+n+1 points and n^2+n+1 lines with the following properties:

1. Any two points determine a line
2. Any two lines determine a point
3. Every point has $n+1$ lines on it, and
4. Every line contains $n+1$ points.

Also, a projective plane of order n only exists when n is a prime power, that is, when $n = p^a$ for a prime p and natural number a [8]. Thus, the smallest projective plane has order 2 and is called the Fano plane.

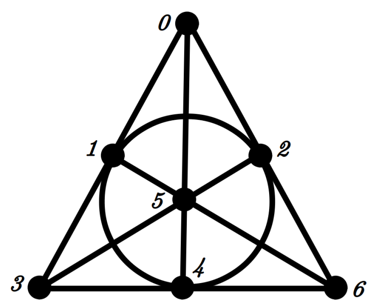


Figure 9: The Fano plane

In the drawing of the Fano plane in Figure 9, the circle is considered to be one line. So, the lines are

$$l_0 = (0,1,3), \quad l_1 = (1,2,4), \quad l_2 = (2,3,5), \quad l_3 = (3,4,6), \quad l_4 = (4,5,0), \quad l_5 = (5,6,1), \quad l_6 = (6,0,2).$$

Then, we can easily see that the Fano plane contains $(2)^2 + (2) + 1 = 7$ points and 7 lines, as well as follows all four of the necessary properties for a finite projective plane. Note that the finite projective plane is not itself a simple graph, nor are points and lines the same as edges and vertices. According to property 3 of finite projective planes, a line must contain more than two points (when $n \geq 2$), unlike an edge which must always contain exactly two vertices in a simple graph. However, we can construct a simple graph by considering the incidence structure of the Fano plane.

Recall the definition of incidence from earlier: an edge and a vertex are *incident* if the vertex is an endpoint of the edge. We now specify that this is called **graph incidence**, because it only applies to graphs.

We adapt this definition slightly for use with point and line structures like the projective planes. In this case, a point and line are **incident** if the point lies on the line. To differentiate between the two definitions, we call this **point-line incidence**. This adapted definition allows us to consider the incidence structure of the Fano plane.

Given the structure S with n points and m lines, we define the **point-line incidence matrix** of S, denoted $I(S)$, to be a $m \times n$ binary matrix where the i, j-entry is 1 if line i is incident with point j, and 0 if they are not incident. The point-line incidence matrix of $PG(2, 2)$, or Fano plane, is shown below in Table 1. In order to emphasize the incidences, blanks are used instead of zeros.

	0	*1*	*2*	*3*	*4*	*5*	*6*
l_0	1	1		1			
l_1		1	1		1		
l_2			1	1		1	
l_3				1	1		1
l_4	1				1	1	
l_5		1				1	1
l_6	1		1				1

Table 1: Incidence structure of the Fano Plane, or $PG(2, 2)$.

Using this point-line incidence matrix, we can construct a simple graph G. First, choose the vertex set of G to be all of the points and lines from the Fano plane, so $V = \{0, \ldots, 6, l_0, \ldots, l_6\}$. Then, add an edge between two vertices if they are point-line incident in the Fano plane, i.e. if i, j-entry of $I(PG(2, 2))$ is 1. Since incidence is a binary relation, edges in G will always consist of one point and one line from the Fano plane and never two points or two lines. This means that G will be bipartite with one partite set consisting of the points and the other of the lines. The construction of G can be seen in Figure 10. We reorder the vertices to show the graph with its traditional drawing. Though we will not prove it, the graph G is exactly the (3,6)-cage, also known as the Heawood Graph [20]. In fact, this process can be generalized to create another Moore graph family, those in Theorem 2(v), corresponding to the infinite family of girth 6 cages.

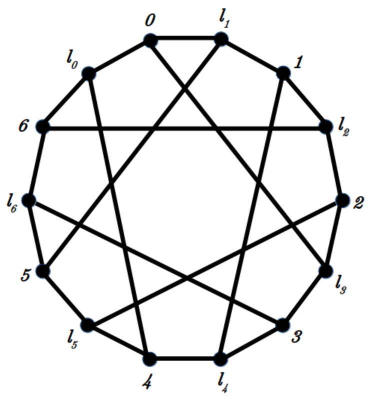


Figure 10: The Heawood graph, or (3,6)-cage, constructed from the incidence matrix of the Fano plane.

Theorem 7 ([20])**.** The incidence graph of a projective plane of order $r-1$ is a Moore graph and a $(r, 6)$-cage.

Here it is crucial to remember that finite projective planes only exist when the order $r - 1$ is a prime power. Thus, this Moore graph family is still infinite, but unlike the previous families, there is not a graph for every $r \geq 2$. Instead there is only a Moore graph when $r - 1$ is a prime power. The first non-existent finite projective plane is that of order 6. Thus, the $(7, 6)$-cage is not formed from the incidence graph of a finite projective plane. Further, the (7,6)-cage is *not* a Moore graph, the first cage we have seen with this property. Upon viewing Appendix A.4, we see that, for this reason, the $(7, 6)$-cage does not follow the symmetrical pattern of the other girth 6 cages.

The finite projective planes are actually a subset of a class of objects called the *generalized polygons* [20]. The finite projective planes are also known as the *generalized triangles*, or 3-gons. We won't explicitly define what generalized polygons, also called n-gons, are but the reader can refer to [20] to learn more.

Just as the incidence graphs of the generalized triangles generate a Moore graph family corresponding to the $(r, 6)$-cages, the generalization of other polygons form the basis of the last two Moore graph families. These families of generalized polygons are known to form Moore graphs because the order of their incidence structures is equal to the Moore bound. In this paper, we will not consider these other two families explicitly other than to note their existence with the following theorems.

Theorem 8 ([20])**.** The incidence graph of a quadrangle, or 4-gon, of order $(r - 1,\ r - 1)$ is a Moore graph and an $(r, 8)$-cage

Theorem 9 ([20])**.** The incidence graph of a hexagon, or 6-gon, of order $(r-1,\ r-1)$ is a Moore graph and an $(r,12)$-cage.

Like the generalized triangles, the quadrangles and hexagons only exists when $r-1$ is a prime power. These three generalized polygons are the only ones known to generate cage families. Thus, there are a total of six known infinite families of Moore graphs and cages, as we have defined them in this section. In Section 7, we will embed these graphs into a *book*, which we introduce in the next section.

4 Book Embeddings

Key words: pages, spine, book thickness, circular embedding, subgraph, Hamiltonian, dispersable.

In 1973, Ollmann proposed a new surface to embed graphs on: *the book* [34]. For a natural number p, we define a ***p*-book**, also called a *book with p pages*, to be a surface in Euclidean 3-space. A p-book consists of p half-planes and a line L as their common boundary. We call the p different half planes **pages** and the line L is called the **spine** [6]. Figure 11 shows the resemblance between a p-book and its namesake.

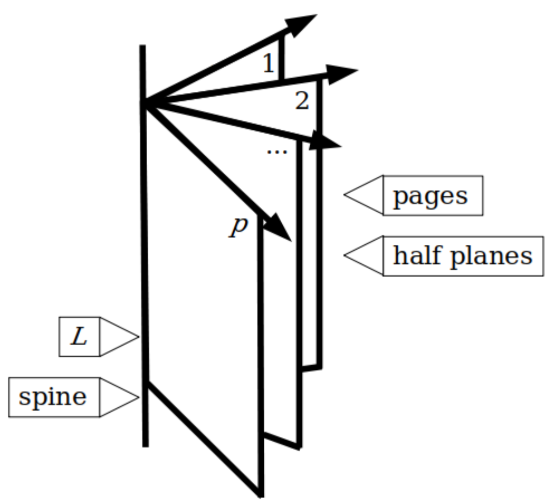


Figure 11: A p-book, or a book with p pages.

With this new surface, it is only natural to consider embeddings graphs onto books, also known as *book embeddings*. To embed a graph in a p-book, we first induce a linear layout, or ordering σ, of the vertices and embed them along the spine L. Then, we assign every edge to exactly one of the p pages in a way so that no two edges on any page cross. If the graph is planar with respect to the p-book (i.e. no crossing edges), then it is p-book embeddable, and it admits a p-page book embedding [18].

Clearly, every finite graph is embeddable in *some* book since it has a finite number of edges, which can be embedded, one per page, on a finite number of pages. So instead, we want to know the smallest book capable of embedding a graph. Thus, we define the **book thickness** of a graph G, denoted $bt(G)$, as the smallest $p \in \mathbb{N}$ such that G is p-book embeddable [6]. Also known as the *page number*, the book thickness is the number of pages in the smallest book in which G can be embedded. In Figure 12(a), we see that K_5 can be embedded on three pages. Be careful, this does not mean $bt(K_5) = 3$, because there could be a two-page or one-page book embedding of K_5. Instead, Figure 12(a) tells us that $bt(K_5) \leq 3$ [6].

As stated before, a book embedding of a graph has a linear layout. The vertices are placed on the spine according to some ordering, σ. Then, each edge of the graph is embedded on exactly one page such that no two edges on the same page cross [23]. It can be difficult to conceptualize the book embedding of a graph. However, we can easily transform the book embedding into a circular embedding to aid our understanding.

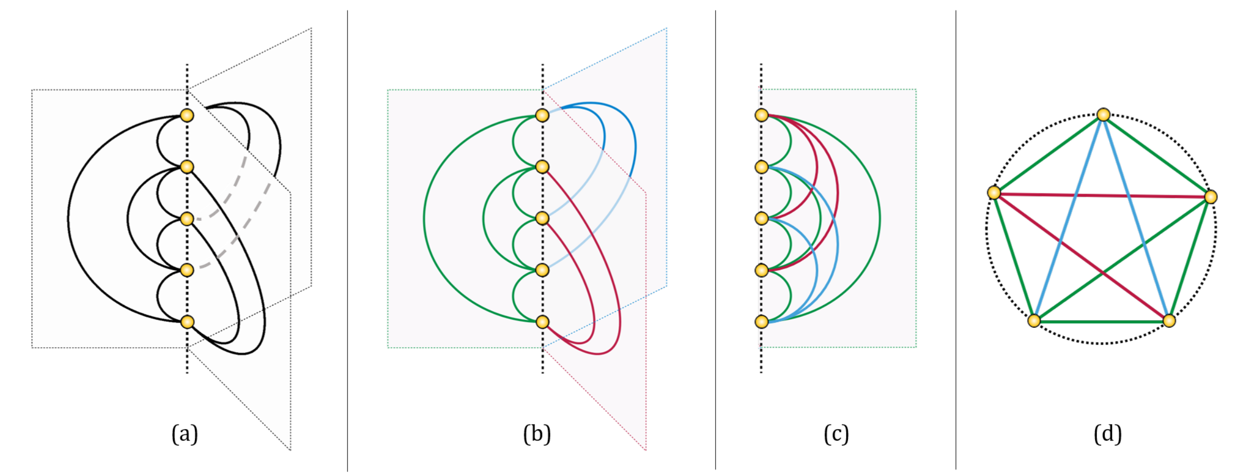


Figure 12: Transforming a book embedding of K_5 into a colored circular embedding.

(a) 3-book embedding of K_5. (b) colored 3-book embedding [41]. (c) colored single-page projection of 3-book embedding. (d) colored circular embedding of K_5.

To transform a book embedding into a circular embedding, begin with a book embedding of a graph, see Figure 12(a). Then, choose a distinct color for each page of the book. So, for a p-book, choose p distinct colors, assign each color to a different page, and color every edge on each page with the page's assigned color

[Fig. 12(b)]. Now, 'close' the book by projecting every page and its edges onto one single page [Fig. 12(c)]. In this single-paged book, all crossing edges must have different colors. Finally, connect the two ends of the spine to form a circle with the edges on the inside [Fig. 12(d)].

Thus, we have created a circular embedding of G where no two edges of the same color cross. Just as easily, we could transform a circular embedding with this property back into a book embedding. The equivalence of the book embedding and edge-colored circular embedding is described in a helpful theorem and corollary for understanding book embeddings.

Theorem 10 ([23])**.** A p-page book embedding of a graph G can be decomposed into p outerplanar subgraphs of G, one for each page, (see Figure 13.)

Corollary 11. A graph G has book thickness $bt(G) = p$ if and only if p is the minimum number of colors required to edge-color a circular embedding of G such that no two edges of the same color cross.

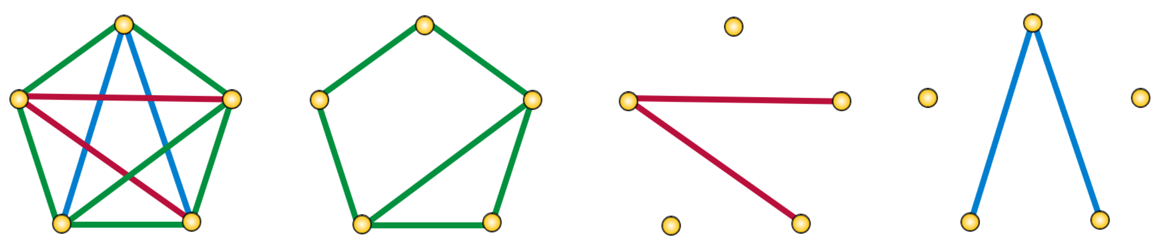


Figure 13: K_5 and its decomposition into three outerplanar subgraphs with the same layout.

Using planar subgraphs in [6], Bernhart and Kainen constructed two important theorems for book embeddings. First, recall the definitions of outerplanar and planar, given in Section 2. Next, consider an outerplanar graph G, which has a circular embedding (i.e. a circular layout such that no two edges cross). Then, we can color all of the edges the same color and no two edges of the same color cross. Thus, by the corollary, G must be one-page embeddable.

