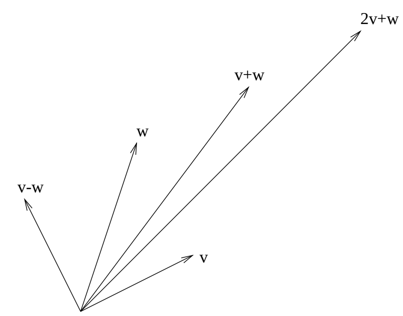
PROBLEM SET 10 SOLUTIONS.

Reading. *Matrices and Transformations*, pp. 1–12.
Supplementary reading. Strang, Chapter 1 and Section 2.1.

1. Draw $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, along with $v + w$, $2v + w$, and $v - w$ in one xy-plane.



2. The vectors $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $w = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ are perpendicular. Thus, v, w and $v + w$ form a right triangle. Check the Pythagorean Theorem, $||v||^2 + ||w||^2 = ||(v + w)||^2$ in terms of the definition of length $||v||$.

$$\begin{aligned} ||u||^2 = 1 \cdot 1 + 2 \cdot 2 &= 5 \\ ||v||^2 = 4 \cdot 4 + (-2) \cdot (-2) &= 20 \\ ||u + v||^2 = 5 \cdot 5 + 0 \cdot 0 &= 25 \end{aligned}$$

3. To save space, I will write column vectors as rows. For $u = (0, 1, 2)$, $v = (1, 3, 0)$, and $w = (1, 0, 4)$, find $||u||$, $||v||$, $||w||$, $u \cdot v$, $u \cdot w$, and $v \cdot w$. Check the Law of Cosines for u and v, as well as the Schwartz inequality for v and w.

$$\begin{aligned} ||u|| &= \sqrt{0 \cdot 0 + 1 \cdot 1 + 2 \cdot 2} = \sqrt{5} \\ ||v|| &= \sqrt{1 \cdot 1 + 3 \cdot 3 + 0 \cdot 0} = \sqrt{10} \\ ||w|| &= \sqrt{1 \cdot 1 + 0 \cdot 0 + 4 \cdot 4} = \sqrt{17} \\ u \cdot v &= 0 \cdot 1 + 1 \cdot 3 + 2 \cdot 0 = 3 \\ u \cdot w &= 0 \cdot 1 + 1 \cdot 0 + 2 \cdot 4 = 8 \\ v \cdot w &= 1 \cdot 1 + 3 \cdot 0 + 0 \cdot 4 = 1 \end{aligned}$$

Law of Cosines for u and v:

$$\begin{aligned} ||u-v||^2 &= ||u||^2 + ||v||^2 - 2u \cdot v \\ &= 5 + 10 - 2 \cdot 3 = 9 \end{aligned}$$

$$\begin{aligned} u - v &= (-1, -2, 2) \\ ||u-v|| &= (-1)\cdot(-1) + (-2)\cdot(-2) + 2 \cdot 2 = 3 \\ ||u-v||^2 = 9 & \end{aligned}$$

Schwartz inequality for v and w:

$$v \cdot w = 1 \leq \sqrt{170} = \sqrt{10} \cdot \sqrt{17} = ||v||||w||$$

4. Solve the systems of equations

$$\begin{cases} 2x + y = 5 \\ x - 3y = 6 \end{cases} \qquad \begin{cases} x + y - z = 2 \\ x - y + 2z = 1 \\ y + 4z = 0 \end{cases}$$

The first system is solved by $x = 3, y = -1$. The second system is solved by $x = \frac{17}{11}, y = \frac{4}{11}, z = -\frac{1}{11}$.

5. Let A, B, C, D, E and F be the matrices below. Find $B + D$, $2E - F$, AC, BC, CB, ACD, EF, FE and CEF. In particular, note that $EF \neq FE$!

