Problem Set 2. Problems from Lecture 2.

Reading. *Quick Calculus*, pp. 50–97.

Supplementary reading. Simmons, Chapter 2, sections 2.1–2.5. Read section 2.6 if you are interested in some applications of the derivative.

1. **Compute the following limits.**

(a) $\lim_{\theta\to 0}\frac{\sin(5\theta)}{\theta}$

$$\begin{aligned}\lim_{\theta\to 0}\frac{\sin(5\theta)}{\theta} &= 5(\lim_{\theta\to 0}\frac{\sin(5\theta)}{5\theta}) \\ &= 5(\lim_{\gamma\to 0}\frac{\sin(\gamma)}{\gamma}) \\ &= 5\end{aligned}$$

Here, we used the fact that $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$, and made the variable substitution $\gamma = 5\theta$.

(b) $\lim_{\theta\to 0}\frac{\sin(3\theta)}{\sin(4\theta)}$

$$\begin{aligned}\lim_{\theta\to 0}\frac{\sin(3\theta)}{\sin(4\theta)} &= \lim_{\theta\to 0}\frac{\frac{\sin(3\theta)}{\theta}}{\frac{\sin(4\theta)}{\theta}} \\ &= \frac{3}{4}\end{aligned}$$

(c) $\lim_{x\to\infty}\frac{x}{x^2+1}$

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{x^2+1} &= \lim_{x\to\infty}\frac{\frac{1}{x}}{1+\frac{1}{x^2}} \\ &= \frac{0}{1} \\ &= 0\end{aligned}$$

(d) $\lim_{x\to\infty}\frac{2x^2+3x}{3x^2-2}$

$$\begin{aligned}\lim_{x\to\infty}\frac{2x^2+3x}{3x^2-2} &= \lim_{x\to\infty}\frac{2+\frac{3}{x}}{3-\frac{2}{x^2}} \\ &= \frac{2}{3}\end{aligned}$$

2. **Where are the following functions discontinuous?**

(a) $\frac{x}{x^2+1}$

(b) $\frac{1}{x^2+x-6}$

(c) $\frac{x^3+x}{x^2+1}$

(d) $\frac{x^2+2x}{x^3+2x^2-x-2}$

Functions (a) and (c) are continuous everywhere. Function (b) is discontinuous at $x = 3$ and $x = -2$. Function (d) is discontinuous at $x = 1$, $x = -1$ and $x = -2$.

3. **Use the definition of the derivative to show that for $f(x) = ax^2 + bx + c$, for constants $a, b, c \in \mathbb{R}$, the derivative is $f'(x) = 2ax + b$.**

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{x} \\
&= \lim_{\Delta x \to 0} \frac{(a(x+\Delta x)^2 + b(x+\Delta x) + c) - (ax^2 + bx + c)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(2ax\Delta x + (\Delta x)^2 + b\Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2ax + \Delta x + b \\
&= 2ax + b
\end{aligned}
$$

4. **Use the definition of the derivative to find the derivative of the function $f(x) = \frac{x}{x+1}$.**

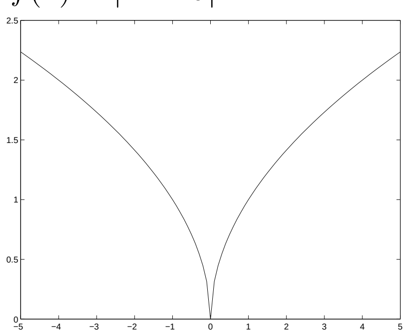
$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{\frac{x+\Delta x}{x+\Delta x+1} - \frac{x}{x+1}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{(x+\Delta x)(x+1)-(x)(x+\Delta x+1)}{(x+\Delta x+1)(x+1)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{(x^2+x\Delta x+x+\Delta x)-(x^2+x\Delta x+x)}{(x^2+x\Delta x+2x+\Delta x+1)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{\Delta x}{(x^2+x\Delta x+2x+\Delta x+1)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{(x^2 + x\Delta x + 2x + \Delta x + 1)} \\
&= \frac{1}{x^2 + 2x + 1}
\end{aligned}
$$

Note that at $x = -1$, the function is undefined and the limit does not exist.

