Exam 2. Problem 1.

a) For chloesky factorization, we can start partition the A & L

$$A = \left[\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right] \quad L = \left[\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right]$$

since $A = LL^T$

$$\left[\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right] = \left[\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right] \left[\begin{array}{c|c} L_{00}^T & l_{10} \\ \hline 0 & \lambda_{11} \end{array}\right] = \left[\begin{array}{c|c} L_{00}L_{00}^T & \\ \hline l_{10}^T L_{00}^T & l_{10}^T l_{10} + \lambda_{11}^2 \end{array}\right]$$

We can then solve for L.

$$L = \left[\begin{array}{c|c} L_{00} = \mathrm{CHOL}(A_{00}) & * \\ \hline l_{10}^T = a_{10}^T L_{00}^{-T} & \lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}} \end{array}\right]$$

Then we have a algorithm:

1. Partition $A \longrightarrow \left(\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$
2. Assume $A_{00} = L_{00}$ is computed by previous iteration)
3. Overwrite $a_{10}^T := l_{10}^T = a_{10}^T L_{00}^{-T}$
4. Overwrite $\alpha_{11} := \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

b). Proof by induction:

1. base case $n = 1$

if size of the matrix $= 1$ $A = \alpha_{11}$ and A is SPD so A is real and positive. if we insist λ_{11} to be positive the $\lambda_{11} = \sqrt{\alpha_{11}}$ is unique and ~~definite~~ defined.

2. Induction Step

if result is true for $n=k$, we will show $n=k+1$ also holds.

Let $A = \mathbb{R}^{k+1} \times \mathbb{R}^{k+1}$ $A = \left[\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right]$ $L = \left[\begin{array}{c|c} L_{00}^T & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right]$

$l_{10}^T =$ ~~[illegible]~~ $a_{10}^T L_{00}^{-T}$ since $A_{00} = L_{00}L_{00}^T$ then L_{00} is unique and defined

Since L_{00} is lower triangular matrix.
If we insist the diagonal entries to be positive,
L_{00}^{-1} is unique and defined therefore L^{-T} is unique and defined. So l_{10}^T is unique and defined.

Lastly, if α_{11}, λ_{11} are positive then $\lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$
l_{10}^T is unique then λ_{11} is unique and defined.

L is the desired chloesky factor for A.

3. By the principal of Mathematical induction, the theorem holds.