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \qquad E = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

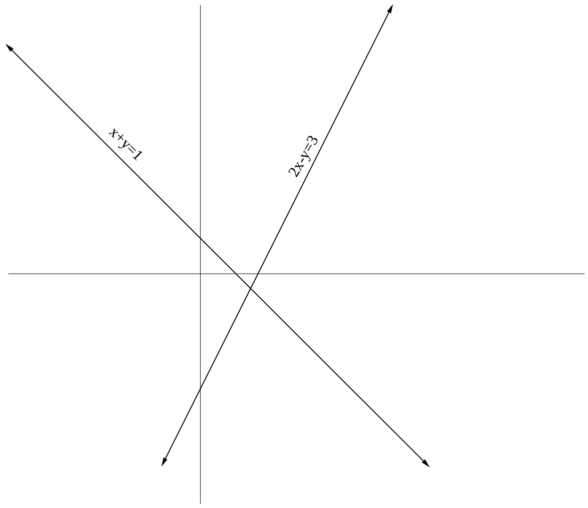
$$B + D = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 0 & 3 \end{bmatrix}, \quad 2E - F = \begin{bmatrix} 3 & 8 \\ 4 & 3 \end{bmatrix}, \quad AC = \begin{bmatrix} 14 & -1 \\ 8 & -1 \\ -1 & -1 \end{bmatrix}$$

$$BC = \begin{bmatrix} -1 & 4 \\ 10 & -5 \end{bmatrix}, \quad CB = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & 0 \\ 3 & 7 & -10 \end{bmatrix}, \quad ACD = \begin{bmatrix} -2 & 27 & 15 \\ -2 & 15 & 9 \\ -2 & -3 & 0 \end{bmatrix}$$

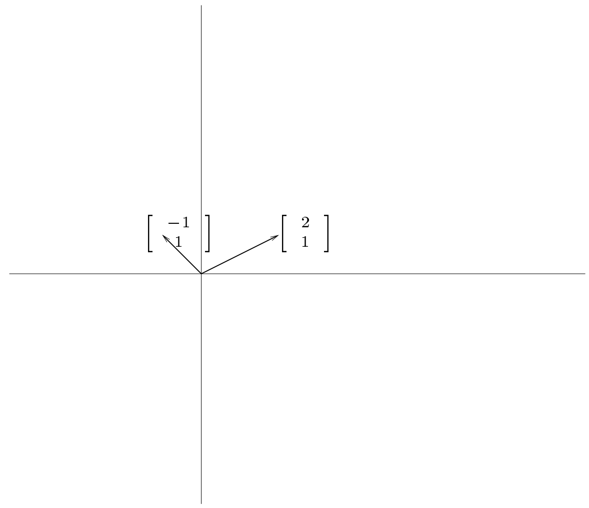
$$EF = \begin{bmatrix} 10 & -4 \\ 5 & -1 \end{bmatrix}, \quad FE = \begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}, \quad CEF = \begin{bmatrix} 10 & -4 \\ 25 & -9 \\ 25 & -11 \end{bmatrix}$$

6. Draw the row and column pictures for

$$\begin{aligned} 2x - y &= 3 \\ x + y &= 1 \end{aligned}$$



The Row Picture



The Column Picture

7. **If you have** 5 **linear equations in** 3 **unknowns, then the row picture shows five** planes. **The column picture is in what dimensional space?** 5 **The equations will have a solution only if the vector on the right hand side is a combination of what?** The columns of the matrix.

8. **Consider the matrix**
$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}.$$
Compute A^2**,** A^3 **and** A^4**. What do you notice about the columns?**
$$A^2 = \begin{bmatrix} .7 & .45 \\ .3 & .55 \end{bmatrix}, A^3 = \begin{bmatrix} .65 & .52 \\ .35 & .475 \end{bmatrix}, A^4 = \begin{bmatrix} .625 & .5625 \\ .375 & .4375 \end{bmatrix}$$
In all cases, the entries in each column sum to one.

9. **What matrix sends** $v = (1, 0)$ **to** $(0, 1)$ **and also sends** $w = (0, 1)$ **to** $(-1, 0)$**? This matrix rotates** $\mathbb{R}^2$ **by** $90°$**.**
The desired matrix is:
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$