Conversely, consider a one-page embeddable graph G'. By the corollary, it takes one color to properly color a circular embedding of G' so that no two edges of the same color cross. Since we only needed one color, that means none of the edges cross. Thus, G' has circular embedding where no two edges cross, so G' is outerplanar. These two results from [6] form a theorem that classifies all one-page-embeddable graphs.

Theorem 12 ([6])**.** A graph G is one-page embeddable if and only if it is outerplanar.

Bernhart and Kainen also classified all two-page-embeddable graphs as planar subhamiltonian graphs. A graph, $G' = (V', E')$, is a **subgraph** of $G = (V, E)$, if V' is a subset of V *and* E' is a subset of E. Another way of obtaining a subgraph is starting with a graph G and deleting a subset of the edges and a subset of the vertices (along with all edges incident to those vertices) to obtain a new graph G'. Also, recall that a graph is *Hamiltonian* if it has cycle that contains every vertex. Then, a graph is **subhamiltonian** if it is a subgraph of a Hamiltonian graph. A graph is **planar subhamiltonian** if it is subhamiltonian and its Hamiltonian supergraph is planar. Now, we are able to classify all two-page embeddings with a theorem.

Theorem 13 ([6])**.** A graph G is two-page embeddable if and only if it is planar subhamiltonian.

Proof. In a book embedding, each page (which is a half plane) is planar. Since each page has the same vertex ordering and the union of two half planes is a plane, it follows that all two page embeddings are planar. Then, consider a graph G with a two-page embedding. We construct a Hamiltonian cycle along the spine using the linear ordering of the vertices, adding any missing edges to form the cycle. Since all two-page embeddings are planar graphs, and we only added edges on the spine, we now have a planar Hamiltonian graph. Thus, G is planar subhamiltonian.

Conversely, suppose G is the subgraph of a planar Hamiltonian graph H. Then, trace the Hamiltonian cycle C in a planar drawing of H. Then, C and all the edges inside C together form an outerplanar graph. Also, all of the edges outside C form another outerplanar graph. Designate each of these outerplanar graphs as a page and then we have found a two-page embedding of G. $\square$

By combining the two previous theorems in [6], we can classify all graphs with very small book thickness. This theorem gives a useful lower bound on the book thickness of non-planar graphs.

Theorem 14 ([6])**.** If a graph G is *not* planar, then $bt(G) \geq 3$.

Also considered in [6] is the book thickness of large planar graphs, including a conjecture of the existence of planar graphs with arbitrarily high book thickness. This conjecture was disproven in [11] where it was shown that all planar graphs can be embedded in 9 pages. Further research in [24], [28], and [44] has lowered this upper bound from 9 to 7, 6, and 4 pages, respectively. There are known planar graphs that require 3 pages, such as the Goldner-Harary graph, shown in Figure 14. However, despite a claim in [44], no planar graph has yet been found that requires exactly 4 pages [4].

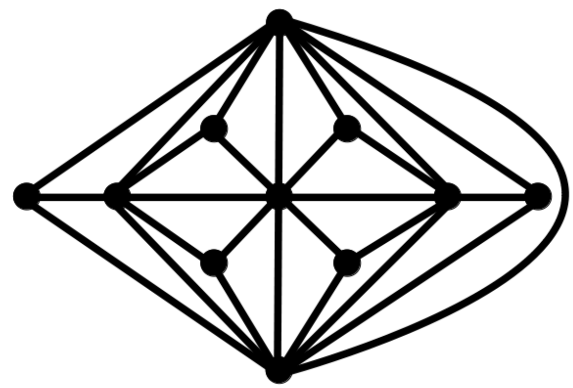


Figure 14: Goldner-Harary graph, a planar graph with book thickness 3.

There is much scholarship on other variations of book embeddings including requirements that subgraphs have a tree structure [4] or pages have structures other than a half plane [35]. One of these variations is called *dispersable* book embeddings. A **dispersable book embedding** of a graph G is a book embedding of G with the extra requirement that the subgraphs induced by the edges on each page are 1-regular, or *matchings*. We can also define the **dispersable book thickness**, denoted $dbt(G)$, to be the minimum number of pages in a dispersable book embedding of G [2]. Figure 15 shows a circular embedding of a graph G, and a dispersable book embedding with $dbt(G) = 3$. Since each page of a dispersable book embedding of G is 1-regular, it easily follows that $dbt(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum vertex degree of G. Additionally, we define a graph G to be **dispersable** if and only if $dbt(G) = \Delta(G)$[2].

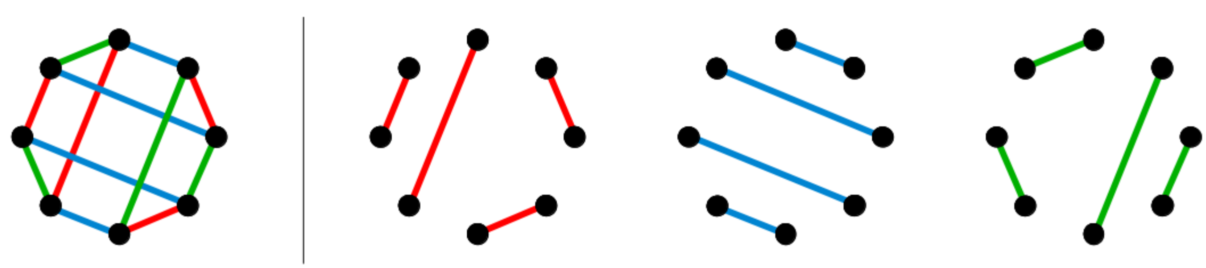


Figure 15: A dispersable book embedding, shown as a circular embedding of a graph (left) and its decomposition into three 1-regular subgraphs, or *matchings* (right).

Just as each graph has a book embedding, each graph also has a dispersable book embedding. As before, if each page contains only one edge of a graph, then the induced subgraphs are all 1-regular so the book embedding is dispersable. However, not all graphs with a dispersable book embedding are *dispersable*, only those graphs where $dbt(G) = \Delta(G)$ are called dispersable [35]. For the remainder of this paper, when we discuss dispersable graphs, we are explicitly referring to those with the property, $dbt(G) = \Delta(G)$.

In their seminal paper on book embeddings, Bernhart and Kainen included a section about dispersability, complete with a theorem and proof about the dispersability of the Cartesian product of graphs, see [6] for reference. Also included in that section was an unproven conjecture.

Conjecture 15 ([6])**.** Every k-regular bipartite graph G is dispersable, with $dbt(G) = k$.

Further research by Overbay [35] proved that not all k-regular graphs are dispersable, meaning bipartiteness was a necessary condition for the conjecture. Overbay also proved the dispersability of several families of graphs including binary cube graphs, trees, even order cycles, even order complete graphs, and the balanced complete bipartite graphs $K_{n,n}$. An astute reader may notice that the last three are part of the Moore graph families discussed earlier.

In 2018, leaders in the field of book embeddings published a paper that found two examples of non-dispersable regular bipartite graphs, disproving the conjecture by Bernhart and Kainen [2]. Since the authors disproved Bernhart and Kainen's conjecture, we now only have proof that certain families of regular bipartite graphs are dispersable, specifically those families considered by Overbay in [35]. The goal of this paper is to further the research done by Overbay, by finding which families of k-regular bipartite graphs are dispersable.

In this section, we introduced book embeddings and dispersable book embeddings from a graph theory perspective. In the next section, we discuss a computer scientist's view on dispersable book embeddings. Then in Section 7, we review the dispersability proofs for the even order cycles, even order complete graphs, and the balanced complete bipartite graphs $K_{n,n}$. We also prove the dispersability of another family of regular bipartite Moore graphs: the incidence graphs of the finite projective plane.

5 Stacks

Key words: data structure, outerplanar, stack number

In this section, we examine the close relationship between dispersable book embeddings and stacks. In computer science, a **stack** is an abstract data structure consisting of a collection of elements and two operations, *push* and *pop*. The *push* operation adds an element to the collection, and the *pop* operation removes the most recently added element that remains in the collection. The stack data structure can be imagined as a physical stack of objects, where adding and removing an object is only possible at the top of the stack and accessing a lower object requires the removal of all objects above it. In this way, the stack follows the last-in first-out (LIFO) principle of data storage [16]. A representation of a stack with a series of push and pop operations is shown below in Figure 16.

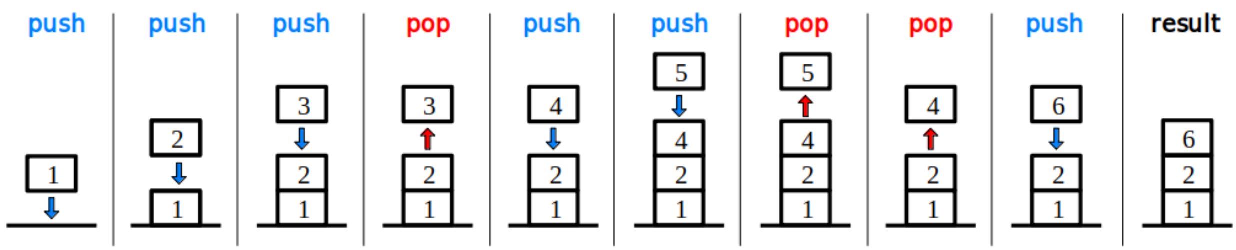


Figure 16: A series of push and pop operations preformed on a stack data structure and the resultant stack.

The concept of a stack was first considered by Alan M. Turing in 1946 as a way of calling and returning subroutines, only the Englishmen used the terms *bury* and *unbury* instead of *push* and *pop* [12]. Stacks are a commonly used data structure in computer science due to their efficient *limited-access* data structure.

In computer science, data structures can have different levels of accessibility. An array is an example of a *direct* or *random access* data structure because each element can be accessed directly and in constant time [42]. Another example of a direct access data structure would be a book (in the traditional paper sense), where every page has equally fast, direct access. Direct access data structures are important for many algorithms, including binary searches, but they require a large amount of data storage to function. In contrast, there are *sequential access* data structures, where each element may only be accessed in a particular, pre-defined order. A real world example of a sequential access data structure would be a roll of film, which must be unraveled to access the later frames, which appear in a pre-determined order. The stack is known as a *limited access* data structure, which is a subset of sequential access structures. In limited access structures, data must still be accessed sequentially but the access can only begin from one place. For a stack, data can only be accessed and changed from the top of the stack [1].

Based on the construction of a stack, shown in Figure 16, it is easy to see why a stack is known as a *linear data structure*. Recall that the ordering σ of the vertices along the spine of a book, has a linear layout [16]. With this in mind, we are able to define the relationship between a stack and a book embedding.

Formally, consider a stack with an arbitrary sequence of push and pop operations. Then, define each operation to be a vertex, placed in a linear ordering according to the sequence. Finally, if any object is the subject of both a push and a pop operation, connect the corresponding vertices in the linear ordering with an edge. In other words, if an object is both added and removed from the stack, draw an edge between the two vertices corresponding to the object [18, 33]. Though we will not provide proof, this resulting graph is outerplanar, or 1-page embeddable [31]. The 1-page embeddable graph in Figure 17 is constructed using this method on the stack from Figure 16.

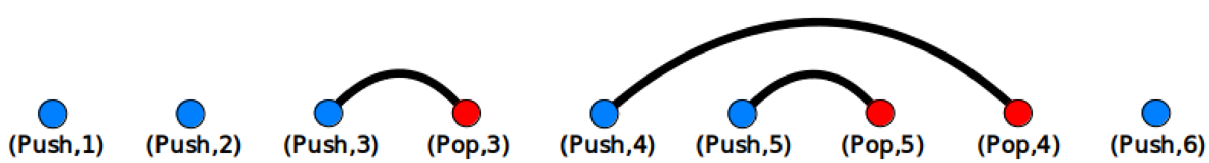


Figure 17: Graph constructed from the stack operations in Figure 16.

Recall that the edges on each page in a dispersable book embedding induce a 1-regular graph. Then, given any n-page dispersable book embedding we can transform each page into a stack using the above method. This gives rise to another name of an n-page dispersable book embedding, the n-stack [18]. For a dispersable graph, we can now consider the number of stacks, or *stack number*, necessary to properly embed the graph. Combining methods defined in this section and Section 4, it becomes clear that the *stack number* of a graph is equivalent to its *dispersable book thickness*.

In mathematics and life, looking at a problem from a different perspective can lead to new insight and solutions that would have been impossible before. The concepts of stacks and book embeddings developed mostly independently of one another, yet when the connection between them was discovered, the wealth of knowledge on the subject exploded. This paper uses book embeddings to focus on the graphical interpretation of the problem, instead of the computer science and data-driven interpretation using stacks. Though we will not mention stacks after this section, keep in mind that the concepts in this paper related to dispersability could have been written about stacks instead of book embeddings, while retaining all of the same information.

6 Difference Sets and Projective Planes

Key words: ring, group, difference, cyclic, Singer, shift matrix, adjacency matrix.

In this section, we examine the fundamental connection between the finite projective planes, defined in Section 3, and a concept known as the *difference set*. The relationship between these two ideas is critical for the proof offered in Section 7. This section uses basic concepts of abstract algebra, borrowed and abbreviated from Marlow Anderson's excellent textbook [3].

Consider a set consisting only of the integers from 0 to $n-1$, inclusive, for a natural number n. This set is also known as the the integers modulo n, and forms the *ring*, $\mathbb{Z}/n\mathbb{Z}$. In abstract algebra, a **ring** is structure consisting of a set of elements and two binary operations that generalize arithmetic. If the reader is unfamiliar with abstract algebra, refer to [3] for a brief introduction. However, for this paper a basic knowledge of modular arithmetic and the definition of a ring will suffice.

