PROBLEM SETS 10 TO 14. DUE AS SPECIFIED.

Problem Set 10: Due 19 September 2000.

Reading. *Matrices and Transformations*, pp. 1–12.
Supplementary reading. Strang, Chapter 1 and Section 2.1.

1. Draw $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, along with $v+w$, $2v+w$, and $v-w$ in one xy-plane.
2. The vectors $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $w = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ are perpendicular. Thus, v, w and $v+w$ form a right triangle. Check the Pythagorean Theorem, $||v||^2 + ||w||^2 = ||(v+w)||^2$ in terms of the definition of length $||v||$.
3. To save space, I will write column vectors as rows. For $u = (0,1,2)$, $v = (1,3,0)$, and $w = (1,0,4)$, find $||u||$, $||v||$, $||w||$, $u \cdot v$, $u \cdot w$, and $v \cdot w$. Check the Law of Cosines for u and v, as well as the Schwartz inequality for v and w.
4. Solve the systems of equations
$$\begin{cases} 2x + y = 5 \\ x - 3y = 6 \end{cases} \qquad \begin{cases} x + y - z = 2 \\ x - y + 2z = 1 \\ y + 4z = 0 \end{cases}$$
5. Let A, B, C, D, E and F be the matrices below. Find $B+D$, $2E-F$, AC, BC, CB, ACD, EF, FE and CEF. In particular, note that $EF \neq FE$!
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \end{bmatrix}$$
$$D = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \qquad E = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$
6. Draw the row and column pictures for
$$\begin{aligned} 2x - y &= 3 \\ x + y &= 1 \end{aligned}$$
7. If you have 5 linear equations in 3 unknowns, then the row picture shows five ___________. The column picture is in what dimensional space? The equations will have a solution only if the vector on the right hand side is a combination of what?
8. Consider the matrix
$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}.$$
Compute A^2, A^3 and A^4. What do you notice about the columns?
9. What matrix sends $v = (1,0)$ to $(0,1)$ and also sends $w = (0,1)$ to $(-1,0)$? This matrix rotates $\mathbb{R}^2$ by $90°$.

Problem Set 11: Due 20 September 2000.

Reading. *Matrices and Transformations*, sections 2-2 and 2-5.
Supplementary reading. Strang, sections 2.1–2.4.

1. Solve the followng systems of equations using Gaussian elimination.

$$\begin{cases} 3x + y + z = 6 \\ x - y - z = -2 \\ 4y + z = 3 \end{cases} \qquad \begin{cases} x + 2y - z = 3 \\ 3x - y + 2z = 3 \\ 2x + y + 4z = 1 \end{cases}$$

2. Consider the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & -3 & -2 \\ 0 & 1 & c \end{bmatrix}.$$

For what value(s) of c can you not perform elimination (without row exchanges) on this matrix?

3. Consider the system of equations,

$$\begin{cases} x + y + 2z = 1 \\ 2x + 2y - z = 1 \\ y + cz = 2 \end{cases}$$

For what values of c does this system have no solutions? one solution? infinitely many solutions?

4. Compute the inverses of the following matrices, using elimination.

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

5. Compute the inverse of

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

For what a and b is there no inverse?

6. Compute the inverse of the general 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

What condition on a, b, c, and d ensures that this will exist?

7. Suppose $Ax = b$ has two solutions v and w (with $v \neq 2$). Then show that $\frac{1}{2}(v + w)$ is also a solution, although $v + w$ is not.
8. In recitation, we saw that for a permutation matrix, P, PA has the same entries as A, but the rows are permuted around. What would you guess happens if we multiply in the other order, that is what does P do when we multiply AP? Check your answer in the 2×2 case.

Problem Set 12: Due 21 September 2000.

Reading. *Matrices and Transformations*, sections 1-3, 1-4, and 2-1.

Supplementary reading. Strang, sections 2.5–2.6, Chapter 5.

1. Factor the following matrices into the form $A = L \cdot U$.
$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 4 \\ 3 & 13 & 7 \end{bmatrix}$$
2. Factor the following symmetric matrices into $A = L \cdot D \cdot L^T$.
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$
3. Given a permutation matrix
$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
what is P^{-1}? Do you recognize this matrix?
4. There are only finitely many ($n!$) $n \times n$ permutation matrices. Use this fact to show that $P^r = I$ for some r.
5. For what value(s) of c can you *not* factorize $A = L \cdot U$?
$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & c & 7 \\ 0 & 1 & 3 \end{bmatrix}$$
6. Compute the determinants of the following matrices.
$$\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & -2 \end{bmatrix}$$
7. Prove that $\det(A^{-1}) = \frac{1}{\det(A)}$.
8. Let
$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 3 & \cdots & 3 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}.$$
Compute $\det(A)$.
9. If you have a block matrix
$$A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix},$$
what is $\det(A)$?

Problem Set 13: Due 22 September 2000.

Reading. *Matrices and Transformations*, none.

Supplementary reading. Strang, sections 3.1–3.2.

