PROBLEM SET 17 SOLUTIONS.

(1) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues of A.
(b) Find the eigenvectors of A.
(c) Diagonalize A: write it as $A = PDP^{-1}$.

ANSWER:

(a) We just have to solve the characteristic equation

$$\det \begin{bmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) = 0.$$

So $\lambda = 1, 2$. Note that in general the eigenvalues of an upper triangular or lower triangular matrix are the diagonal entries.

(b) To find the eigenvector corresponding to λ, we have to find the null space of $A - \lambda I$. There are two cases.

- $\lambda = 1$. Here we have to solve

$$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So the eigenvector is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- $\lambda = 2$. Here we have to solve

$$\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So the eigenvector is

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(2) Consider the matrix

$$A = \begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues of A. Just kidding! The eigenvalues are -1, 1 and 2. Two eigenvectors are

$$v = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad w = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Check that these are eigenvectors. What are the corresponding eigenvalues?

ANSWER:

$$\begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

So the eigenvalue corresponding to v is -1.

$$\begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

So the eigenvalue corresponding to w is 1.

(b) Find a third eigenvector corresponding to the third eigenvalue.

ANSWER: We need to find an element of the null space of $A - 2I$, in other words a solution to

$$\begin{bmatrix} -5 & 4 & -4 \\ -3 & 3 & -3 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can solve this using row reduction, or just observe that the third column is just -1 times the second column, and so from the column definition of matrix multiplication we know immediately that the eigenvector must be,

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(Note: if you know about vector calculus, another neat way to find the eigenvector is to take the cross-product of any two rows of $A - \lambda I$.)

(3) Suppose that A is an $n \times n$ matrix, and that $A^2 = A$. What can you say, then, about the eigenvalues of A?

ANSWER: If λ is an eigenvalue of A. Then

$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

Where $\overrightarrow{v}$ is an eigenvector corresponding to λ. Therefore

$$A^2\overrightarrow{v} = AA\overrightarrow{v} = A\lambda\overrightarrow{v} = \lambda A\overrightarrow{v} = \lambda^2\overrightarrow{v}$$

And so $A^2 = A$ implies $A^2\overrightarrow{v} = A\overrightarrow{v}$. So it must be that

$$\lambda = \lambda^2$$

And λ must be 0 or 1.

(4) Suppose A is a 3×3 matrix with eigenvalues 1, 2 and 3. If v_1 is an eigenvector for the eigenvalue 1, v_2 for 2, and v_3 for 3, then what is $A(v_1 + v_2 - v_3)$?

ANSWER:

$$A(\overrightarrow{v_1} + \overrightarrow{v_2} - \overrightarrow{v_3}) = A\overrightarrow{v_1} + A\overrightarrow{v_2} - A\overrightarrow{v_3} = 1\overrightarrow{v_1} + 2\overrightarrow{v_2} - 3\overrightarrow{v_3}$$