Just as with modular arithmetic, we can visualize the elements of the ring $\mathbb{Z}/n\mathbb{Z}$ as existing on a circle. The ring $\mathbb{Z}/7\mathbb{Z}$ is shown as such in Figure 18. For now, we limit n to be odd, but give the justification later.

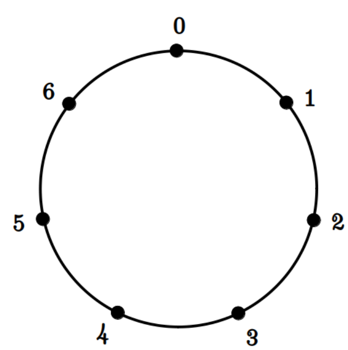


Figure 18: The ring $\mathbb{Z}/7\mathbb{Z}$ shown on a circle.

Given any two elements of $\mathbb{Z}/n\mathbb{Z}$, we can determine the *difference* between them. For example, in the ring $\mathbb{Z}/7\mathbb{Z}$, the difference between the elements 2 and 4 is,

$$4 - 2 \equiv 2 \pmod 7.$$

We could also consider the difference between elements 4 and 2 (i.e. switching the order). In this case,

$$2 - 4 \equiv -2 \equiv 5 \pmod 7.$$

These two unique differences, 2 and 5, between the elements arise because we could consider starting at either one of the elements. Note that we will always enumerate vertices and measure difference in the clockwise direction. Then, since the length of the circle is 7, it follows that the two differences sum to 7. In fact, the sum of the two differences between elements i and j is always,

$$(i - j \pmod n\,) + (\,j - i \pmod n) \equiv i - j + j - i \pmod n \equiv 0 \pmod n.$$

Instead of considering the difference between just one pair of elements, we could consider all the possible differences in $\mathbb{Z}/7\mathbb{Z}$. These are shown below in Table 2, where the (i, j)-entry is the difference starting at element i and going to element j. Also, notice that the (i, j) and (j, i)-entries sum to 7 for all i, j, just as we proved above.

	0	***1***	***2***	***3***	***4***	***5***	***6***
0	0	1	2	3	4	5	6
1	6	0	1	2	3	4	5
2	5	6	0	1	2	3	4
3	4	5	6	0	1	2	3
4	3	4	5	6	0	1	2
5	2	3	4	5	6	0	1
6	1	2	3	4	5	6	0

Table 2: Pairwise differences in $\mathbb{Z}/7\mathbb{Z}$.

Looking at the Table 2, we see that the differences between pairs of elements are not unique. That is, many distinct pairs of elements have the same difference. The equivalency,

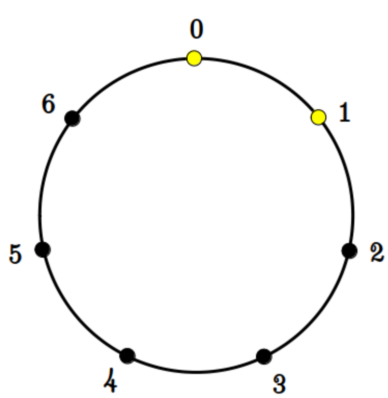
$$i - j \equiv i - j + 1 - 1 \equiv (i + 1) - (j + 1) \mod n$$

shows that for all i, j, we have $(i, j) \equiv (i + 1, j + 1) \mod n$. This equivalency creates the diagonal lines of equal values visible in Table 2. An alternative explanation for why the diagonal lines form is that the rotation of all elements on the circle does not change the differences between them.

Instead of considering *all* of the elements in $\mathbb{Z}/n\mathbb{Z}$, what if we considered only a subset of them? Can we choose k of the n elements in $\mathbb{Z}/n\mathbb{Z}$ such that all of the pairwise differences are unique? This was the fundamental question first considered formally by Bose in 1939 [10].

Before continuing, recall earlier that we required n to be odd. We also showed that the two differences between any pair of elements sum to n. Thus, we chose n to be odd to ensure that the two differences from any single pair of elements cannot be equal.

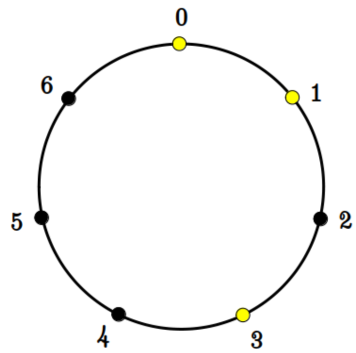
Continuing with the ring $\mathbb{Z}/7\mathbb{Z}$ as an example, let's try and find a subset D of the elements where all of the pairwise differences are unique. We begin by choosing the value of k, which is the order of D. Clearly, $k \geq 2$ because we need at least two elements to be able to calculate a difference. Also, the case where $k = 2$ is trivial because there are only two differences, which cannot be equal. Thus for $k = 2$, any two elements chosen will yield a subset with unique pairwise differences. One solution of the trivial $k = 2$ case, along with its difference table, is given in Figure 19. In this solution, $D = \{0, 1\}$.



	0	*1*
0	0	1
1	6	0

Figure 19: A trivial solution for $k = 2$ on the ring $\mathbb{Z}/7\mathbb{Z}$ and its distance table.

Due to the cyclic nature of the problem, all elements chosen first are identical up to rotation. Then, we can reduce the computational complexity by always selecting 0 to be the first element in our subset. However, the problem still becomes more complex when considering larger values of k. Below, we give a solution for $k = 3$, in Figure 20. In this case, $D = \{0, 1, 3\}$.



	0	***1***	***3***
0	0	1	3
1	6	0	2
3	4	5	0

Figure 20: A solution for $k = 3$ on the ring $\mathbb{Z}/7\mathbb{Z}$ and its difference table.

Notice that in the difference table for the $k = 3$ case, all of the integers from 0 to $n - 1$ are included. So, there is no way to add another element while keeping all of the pairwise differences unique. (This will be important in a moment!) Thus, for the ring $\mathbb{Z}/7\mathbb{Z}$, $k = 2$ or $k = 3$.

We are now able to formalize this section so far with a few short definitions. When discussing pairwise unique differences, what we were actually considering are called *difference sets.* We chose to explain the process for finding a difference set first, as opposed to just giving the definition for several reasons. First, the definitions are abstract and difficult to understand without a visual basis for what is happening. Thus, the process described above helps illuminate the definitions in a more meaningful way. Second, to obtain the final theorem of this paper, discussed in Section 7, we used properties of difference sets before even knowing what they were. Hence, we wanted to show how we arrived at our theorem in a more organic way.

Explicitly, a **(n, k, λ)-difference set** is a subset D with order k, of a group G with order n, such that every element of G can be expressed as a product of elements of D in exactly λ ways. Also, if G is an abelian group written in additive notation, then each element is expressed as a *difference* rather than a product [22].

Although this definition is dense, our example from this section can help to simplify it. We started with a ring $\mathbb{Z}/n\mathbb{Z}$, which of course is also a group, so $G = \mathbb{Z}/n\mathbb{Z}$ is a group with order n. Then, we took a k-element subset D, from G. Since G is an abelian group written in additive notation, we use the *difference* version of the definition. Then, by our example, each element in G was expressed as the difference of two elements in D, as shown by the Table in Figure 20. Though we will not prove it, each element can only be expressed in exactly one difference, thus $\lambda = 1$.

Referring back to the $k = 2$ and $k = 3$ examples from earlier, we noted that only in the $k = 3$ case could each integer from 0 to $n-1$ be expressed as a difference. Thus, the $k = 2$ case is *not* a difference set, because not every element in G can be expressed as a difference of elements in D.

Difference sets can also have the property of being **cyclic**, if such a criteria applies. Also, if $\lambda = 1$, the difference set is called **simple** [45]. Thus, we have shown the existence a $(7, 3, 1)$-simple, cyclic difference set. Since all of our difference sets in this paper are cyclic and simple, we will just call this the $(7, 3, 1)$-difference set, for simplicity's sake. In general, proving the existence of difference sets is difficult [45] and beyond the scope of this paper. So for the remainder of the paper we limit ourselves to known difference sets.

There are many known types of difference sets [22], but in this paper we focus on the *Singer* difference sets. First proposed by James Singer in 1938, a (q, d)-Singer difference set has (n, k, λ) parameters,

$$n = \frac{q^d - 1}{q - 1}, \quad k = \frac{q^{d-1} - 1}{q - 1}, \quad \lambda = \frac{q^{d-2} - 1}{q - 1},$$

where $d \geq 3$, and q is a prime power [37]. If we choose to set $d = 3$, then

$$\begin{aligned} n &= \frac{q^3 - 1}{q - 1} = \frac{(q-1)(q^2 + q + 1)}{q - 1} = q^2 + q + 1, \\ k &= \frac{q^{3-1} - 1}{q - 1} = \frac{q^2 - 1}{q - 1} = \frac{(q-1)(q+1)}{q - 1} = q + 1, \\ \lambda &= \frac{q^{3-2} - 1}{q - 1} = \frac{q^1 - 1}{q - 1} = \frac{q - 1}{q - 1} = 1. \end{aligned}$$

That is, a $(q, 3)$-Singer difference set has (n, k, λ) parameters $(q^2 + q + 1,\ q + 1,\ 1)$. Notice that setting $q = 2$ gives us the $(7, 3, 1)$-difference set that we found earlier in this section! Table 3 lists more (n, k, λ) parameters which correspond to some $(q, 3)$-Singer difference sets, but not all, as there are infinitely many of them.

Let's again recall what these difference sets actually tell us. Given the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$, with n elements, we can choose a k-element subset D, such that each element in G can be expressed as the difference of pairs of elements in D in exactly λ ways. Since we are only considering cases where $\lambda = 1$, this means

$q = p^a$	d	(n, k, λ)
$2 = 2^1$	3	$(7, 3, 1)$
$3 = 3^1$	3	$(13, 4, 1)$
$4 = 2^2$	3	$(21, 5, 1)$
$5 = 5^1$	3	$(31, 6, 1)$
$7 = 7^1$	3	$(57, 8, 1)$
$8 = 2^3$	3	$(73, 9, 1)$
$9 = 3^3$	3	$(91, 10, 1)$
$11 = 11^1$	3	$(133, 12, 1)$
$13 = 13^1$	3	$(183, 14, 1)$
$16 = 2^4$	3	$(273, 17, 1)$

Table 3: Parameters for $(q, 3)$-Singer difference sets.

that each pairwise difference is unique. Perhaps more difficult than proving the existence of these difference sets is determining which of the k elements should be in the subset D to ensure a valid difference set.

In [37], Singer details the process for determining these subsets, though we will not. Instead, recall that for one possible $(7, 3, 1)$-difference set, we could have $D = \{0, 1, 3\}$. In Table 4, we list one difference set of $\mathbb{Z}/n\mathbb{Z}$ corresponding to each of the $(q, 3)$-Singer parameters from Table 3 above. Notice that D always contains the elements 0 and 1. Since the difference sets are cyclic, any element picked first is identical. So, we can always choose 0 to be in D. Similarly, we can always add the element 1 since there must be *some* unique pairwise difference equal to 1. Thus, every difference set can be written equivalently to contain the elements 0 and 1. Finally, we noticed that D contains the element 3 in every case except the $(4, 3)$-Singer case. We are unsure why this case would be different but could not find any literature referencing this.

Over the last half decade, a considerable amount of scholarship has been done regarding the relationship between difference sets and finite projective planes (see [5],[9],[26],[32],[45]). One of the most important discoveries, detailed in [45], shows that projective planes and other finite projective geometries can be generated from various types of difference sets.

(n,k,λ)	D
$(7,3,1)$	$\{0,1,3\}$
$(13,4,1)$	$\{0,1,3,9\}$
$(21,5,1)$	$\{0,1,4,14,16\}$
$(31,6,1)$	$\{0,1,3,8,12,18\}$
$(57,8,1)$	$\{0,1,3,13,32,36,43,52\}$
$(73,9,1)$	$\{0,1,3,7,15,31,36,54,63\}$
$(91,10,1)$	$\{0,1,3,9,27,49,56,61,77,81\}$
$(133,12,1)$	$\{0,1,3,12,20,34,38,81,88,94,104,109\}$
$(183,14,1)$	$\{0,1,3,16,23,28,42,76,82,86,119,137,154,175\}$
$(273,17,1)$	$\{0,1,3,7,15,31,63,90,116,127,136,181,194,204,233,238,255\}$

Table 4: Simple difference sets and corresponding subsets D, with order k.

In [45], Theorem 1.9 states that the incidence matrix of $PG(2,q)$ can be generated from the $(q,3)$-Singer difference set using the following method. Define D as a $(q,3)$-Singer difference set with corresponding (n,k,λ) parameters. Then, in an $n \times n$ matrix, label the columns from 0 to $n-1$ and the rows from l_0 to l_{n-1}. In the first row, let $(l_0, j) = 1$ if $j \in D$. In the second row, let $(l_1, j+1) = 1$ if $j \in D$. Repeat this process, so in the ith row, we let $(l_{i-1}, j+i-1) = 1$ if $j \in D$. Let all other entries be 0. The resulting matrix is exactly the incidence structure of the finite projective plane of order $q-1$. An example of this matrix for $PG(2,2)$ is shown in Table 5, where $D = \{0,1,3\}$. This is exactly the same table seen in Section 3, where we generated the incidence structure graphically instead.

