PROBLEM SETS 15 TO 17. DUE AS SPECIFIED.

Problem Set 15: Due 26 September 2000.

Supplementary reading. Strang, last two sections of Chapter 3.
Reading for PS 16. Strang, Chapter 4 and section 7.3.
Reading for PS17. *Matrices and Transformations,* sections 4–1 to 4–4. Strang, sections 6.1 and 6.2.

1. Which of the following sets of vectors span $\mathbb{R}^3$?
   (a) $(1, 2, 0$ and $(0, -1, 1)$.
   (b) $(1, 1, 0)$, $(0, 1, -2)$, and $(1, 3, 1)$.
   (c) $(-1, 2, 3)$, $(2, 1, -1)$, and $(4, 7, 3)$.
   (d) $(1, 0, 2)$, $(0, 1, 0)$, $(-1, 3, 0)$, and $(1, -4, 1)$.
2. Which of the following sets of vectors span $P_3 = \{at^3 + bt^2 + ct + d\}$?
   (a) $t + 1$, $t^2 - t$, and t^3.
   (b) $t^3 + t$ and $t^2 + 1$.
   (c) $t^2 + t + 1$, $t + 1$, 1, and t^3.
   (d) $t^3 + t^2$, $t^2 - t$, $2t + 4$, and $t^3 + 2t^2 + t + 4$.
3. Are the following sets of vectors linearly dependent or independent? If they are dependent, write one as a linear combination of the others.
   (a) $(1, 2, 0)$ and $(0, -1, 1)$ in $\mathbb{R}^3$.
   (b) $(-1, 2, 3)$, $(2, 1, -1)$, and $(4, 7, 3)$ in $\mathbb{R}^3$.
   (c) $(1, 2)$, $(2, 3)$, and $(8, -2)$ in $\mathbb{R}^2$.
   (d) $t^2 + 2t + 1$, $t^3 - t^2$, $t^3 + 1$, and $t^3 + t + 1$ in P_3.
4. What is the dimension of the following spaces?
   (a) The set of 2×2 symmetric matrices, $A = A^T$.
   (b) The set of 2×2 matrices
   $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
   with $a + d = 0$.
   (c) The set $\{(x, y, x - 3y, 2y - x) \mid x, y \in \mathbb{R}\}$ inside of $\mathbb{R}^4$.
5. What is the column space and row space of the matrix
   $$A = \begin{bmatrix} 1 & 3 & 5 & -2 \\ 2 & -1 & 3 & -4 \\ -1 & 4 & 2 & 2 \end{bmatrix}?$$
6. Find an (infinite) basis for the space of all polynomials
   $$\mathcal{P} = \{a_n x^n + z_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid \text{ for all } n\}.$$
7. Suppose $\{v_1, \ldots, v_n\}$ spans a vector space V, and suppose that v_n is a linear combination of v_1 through v_{n-1}. Then show that $\{v_1, \ldots, v_{n-1}\}$ spans V as well.
8. If A is a 4×6 matrix, show that the columns of A are linearly dependent.

9. Compute
$$\begin{bmatrix} .1 & .95 \\ .9 & .05 \end{bmatrix}^n,$$
for $n = 3$, 5 and 100 using methods from recitation.

Problem Set 16: Due 27 September 2000.

1. Check that $(1,1,0)$, $(0,1,1)$, and $(1,0,1)$ form a basis for $\mathbb{R}^3$. Transform this basis into an orthonormal basis using the Gram-Schmidt algorithm. Check that the resulting vecotrs are indeed orthogonal!
2. Check that the vectors $(1,1,1,1)$, $(1,1,1,0)$, $(1,1,0,0)$ and $(1,0,0,0)$ form a basis of $\mathbb{R}^4$. Use the Gram-Schmidt algorithm to make this into an orthogonal basis. Don't worry about normalizing this basis – the result is already a little messy!
3. Consider the orthonormal vectors

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \quad \& \quad v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

in $\mathbb{R}^3$. Find some other vector

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

such that v_1, v_2, and b are a basis of $\mathbb{R}^3$. Then use the Gram-Schmidt algorithm to make your basis into an orthogonal one.

