Appendix E

Mason's Gain Formula

Mason's gain formula is a very convenient tool to obtain transfer functions of complicated structures. It can be applied to any signal flow graph (see below), for example, the coupled form structure reproduced in Fig. E-1(a). We now state this result, and present some examples of its use. Further examples and details can be found in B. C. Kuo [1975]. For orginal work and theoretical developments see S. J. Mason [1953,1956].

Terminology

A signal flow graph is a collection of *nodes* interconnected by *directed branches.* In Fig. E-1(a) the nodes are indicated by circled numbers. Each node is associated with a signal, for example, node 1 is associated with

$$s_1(n) = u(n) - (R\sin\theta)y(n) + (R\cos\theta)s_2(n),$$

where $s_2(n)$ is the signal associated with node 2. Each branch has a gain, for example, the branch from node 1 to node 2 in the above example has gain z^{-1}. The gain is indicated adjacent to the branch. Whenever a branch is unlabeled, its gain is assumed to be unity.

The input and output nodes of the graph are typically labeled as $u(n)$ and $y(n)$ respectively. Notice that the signal associated with node 4 is also the output, i.e., $s_4(n) = y(n)$.

Paths. A path is merely a succession of branches, directed the same way. The *path gain* is the product of individual branch gains. (This explains why the gains are always indicated in terms of z-transforms rather than in time domain). The symbol $P_k(z)$ denotes the gain for the kth path. In Figs. E-1(b),(c) we have indicated some paths of the flow graph of Fig. E-1(a). These paths have gains

$$P_1(z) = -R\sin\theta,\quad P_2(z) = z^{-2}R\sin\theta. \tag{E.1}$$

If a path starts from the input node and ends at the output node *with no node encountered more than once*, it is said to be a *forward path.* In the above example $P_2(z)$ is a forward path. A signal flow graph can have many forward paths.

Deleting a path. The operatin of 'path deletion' is central to Mason's gain formula. Deleting a path $P_k(z)$ means (i) delete all branches in the path, and (ii)

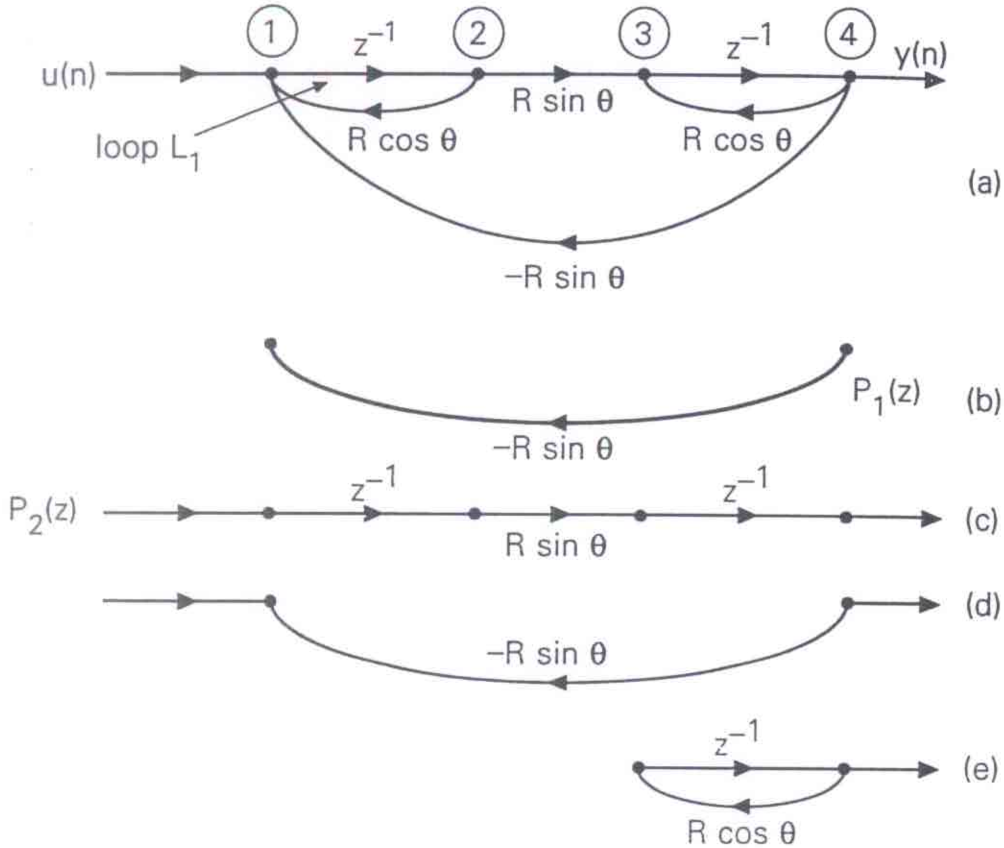

Figure E-1 (a) Signal flow graph of the coupled form structure, (b), (c) two examples of 'paths', (d) graph that remains after deleting the path '$R \sin \theta$.' (e) graph that remains after deleting the path z^{-1} which connects nodes 1 and 2.

delete all branches that touch any node on the path. Figs. E-1(d),(e) show examples of the remaining graph after deleting certain paths from Fig. E-1(a). Note that if we delete the horizontal path which connects nodes 1 and 4 in Fig. E-1(a), the resulting graph is empty!

