Problem Set 4 Solutions

1. (a)
$$y(x) = [\cos(x)\sin(x)]^5 = \frac{1}{32}[\sin(2x)]^5$$
$$y'(x) = 5[\cos(x)\sin(x)]^4(\cos^2(x) - \sin^2(x)) = \frac{5}{16}\sin^4(2x)\cos(2x)$$

   (b)
$$y'(x) = \frac{1}{\ln(5)}\left[\frac{4x}{2x^2-6} + \frac{1}{x+7}\right]$$

   (c)
$$y'(x) = -8x\tan(4x^2)$$

   (d)
$$y'(x) = e^{\tan(x)}\sec^2(x)$$

   (e)
$$y'(x) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$$

2. (a)
$$y = x^3 + x^2 + 5x + 4$$
$$y' = 3x^2 + 2x + 5, \text{ which doesn't vanish for any real } x$$
$$y'' = 6x + 2, \text{ which vanishes at } x = -\frac{1}{3}$$

The graph has no minima, since y' doesn't vanish, and it is always increasing. for $x < -\frac{1}{3}$, the graph curves down (it's concave), and for $x > -\frac{1}{3}$ it curves up (it's convex).

   (b)
$$y = e^{x^2}$$
$$y' = 2xe^{x^2}$$
$$y'' = (4x^2 + 2)e^{x^2}$$

The derivative vanishes just at $x = 0$. The value of y'' is greater than 0 everywhere so the graph curves up (it's convex).

(c)

$$y = \frac{x-3}{x^3-3x^2-9x+27} = \frac{x-3}{(x-3)(x^2-9)} = \frac{1}{x^2-9} = \frac{1}{6}(\frac{1}{x-3} - \frac{1}{x+3})$$
$$y' = -\frac{2x}{(x^2-9)^2} = \frac{1}{6}(-\frac{1}{(x-3)^2} + \frac{1}{(x+3)^2})$$
$$y'' = \frac{1}{3}(\frac{1}{(x-3)^3} - \frac{1}{(x+3)^3})$$

The derivative vanishes only at $x = 0$. The second derivative is negative at $x = 0$, so there is a local maximum there. The function is increasing for negative x and decreasing for positive x. There are vertical asymptotes at $x = \pm 3$. The function tends to 0 as x goes to $\pm\infty$.

3. Let x denote the number of dollars the price is reduced. The new sale price is $16 - x$ dollars and the profit per book is $10 - x$ dollars. The total number of books sold is estimated to be $180 + 30x$ so the total profit is

$$\text{Profit } = (180 + 30x)(10 - x).$$

The derivative is

$$\text{Profit}' = -(180 + 30x) + (10 - x)30$$

and it vanishes at $x = 2$. The optimal price of the book is therefore

$$\text{Best Price } = \$14$$

(if x didn't work out to be a whole number we would have to round up or down, depending on which gave the most profit)

4. If the Height of the box is x centimeters, then the base of the box will have dimensions $(10-2x)$ by $(20-2x)$. So the total volume is,

$$\text{Volume}(x) = x(10-2x)(20-2x) = 4(x^3 - 15x^2 + 50x)$$

Differentiating,

$$\text{Volume}'(x) = 4(3x^2 - 30x + 50)$$

Using the quadratic formula, we find this vanishes when

$$x = 5 \pm \frac{5}{3}\sqrt{3}$$

Since $10-2x$ has to be positive, x must be less than 5, so x must be $5 - \frac{5}{3}\sqrt{3}$ which is approximately 2.11. Plugging in to the volume formula, we get

$$\text{Max. Volume } = 192.45 \text{ cm}^3$$

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