1. Consider the set $M_2 = \{2 \times 2 \text{ matrices}\}$ as a vector space. Let
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$$
   (a) Name a subspace containing A but not B.
   (b) Name a subspace containing B but not A.
   (c) Is there a subspace containing A and B but not the 2×2 identity matrix?
2. Consider $\mathbb{R}^2$ as a vector space. Which of the following are subspaces and which are not? If not, why not?
   (a) $\{(a, a^2 \mid a \in \mathbb{R}\}$
   (b) $\{(b, 0) \mid b \in \mathbb{R}\}$
   (c) $\{(0, c) \mid c \in \mathbb{R}\}$
   (d) $\{(m, n) \mid m, n \in \mathbb{Z}\}$
   (e) $\{(d, e) \mid d, e \in \mathbb{R},\ d \cdot e = 0\}$
   (f) $\{(f, f) \mid f \in \mathbb{R}\}$
3. Show that for some $b \neq 0$, the solution set $\{x \mid Ax = b\}$ does **not** form a subspace. (Hint: look at Problem set 11, problem number 7.)
4. Considerthe set $M_n = \{n \times n \text{ matrices}\}$ as a vector space. Which of the following are subspaces?
   (a) The symmetric matrices, $S = \{A \mid A^{\mathrm{T}} = A\}$
   (b) The non-symmetric matrices, $NS = \{A \mid A^{\mathrm{T}} \neq A\}$
   (c) The *skew-symmetric* matrices, $S = \{A \mid A^{\mathrm{T}} = -A\}$
5. Describe the column spaces of the following matrices.
$$C = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$
6. Describe the null-space for the following matrices.
$$E = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 2 & -4 \\ -1 & 1 & 3 \\ 1 & 5 & -5 \end{bmatrix}$$
7. Let P be the plane in $\mathbb{R}^3$ defined by the equation
$$x - y - z = 3.$$
   Find two vectors in P and show that their sum is not in P.
8. (a) Find a subset $W \subseteq \mathbb{R}^2$ where, for $v, w \in W$, $v + w \in W$, but cv is not necessarily in W.
   (b) Find a subset $W \subseteq \mathbb{R}^2$ where, for $v, w \in W$, $cv \in W$, but $v + w$ is not necessarily in W.
9. Let A and B be any $n \times n$ matrices. If $v \in N(B)$, show that $v \in N(A \cdot B)$. If A is invertible, show that if $v \in N(A \cdot B)$, then $v \in N(B)$.

Problem Set 14: Due 25 September 2000.

Reading. *Matrices and Transformations*, sections 2-4 and 2-5.
Supplementary reading. Strang, sections 3.3–3.5.

1. Find the reduced row echelon form of the following matrices.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

2. Compute the ranks of the following matrices.

$$E = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & 4 \\ -1 & -1 & -1 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 2 & -1 & -3 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -2 & 1 & 4 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ -1 & -3 \end{bmatrix}$$

3. Find the complete solution to the equation

$$\begin{bmatrix} 1 & 3 & 2 & 4 & -3 \\ 2 & 6 & 0 & -1 & -2 \\ 0 & 0 & 6 & 2 & -1 \\ 1 & 3 & -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 12 \\ -6 \end{bmatrix}$$

4. Find a matrix with the following property, or say why you cannot have one.
   (a) The complete solution to $B \cdot x = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
   (b) The complete solution to $C \cdot x = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$.

5. The complete solution to

$$A \cdot x = b$$

   is

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

   What is A?

6. Consider the set $P_2 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$ of all polynomials of degree less than or equal to 2.

(a) Show that this is a vector space.
(b) Show that the function

$$L(p(x)) = \int_0^1 p(x)\ dx$$

is a linear transformation $L : P_2 \to \mathbb{R}$.
(c) What is the null sapce of L? (That is, what polynomials does L map to 0?)
(d) What is the range of L?

7. (a) Mike, Shai, and Tara all decide that they are unhappy with the color scheme at ADUni, and they do something about it. They go down to the paint store, and each buy some paint. Mike wants to paint the lab aqua, so he buys one gallon of red paint, six gallons of blue paint, and one gallon of yellow paint. He spends \$44.00. Shai, on the other hand, wants to paint the front office green. He buys no red paint, two gallons of blue paint and three gallons of yellow paint. He spends \$24.00. Tara finally decides to paint the classroom purple. She buys one gallon of red paint and five gallons of blue paint, and spends \$33.00. How much does each color of paint cost?
   (b) What is wrong with your answer to the previous problem? When Mike, Shai and Tara compare receipts, they realize that one of them was charged \$4.00 too little. Who was it?
8. Heather and Tony decide to start a cookie business. Heather is going to contribute chocolate chip cookies and Tony is going to make Tony's Special Secret Recipe cookies. Heather's cookies take $\frac{3}{4}$ of an hour of preparation time, and one hour in the oven (to make 100 cookies). Tony's Special Secret Recipe cookies take one hour of prep time and a full two hours in the oven (for 100 cookies). Combined, Heather and Tony are willing to put in 30 hours of prep time, and 50 hours of oven time. They figure they can make \$60.00 on each 100 of Heather's cookies and \$90.00 on each 100 of Tony's cookies. How many cookies of each type do they make in order to maximize profits? (Hint: Set up a linear programming problem like Shai showed in recitation, and solve it geometrically by graphing the constraints and determining on which extreme values the profits are maximal.)