Minimize $\sum_{i=1}^{n} \sum_{j=1}^{k} (y_{ij} - \hat{y}_{ij})^2$ s.t. $\sum_{j=1}^{k} \gamma_j = 1$

Where $\underset{n\times k}{\hat{Y}} = \underset{n\times k}{X} \odot (1_n \underset{k\times 1}{\underline{\gamma}}')$

For $k=2$, $\underline{\gamma} = \begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$

$$\mathcal{L} = \sum_{i=1}^{n}\sum_{j=1}^{k} (y_{ij} - x_{ij}\gamma_j)^2 + \lambda\left(1 - \sum_{j=1}^{k} \gamma_j\right)$$

FOC: $$\frac{\partial \mathcal{L}}{\partial\gamma_1} = \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{k} \frac{\partial}{\partial \gamma_1}(y_{ij} - x_{ij}\gamma_j)^2}_{= 2(y_{i1} - x_{i1}\gamma_1)(-x_{i1})} - \lambda = 0$$

$$= \sum_{i=1}^{n} 2(y_{i1} - x_{i1}\gamma_1)(-x_{i1}) = \lambda = \sum_{i=1}^{n} 2(y_{i2}-x_{i2}\gamma_2)(-x_{i2}) \quad \text{(by analogy)}$$

$$\Rightarrow 2\left[\gamma_1 \sum_{i=1}^{n} x_{i1}^2 - \sum_{i=1}^{n} y_{i1}x_{i1}\right] = \lambda = 2\left[\gamma_2 \sum_{i=1}^{n} x_{i2}^2 - \sum_{i=1}^{n} y_{i2}x_{i2}\right]$$

$$\Rightarrow \boxed{\gamma_1^* = \left(\frac{\lambda}{2} + \sum_{i=1}^{n} y_{i1}x_{i1}\right)\left(\sum_{i=1}^{n} x_{i1}^2\right)^{-1}}$$
$$\Rightarrow \boxed{\gamma_2^* = \left(\frac{\lambda}{2} + \sum_{i=1}^{n} y_{i2}x_{i2}\right)\left(\sum_{i=1}^{n} x_{i2}^2\right)^{-1}}$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} \Rightarrow 1 = \sum_{j=1}^{k} \gamma_j^* \Rightarrow \boxed{\gamma_2^* = 1-\gamma_1^*}$$

$$\Rightarrow 1-\gamma_1^* = \left(\frac{\lambda}{2} + \sum_{i=1}^{n} y_{i2}x_{i2}\right)\left(\sum_{i=1}^{n} x_{i2}^2\right)^{-1}$$

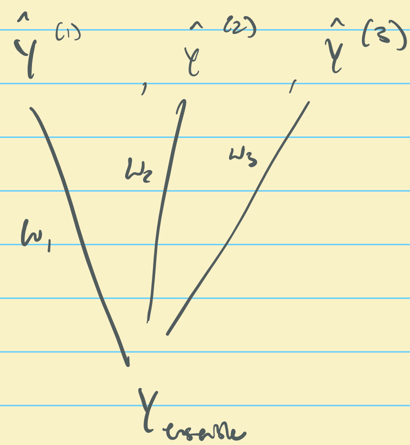
$$\Rightarrow \gamma_1^* = 1 - \left(\frac{\lambda}{2} + \sum_{i=1}^{n} y_{i2}x_{i2}\right)\left(\sum_{i=1}^{n} x_{i2}^2\right)^{-1} = \left(\frac{\lambda}{2} + \sum_{i=1}^{n} y_{i1}x_{i1}\right)\left(\sum_{i=1}^{n} x_{i1}^2\right)^{-1}$$

$$\Rightarrow 1 - \frac{\lambda}{2}\left(\sum_{i=1}^{n} x_{i2}^2\right)^{-1} - \left(\sum_{i=1}^{n} y_{i2}x_{i2}\right)\left(\sum_{i=1}^{n} x_{i2}^2\right)^{-1} = \frac{\lambda}{2}\left(\sum_{i=1}^{n} x_{i1}^2\right)^{-1} + \left(\sum_{i=1}^{n} y_{i1}x_{i1}\right)\left(\sum_{i=1}^{n} x_{i1}^2\right)^{-1}$$

