PROBLEM SET 14 SOLUTIONS

(1) Find the reduced row echelon form of the following matrices.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

ANSWER:
(a)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

(2) Compute the ranks of the following matrices.

$$E = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & 4 \\ -1 & -1 & -1 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 2 & -1 & -3 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -2 & 1 & 4 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ -1 & -3 \end{bmatrix}$$

ANSWER:

(a)

$$\begin{bmatrix} 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & 4 \\ -1 & -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & -1 & 1 & -6 \\ 0 & 1 & -1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Two pivots so E has rank 2

(b)

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 2 & -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Two pivots, so F has rank 2

(c)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -2 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 13 \end{bmatrix}$$

Three pivots, so G has rank 3

(d)

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \\ -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & -7 \\ 0 & 0 \end{bmatrix}$$

Two pivots, so H has rank 2

(3) Find the complete solution to the equation

$$\begin{bmatrix} 1 & 3 & 2 & 4 & -3 \\ 2 & 6 & 0 & -1 & -2 \\ 0 & 0 & 6 & 2 & -1 \\ 1 & 3 & -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 12 \\ -6 \end{bmatrix}$$

ANSWER:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 2 & 4 & -3 & -7 \\ 2 & 6 & 0 & -1 & -2 & 0 \\ 0 & 0 & 6 & 2 & -1 & 12 \\ 1 & 3 & -1 & 4 & 2 & -6 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 3 & 2 & 4 & -3 & -7 \\ 0 & 0 & -4 & -9 & 4 & 14 \\ 0 & 0 & 6 & 2 & -1 & 12 \\ 0 & 0 & -3 & 0 & 5 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccccc|c} 1 & 3 & 2 & 4 & -3 & -7 \\ 0 & 0 & -4 & -9 & 4 & 14 \\ 0 & 0 & 0 & 46 & -20 & -132 \\ 0 & 0 & 0 & -27 & -8 & 38 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 3 & 2 & 4 & -3 & -7 \\ 0 & 0 & -4 & -9 & 4 & 14 \\ 0 & 0 & 0 & 23 & -10 & -66 \\ 0 & 0 & 0 & -27 & -8 & 38 \end{array}\right] \to$$

$$\left[\begin{array}{ccccc|c} 1 & 3 & 2 & 4 & -3 & -7 \\ 0 & 0 & -4 & -9 & 4 & 14 \\ 0 & 0 & 0 & 23 & -10 & -66 \\ 0 & 0 & 0 & 0 & -454 & -908 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 3 & 2 & 4 & -3 & -7 \\ 0 & 0 & -4 & -9 & 4 & 14 \\ 0 & 0 & 0 & 23 & -10 & -66 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \to$$

$$\left[\begin{array}{ccccc|c} 1 & 3 & 2 & 4 & 0 & -1 \\ 0 & 0 & -4 & -9 & 0 & 6 \\ 0 & 0 & 0 & 23 & 0 & -46 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 3 & 2 & 4 & 0 & -1 \\ 0 & 0 & -4 & -9 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \to$$

$$\left[\begin{array}{ccccc|c} 1 & 3 & 2 & 0 & 0 & 7 \\ 0 & 0 & -4 & 0 & 0 & -12 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 3 & 2 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \to$$

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

So $x_5 = 2$, $x_4 = -2$, $x_3 = 3$, $x_2 = C$, $x_1 = 1 - 3C$, which may be rewritten as,

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \\ 2 \end{bmatrix} + C \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(4) Find a matrix with the following property, or say why you cannot have one.

(a) The complete solution to $B \cdot x = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) The complete solution to $C \cdot x = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$.

ANSWER:

(a)

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 4 & 0 \end{bmatrix}$$

or any other

$$\begin{bmatrix} 1 & a \\ 2 & b \\ 4 & c \end{bmatrix}$$

as long as the second column is not a multiple of the first.

(b) There is no such matrix. It would have to have two rows and three columns. This would mean there would have to be at least one free variable, and so there would have to be more than one vector in the complete solution.

(5) The complete solution to

$$A \cdot x \;=\; b$$

is

$$x \;=\; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

What is A?

ANSWER: There is some ambiguity in the question. It is not clear what the dimensions of b are. Assuming that b equals

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Then the matrix A will have the form

$$\begin{bmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 0 \\ \vdots & \vdots & \vdots \\ b_n & 0 & 0 \end{bmatrix}.$$

Note that the second and third columns are zero because the vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are in the null space of A.

(6) Consider the set $P_2 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$ of all polynomials of degree less than or equal to 2.

(a) Show that this is a vector space.

ANSWER: We have to show two things. First, that any constant multiple of a polynomial of degree at most two is again of degree at most two. This is easy since

$$d(ax^2 + bx + c) = (da)x^2 + (db)x + (dc),$$

which is a polynomial of degree at most two. Second, we have to show that the sum of any two polynomials of degree at most two is again a polynomial of degree at most two. Again this is easy since

$$(ax^2 + bx + c) + (dx^2 + ex + f) = (a + d)x^2 + (b + e)x + (c + f),$$

which is a polynomial of degree at most two. Therefore, P_2 satisfies the axioms of a vector space.

