PROBLEM SET 6. DUE WEDNESDAY, 13 SEPTEMBER

Reading. *Quick Calculus*, pp. 151–157; 171–190.
Supplementary reading. Simmons, Chapter 6.

(1) Compute the following integrals by substitution.

(a) $\int (1+\frac{1}{x})^2 \frac{1}{x^2}\ dx$
Substituting $u = 1+\frac{1}{x} = 1 + x^{-1},\ du = -x^{-2}$:

$$\int (1+\frac{1}{x})^2 \frac{1}{x^2}\ dx = -\int u^2\ du$$
$$= -\frac{u^3}{3} + C$$
$$= -\frac{(1+\frac{1}{x})^3}{3} + C$$

(b) $\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx$
Substituting $u = \cos(x),\ du = -\sin(x)\ dx$:

$$\begin{aligned}\int \tan(x)\ dx &= \int \frac{\sin(x)}{\cos(x)}\ dx \\ &= -\int \frac{du}{u} \\ &= -\ln u + C \\ &= -\ln\cos(x) + C\end{aligned}$$

(c) $\int \cos(x)\cos(\sin(x))\ dx$
Substituting $u = \sin(x),\ du = \cos(x)\ dx$:

$$\begin{aligned}\int \cos(x)\cos(\sin(x))\ dx &= \int \cos(u)\ du \\ &= \sin(u) + C \\ &= \sin(\sin(x)) + C\end{aligned}$$

(2) Compute the following two trigonometric integrals.

(a) Remember that $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Use this to compute

$$\int \sin^2(x)\ dx.$$

$$\begin{aligned}
\int \sin^2(x)\ dx &= \int (\frac{1}{2} - \frac{1}{2}\cos(2x)) \\
&= \int \frac{1}{2}\ dx - \int \frac{1}{2}\cos(2x)\ dx \\
&= \frac{x}{2} - \frac{1}{4}\sin(2x) + C
\end{aligned}$$

(b) Now use the fact that $\cos^2(x) = 1 - \sin^2(x)$ to compute

$$\int \cos^2(x)\ dx.$$

$$\begin{aligned}
\int \cos^2(x)\ dx &= \int 1 - \sin^2(x)\ dx \\
&= \int\ dx - \int \sin^2(x)\ dx \\
&= x - \frac{x}{2} - \frac{1}{4}\sin(2x) + C \\
&= \frac{x}{2} - \frac{1}{4}\sin(2x) + C
\end{aligned}$$

(3) **Use right-hand Riemann sums (in this case, the same as upper Riemann sums) to show that the area under the graph of $y = x^3$ from $x = 0$ to $x = b$ is $\frac{b^4}{4}$.**

We divide the interval $[0, b]$ into n intervals, with endpoints $0, \frac{b}{n}, \frac{2b}{n}, \ldots, \frac{nb}{n} = b$. For i between 1 and n, the i'th rectangle has width $\frac{b}{n}$, and height $(\frac{ib}{n})^3$. Therefore, the total area under the n rectangles is:

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} \frac{b}{n} \cdot (\frac{ib}{n})^3 \\
&= \frac{b^4}{n^4} \sum_{i=1}^{n} i^3 \\
&= \frac{b^4}{n^4}(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4})
\end{aligned}$$

Taking the limit as n tends to infinity, the $\frac{n^3}{2}$ and $\frac{n^2}{4}$ terms vanish, as they are dominated by the n^4 in the denominator, and we are left with $\frac{b^4}{4}$.

(4) **Each of the following functions has one arch above the x-axis. Find the area of the region under that arch.**

(a) $f(x) = 9 - x^2$

The limits of integration are 3 and -3 (see figure). Therefore, the area is:

$$\int_{-3}^{3} (9 - x^2)\ dx = 9x - \frac{x^3}{3}]_{-3}^{3}$$
$$= (27 - 9) - (-27 + 9)$$
$$= 72$$

(b) $f(x) = x^3 - 9x$
The limits of integration are -3 and 0 (see figure). Therefore, the area is:

$$\int_{-3}^{0} (x^3 - 9x)\ dx = \frac{x^4}{4} - \frac{9}{2}x^2]_{-3}^{0}$$
$$= 0 - (\frac{81}{4} - \frac{81}{2})$$
$$= \frac{81}{4}$$

