Modelling and Simulation: Theory and Examples

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1. Random Process

- It is better to think of a random process in terms of a function rather than a variable.
- In a random function, say, `random(x)`, x is the sample space of random outcomes that are possible after performing a certain stochastic process such as tossing a fair coin.
- Let's say we are tossing a pair of coins. The outcome does not have to be a number. It can be heads or tails. However, the number of possible outcomes lies in a certain range, say, `range(x)`.

$$\begin{align} random(x) &= \begin{cases} Coin 1 &= [H, H, T, T]\\ Coin 2 &= [H, T, H, T] \end{cases} \\ range(x) &= [1, 2, 3, 4] \end{align}$$

- x outputs an outcome from `range(x)` using a probability distribution that represents a likelihood of occurrences of events within the sample space.
- In the above coin-tossing experiment, the sum of the probabilities of all outcomes should add up to 1 because a coin toss will always yield some output.

$P_{\text{1 or 2 or 3 or 4}} = P_1 + P_2 + P_3 + P_4 = 1$

- x can be discrete or continuous.

1.1. Discrete random variable

- A **discrete random variable** can take a distict or separate values.
- Discrete random variable is **countably infinite**
- A coin toss

$$x = \begin{cases} Head &= [1]\\ Tail &= [0] \end{cases}$$

- The year that a random student in a school was born
- The number of mosquitos born yesterday on earth
- Winning time of random athrets to the nearest 2 decimals
- The number of cars passing at a certain junction at a certain time
- A discrete random variable can be countable finite or approach infinite values. At least you can list a few specific values.
- The probability distribution of a discrete random variable is called **Probability Mass Function (PMF)**

1.2. Continuous random variable

- A **continuous random variable** can take any value in an interval.
- Continuous random variable is **uncountably infinite**

$$x = \begin{cases} \textbf{Exact mass}\text{ of random organisms on earth} \end{cases}$$

- Even if you could find a range of this mass, the exact mass of a certain organism at a given time could have several decimal points…
- Precisely exact winning time of random athrets
- Temperature of a random place in a year measured at a high precision
- **A real-world example of a true case of a continuous random variable is rare.**

- A continuous random variable can be uncountable infinite values. You cannot list a few specific values.
- The probability distribution of a continuous random variable is called **Probability Density Function (PDF)**
- Each variable can take on a different value from a probability distribution.
- A **random process** can be discrete or continuous depending on whether its member variables are discrete or continuous.

1.3. Gambler's Ruin

> A gambler has KES 500. They can only bet in increments of KES 100. They can only win or lose KES 100 per bet. They will keep gambling until they either lost all their money (KES 0) or win KES 1000. Simulate this gambling situation.

- *Click to reveal/hide the solution.*

\[\text{gambling outcome, } x = \begin{cases} \text{Win KES 100 (P = 50%)} &= [+100] \\ \text{Lose KES 100 (P = 50%)} &= [-100] \end{cases}\]

```python
import numpy as np
import random
import matplotlib.pyplot as plt

def gamble():
    start_money = 500
    win_money = 1000
    bet_size = 100
    win_lose = 100
    gambling_range = range(win_lose,win_money)
    simul_gamble = []
    while start_money in gambling_range:
        win_lose = bet_size*random.randrange(-1, 2, step=2)
        start_money += win_lose
        simul_gamble.append(start_money)
    return simul_gamble

plt.plot(gamble(), marker = 'o')
```

2. Makov Process

The future evolution of the process depends only on the instant observation of the system and not on the past.

2.1. A 2 state markov chain

> Assume you have 2 shirts, white and blue. Their state space is $S = \{W,B\}$.
>
> - The probability of you wearing a white shirt and continue wearing the white shirt is 0.7.
> - The probability of changing the white shirt to a blue shirt is 0.3.
> - Once you're wearing a blue shirt, the probability of you continue wearing the blue shirt is 0.4.
> - The probability of changing the blue shirt to the white shirt is 0.6.