(b) Show that the function

$$L(p(x)) = \int_0^1 p(x)\ dx$$

is a linear transformation $L : P_2 \to \mathbb{R}$.

ANSWER: By definition, L takes in a polynomial and spits out a number. In fact,

$$L(ax^2 + bx + c) = \frac{a}{3} + \frac{b}{2} + c$$

What we have to show is that it does so in a linear way. The easy answer is to say that $\frac{a}{3} + \frac{b}{2} + c$ is a linear function of the coefficients, end of story. But just for the sake of it, let's also see how to do this by checking the rules for a linear transformation. We need check two things. First the multiplicativity of L, *i.e.* that

$$L(d(ax^2 + bx + c)) = dL(ax^2 + bx + c)$$

This is easy to check by computation

$$L((da)x^2 + (db)x + (dc)) = \frac{da}{3} + \frac{db}{2} + dc$$

and

$$dL(ax^2 + bx + c) = d(\frac{a}{3} + \frac{b}{2} + c)$$

Second, the additivity of L, *i.e.* that

$$L((ax^2 + bx + c) + (dx^2 + ex + f)) = L(ax^2 + bx + c) + L(dx^2 + ex + f)$$

This follows since the left hand side is

$$L((a+d)x^2 + (b+e)x + (c+f)) = \frac{a+d}{3} + \frac{b+e}{2} + (c+f)$$

and since the right hand side is

$$(\frac{a}{3} + \frac{b}{2} + c) + (\frac{d}{3} + \frac{e}{2} + f)$$

Therefore, L is a linear transformation.

(c) What is the null space of L? (That is, what polynomials does L map to 0?)

ANSWER: All the polynomials $ax^2 + bx + c$ that satisfy

$$\frac{a}{3} + \frac{b}{2} + c = 0$$

(d) What is the range of L?

ANSWER: The range is all the possible values that $\frac{a}{3} + \frac{b}{2} + c$ could take, so

$$\text{Range } = \mathbb{R},$$

since a, b, c are arbitrary.

(7) (a) Mike, Shai, and Tara all decide that they are unhappy with the color scheme at ADUni, and they do something about it. They go down to the paint store, and each buy some paint. Mike wants to paint the lab aqua, so he buys one gallon of red paint, six gallons of blue paint, and one gallon of yellow paint. He spends \$44.00. Shai, on the other hand, wants to paint the front office green. He buys no red paint, two gallons of blue paint and three gallons of yellow paint. He spends \$24.00. Tara finally decides to paint the classroom purple. She buys one gallon of red

paint and five gallons of blue paint, and spends \$33.00. How much does each color of paint cost?

ANSWER: Let r be the cost per gallon of the red paint, b be the cost per gallon of the blue paint, y be the cost per gallon of the yellow paint. We get the following three equations for each person's expenses

$$\begin{aligned} 1r + 6b + 1y &= 44 \\ 0r + 2b + 3y &= 24 \\ 1r + 5b + 0y &= 33, \end{aligned}$$

Solving the system, we get that

$$y = 2, b = 9, r = -12$$

(b) What is wrong with your answer to the previous problem? When Mike, Shai and Tara compare receipts, they realize that one of them was charged \$4.00 too little. Who was it?

ANSWER:
Old Economy Answer: The store is paying people \$12 per gallon to take its red paint. Something doesn't make sense. We can figure out who was undercharged by trial and error. We have to add \$4 to the right hand side of one of the three equations above, and then solve it again. After some computation, we find that the only way we can get a nonzero solution is if we add \$4 to the second equation. Therefore Shai was overcharged.

New Economy Answer: Nothing is wrong with the answer to the previous problem.

(8) Heather and Tony decide to start a cookie business. Heather is going to contribute chocolate chip cookies and Tony is going to make Tony's Special Secret Recipe cookies. Heather's cookies take $\frac{3}{4}$ of an hour of preparation time, and one hour in the oven (to make 100 cookies). Tony's Special Secret Recipe cookies take one hour of prep time and a full two hours in the oven (for 100 cookies). Combined, Heather and Tony are willing to put in 30 hours of prep time, and 50 hours of oven time. They figure they can make \$60.00 on each 100 of Heather's cookies and \$90.00 on each 100 of Tony's cookies. How many cookies of each type do they make in order to maximize profits? (HINT:

Set up a linear programming problem like Shai showed in recitation, and solve it geometrically by graphing the constraints and determining on which extreme values the profits are maximal.)

ANSWER: Let H stand for the number of batches (of 100) of Heather's cookies and T stand for the number of batches of Tony's cookies. The objective function we are trying to maximize is

$$\text{PROFIT} = 60H + 90T$$

The constraints we have are

POSITIVITY	$H, T \geq 0$
PREP TIME	$\frac{3}{4}H + T \leq 30$
OVEN TIME	$H + 2T \leq 50$

The region described by the inequalities is four sided and has corners at $(H, T) = (0, 0)$, $(40, 0)$, $(0, 25)$, $(20, 15)$. By the principles of linear programming, profit will be maximized at one of these corner points. Plugging in the different possibilities gives profits of \$0, \$2400, \$2250, \$2550, respectively. Therefore profits are maximized with 20 batches (= 2000) of Heather's cookies and 15 batches (= 1500) of Tony's.