$$\Rightarrow 1 - \left(\sum_{i=1}^{n} y_{i2}x_{i2}\right)\left(\sum_{i=1}^{n} x_{i2}^2\right)^{-1} - \left(\sum_{i=1}^{n} y_{i1}x_{i1}\right)\left(\sum_{i=1}^{n} x_{i1}^2\right)^{-1} = \frac{\lambda}{2}\left(\sum_{i=1}^{n} x_{i2}^2\right)^{-1} + \frac{\lambda}{2}\left(\sum_{i=1}^{n} x_{i1}^2\right)^{-1}$$

$$\Rightarrow \frac{\lambda}{2} = \frac{1 - \left(\sum_{i=1}^{n} y_{i2}x_{i2}\right)\left(\sum_{i=1}^{n} x_{i2}^2\right)^{-1} - \left(\sum_{i=1}^{n} y_{i1}x_{i1}\right)\left(\sum_{i=1}^{n} x_{i1}^2\right)^{-1}}{\left(\sum_{i=1}^{n} x_{i2}^2\right)^{-1} + \left(\sum_{i=1}^{n} x_{i1}^2\right)^{-1}}$$

$$\boxed{\begin{gathered}\frac{\lambda}{2} = \frac{1 - \sum_{j=1}^{k}\left[\left(\sum_{i=1}^{n} y_{ij}x_{ij}\right)\left(\sum_{i=1}^{n} x_{ij}^2\right)^{-1}\right]}{\sum_{j=1}^{k}\left[\left(\sum_{i=1}^{n} x_{ij}^2\right)^{-1}\right]} \\ \gamma_\ell^* = \left(\frac{\lambda}{2} + \sum_{i=1}^{n} y_{i\ell}x_{i\ell}\right)\left(\sum_{i=1}^{n} x_{i\ell}^2\right)^{-1}\end{gathered}}$$



$$\underset{n \times k}{\hat{Y}_{\text{Ensemble}}} = \underset{n \times k}{\hat{Y}^{(1)}} \odot \left(\underset{n \times k}{[1_n \cdots 1_n]} w_1 \right) + \cdots + \hat{Y}^{(2)} \odot \left([1_n \cdots 1_n] w_k \right)$$

$$= \sum_{m=1}^{M} \hat{Y}^{(m)} \odot \left([1_n \cdots 1_n] w_m \right)$$

$$\underset{\underline{w}}{\arg\min} \left\| Y - \hat{Y}_{\text{Ensemble}} \right\|_F^2 \equiv \sum_{i=1}^{n} \sum_{j=1}^{k} \left(Y_{ij} - \sum_{m=1}^{M} w_m \hat{Y}_{ij}^{(m)} \right) \quad s.t. \quad \sum_{m=1}^{M} w_m = 1$$

$$\mathcal{L} = \sum_{i=1}^{n} \sum_{j=1}^{k} \left(Y_{ij} - \sum_{m=1}^{M} w_m \hat{Y}_{ij}^{(m)} \right)^2 + \lambda \left(1 - \sum_{m=1}^{M} w_m \right)$$

FOC: For $l = 1, \ldots, M$

$$\frac{\partial \mathcal{L}}{\partial w_l} = \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{\partial}{\partial w_l} \left(Y_{ij} - \sum_{m=1}^{M} w_m \hat{Y}_{ij}^{(m)} \right)^2 - \lambda = 0$$

$$\Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{k} 2 \left(Y_{ij} - \sum_{m=1}^{M} w_m \hat{Y}_{ij}^{(m)} \right) \left(- \hat{Y}_{ij}^{(l)} \right) = \lambda$$

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \left(-Y_{ij} \hat{Y}_{ij}^{(l)} + \hat{Y}_{ij}^{(l)} \sum_{m=1}^{M} w_m \hat{Y}_{ij}^{(m)} \right) = \frac{\lambda}{2}$$