The process described in [45] and seen in Table 5 creates a matrix with multiple diagonals functioning mod n. Each row is created by shifting the above row to the right by one place, earning the name of a **shift matrix**. Then, each element of D defines a distinct diagonal in $I(PG(2,q))$. So, the number of diagonals is exactly equal to the order of D. We denote the k-th diagonal as P_k and define them explicitly. As mentioned earlier, every subset that we are considering contains elements 0 and 1. Thus, for any $(q,3)$-Singer difference

	0	*1*	*2*	*3*	*4*	*5*	*6*
$\boldsymbol{l_0}$	1	1		1			
$\boldsymbol{l_1}$		1	1		1		
$\boldsymbol{l_2}$			1	1		1	
$\boldsymbol{l_3}$				1	1		1
$\boldsymbol{l_4}$	1				1	1	
$\boldsymbol{l_5}$		1				1	1
$\boldsymbol{l_6}$	1		1				1

Table 5: Incidence structure of $PG(2,2)$ generated from $(7,3,1)$ difference set.

set, $I(PG(2,q))$ has the diagonals P_0 and P_1, along with the other $q-1$ diagonals P_j, for $j \in D$ where

$$P_0 = \{(i, l_i) \mid 0 \leq i \leq n-1\}, \qquad P_1 = \{(i+1, l_i) \mid 0 \leq i \leq n-1\}, \qquad P_j = \{(i+j, l_i) \mid 0 \leq i \leq n-1\}.$$

Just as we did earlier, we want to define a graph from the incidence structure, except this time we will use an *adjacency matrix*. Once again, we define our vertex set to be all of the points and lines from the incidence structure. Then, we define an **adjacency matrix** of a graph G with order m, to be an $m \times m$, binary matrix where the (i,j)-entry is 1 whenever vertex i is adjacent to vertex j and 0 otherwise.

By definition, the finite projective plane $PG(2,q)$ has q^2+q+1 points and q^2+q+1 lines. So, the graph generated from this structure must have $2(q^2+q+1)$ vertices. Thus, this graph has a $2(q^2+q+1) \times 2(q^2+q+1)$ adjacency matrix. In particular, the vertex set V is defined as, $V = \{0, \ldots, n-1, l_0, \ldots, l_{n-1}\}$.

For any adjacency matrix generated in this manner, the top left and bottom right quadrant will be all zeros since points cannot be incident to points and lines cannot be incident to lines in a projective plane. The other quadrants give the incidence structure, with $I(PG(2,q))$ in the bottom left quadrant and $I(PG(2,q))^T$ in the top right quadrant. The adjacency matrix shown in Figure 21 demonstrates these properties.

To gain a different perspective on this structure, we could rearrange the columns and rows of the adjacency matrix using elementary matrix operations. Instead of using the original vertex ordering,

	0	1	2	3	4	5	6	L_0	L_1	L_2	L_3	L_4	L_5	L_6
0								1				1		1
1								1	1				1	
2									1	1				1
3								1		1	1			
4									1		1	1		
5										1		1	1	
6											1		1	1
L_0	1	1		1										
L_1		1	1		1									
L_2			1	1		1								
L_3				1	1		1							
L_4	1				1	1								
L_5		1				1	1							
L_6	1		1				1							

Figure 21: The adjacency matrix generated from $I(PG(2,2))$.

$\sigma = \{0, \dots, n-1, l_0, \dots, l_{n-1}\}$, we use row and column transpositions to get the new ordering,

$$\sigma' = \{0, l_{n-1}, 1, l_{n-2}, 2, l_{n-3}, \dots, n-3, l_2, n-2, l_1, n-1, l_0\},$$

which we use for both the columns and the rows of the adjacency matrix. The new ordering helps simplify and emphasize the diagonals in the adjacency matrix. The diagonals are still defined as, $P_j = \{(i+j, l_i) \mid 0 \leq i \leq n-1\}$ for $j \in D$, but now the adjacency matrix consists only of the $q+1$ diagonals, without any quadrant distinction. The new adjacency matrix for the graph generated from $I(PG(2,2))$ is shown in Figure 22.

	0	L_6	1	L_5	2	L_4	3	L_3	4	L_2	5	L_1	6	L_0
0		1				1								1
L_6	1				1								1	
1				1								1		1
L_5			1								1		1	
2		1								1		1		
L_4	1								1		1			
3								1		1				1
L_3							1		1				1	
4						1		1				1		
L_2					1		1				1			
5				1		1				1				
L_1			1		1				1					
6		1		1				1						
L_0	1		1				1							

Figure 22: An adjacency matrix generated from $I(PG(2,2))$ which is similar to the one in Figure 21.

Using the strategies described in this section, the adjacency matrix generated from $I(PG(2,q))$ can be represented as a $2(q^2+q+1) \times 2(q^2+q+1)$ matrix with $q+1$ diagonals. In Section 7, we will use this adjacency matrix to prove a theorem about the dispersability of girth 6 cages.

In this section, we used difference sets to create incidence matrices and then adjacency matrices for incidence graphs of the finite projective planes. The construction of these matrices was cumbersome, but it will greatly simplify the proof in the next section.

7 Dispersable Book Embeddings of Moore Graphs

In her Ph.D thesis, Shannon Overbay proved the dispersability of many families of regular, bipartite graphs. In this section, we review some of these proofs, specifically those relating to the Moore graphs that we discussed earlier. Then, we present a new theorem and proof on the dispersability of a different regular, bipartite graph family: the incidence graphs of the finite projective planes, also known as the girth 6 cages or the Moore Graphs from Theorem 2(iv).

An additional dispersability requirement, proved by Overbay, states that regular graphs can only be dispersable if they have even order. Because of the requirement that $dbt(G) = \Delta(G)$ in dispersable graphs, and because each vertex in a regular graph has equal degree, there can be no degree 0 vertices on any page in a dispersable book embedding of a regular graph. Since there can be no vertices with degree 0 on any page, then there must be an even number of vertices in the graph.

Recall that the $(2,g)$-cages are 2-regular g-cycles. Then, [35] gives the following theorems on the dispersability of cages.

Theorem 16 ([35])**.** If $n \geq 1$, then C_{2n} is dispersable.

Proof. Begin by placing the vertices along the spine in the order of the cycle. Then, the edges of the cycle are assigned to one of two pages, alternating, as we move around the cycle. Since there are an even number of edges, the process terminates with an equal number of edges on both pages, which are 1-regular. Thus, we have embedded C_{2n} on 2 pages and $\Delta(C_{2n}) = 2$, so C_{2n} is dispersable. □

Theorem 17 ([35])**.** If $n \geq 2$, then $K_{n,n}$ is dispersable.

Proof. We already determined that $K_{n,n}$ is n-regular, so $\Delta(K_{n,n}) = n$. For the graph, we denote the red vertices in partite set A as $a_1, a_2, \ldots, a_n$ and the blue vertices in partite set B as $b_1, b_2, \ldots, b_n$. Then, the vertices are placed, alternating between partite sets, along the circle or spine in the following order: $a_1, b_n, a_2, b_n - 1, \ldots, a_k, b_{n-k+1}, \ldots, a_n, b_1$. Figure 23(a) depicts this ordering on the circle. Next, we define the first page of our book embedding to contain the edges, (a_i, b_i) for $1 \leq i \leq n$. Clearly, the subgraph induced by the first page, seen in Figure 23(b), is 1-regular and contains no crossing edges, so it is a valid page for our dispersable book embedding.

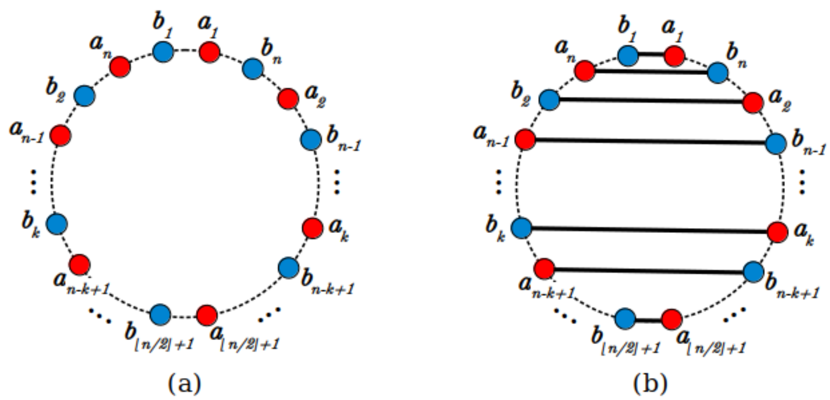


Figure 23: The vertex ordering (a) and first page (b) of dispersable book embedding for $K_{n,n}$.

Recall that in a complete bipartite graph, each vertex is adjacent to all the vertices in the opposite partite set. Next, we perform a somewhat strange operation. First, lock the vertices in place. Then rotate the parallel edges clockwise, about the center, by one vertex. The resulting graph, which we define as page 2, is of course still 1-regular, only this time the edges are defined as, (a_i, b_{i-1}) for $1 \leq i \leq n$ with indices computed mod n. Clearly, no edge is on multiple pages and each edge is still incident to one vertex from each partite set. Repeat this process to get n dispersable pages, where (a_i, b_{i-p+1}) defines all of the edges on page p. The composition of these n pages yields a graph where the arbitrary vertex a_i has degree n and is adjacent to all of the b vertices. So this composition of n pages is exactly the complete, bipartite graph $K_{n,n}$. Thus, $dbt(K_{n,n}) = n = \Delta(K_{n,n})$, so $K_{n,n}$ is dispersable. $\square$

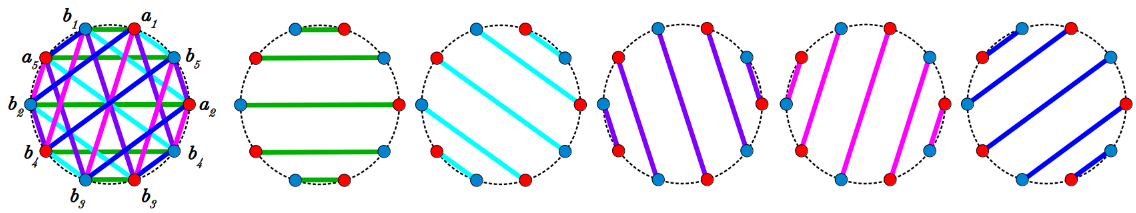


Figure 24: The graph $K_{5,5}$ is dispersable and has dispersable book thickness 5.

An example of the previous theorem is given in Figure 24, demonstrating the dispersability of $K_{5,5}$. The proof of the previous theorem is adapted slightly from the one given in [35] to highlight an interesting fact. The subgraphs produced with the method used in the proof all look identical, up to a rotation. We see this again in the next theorem.

Theorem 18. The graph G generated from the incidence structure of $PG(2,q)$ is dispersable.

Proof. Recall that the graph G generated from $I(PG(2,q))$ has a $2(q^2+q+1)\times 2(q^2+q+1)$ adjacency matrix with $q+1$ diagonals. The diagonals are defined mod n and the k-th diagonal is $P_k = \{(i+j, l_i) \mid 0 \le i \le n-1\}$ for $j \in D$, the $(q,3)$-Singer difference set that defines $PG(2,q)$. Note that each edge of G corresponds to an entry on one of the $q+1$ diagonals, as defined by the shift matrix theorem from [45].

For the book embedding of G, we set the vertex ordering to be

$$\sigma' = \{0, l_{n-1}, 1, l_{n-2}, 2, l_{n-3}, \ldots, n-3, l_2, n-2, l_1, n-1, l_0\}$$

in the clockwise direction. Notice that this is the same vertex ordering we chose for our adjacency matrix in Section 6. We also know that D can always be selected to contain the elements 0 and 1. Then, the diagonals $P_0 = \{(i, l_i)\}$ and $P_1 = \{(i+1, l_i) \mid 0 \le i \le n-1\}$ define all of the edges on page p_0 and p_1, respectively. The drawings in Figure 25 show that pages p_0 and p_1 are both planar and 1-regular.

Notice that P_1 can be obtained from P_0 by rotating all of the edges (while keeping the vertices frozen) clockwise by one vertex about the center. The result is still a planar, 1-regular graph but each edge in the edge set went from (i, l_i) to $(i+1, l_i)$.

Consider an arbitrary diagonal P_k in the adjacency matrix. Then, $P_k = \{(i+j, l_i) | 0 \le i \le n-1\}$ for $j \in D$ defines the page P_k which is a planar, 1-regular graph with circular layout. Further, P_k can be obtained by rotating the edges of P_0 clockwise by k vertex places.

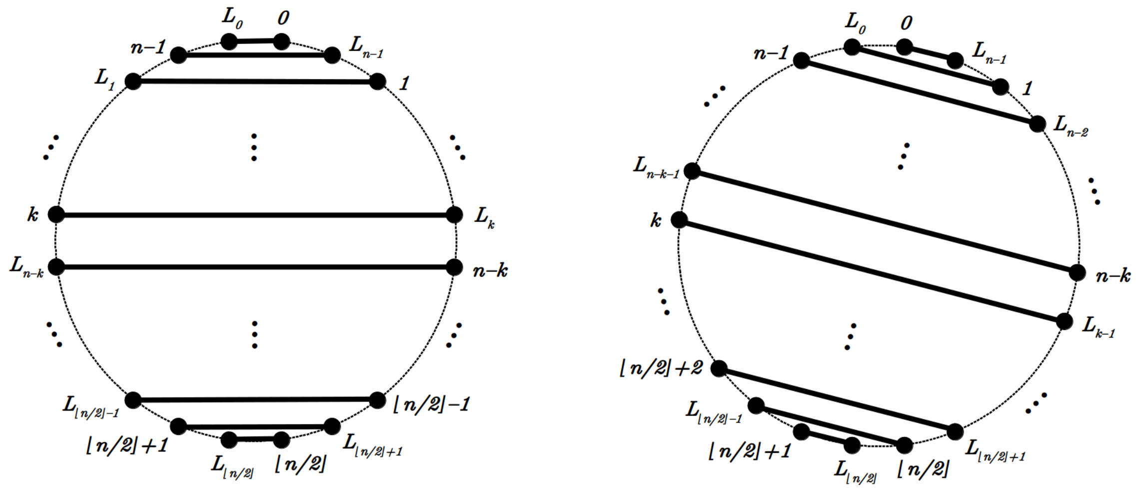


Figure 25: The first two pages of a dispersable book embedding of G.