- *Click to reveal/hide the solution.*

$$\text{2 Shirts} = \begin{cases} P(W|W) &= 0.7\\ P(B|W) &= 0.3\\ P(W|B) &= 0.6\\ P(B|B) &= 0.4 \end{cases}$$

- Suppose you repeat this for several days. Let's say today is Monday and you've already decided that you'll wear a white shirt. $S = \{1,0\}$
- What is the probability of you wearing a white or blue shirt on Tuesday through Friday?

(i) At the end of Monday: Mon $= S \times \text{trans\_matrix}$
(ii) At the end of Tuesday: Tue $= Mon \times \text{trans\_matrix}$
(iii) At the end of Wednesday: Wed $= Tue \times \text{trans\_matrix}$

```python
nodes = ['White', 'Blue']
trans_matrix = np.array([[0.7,0.3],[0.6,0.4]])
pd.DataFrame(trans_matrix, columns=nodes, index=nodes)
```

```python
S = np.array([[1,0]])
Mon = np.matmul(S,trans_matrix)
Mon
```

```python
Tue = np.matmul(Mon,trans_matrix)
Tue
```

- If you are not decided, you can set $S = [0.5, 0.5]$ or introduce randomness.

2.2. A 3 state markov chain

There are 3 states:
- Home
- Bar
- Back Home

- You only have one bar you go to if you want to go out.
- From state **Home** you can only go out. The first assumption is that you have to go out to be in a bar.
- From the **Bar**, you can stay in the bar or go back home.
- For each time step, you have a 50% probability of going back home and a 50% probability of staying.
- When you are **Back Home**, the only thing that you could do is stay home. It means you won't go out again.

- *Click to reveal/hide the solution.*

```python
import numpy as np
import pandas as pd

nodes = ['Home', 'Bar', 'Back Home']
trans_matrix = np.array([[0,1,0],[0,0.5,0.5],[0,0,1]])
pd.DataFrame(trans_matrix, columns=nodes, index=nodes)

# You're in the bar. Where can you go next?
# All nodes: np.arange(0,len(nodes),1)
# Bar transitions: trans_matrix[1]
print(np.random.choice(np.arange(0,len(nodes),1), p=trans_matrix[1]))
```

3. Differentiation and Integration

3.1. Using Python to find derivatives and integrals .

Use the `Python` `sympy` package to solve the following

(a) $\frac{dy}{dx} = x^2$
(b) $\frac{dy}{dx} = cos(x)$
(c) $\frac{dy}{dx} = e^{x^2}$
(d) $\frac{d^3y}{dx^3} = x^4$
(e) $\int cos(x)$
(f) $\int_0^\infty e^{-x}$
(g) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{- x^{2} - y^{2}}\, dx\, dy$

Use the `Python` `scipy` package to solve and plot the following

(a) $\frac{dy(t)}{dt} = -k \ y(t)$
$k = 0.3; y_0 = 5$
(b) $\frac{dy(t)}{dt} = ke^t + y$

- *Click to reveal/hide the solution.*

3.2. Spread of virus using the SIR Model.

A new infectious flu virus is discovered to have infected 10% of a community. The rate of infection is 0.35 and that of recovery is 0.1. Given that no one has recovered so far, use the SIR Model to simulatet the spread of this virus.

- *Click to reveal/hide the solution.*

4. Monte Carlo Simulation

Sometimes it is difficult to obtain an analytical solution of a problem. Monte carlo methods rely on repeated random sampling to obtain numerial results. Randomness may be used to solve some deterministic problems.

4.1. Calculate the aproximate value of π.

> Use the monte carlo method to evaluate
>
> (a) the approximate value of π.
> (b) the approximate area of a triangle.
> (c) $\int_{-3}^{3} e^{x^2}$
> (d) $\int_0^{15} \Big(\big(7x^3 + 20x^2 + 45x + 5 \big)^{\frac{1}{3}} \Big) e^{-\frac{1}{5}x} dx$

- *Click to reveal/hide the solution.*