PROBLEM SET 16 SOLUTIONS

by Michael Allen

(1) Check that $(1,1,0)$, $(0,1,1)$, and $(1,0,1)$ form a basis for $\mathbb{R}^3$. Transform this basis into an orthonormal basis using the Gram-Schmidt algorithm. Check that the resulting vectors are indeed orthogonal!

Answer: The three vectors will form a basis for $\mathbb{R}^3$ iff they are linearly independent. To determine this, we can simply place them into the columns of a matrix, and reduce it to find the pivots.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since there are three pivots, the vectors are linearly independent and form a basis for $\mathbb{R}^3$.

Now, we can use Gram-Schmidt to find an orthogonal basis:

$$w1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$w2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$w3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

And then normalize:

$$w1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$w2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$w3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

(2) Check that the vectors $(1,0,0,0)$, $(1,1,0,0)$, $(1,1,1,0)$ and $(1,1,1,1)$ form a basis of $\mathbb{R}^4$. Use the Gram-Schmidt algorithm to make this into an orthonormal basis.

Answer: As in the last problem, we can determine if the vectors form a basis by putting the vectors into the columns of a matrix and counting the pivots.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There are four pivots, so everything is okay. Now, we use Gram-Schmidt on them (or just do it by inspection), and we find that the orthonormal basis is:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(3) Consider the orthonormal vectors

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \quad \& \quad v_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

in $\mathbb{R}^3$. Find some other vector

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

such that v_1, v_2, and b are a basis of $\mathbb{R}^3$. Then use the Gram-Schmidt algorithm to make your basis into an orthogonal one.

Answer: As long as we choose a vector which is linearly independent of the two given, we will have a valid basis, so there are many possible answers. So, let us choose $b = (1,0,0)$.

Then, when we perform Gram-Schmidt on the basis, we will get the following:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(4) Check that the matrix

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

is an orthogonal matrix by checking that $Q \cdot Q^T = I$. Also, check that $||Q \cdot v|| = ||v||$ for the vector $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Answer:

$$Q \cdot Q^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$||Q \cdot v|| = \| \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \| = \| \begin{bmatrix} 2\cos\theta - \sin\theta \\ 2\sin\theta + \cos\theta \end{bmatrix} \|$$

$$= \sqrt{5\sin^2\theta + 5\cos^2\theta} = \sqrt{5} = \sqrt{2^2 + 1^2} = \|v\|$$

(5) We have the following theorem.

Theorem 1. *If $\{v_1, \dots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then for any $v \in \mathbb{R}^n$, we can write*

$$v = c_1 v_1 + \cdots c_n v_n,$$

where $c_i = v \cdot v_i$ $(1 \leq i \leq n)$, where $\cdot$ is the dot product.

(a) Use this theorem to write the vector $(3, 2)$ as linear combinations of the vectors

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \& \quad \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Answer:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{5}{\sqrt{2}}$$

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{-1}{\sqrt{2}}$$

$$\frac{5}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{-1}{\sqrt{2}} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(b) Use this theorem to write $(1, 2, -1)$ in terms of the basis

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Answer: Since the basis is not orthonormal, the theorem does not apply to this problem, and we are stuck.

(6) Consider the two bases of $\mathbb{R}^3$,

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and

$$\hat{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The change of basis matrix $M_{\hat{B}}^{B}$ is

$$M_{\hat{B}}^{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Compute $M_{B}^{\hat{B}}$.

Answer: $M_{B}^{\hat{B}}$ is just the inverse of $M_{\hat{B}}^{B}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

(7) Now consider the two bases

$$\hat{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \And \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\},$$

and

$$\tilde{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \And \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

Compute the matrices $M_{\tilde{B}}^{\hat{B}}$ and $M_{\hat{B}}^{\tilde{B}}$.

Answer:

$$M_{\tilde{B}}^{\hat{B}} = \tilde{B}\hat{B}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 4 \\ -1 & 3 & -3 \\ 2 & -3 & 1 \end{bmatrix}$$

$$M_{\hat{B}}^{\tilde{B}} = M_{\tilde{B}}^{\hat{B}-1} = \frac{1}{4} \begin{bmatrix} 6 & 8 & 6 \\ 5 & 6 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$