Exam 2. Problem 2

Base case

when $n=1$, $A=[\alpha_{11}]$ and bordered algorithm will run for one iteration. $\check{L}=[1]$ $\check{U}=[\alpha_{11}]$ as the result.

$$\Delta A=[0] \leq \gamma_1 |\check{L}||\check{U}| \quad \text{since } \gamma_1 \geq 0$$

Induction step

Assume $\quad A_{00}+\Delta A_{00}=\check{L}_{00}\check{U}_{00} \quad$ with $\quad \Delta A_{00} \leq \gamma_n |\check{L}_{00}||\check{U}_{00}|$

then for $A=\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$

$$\begin{bmatrix} \check{L}_{00} & 0 \\ \check{l}_{10}^T & 1 \end{bmatrix}\begin{bmatrix} \check{U}_{00} & \check{u}_{01} \\ 0 & \check{v}_{11} \end{bmatrix} = \begin{bmatrix} \check{L}_{00}\check{U}_{00} & \check{L}_{00}\check{u}_{01} \\ \check{l}_{10}^T\check{U}_{00} & \check{l}_{10}^T\check{u}_{01}+\check{v}_{11} \end{bmatrix}$$

To prove

$$\gamma_{n+1}\left|\begin{bmatrix} \check{L}_{00} & 0 \\ \check{l}_{10}^T & 1 \end{bmatrix}\right|\left|\begin{bmatrix} \check{U}_{00} & \check{u}_{01} \\ 0 & \check{v}_{11} \end{bmatrix}\right| = \begin{bmatrix} \gamma_{n+1}|\check{L}_{00}||\check{U}_{00}| & \gamma_{n+1}|\check{L}_{00}||\check{u}_{01}| \\ \gamma_{n+1}|\check{l}_{10}^T||\check{U}_{00}| & \gamma_{n+1}|\check{l}_{10}^T||\check{v}_{11}| \end{bmatrix}$$

We need to prove the following conclusion.

1. $A_{00}+\Delta A_{00}=\check{L}_{00}\check{U}_{00}$ with $|\Delta A_{00}| \leq \gamma_{n+1}|\check{L}_{00}||\check{U}_{00}|$
   this is proved by the fact that $\gamma_{n+1} \geq \gamma_n$

2. $a_{01}+\delta a_{01}=\check{L}_{00}\check{u}_{01}$ with $|\delta a_{01}| \leq \gamma_{n+1}|\check{L}_{00}||\check{u}_{01}|$
   we first need to prove a lemma:

~~Lemma~~

Lemma 1.

Let $L\in\mathbb{R}^{n\times n}$ be a non-singular lower triangular matrix and $\check{x}$ be the computed result of of $Lx=y$ than there is δy such that:

$$y+\delta y = L\check{x} \quad \text{with } |\delta y| \leq \gamma_n |L||\check{x}|$$

According to Collary 6.4.1.4 of the book.

$$(L+\Delta L)\check{x} = y \text{ where } |\Delta L| \leq \gamma_n |L|$$

let $\delta y = -\Delta L\check{x}$ then we have $y + \delta y = L\check{x}$ with $|\delta y| = |\Delta L|\,|\check{x}|$
$\leq \gamma_n |L|\,|\check{x}|$

Then for conclusion 2. $L_{00} \in \mathbb{R}^{n \times n}$ is a non-singular lower triangular matrix and $\check{u}_{01}$ is the computed result when this bordered algorithm solve $L_{00} u_{01} = a_{01}$ according to lemma 1,

$$a_{01} + \delta a_{01} = \check{L}_{00}\check{u}_{01} \text{ with } |\delta a_{01}| \leq \gamma_n |\check{L}_{00}|\,|\check{u}_{01}| \leq \gamma_{n+1} |\check{L}_{00}|\,|\check{u}_{01}|$$

Conclusion 3

$$a_{10}^T + \delta a_{10}^T = \check{l}_{10}^T \check{u}_{00} \text{ with } |\delta a_{10}^T| \leq \gamma_{n+1} |\check{l}_{10}^T|\,|\check{u}_{00}|$$

as l_{10}^T is the computed result by solving $l_{10}^T u_{00} = a_{10}^T$

Thus l_{10} is the result of $u_{00}^T l_{10} = a_{10}$

since u_{00}^T is non-singular lower triangular matrix

then according to Lemma 1:

$$a_{10} + \delta a_{10} = u_{00}^T \check{l}_{10} \text{ with } |\delta a_{10}| \leq \gamma_n |\check{u}_{00}|\,|\check{l}_{10}|$$

Finally we have:

$$a_{10}^T + \delta a_{10}^T = \check{l}_{10}^T \check{u}_{00} \text{ with } |a_{10}^T| \leq \gamma_n |\check{l}_{10}^T|\,|\check{u}_{00}| \leq \gamma_{n+1} |\check{l}_{10}^T|\,|\check{u}_{00}|$$

Conclusion 4.

$$\alpha_{11} + \delta\alpha_{11} = \check{l}_{10}^T \check{u}_{01} + \check{\upsilon}_{11} \text{ with } |\delta\alpha_{11}| \leq \gamma_{n+1}(|\check{l}_{10}^T|\,|\check{u}_{01}| + |\check{\upsilon}_{11}|)$$

$\check{\upsilon}_{11}$ is the computed result of bordered algorithm step $\upsilon_{11} = \alpha_{11} - l_{10}^T u_{01}$

According to Reference [1] there exist $\delta\alpha_{11}$ such

$$\check{\upsilon}_{11} = \alpha_{11} - \check{l}_{10}^T \check{u}_{01} + \delta\alpha_{11} \text{ with } \delta\alpha_{11} \leq \gamma_n(|\check{l}_{10}^T|\,|\check{u}_{01}| + |\check{\upsilon}_{11}|)$$

Lastly, $\alpha_{11} + \delta\alpha_{11} = \check{l}_{10}^T \check{u}_{01} + \check{\upsilon}_{11}$ with $|\delta\alpha_{11}| \leq \gamma_{n+1}(|\check{l}_{10}^T|\,|\check{u}_{01}| + |\check{\upsilon}_{11}|)$

By proving the 4 conclusion we prove the induce step for our question.

Refarrence :

[1] Paolo Bientinesi, Robert A. van de Geijn, Goal oriented and Modular Stability Analysis. SIAM Journal on Matrix Analysis and Applications, Volume 32 Issue 1, Feb 2011