PROBLEM SET 7 SOLUTIONS.

(1) (3pts) Compute the following definite integrals.

(a) $\int_0^1 x(x^2+2)^3 \, dx$

ANSWER: Letting $u = x^2 + 2$ we have

$$\frac{1}{2}\int_2^3 u^3 \, du = \frac{65}{8} \approx 8.1.$$

(b) $\int_0^1 xe^x \, dx$

ANSWER: Integrate by parts with $u = x$ and $dv = e^x \, dx$ to get

$$(xe^x - e^x)\Big|_0^1 = 1.$$

(c) $\int_0^2 \sqrt{4-x} \, dx$

ANSWER: Use the substitution $u = 4-x$. Then $dx = -du$ and the new limits are from $u = 4$ to $u = 2$. We then get

$$\frac{2}{3}u^{\frac{3}{2}}\Big|_4^2 = \frac{16-4\sqrt{2}}{3} \approx 3.5.$$

(2) (3pts) Find the geometric area of the following functions on the corresponding interval.

(a) $f(x) = 6 - 3x^2$ on $[0,2]$

ANSWER: The integral of f is $6x - x^3$. The function f is positive from 0 to $\sqrt{2}$ and negative from $\sqrt{2}$ to 2, so to find the geometric area we need to evaluate

$$(6x - x^3)\Big|_0^{\sqrt{2}} - (6x - x^3)\Big|_{\sqrt{2}}^2 = 8\sqrt{2} - 4 \approx 7.3.$$

(b) $f(x) = 3x^2 - 3$ on $[0,3]$

ANSWER: The integral of f is $x^3 - 3x$. In this case the function f is negative from 0 to 1 and positive from 1 to 3 so we compute

$$-(x^3 - 3x)\Big|_0^1 + (x^3 - 3x)\Big|_1^3 = 22.$$

(c) $f(x) = 9x^2 - 36$ on $[0,4]$

ANSWER: The integral of f is $3x^3 - 36x$. Here f is negative from 0 to 2 and positive from 2 to 4 so the area is just

$$-(3x^3 - 36x)\Big|_0^2 + (3x^3 - 36x)\Big|_2^4 = 144.$$

(3) (8pts) Compute the following integrals using integration by parts.

(a) $\int \frac{\ln(x)}{x}\, dx$

ANSWER: You can do this integral by integration by parts (see below), but its much easier to just substitute $u = \ln(x)$, because then $du = \frac{1}{x}\, dx$ and the integral just becomes

$$\int u du = \frac{u^2}{2} + C = \frac{1}{2}(\ln(x))^2 + C.$$

The integration by parts method is interesting however, because it it is an example of an integration by parts that does not yield the answer directly, but rather implicitly: Let $u = \ln(x)$, $dv = \frac{1}{x}\, dx$. Then $du = \frac{1}{x}\, dx$ and $v = \ln x$, so

$$\int \frac{\ln(x)}{x}\, dx = \ln(x)\ln(x) - \int \frac{\ln(x)}{x}\, dx$$

Therefore $2\int \frac{\ln(x)}{x}\, dx = (\ln(x))^2$, and we get the same answer as we do by substitution, even though we never directly computed the integral.

(b) $\int x^2 e^x\, dx$ (You will have to do the process twice in this example.)

ANSWER: Let $u = x^2$ and $dv = e^x\, dx$. The first integration by parts gives us $x^2 e^x - 2\int x e^x\, dx$, which must be integrated by parts again, as in 1(b). The final answer is $(x^2 - 2x + 2)e^x + C$

(c) $\int x e^{ax}\, dx$ for a real number a

ANSWER: Let $u = x$, $dv = e^{ax}\, dx$.

$$\int x e^{ax}\, dx = \frac{x e^x}{a} - \frac{1}{a}\int e^{ax}\, dx = (\frac{x}{a} - \frac{1}{a^2})e^{ax} + C.$$

(d) $\int (\ln(x))^2\, dx$

ANSWER: Let $u = (\ln(x))^2$, $dv = dx$. Then $du = 2\frac{\ln(x)}{x}\, dx$ and $v = x$. So,

$$\int (\ln(x))^2\, dx = x(\ln(x))^2 - 2\int \ln(x)\, dx.$$

We can now evaluate $\int \ln(x)\, dx$ in a similar way, letting $u = \ln(x)$, $dv = dx$. Integration by parts gives $\int \ln(x)\, dx = x\ln(x) - \int 1\, dx$. The final answer is:

$$\int (\ln(x))^2\, dx = x(\ln(x))^2 - 2x\ln(x) + 2x + C.$$

(4) (3pts) Find the (geometric) area between the following curves and the x-axis.

(a) $f(x) = 27 - 3x^2$

ANSWER: The indefinite integral is $27 - x^3$, and the curve f is above the x-axis between $x = -3$ and $x = 3$, so

$$\text{Geometric Area} = (27x - x^3)\Big|_{-3}^{3} = 108.$$

(b) $f(x) = 12 - \frac{3}{4}x^2$

ANSWER: The indefinite integral is $12x - \frac{1}{4}x^3$, and the curve f is above the x-axis between $x = -4$ and $x = 4$, so,

$$\text{Geometric Area } = (12x - \frac{1}{4}x^3)\Big|_{-4}^{4} = 64.$$

(c) $f(x) = -2x - \frac{x^2}{2}$

ANSWER: The indefinite integral is $-x^2 - \frac{1}{6}x^3$, and the curve f is above the x-axis between $x = -4$ and $x = 0$, so,

$$\text{Geometric Area } = (-x^2 - \frac{1}{6}x^3)\Big|_{-4}^{0} = \frac{16}{3} \approx 5.3.$$

(5) (3pts) Find the area of the region bounded by the two curves given.

(a) $f(x) = \cos(x)$ and $g(x) = \sin(2x)$ on $[0, \frac{\pi}{2}]$ (Hint: $f(x) = g(x)$ when $x = \frac{\pi}{6}$.)

ANSWER: The function $\cos(x)$ is greater than $\sin(2x)$ on the interval $[0, \frac{\pi}{6}]$ and less than $\sin(2x)$ on $[\frac{\pi}{6}, \frac{\pi}{2}]$, so the area is,

$$\int_0^{\frac{\pi}{6}} (\cos(x) - \sin(2x))\, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin(2x) - \sin(x))\, dx = \frac{1}{2}.$$

(b) $f(x) = x^2 - 4x$ and $g(x) = 2x$

ANSWER: The two curves intersect where $x^2 - 4x - 2x = 0$, i.e. $x = 0, 6$, and so the region bounded by the two curves is between $x = 0$ and $x = 6$. In that interval $g(x) \geq f(x)$, so the area of the region is

$$\int_0^6 (2x - x^2 + 4x\, dx) = 36.$$

(c) $f(x) = 7 - x^2$ and $g(x) = 2$

ANSWER: The two curves intersect at $x = \pm\sqrt{5}$. In the interval $[-\sqrt{5}, \sqrt{5}]$, $f(x) \geq g(x)$, so the area of the bounded region is,

$$\int_{-\sqrt{5}}^{\sqrt{5}} (7 - x^2 - 2)\, dx = \frac{20\sqrt{5}}{3} \approx 14.9.$$