$$- \sum_{i=1}^{n} \sum_{j=1}^{k} Y_{ij} \hat{Y}_{ij}^{(l)} + \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\hat{Y}_{ij}^{(l)} \sum_{m=1}^{M} w_m \hat{Y}_{ij}^{(m)} \right) = \frac{\lambda}{2}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \left(\hat{Y}_{ij}^{(l)} \sum_{m=1}^{M} w_m \hat{Y}_{ij}^{(m)} \right) = \frac{\lambda}{2} + \sum_{i=1}^{n} \sum_{j=1}^{k} Y_{ij} \hat{Y}_{ij}^{(l)}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \left[\hat{Y}_{ij}^{(l)} \left(w_l \hat{Y}_{ij}^{(l)} + \sum_{m \neq l}^{M} w_m \hat{Y}_{ij}^{(m)} \right) \right] = \frac{\lambda}{2} + \sum_{i=1}^{n} \sum_{j=1}^{k} Y_{ij} \hat{Y}_{ij}^{(l)}$$

$$w_l \sum_{i=1}^{n} \sum_{j=1}^{k} \hat{Y}_{ij}^{(l)2} + \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\hat{Y}_{ij}^{(l)} \sum_{m \neq l}^{M} w_m \hat{Y}_{ij}^{(m)} \right) = \frac{\lambda}{2} + \sum_{i=1}^{n} \sum_{j=1}^{k} Y_{ij} \hat{Y}_{ij}^{(l)}$$

$$w_l \sum_{i=1}^{n} \sum_{j=1}^{k} \hat{Y}_{ij}^{(l)2} = \left(\frac{\lambda}{2} + \sum_{i=1}^{n} \sum_{j=1}^{k} Y_{ij} \hat{Y}_{ij}^{(l)} \right) - \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\hat{Y}_{ij}^{(l)} \sum_{m \neq l}^{M} w_m \hat{Y}_{ij}^{(m)} \right)$$

$$\Rightarrow w_l^* = \left[\left(\frac{\lambda}{2} + \sum_{i=1}^{n} \sum_{j=1}^{k} Y_{ij} \hat{Y}_{ij}^{(l)} \right) - \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\hat{Y}_{ij}^{(l)} \sum_{m \neq l}^{M} w_m \hat{Y}_{ij}^{(m)} \right) \right] \left(\sum_{i=1}^{n} \sum_{j=1}^{k} \hat{Y}_{ij}^{(l)2} \right)^{-1}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} \Rightarrow \sum_{m=1}^{M} w_m = 1 \Rightarrow \sum_{q=1}^{Q} \left[\left(\frac{\lambda}{2} + \sum_{i=1}^{n} \sum_{j=1}^{k} Y_{ij} \hat{Y}_{ij}^{(q)} \right) - \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\hat{Y}_{ij}^{(q)} \sum_{m \neq q}^{M} w_m \hat{Y}_{ij}^{(m)} \right) \right] \left(\sum_{i=1}^{n} \sum_{j=1}^{k} \hat{Y}_{ij}^{(q)2} \right)^{-1} = 1$$

$$\sum_{q=1}^{Q} \frac{\lambda}{2} \left(\sum_{i=1}^{n} \sum_{j=1}^{k} \hat{Y}_{ij}^{(q)2} \right)^{-1} + \sum_{q=1}^{Q} \left[\left(\sum_{i=1}^{n} \sum_{j=1}^{k} Y_{ij} \hat{Y}_{ij}^{(q)} \right) - \left(\sum_{i=1}^{n} \sum_{j=1}^{k} \hat{Y}_{ij}^{(q)} \sum_{m \neq q}^{M} w_m \hat{Y}_{ij}^{(q)} \right) \right] \left(\sum_{i=1}^{n} \sum_{j=1}^{k} \hat{Y}_{ij}^{(q)2} \right)^{-1} = 1$$

$$\Rightarrow \frac{\lambda}{2} = \left(1 - \sum_{q=1}^{Q} \left[\left(\sum_{i=1}^{n} \sum_{j=1}^{k} Y_{ij} \hat{Y}_{ij}^{(q)} \right) - \left(\sum_{i=1}^{n} \sum_{j=1}^{k} \hat{Y}_{ij}^{(q)} \sum_{m \neq q}^{M} w_m \hat{Y}_{ij}^{(q)} \right) \right] \left(\sum_{i=1}^{n} \sum_{j=1}^{k} \hat{Y}_{ij}^{(q)2} \right)^{-1} \right) \left(\sum_{q=1}^{Q} \left(\sum_{i=1}^{n} \sum_{j=1}^{k} \hat{Y}_{ij}^{(q)} \right)^{-1} \right)^{-1}$$