The adjacency matrix of G has $q+1$ distinct diagonals mod n. Each diagonal generates a planar, 1-regular page, as we've described above, giving a total of $q+1$ pages. Since the sum of each row and column in the adjacency matrix is $q+1$, then G is a $(q+1)$-regular graph with maximum degree, $\Delta = q+1$. Thus, $dbt(G) = \Delta(G)$, and G is dispersable. $\square$

We have now shown that all even order Moore graphs from four of the six infinite families are dispersable. The book embeddings for the incidence graphs of the finite projective planes can be seen in Appendix B. After [2] proved that not all regular bipartite graphs are dispersable, we have added to Overbay's research, showing that another family of regular bipartite graphs is dispersable.

8 Future Work

Throughout this process there were many questions left unanswered due to either difficulty or time. This section addresses a few of these in the hope that future research may explore some of these concepts.

The most obvious question left unanswered concerns the dispersability of the girth 8 and 12 cages, the last two known infinite cage families. We began research on the dispersability of the finite projective quadrangles, but we have been unable to make significant progress so far. Experimentally, we found that the $(3,8)$-cage is dispersable, but the order of the other projective quadrangles grows too quickly to check any more with current SAT solvers. A key insight leading to the final proof in Section 7 was the connection between Singer difference sets and projective planes. Similarly, there are other types of difference sets that can be used to generate quadrangles [17]. Whether any of these sets could be used to generate the girth 8 cages, or if dispersabilty holds, would be an interesting topic to explore. Several other works have investigated the connection between quadrangles and difference sets which may be helpful for further research, these include [21], [29], and [30].

Another question that arose during our research was about the dispersability of the girth 6 cages that do not correspond to the finite projective planes. The first cage identified in the paper with this property was the (7,6)-cage, seen in Appendix A.4, although there are infinitely many of them. Throughout our research, we were unable to determine whether or not the (7,6)-cage was dispersable. Due to its large size, the SAT solvers were unable to satisfy any of the dispersability formulas. We were also unable to find any new methods for calculating the dispersability of other graphs. An extremely lofty goal for future research could be to find a method for determining the dispersability of any given graph, without the use of SAT solvers or other brute force methods.

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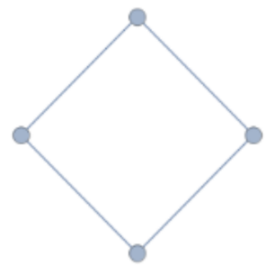
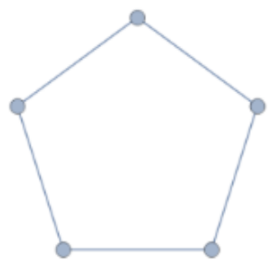
A Moore Graph/Cage Drawing Appendix

A.1 Cycle Graphs; Cages of Regularity 2

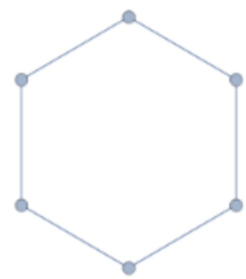
C_3 --- (2,3)-Cage



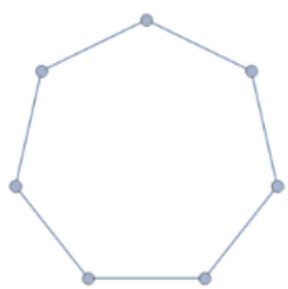
C_4 --- (2,4)-Cage



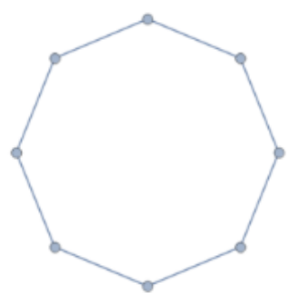
C_5 --- (2,5)-Cage



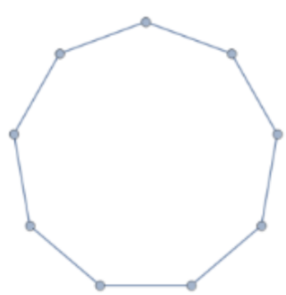
C_6 --- (2,6)-Cage



C_7 --- (2,7)-Cage



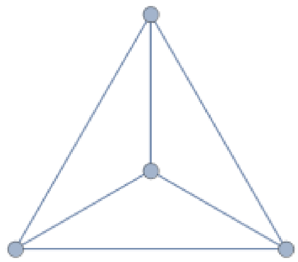
C_8 --- (2,8)-Cage



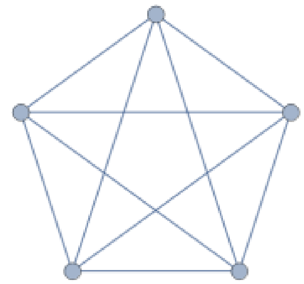
C_9 --- (2,9)-Cage

C_{10} --- (2,10)-Cage

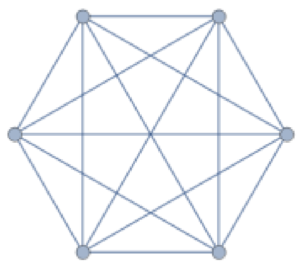
A.2 Complete Graphs; Cages of Girth 3



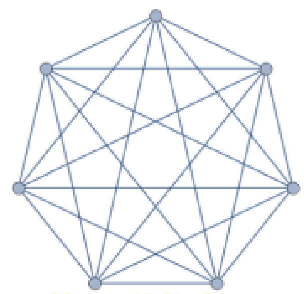
K_4 --- (3,3)-Cage



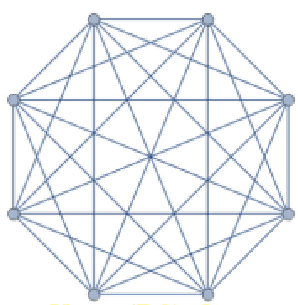
K_5 --- (4,3)-Cage



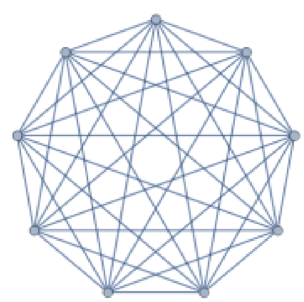
K_6 --- (5,3)-Cage



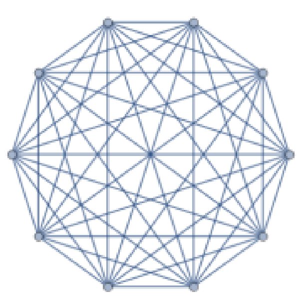
K_7 --- (6,3)-Cage



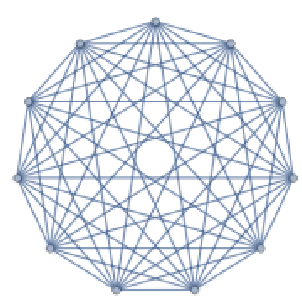
K_8 --- (7,3)-Cage



K_9 --- (8,3)-Cage

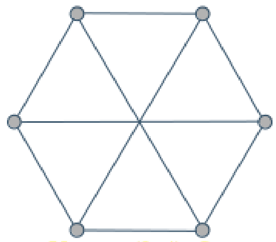


K_{10} --- (9,3)-Cage

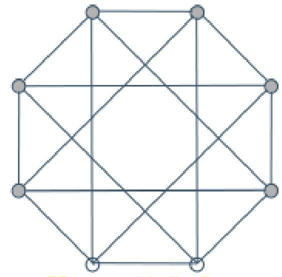


K_{11} --- (10,3)-Cage

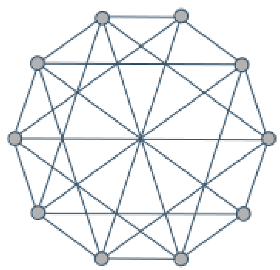
A.3 Complete Bipartite Graphs; Cages of Girth 4



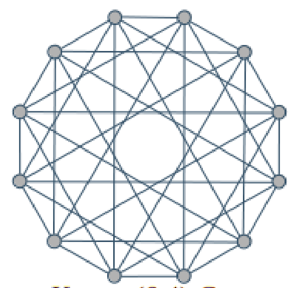
$K_{3,3}$ --- (3,4)-Cage



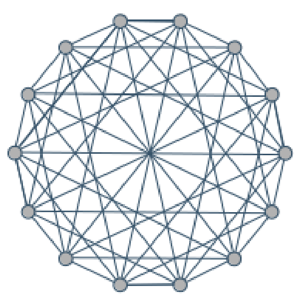
$K_{4,4}$ --- (4,4)-Cage



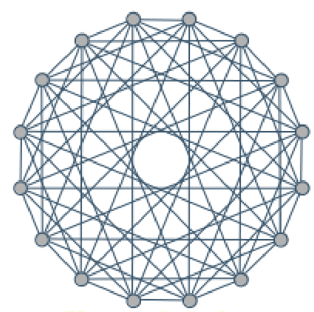
$K_{5,5}$ --- (5,4)-Cage



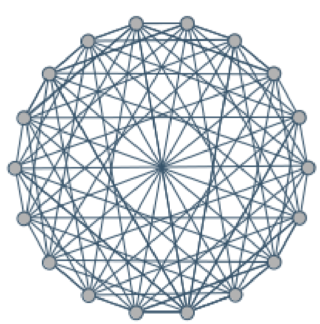
$K_{6,6}$ --- (6,4)-Cage



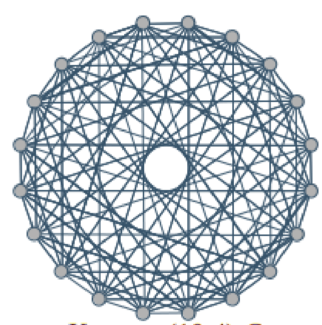
$K_{7,7}$ --- (7,4)-Cage



$K_{8,8}$ --- (8,4)-Cage

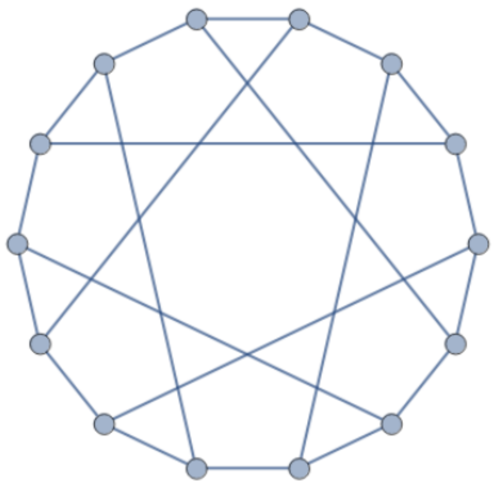


$K_{9,9}$ --- (9,4)-Cage

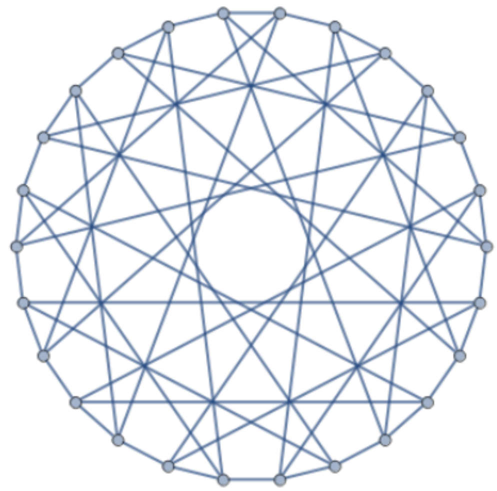


$K_{10,10}$ --- (10,4)-Cage

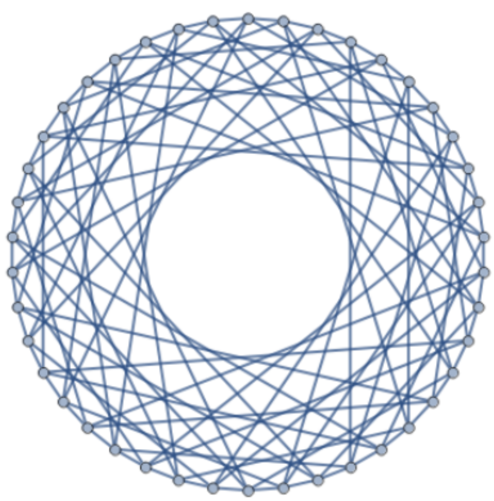
A.4 Incidence Graphs of Finite Projective Planes and Other Cages of Girth 6