Loops. A path which starts and ends at the same node with no other node encountered more than once is said to be a *loop*. The product of the branch gains is said to be the *loop gain*. In the above example, the loop labeled L_1 has loop gain $z^{-1} R \cos \theta$. A signal flow graph can have any number of loops. Two loops are said to be *touching* if they share a common node, and *nontouching* if they do not. Figure E-2 shows examples of touching and nontouching loops.

Determinant of a Signal Flow Graph

The determinant of a signal flow graph is defined as

$$\begin{aligned}\Delta(z) = 1 &- \text{sum of all loop gains} \\ &+ \text{sum of products of gains of pairs of nontouching loops} \\ &- \text{sum of products of gains of triples of nontouching loops} \\ &+ \dots\end{aligned} \tag{E.2}$$

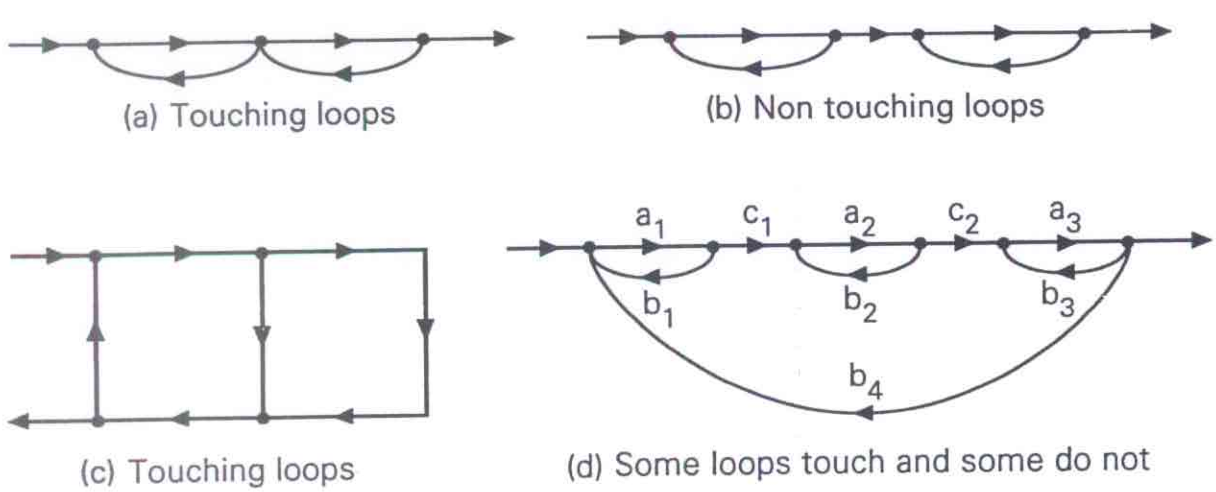

Figure E-2 Examples of touching and nontouching loops. (a) Touching, (b) nontouching, (c) touching, and (d) both types.

To demonstrate this, consider Fig. E-2(d). We have

Loops: $a_1b_1,\ a_2b_2,\ a_3b_3,\ a_1c_1a_2c_2a_3b_4$.

Nontouching pairs: $(a_1b_1, a_2b_2),\ (a_1b_1, a_3b_3),\ (a_2b_2, a_3b_3)$.

Nontouching triples: (a_1b_1, a_2b_2, a_3b_3).

So the determinant is

$$\begin{aligned}\Delta(z) = 1 &- (a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1a_2c_2a_3b_4) \\ &+ (a_1b_1a_2b_2 + a_1b_1a_3b_3 + a_2b_2a_3b_3) \\ &- a_1b_1a_2b_2a_3b_3.\end{aligned} \tag{E.3}$$

Notice that a graph with no loops has $\Delta = 1$. This is true, in particular, for the "empty graph," that is, graph with no nodes.

Mason's formula

Let $H(z)$ denote the transfer function $Y(z)/U(z)$ of a signal flow graph. Define the following notations:

1. $\Delta(z) =$ determinant of the graph.
2. $L =$ number of forward paths, with $P_k(z), 1 \le k \le L$ denoting the forward path gains.
3. $\Delta_k(z) =$ determinant of the graph that remains after deleting the kth forward path $P_k(z)$.

Then the transfer function is

$$H(z) = \frac{\sum_{k=1}^{L} P_k(z)\Delta_k(z)}{\Delta(z)} \qquad \text{(Mason's formula)}. \tag{E.4}$$

Example E.1

Consider the direct form structure of Fig. E-3. We have only one loop, with gain az^{-1}. So the determinant $\Delta(z) = 1 - az^{-1}$. There are two forward paths, with gains $P_1(z) = b_0$ and $P_2(z) = b_1 z^{-1}$. If either of these paths is deleted, the result is the empty graph, so that $\Delta_1(z) = \Delta_2(z) = 1$. Applying (E.4) we obtain

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1}}{1 - az^{-1}}. \qquad (E.5)$$

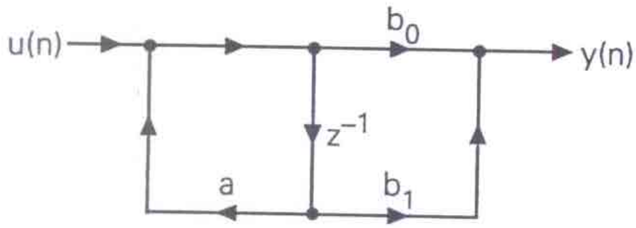

Figure E-3 First order direct form structure.

