ArsDigita University
Month 2: Discrete Mathematics - Professor Shai Simonson

Problem Set 6 – Combinatorics and Discrete Probability

1. Assume someone is throwing three dice of different colors.

    a. How many ways are there to roll the dice?
    b. Make a chart showing how many of these rolls, have *i* as the second highest and *j* as the highest value, for $1 \leq i \leq j \leq 6$. (Feel free to write a program to do it, if that will be faster).
    c. Sum up the columns and rows of your chart and state an interesting theorem about the number of rolls whose second highest die value is *m*.
    d. A generalization of this theorem for *n* dice, where *n* is an odd number, is that the number of rolls of *n* dice whose median value die equals *m* is the same as the number of rolls of *n* dice whose median value die equals $7-m$. Prove this theorem.

2. In the game of Risk, one player, the attacker, throws three dice. The defender chooses to roll either one die or two dice. If the defender rolls two dice, then the two highest rolls of the attacker are compared with the rolls of the defender. The higher is compared against the higher and the lower against the lower, with a tie going to the defender. In every battle with two dice, the defender can win two, lose two or split. If the defender rolls just one die, then only the highest roll of the attacker is compared to it, and again a tie goes to the defender. Here the defender can win one or lose one.

    a. Let *x* and *y* be the highest and second highest rolls of the attacker respectively. What is the probability of winning/losing one fight when the defender is rolling one die? What is the probability of winning/losing two fights when the defender is rolling two dice? What is the expected win/loss total in each case?
    b. If the defender can see the attackers roll, before he must choose to roll one or two dice, then describe the defenders strategy based on the attacker's roll. That is, under what circumstances would the defender wish to roll one die, and under what circumstances would he roll two? Use 1a.
    c. If the defender must decide in advance how many dice to roll, then what is the expected result over *n* battles, when he always chooses to roll one die?
    d. Same question when he always chooses to roll two dice? (You can write a program to help you, or you can calculate this by hand, using 2a and 1b).

3. What is the chance, in a room of *n* people, to find three people or more with the same birthday? For what *n* is this value closest to 50%?

4. A password is created with eight characters each of which is between the letters *a* and *z* inclusive.
    a. How many different passwords are possible?
    b. If no duplicate letters are allowed, then how many passwords are possible?
    c. In each case, if 2,000,000 random attempts are made by a hacker to guess the password, what's the chance that he cracks it?
    d. If it is known that the password is one of the 3,300,000 entries in a list of words and proper names, and again the hacker tries 2,000,000 random attempts to crack it. What's the chance of his success?

5. Assume one person out of 10,000 has AIDS, and there is a test in which 2.5% of all people test positive for the disease although they do not really have it. If you test negative on this test, then you definitely do not have AIDS. What is the chance of having the disease, assuming you test positive for it?

6. Assume that you have two dice, one of which is *fair*, and the other is biased toward landing on six, so that 1/4 of the time it lands on six, and 1/6 of the time it lands on each of 2, 3, 4 and 5, and 1/12 of the time on 1. You choose a die at random, and spin it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die you chose is the *fair* die?

7. A round-robin tournament is one where each player plays each of the other players exactly once. Prove that if no person loses all his games, then there must be two players with the same number of wins.

8. Each of two disks has one megabyte of bits around its perimeter, half of which are ones and half of which are zeros. Prove that no matter how the bits are arranged, they can be placed on top of each other, so that half a megabyte of bits match up. (Hint: Count how many total matches as you rotate the top disk around the bottom, and make a pigeonhole argument).

9. How many different collections of six integers are there (duplicates allowed), where each integer is between 0 and 8, and the sum equals 20? (Use Inclusion/Exclusion).

10. Three computer tasks each with 5 ordered parts, are being multitasked by my PC. Assume, that the choice of which task to work on next is chosen randomly, then what is the probability that after all 15 parts are complete, five parts of one task were executed consecutively? (Use Inclusion/Exclusion).

11. A password requires that you use a sequence of upper-case characters, lower-case characters and digits. A valid password must be at least 10 characters long, and it must contain at least one character from each of the three sets of characters.

    a. If you generate 10 random characters from the union of the three sets of characters, what is the probability that you will get a valid password?
    b. Same question when a valid password must contain at least two characters from each of the three sets of characters.

12. Order of Growth for the Catalan Numbers

    Recall that the number of different binary trees with *n* nodes, and the number of different ways to make a balanced arrangement of *n* pairs of parenthesis, and the number of different ways to multiply $n+1$ square matrices, all equal the *n*th Catalan number.

    In class we proved that the chance of a $2n+1$ series going all the way equals $C(2n,n)/2^{2n} = (1\times 3\times 5\times\ldots\times(2n-1))/(2\times 4\times 6\times\ldots\times 2n)$. Use this result and Stirling's approximation for $n!$ to show that the *n*th Catalan number $C(2n, n)/(n+1) = (4^n/\sqrt{\pi}n^{3/2})(1 + O(1/n))$.