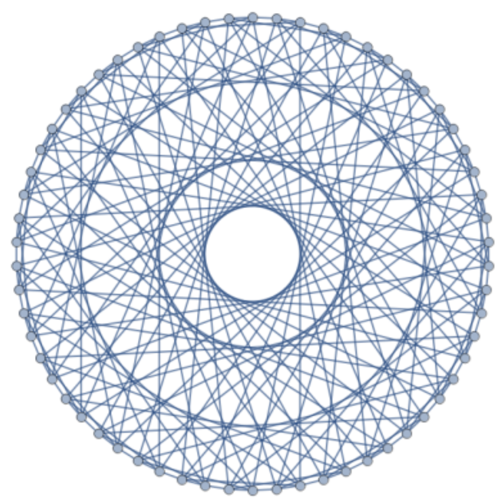
Heawood Graph --- (3,6)-Cage



(4,6)-Cage

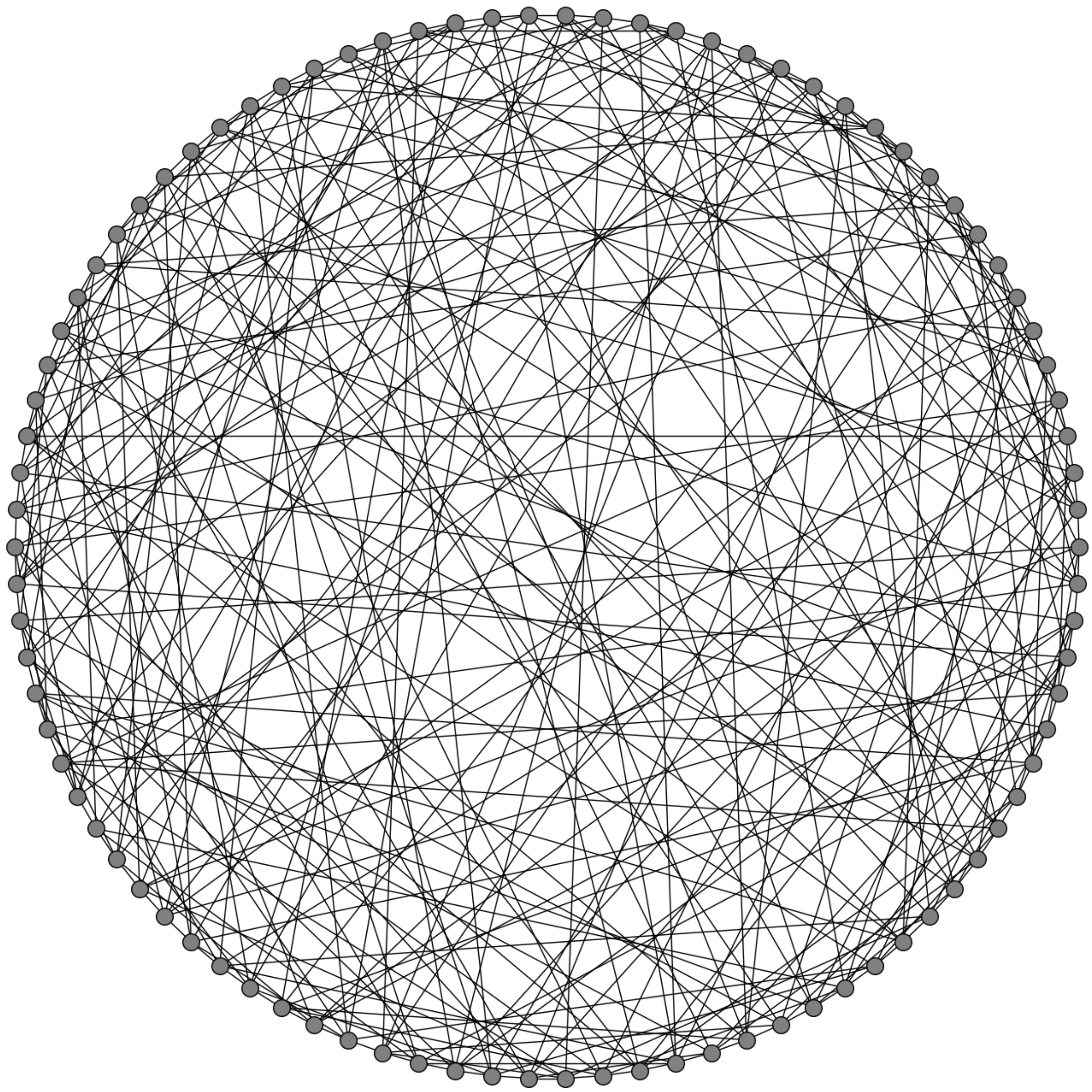


(5,6)-Cage

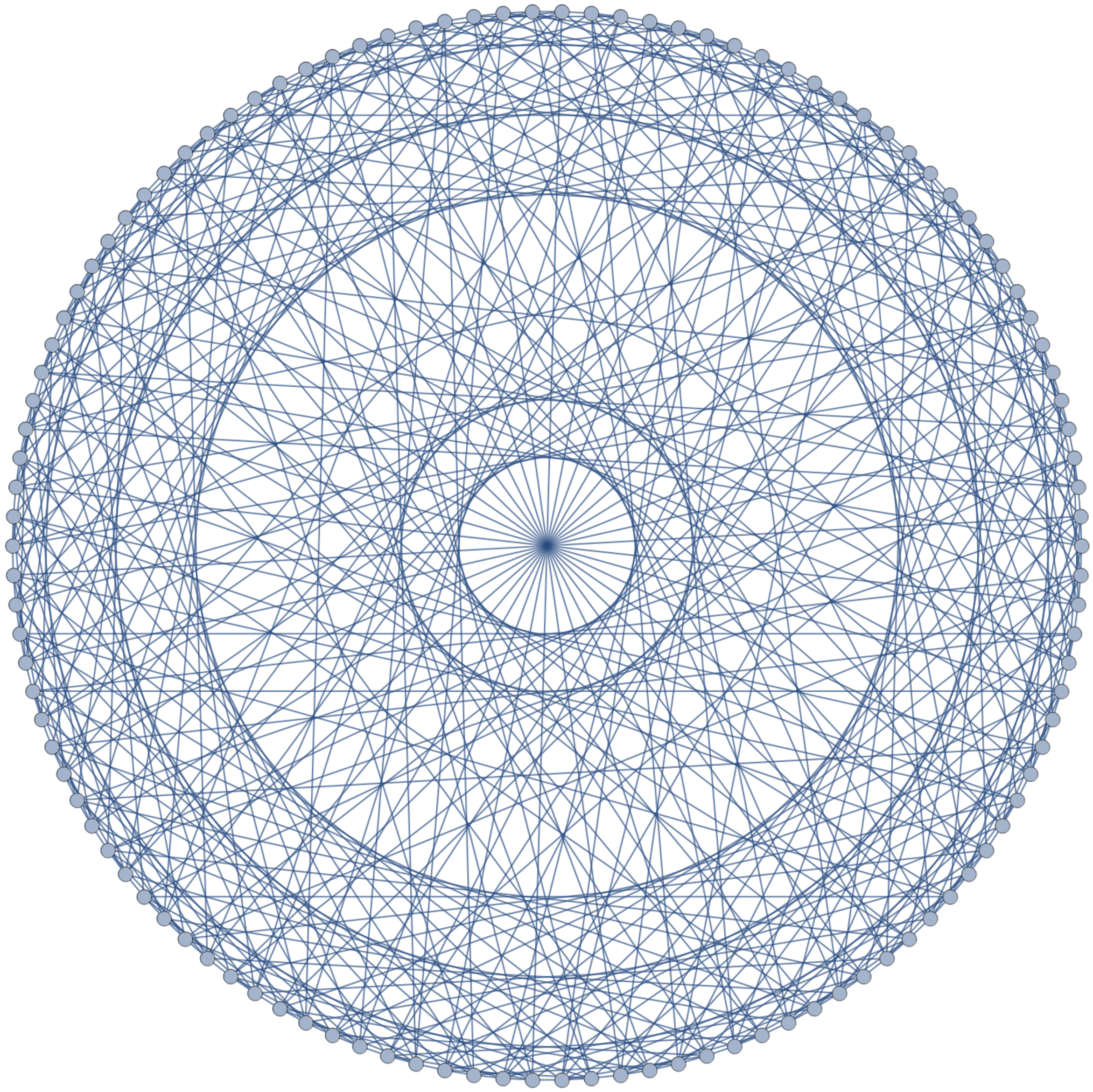


(6,6)-Cage

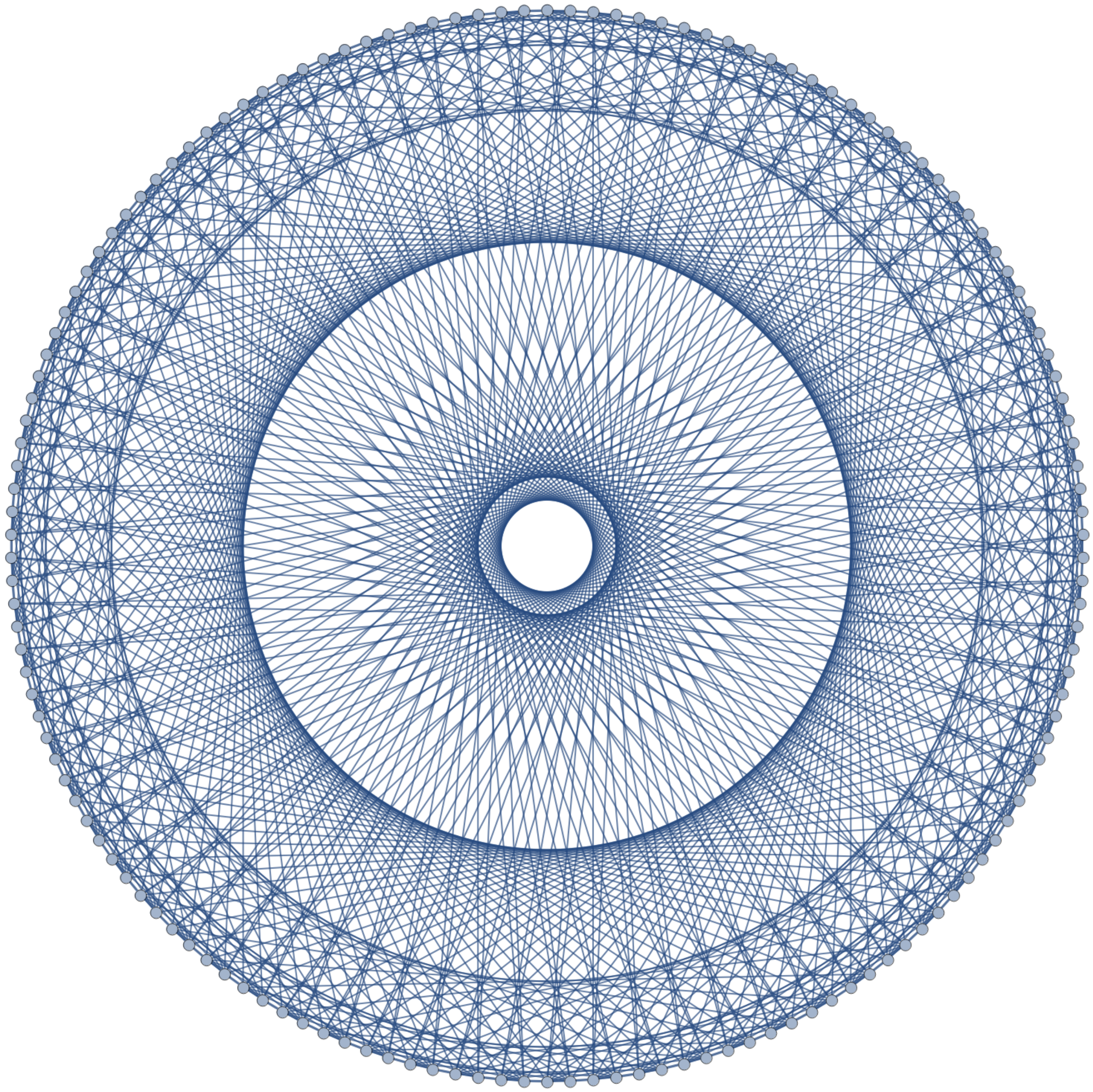
*Appendix A.4 is continued on the following pages