Example E.2.

Consider the coupled form structure shown in Fig. E-1(a). For this we have:

Loops: $z^{-1}R\cos\theta, z^{-1}R\cos\theta$, and $-z^{-2}R^2\sin^2\theta$.

Nontouching pairs: $z^{-1}R\cos\theta, z^{-1}R\cos\theta$.

There are no nontouching triples and so on. Using (E.2) the determinant of the structure is

$$\Delta(z) = 1 - 2z^{-1}R\cos\theta + z^{-2}R^2. \qquad (E.6)$$

There is only one forward path, with $P_1(z) = z^{-2}R\sin\theta$. If this is deleted, the result is an empty graph, and we obtain $\Delta_1(z) = 1$. Using (E.4) we therefore obtain

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z^{-2}R\sin\theta}{1 - 2z^{-1}R\cos\theta + z^{-2}R^2}. \qquad (E.7)$$

Example E.3.

We now consider the lattice structure of Fig. E-4, which provides an exampe where $\Delta_k(z)$ are nontrivial. Here $\widehat{k}_i$ stands for $\sqrt{(1-k_i^2)}$, and $-1 < k_i < 1$. First let us list the loops:

Loops: $-k_1k_2z^{-1}, -k_1z^{-1}$, and $-k_2\widehat{k}_1^2z^{-2}$.

Nontouching pairs: $(-k_1k_2z^{-1}, -k_1z^{-1})$.

Using (E.2) we then obtain the determinant

$$\Delta(z) = 1 + k_1(k_2+1)z^{-1} + k_2z^{-2}, \quad \text{using } \widehat{k}_i = \sqrt{1-k_i^2}. \qquad (E.8)$$

Next, we have three forward paths. The gains are

$$P_1(z) = k_2, \quad P_2(z) = k_1\widehat{k}_2^2z^{-1}, \quad P_3(z) = \widehat{k}_1^2\widehat{k}_2^2z^{-2}.$$

The graph that remains after deletion of the forward path $P_k(z)$ is shown in Fig. E-5 for $k = 1, 2$. (If $P_3(z)$ is deleted, the remaining graph empty.) The determinants of these 'remaining' graphs are

$$\Delta_1(z) = \Delta(z),\ \Delta_2(z) = 1 + k_1 z^{-1},\ \Delta_3(z) = 1. \tag{E.9}$$

Substituting these into (E.4) we arrive at the transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \frac{k_2 + k_1(k_2+1)z^{-1} + z^{-2}}{1 + k_1(k_2+1)z^{-1} + k_2 z^{-2}}. \tag{E.10}$$

which is an allpass function, since k_i are real.

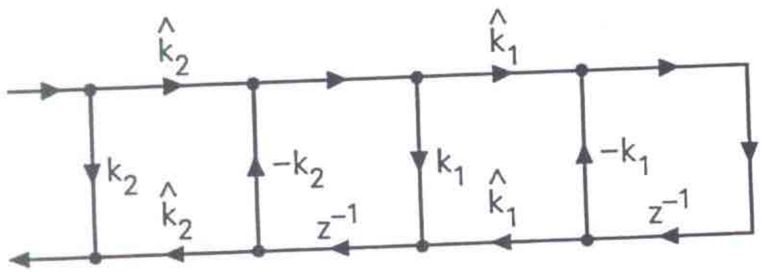

Figure E-4 The lattice structure example.

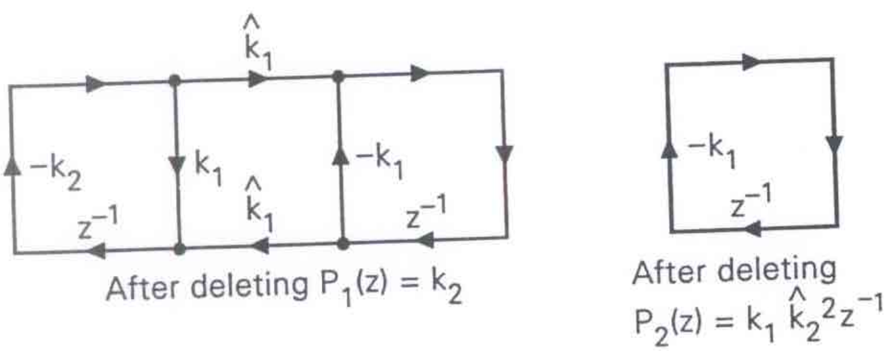

Figure E-5 Graphs that remain after deletion of certain paths from Fig. E-4.