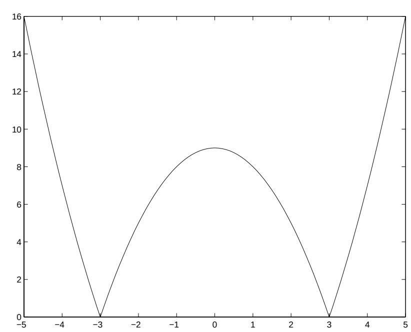
5. **Use the definition of the derivative and the angle summation formula to compute the derivative of $f(\theta) = \cos(\theta)$.**

$$\begin{aligned}
f'(\theta) &= \lim_{\Delta\theta \to 0} \frac{\cos(\theta + \Delta\theta) - \cos(\theta)}{\Delta\theta} \\
&= \lim_{\Delta\theta \to 0} \frac{(\cos(\theta)\cos(\Delta\theta) - \sin(\theta)\sin(\Delta\theta)) - \cos(\theta)}{\Delta\theta} \\
&= \lim_{\Delta\theta \to 0} \frac{\cos(\theta)(\cos(\Delta\theta) - 1) - \sin(\theta)\sin(\Delta\theta)}{\Delta\theta} \\
&= \lim_{\Delta\theta \to 0} [\cos(\theta)\frac{\cos(\Delta\theta) - 1}{\Delta\theta} - \sin(\theta)\frac{\sin(\Delta\theta)}{\Delta\theta}] \\
&= \cos(\theta) \lim_{\Delta\theta \to 0} \frac{\cos(\Delta\theta) - 1}{\Delta\theta} - \sin\theta \lim_{\Delta\theta \to 0} \frac{\sin(\Delta\theta)}{\Delta\theta} \\
&= \cos(\theta)(0) - \sin(\theta)(1) \\
&= -\sin(\theta)
\end{aligned}$$

6. **Sketch the graph of the following two functions. For each, state where it is not differentiable.**
   (a) $f(x) = \sqrt{|x|}$**.**
   (b) $f(x) = |x^2 - 9|$**.**



$f(x) = \sqrt{|x|}$



$f(x) = |x^2 - 9|$

$f(x) = \sqrt{|x|}$ is discontinuous only at $x = 0$. $f(x) = |x^2 - 9|$ is discontinuous at $x = 3$ and $x = -3$.

7. **Let** $f(x) = \begin{cases} x^2 & \textbf{if } x \le -1, \\ mx + b & \textbf{if } x > -1. \end{cases}$ **What must** m **and** b **be for** $f(x)$ **to be differentiable at all points?**

   At $x = -1$, we need the two "branches" of the function to have identical derivatives and values. Noting that $\frac{d}{dx}x^2 = 2x$ and $\frac{d}{dx}(mx+b) = m$, the first condition (equivalence of derivatives) requires that these be equal when $x = -1$, or, equivalently, that $m = -2$. Then, using $m = -2$, the second condition (equivalence of values) implies that $b = 3$.

8. **A penny is dropped off a ledge on the World Trade Center in New York City. The ledge is** 1024 **feet above the ground. The penny falls a distance of** $s = 16t^2$ **feet in** t **seconds.**
   (a) **How long does the penny fall before it hits the ground?**

$$16t^2 = 1024 \to t^2 = 64 \to t = 8$$

(b) **What is the average velocity at which the penny falls during the first three seconds?**
The average velocity AV is given by

$$AV = \frac{s(3) - s(0)}{t(3) - t(0)} = \frac{16(3^2) - 0}{3 - 0} = \frac{144}{3} = 46$$

The average velocity during the first three seconds is 46 feet per second.

9. **With the same situation as in Problem 8, answer the following questions.**
(a) **What is the average velocity at which the penny falls during the last four seconds?**
The average velocity AV is given by

$$AV = \frac{s(8) - s(4)}{t(8) - t(4)} = \frac{16(8^2) - 16(4^2)}{8 - 4} = \frac{768}{4} = 192$$

The average velocity during the last four seconds is 192 feet per second.

(b) **What is the instantaneous velocity of the penny when it hits the ground?**
At any given time, the instantaneous velocity is given by $v(t) = 32t$ (velocity is the derivative of distance with respect to time). Therefore, at $t = 8$, the instantaneous velocity is $32(8) = 256$.

10. **An oil tank is to be drained for cleaning. There are V gallons of oil left in the tank after t minutes of draining, where $V = 50(40 - t)^2$.**
(a) **What is the average rate at which oil drains out of the tank during the first 20 minutes?**
The average rate of draining AR is given by

$$AR = \frac{V(20) - V(0)}{20 - 0} = \frac{50(40 - 20)^2 - 50(40 - 0)^2}{20} = -3000$$

The average rate of drainage during the first 20 minutes is 3,000 gallons per minute.

(b) **What is the rate at which oil is flowing out of the tank 20 minutes after draining begins?**
The instantanteous change in the amount of oil in the tank at time t (for $0 \leq t \leq 40$) is

$$\frac{dV}{dt} = -100(40 - t)$$

At $t = 20$, the rate of change is -2,000 gallons per minute; 2,000 gallons per minute are flowing out of the tank.