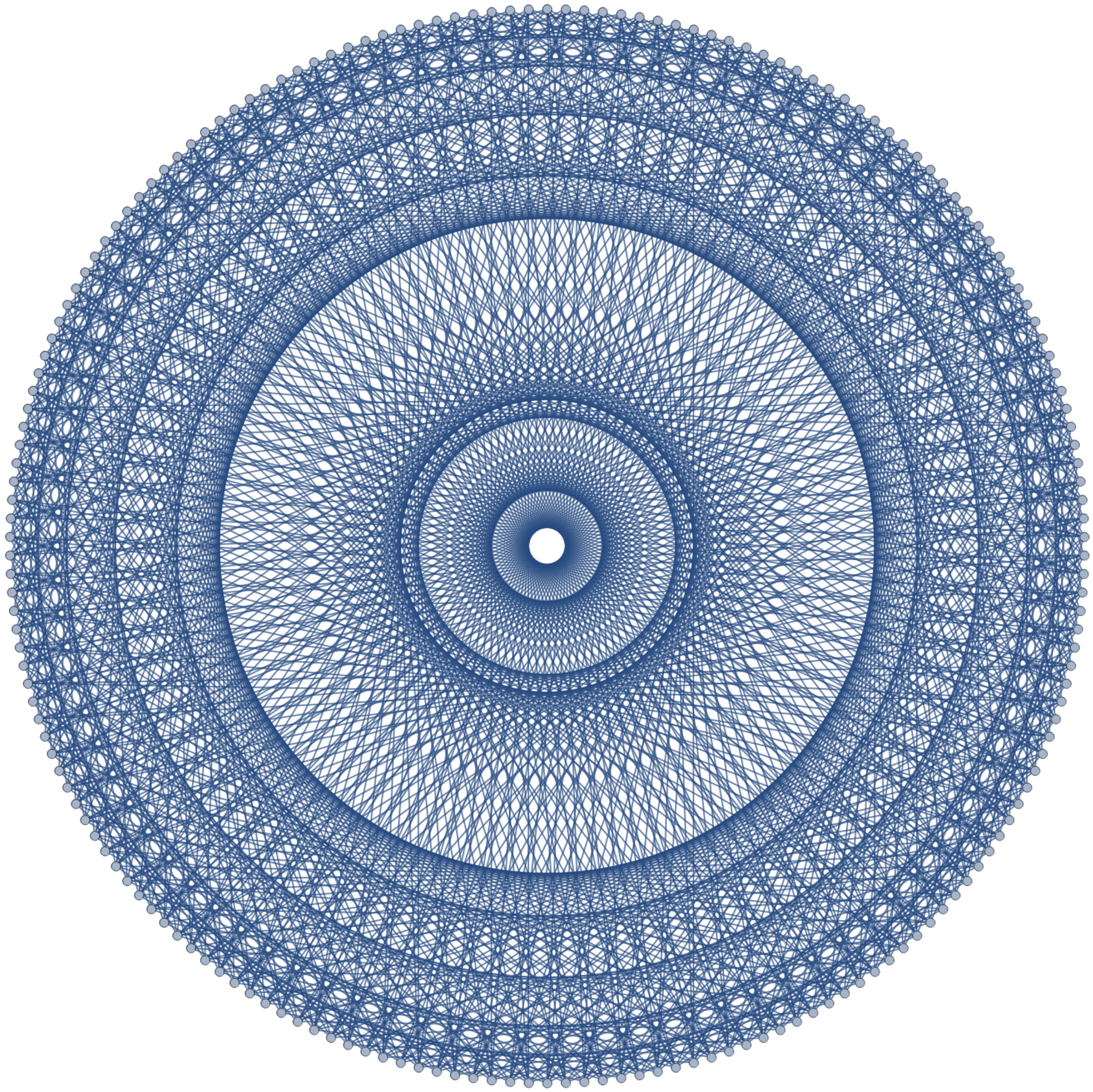
(7,6)-Cage



(8,6)-Cage

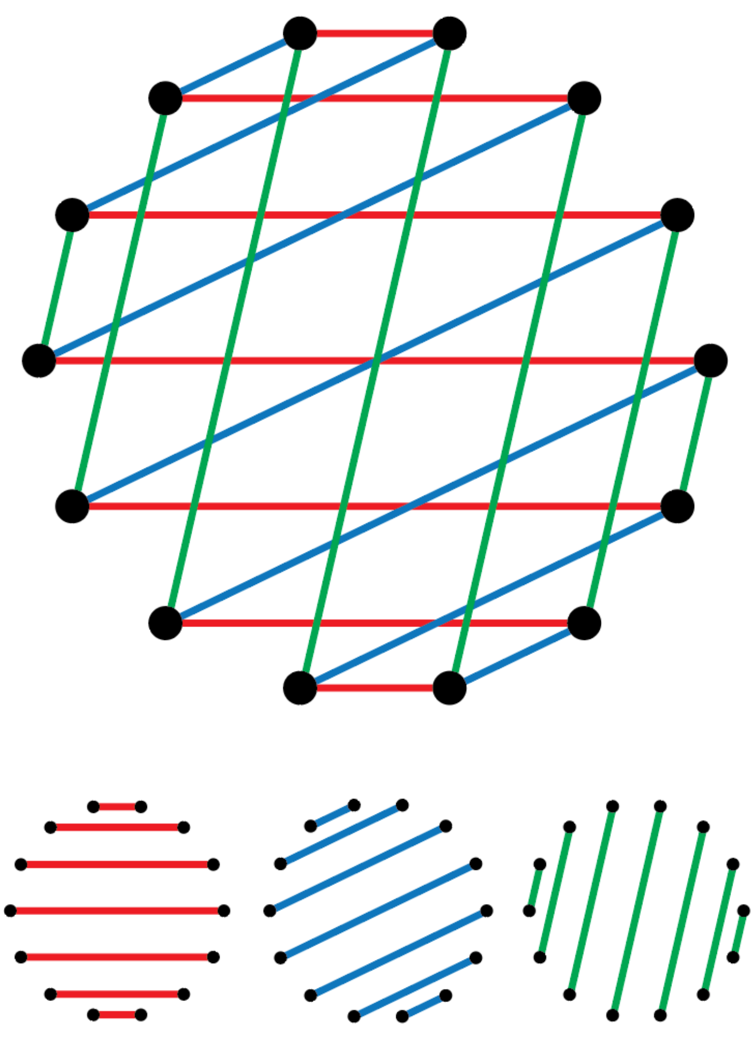


(9,6)-Cage



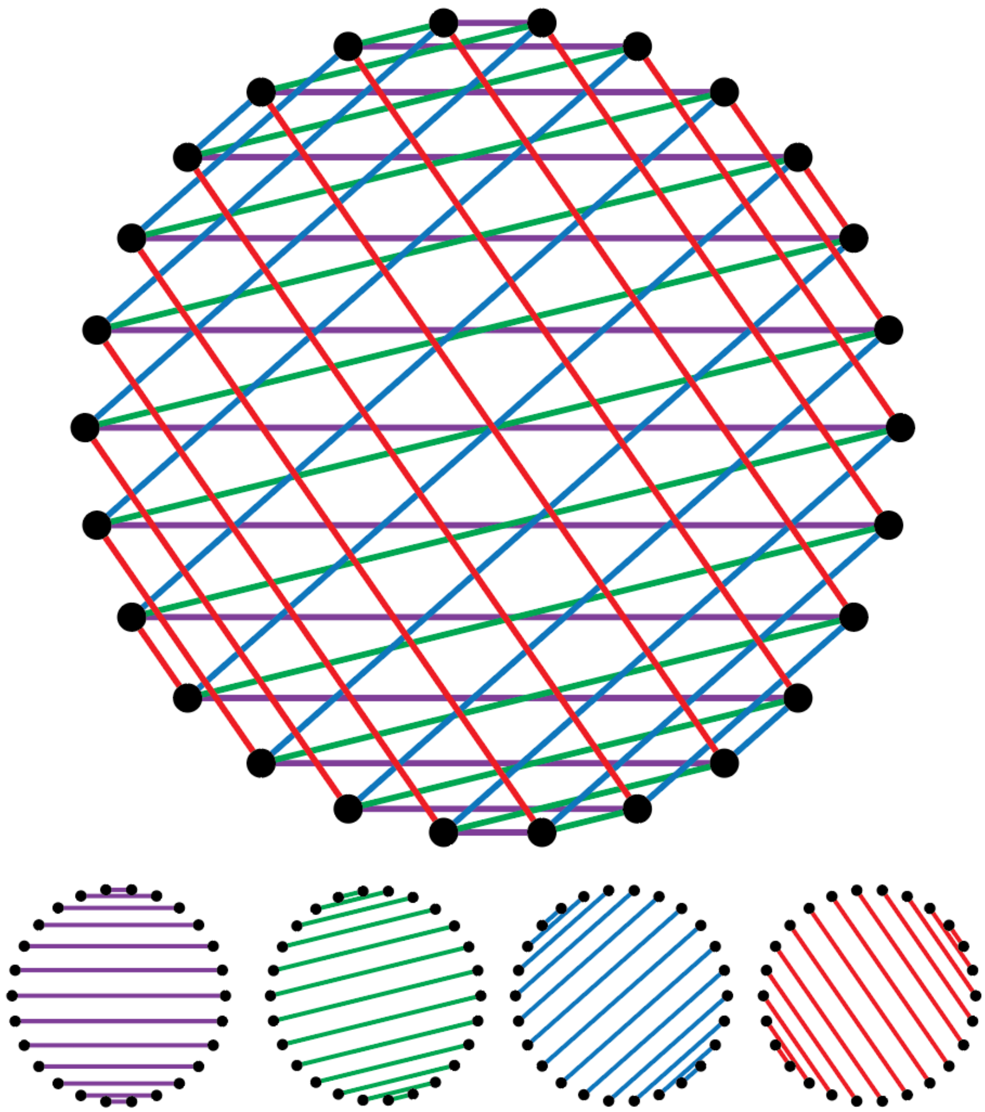
(10,6)-Cage

B Book Embeddings of Moore Graphs/Cages



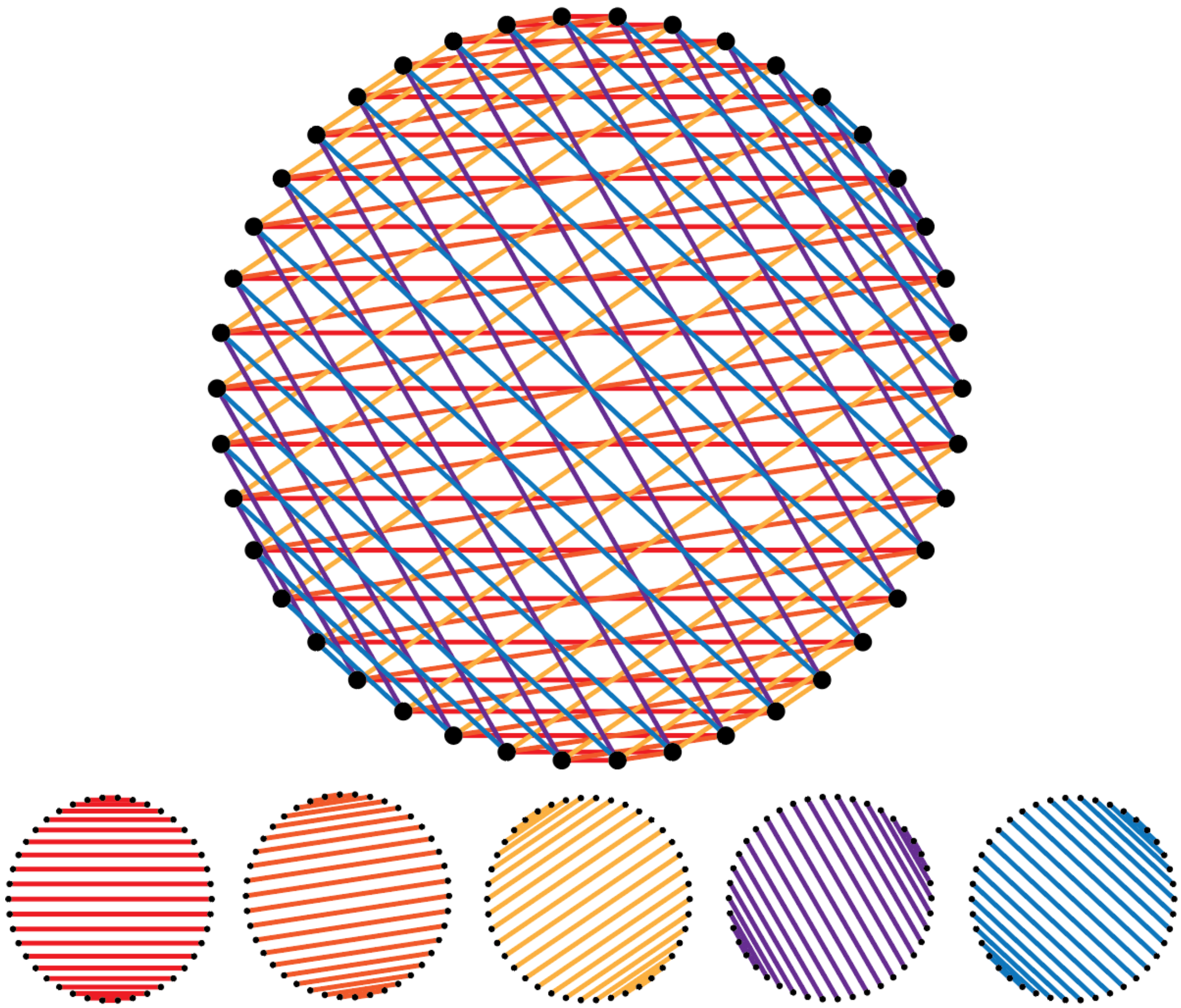
Dispersable Book Embedding of (3,6)-Cage.

All of the edges on a single page are given the same color.



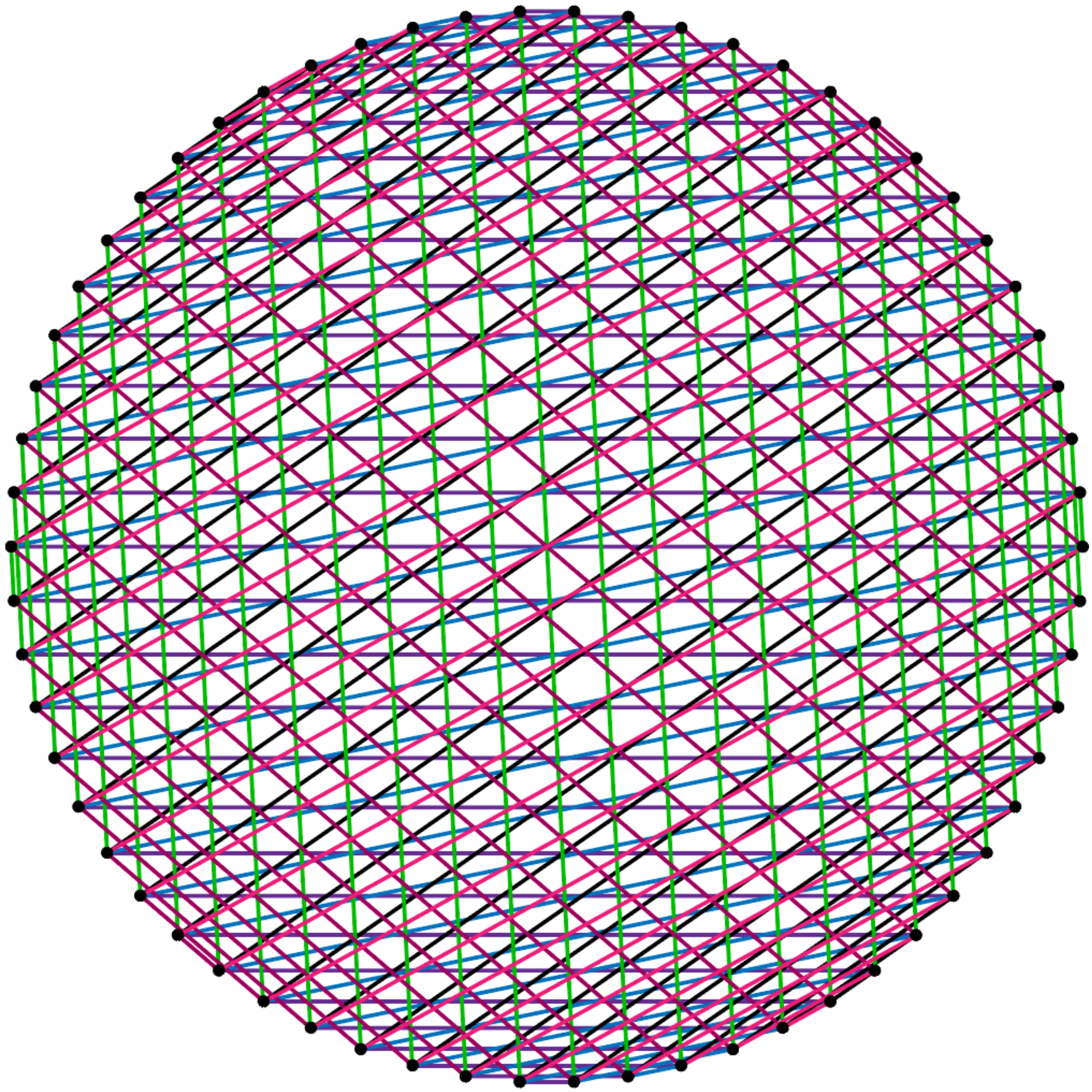
Dispersable Book Embedding of (4,6)-Cage.