(c) $f(x) = 4x - x^3$
The limits of integration are 0 and 2 (see figure). Therefore, the area is:

$$\int_{0}^{2} (4x - x^3)\ dx = 2x^2 - \frac{x^4}{4}]_{0}^{2}$$
$$= (8 - 4) - 0$$
$$= 4$$

$f(x) = 9 - x^2$ $\qquad$ $f(x) = x^3 - 9x$ $\qquad$ $f(x) = 4x - x^3$

(5) Evaluate the following definite integrals using the Fundamental Theorem of Calculus. (This is their *algebraic area*.)

(a) $\int_0^{2\pi} \sin(x)\ dx$

$$\int_{0}^{2\pi} \sin(x)\ dx = -\cos(x)]_{0}^{2\pi}$$
$$= -1 - (-1)$$
$$= 0$$

(b) $\int_{-3}^{2} x^4 + 2x^3 - 5x^2 - 6x\ dx$

$$\begin{aligned}\int_{-3}^{2} (x^4 + 2x^3 - 5x^2 - 6x)\ dx &= \frac{x^5}{5} + \frac{x^4}{2} - \frac{5}{3}x^3 - 3x^2]_{-3}^{2} \\ &= (\frac{32}{5} + 4 - \frac{40}{3} - 12) - (-\frac{243}{5} + \frac{81}{4} + 45 - 27) \\ &= -26 - \frac{32}{5} - \frac{40}{3} + \frac{243}{5} - \frac{81}{4} \\ &\approx -17.3833\end{aligned}$$

(c) $\int_{0}^{\frac{3\pi}{2}} \cos(x)\ dx$

$$\begin{aligned}\int_{0}^{\frac{3\pi}{2}} \cos(x)\ dx &= \sin(x)]_{0}^{\frac{3\pi}{2}} \\ &= -1 - 0 \\ &= -1\end{aligned}$$

(6) Compute the geometric area of the following functions on the corresponding intervals. These are the same functions and intervals as in the previous problem. Note the difference between geometric and algebraic area!

In all cases, we take the area above the x-axis, and subtract the area below it.

(a) $f(x) = \sin(x)$ **on** $[0, 2\pi]$

$$\begin{aligned}A = \int_{0}^{\pi} \sin(x)\ dx - \int_{\pi}^{2\pi} \sin(x)\ dx &= -\cos(x)]_{0}^{\pi} - (-\cos(x))]\pi^{2\pi} \\ &= (0 - (-1)) - (-1 - 0) \\ &= 2\end{aligned}$$

(b) $f(x) = x^4 + 2x^3 - 5x^2 - 6x = x(x-2)(x+1)(x+3)$ on $[-3, 2]$

$$A = -\int_{-3}^{-1} f(x)\ dx - \int_{-1}^{0} f(x)\ dx + \int_{0}^{2} f(x)\ dx$$

In this case, we can save work by taking the negative of the algebraic area (computed in the previous problem), and readding twice the piece between -1 and 0, whose area is:

$$\begin{aligned}
\int_{-1}^{0}(x^4+2x^3-5x^2-6x)\ dx &= \frac{x^5}{5}+\frac{x^4}{2}-\frac{5}{3}x^3-3x^2]_{-1}^{0} \\
&= 0-(-\frac{1}{5}+\frac{1}{2}+\frac{5}{3}-3) \\
&\approx .6333
\end{aligned}$$

Therefore, the total area is approximately 18.65.

(c) $f(x)=\cos(x)$ on $[0,\frac{3\pi}{2}]$

$$\begin{aligned}
A=\int_{0}^{\frac{pi}{2}}\cos(x)\ dx-\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos(x)\ dx &= \sin(x)]_{0}^{\frac{pi}{2}}-\sin(x)]_{\frac{\pi}{2}}^{\frac{3pi}{2}} \\
&= (1-0)-(-1-1) \\
&= 3
\end{aligned}$$

$f(x)=\sin(x)$ $\qquad$ $f(x)=x^4+2x^3-5x^2-6x$ $\qquad$ $\cos(x)$