PROBLEM SET 5. DUE TUESDAY, 12 SEPTEMBER

Reading. *Quick Calculus*, pp. 138–142; 148–167.
Supplementary reading. Simmons, Sections 5.1–5.3.

(1) **(2pts) Approximate the following numbers, using the tangent line approximation.**
   (a) $\sqrt[3]{28}$
   $(x+\delta x)^{\frac{1}{3}} \approx x^{\frac{1}{3}} + (\delta x)(\frac{1}{3}x^{-\frac{2}{3}})$
   Substituting $x = 27$ and $\delta x = 1$, we get:
$$28^{\frac{1}{3}} \approx 27^{\frac{1}{3}} + (1)(\frac{1}{3}27^{-\frac{2}{3}}) \approx 3.037$$
   (b) $\sqrt{102}$ $(x+\delta x)^{\frac{1}{2}} \approx x^{\frac{1}{2}} + (\delta x)(\frac{1}{2}x^{-\frac{1}{2}})$
   Substituting $x = 100$ and $\delta x = 2$, we get:
$$102^{\frac{1}{2}} \approx 100^{\frac{1}{2}} + (2)(\frac{1}{2}100^{-\frac{1}{2}}) = 10.1$$

(2) (2pts) **Find the Taylor series (at $x = 0$) for $f(x) = \frac{1}{1-x}$.**

The n'th derivative of $f(x) = \frac{1}{1-x} = (1-x)^{-1}$ is $-1^n n!(1-x)^{-n}$. The n!'s in the derivatives cancel the n!'s in the denominators of the terms of the Taylor series, and at $x = 0$, $(1-x)^{-n} = 1$ for all n. Therefore, our Taylor series is given by:

$$\begin{aligned} f(x) &\approx 1 - x + x^2 - x^3 + x^4 + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \end{aligned}$$

(3) (4pts) **A sphere of radius r has volume**
$$V(r) = \frac{4}{3}\pi r^3,$$
and surface area
$$A(r) = 4\pi r^2.$$
Approximate the volume and surface area of a sphere of radius 7.02cm. You can check your answer by using a calculator to compute the volume and surface area exactly.

Using linear approximations, we have:
$$V(r+\delta r) \approx \frac{4}{3}\pi r^3 + (\delta r)(4\pi r^2), \quad A(r+\delta r) \approx 4\pi r^2 + (\delta r)(8\pi r)$$
Substituting $r = 7$ and $\delta r = .02$:

$$V(7.02) \approx \frac{4}{3}(\pi)(7^3) + .02(4\pi)(7^2) \approx 1449\ cm^3$$
$$A(7.02) \approx 4\pi(7^2) + .02(8\pi)(7) \approx 619.3\ cm^2$$

In contrast, the volume and area computed directly via calculator are approximately 1449 cm^3 and 619.2 cm^2 respectively. These approximations are quite good.

(4) **(4pts) Compute the following integrals.**

(a) $\int x^3\ dx$

$$\int x^3\ dx = \frac{1}{4}x^4 + C$$

(b) $\int \sin(x)\ dx$

$$\int \sin(x)\ dx = \cos(x) + C$$

(c) $\int e^x\ dx$

$$\int e(x)\ dx = e(x) + C$$

(d) $\int \sqrt{x}\ dx$

$$\int \sqrt{x}\ dx = \int x^{\frac{1}{2}} = \frac{2}{3}x^{\frac{3}{2}}$$

(5) **(4pts) Compute the following integrals by substitution, using the substitution given.**

(a) $\int \sqrt{5+7x}\ dx,\ u = 5+7x$

$u = 5+7x \rightarrow du = 7dx \rightarrow dx = \frac{1}{7}du$

$$\begin{aligned} \int \sqrt{5+7x}\ dx &= \frac{1}{7}\int \sqrt{u}\ du \\ &= \frac{1}{7}(\frac{2}{3}u^{\frac{3}{2}}) + C \\ &= \frac{3}{14}(5+7x)^{\frac{3}{2}} + C \end{aligned}$$

(b) $\int \frac{2x}{\sqrt{3+x^2}}\ dx,\ u = \sqrt{3+x^2}$

$u = \sqrt{3+x^2} \rightarrow du = \frac{1}{2}(3+x^2)^{-\frac{1}{2}}(2x)dx$

$$\begin{aligned} \int \frac{2x}{\sqrt{3+x^2}}\ dx &= \int du \\ &= u + C \\ &= \sqrt{3+x^2} + C \end{aligned}$$

(c) $\int 2xe^{x^2}\ dx,\ u = x^2$
$u = x^2 \to du = 2x\ dx$

$$\begin{aligned} \int 2xe^{x^2}\ dx &= \int e^u\ du \\ &= e^u + C \\ &= e^{x^2} + C \end{aligned}$$

(d) $\int \frac{dx}{(x-4)^5},\ u = x - 4$
$u = x - 4 \to du = dx$

$$\begin{aligned} \int \frac{dx}{(x-4)^5} &= \int \frac{du}{u^5} \\ &= u^{-5}\ du \\ &= -\frac{1}{4}u^{-4} + C \\ &= -\frac{1}{4}(x-4)^{-4} + C \end{aligned}$$

(6) (4pts) **Integrals satisfy**

$$\int (f(x) + g(x))\ dx = (\int f(x)\ dx) + (\int g(x)\ dx),$$

just like derivatives do. Again like derivatives, they do not satisfy a simple product rule:

$$\int (f(x) \cdot g(x))\ dx \neq (\int f(x)\ dx) \cdot (\int g(x)\ dx),$$

Check that this is indeed not true by using $f(x) = x$ and $g(x) = x$, and computing both sides of the above equation.

For simplicity, we take as 0 all arbitrary constants that arise from indefinite integration.

$$
\begin{aligned}
\int (f(x) \cdot g(x)) \; dx &= \int (x \cdot x) \; dx \\
&= \int x^2 \; dx \\
&= \frac{1}{3} x^3 \\
&\neq x^4 \\
&= (x^2) \cdot (x^2) \\
&= (\int x \; dx) \cdot (\int x \; dx) \\
&= (\int f(x) \; dx) \cdot (\int g(x) \; dx)
\end{aligned}
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