All of the edges on a single page are given the same color.



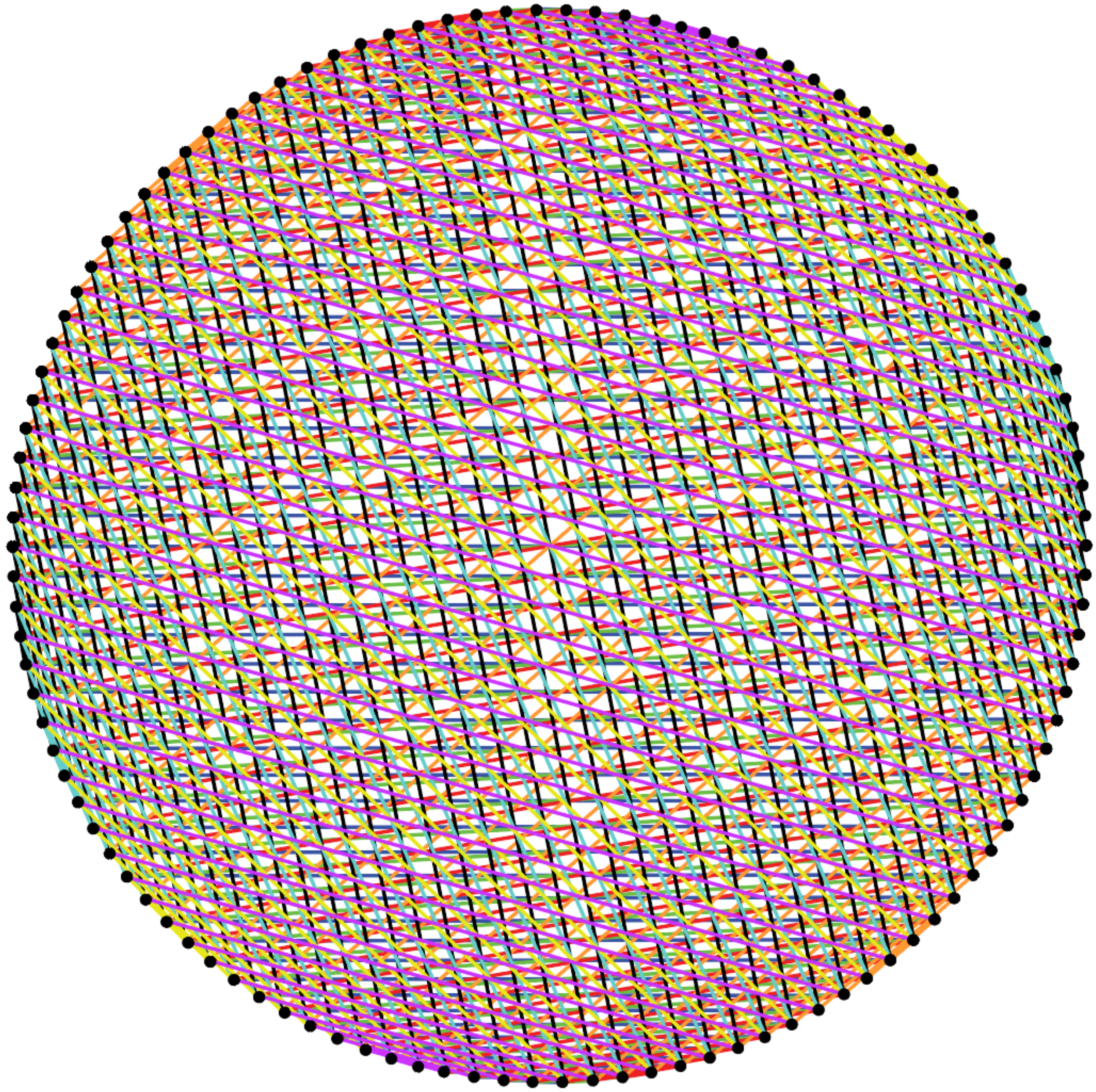
Dispersable Book Embedding of (5,6)-Cage.

All of the edges on a single page are given the same color.



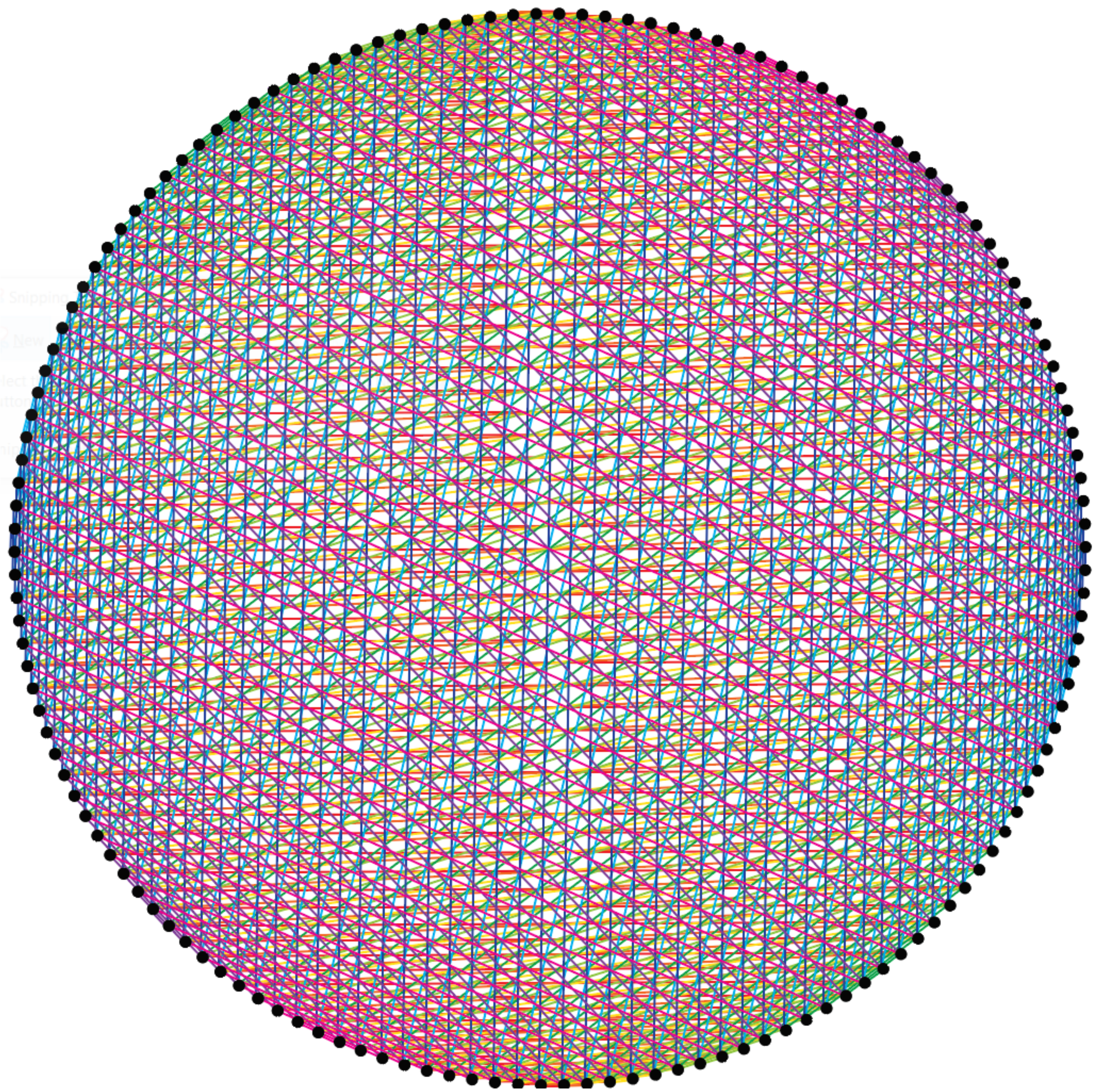
Dispersable Book Embedding of (6,6)-Cage.

All of the edges on a single page are given the same color.



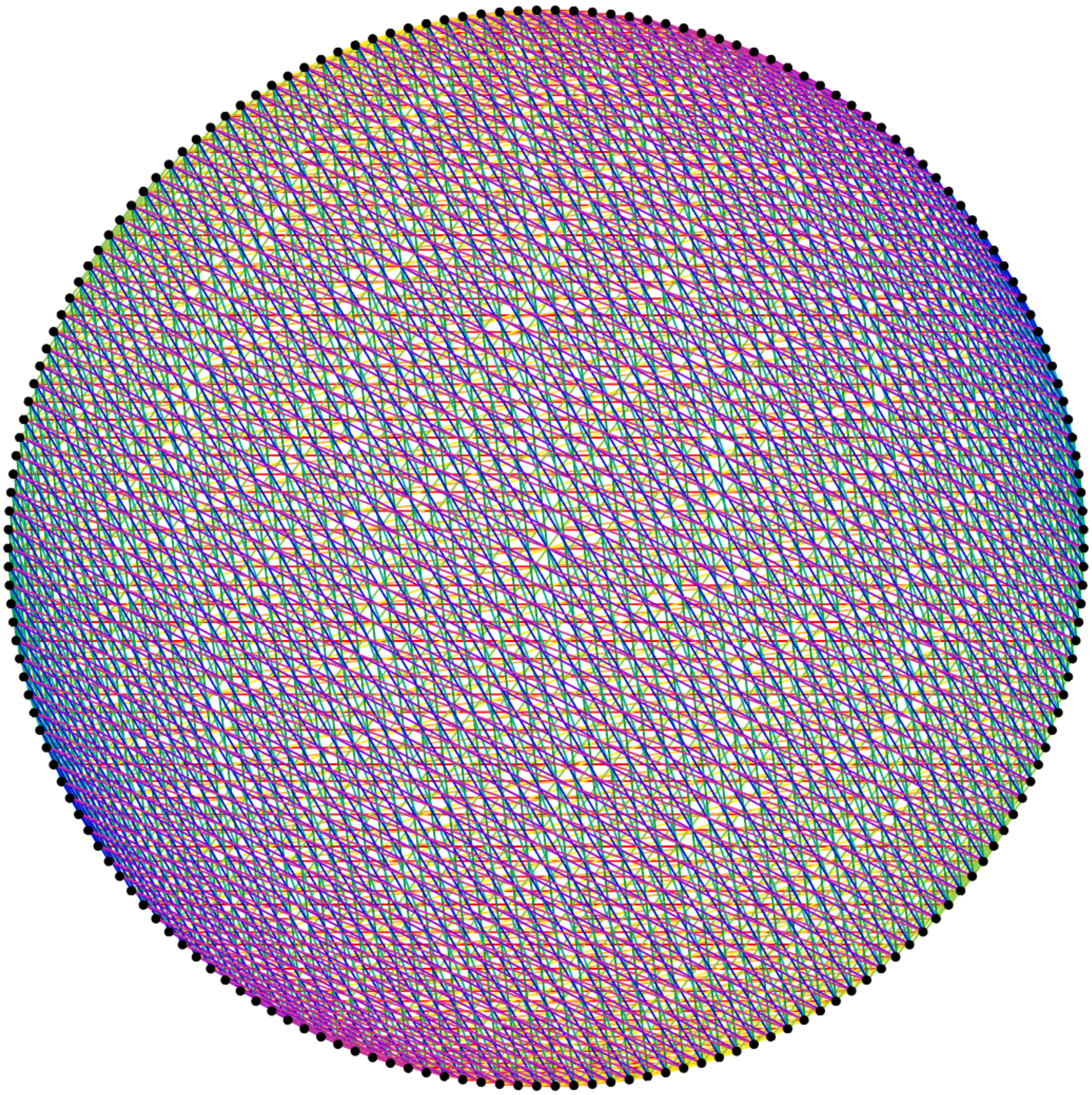
Dispersable Book Embedding of (8,6)-Cage.

All of the edges on a single page are given the same color.



Dispersable Book Embedding of (9,6)-Cage.

All of the edges on a single page are given the same color.



Dispersable Book Embedding of (10,6)-Cage.

All of the edges on a single page are given the same color.