ArsDigita University
Month 2: Discrete Mathematics - Professor Shai Simonson

Problem Set 5 – Combinatorics and Counting

1. Given ten points in the plane with no three collinear,

   a. How many different segments joining two points are there?
   b. How many ways are there to choose a directed path of length two through three distinct points?
   c. How many different triangles are there?
   d. How many ways are there to choose 4 segments?
   e. If you choose 4 segments at random, what is the chance that some three form a triangle?

2. Forty equally skilled teams play a tournament in which every team plays every other team exactly once, and there are no ties.

   a. How many different games were played?
   b. How many different possible outcomes for these games are there?
   c. How many different ways are there for each team to win a different number of games?

3. Let $C(n,k)$ be the number of ways to choose k objects from a set of n. Prove by a combinatorial argument:

   a. $C(n,0) + C(n, 1) + \ldots + C(n, n) = 2^n$.
   b. $C(n,m)C(m,k) = C(n,k)C(n-k, m-k)$.
   c. $C(n, n-k) = C(n, k)$.
   d. $C^2(n,0) + C^2(n, 1) + \ldots + C^2(n, n) = C(2n, n)$. (Hint: Use *c*).

4. Prove the following combinatorial identities by formulae or mathematical induction

   a. $C(n+1, k+1) = C(n, k+1) + C(n, k)$.
   b. $C(r, r) + C(r+1, r) + C(r+2, r) + \ldots + C(n, r) = C(n+1, r+1)$.
   c. Using the identity in 4b above, derive a formula for the sum $(1)(2)(3) + (2)(3)(4) + \ldots + (n-2)(n-1)(n)$.

5. If you have $2n$ socks in a drawer, n white and n black, and you reach in to choose 2 socks at random,

   a. How many ways are there to choose?
   b. How many of these ways result in getting a pair of the same color?
   c. Write a simple closed form formula in terms of n for the chance choosing a matching pair of socks from a drawer with n white and n black socks.

6. A few short problems:
    a. How many ways are there to choose a president, vice president, secretary and treasurer from 9 people?
    b. How many ways can 13 identical balls be distributed into 3 distinct boxes?
    c. How many numbers greater than 3,000,000 can be formed from permutations of 1,2,2,4,6,6,6?
    d. How many nine-digit numbers with twice as many odd digits as even digits? (leading zeros are allowed)
    e. How many passwords can be created in the form $[A\text{-}Z][a\text{-}z]^9[0,1]^6$? (That is, a capital letter followed by 9 lowercase letters followed by 6 bits).

7. Poker:
    a. How many different 5-card Poker hands are there?
    b. How many of these are 1 pair?
    c. How many of these are a flush (all one suit)?
    d. How many are a full house (3 of a kind and a pair)?

8. How many ways are there to distribute eight balls into six distinct boxes with at least one ball in each box if:
    a. The balls are identical?
    b. The balls are distinct?

9. How many ways are there to distribute eight balls into six distinct boxes with at most four balls in the first two boxes if:
    a. The balls are identical?
    b. The balls are distinct?

10. Fibonacci in Pascal's Triangle.
    Prove by induction that the *n*th Fibonacci number F_n equals $C(n, 0) + C(n\text{-}1, 1) + C(n\text{-}2, 2) + \ldots + C(\lceil n/2 \rceil, \lfloor n/2 \rfloor)$. You should assume that $F_0 = F_1 = 1$.

11. Given a deck of 52 cards, how many ways are there to choose a set of four cards in order followed by one additional card? Three in order followed by two additional unordered cards? Two? One?

12. There's a new screen saver that displays a random rectangular piece of an *n* by *n* checkerboard.
    a. How many rectangles are there in a checkerboard of size 1? 2? 3? 4?
    b. How many squares are there in a checkerboard of size 1? 2? 3? 4?
    c. Guess a general formula for the number of squares and rectangles. Put each in closed form in terms of *n*.
    d. Prove your formulas are true either by induction or using a combinatorial argument.
    e. What's the chance that the rectangle displayed is a square? Give a simplified closed form in terms of *n*.
    f. Although the number of squares and rectangles increase without bound as *n* increases, what happens to the ration of squares to rectangles?