4. Check that the matrix

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

is an orthogonal matrix by checking that $Q \cdot Q^T = I$. Also, check that $||Q \cdot v|| = ||v||$ for the vector $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

5. We have the following theorem.

Theorem 1. *If $\{v_1, \dots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then for any $v \in \mathbb{R}^n$, we can write*

$$v = c_1 v_1 + \cdots c_n v_n,$$

where $c_i = v \cdot v_i$ $(1 \leq i \leq n)$, where $\cdot$ is the dot product.

(a) Use this theorem to write the vector $(3,2)$ as linear combinations of the vectors

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \& \quad \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

(b) Use this theorem to write $(1,2,-1)$ in terms of the basis

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(c) (Optional) Prove the above theorem.

6. Consider the two bases of $\mathbb{R}^3$,

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and

$$\hat{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \ \& \ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The change of basis matrix $M_{\hat{B}}^{B}$ is

$$M_{\hat{B}}^{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Compute $M_{B}^{\hat{B}}$.

7. Now consider the two bases

$$\hat{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \ \& \ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\},$$

and

$$\tilde{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \ \& \ \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

Compute the matrices $M_{\tilde{B}}^{\hat{B}}$ and $M_{\hat{B}}^{\tilde{B}}$.

Problem Set 17: Due 28 September 2000.

Turn in problems 1 through 4. Use the rest as review for the final exam.

1. Consider the matrix
$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$
   (a) Find the eigenvalues of A.
   (b) Find the eigenvectors of A.
   (c) Diagonalize A: write it as $A = PDP^{-1}$.
2. Consider the matrix
$$A = \begin{bmatrix} -10 & 22 & -9 \\ 2 & -5 & 3 \\ 18 & -36 & 17 \end{bmatrix}.$$
   (a) Find the eigenvalues of A. Just kidding! The eigenvalues are -1, -1 and 2. Two eigenvectors are
$$v = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad w = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$
   Check that these are eigenvectors. What are the corresponding eigenvalues?
   (b) Find a third eigenvector corrseponding to the third eigenvalue.
3. Suppose that A is an $n \times n$ matrix, and that $A^2 = A$. What can you say, then, about the eigenvalues of A?
4. Suppose A is a 3×3 matrix with eigenvalues 1, 2 and 3. If v_1 is an eigenvector for the eigenvalue 1, v_2 for 2, and v_3 for 3, then what is $A \cdot (v_1 + v_2 - v_3)$?

Review Problems

This is not intended to be a practice exam, but merely some problems to review the major concepts we covered in the course. You should understand how to do these problems, but you should also review your old problem sets, exams, and lecture notes.

1. Graph the follwing functions.
   (a) $y = -x^3 - 2x^2 + 5x - 6$
   (b) $y = \frac{(x-2)(x-1)}{(x+2)}$
   (c) $y = \sin(\theta) + \cos(\theta)$
   (d) $y = e^x + e^{2x}$
2. Suppose that $f(t)$ is a function describing the position of a particle at time t. In this case, what does the derivative $f'(t)$ mean?
3. Suppose that $y = f(x)$ is a function. What does $f'(x)$ mean graphically?
4. Take the derivative of the following functions.
   (a) $y = \sin(\theta)\cos(\theta)$
   (b) $y = \sqrt{x^2 + 2x - 3}$
   (c) $y = \frac{x-4}{x-2}$
   (d) $y = \ln(x) \cdot \sin(x)$
5. ArsDigita University is going to build a new building. It is to have floor area 3500 square meters. It will be a rectangle with three solid brick walls and a glass front (with a beautiful etched ADUni logo). The glass costs 1.8 times as much as the brick wall per linear foot. What dimensions of the building will minimize the costs of materials for the walls and front? (Of course, we are ignoring all other factors in the price –

expensive roofs and floors are no problem. Also, we should have a place for bathrooms this time!)

6. Compute the following integrals.
   (a) $\int \sin(x)\cos(x)\ dx$
   (b) $\int \frac{1}{\sqrt{3x-4}}\ dx$
   (c) $\int 2xe^x\ dx$
   (d) $\int x^3 - 4x - 2\ dx$
   (e) $\int 7x^6 \ln(x)\ dx$
7. Evaluate the following definite integrals using the Fundamental Theorem of Calculus.
   (a) $\int_0^{2\pi} \cos(x)\ dx$
   (b) $\int_{-3}^{2} x^4 + 2x^3 - 5x^2 - 6x\ dx$
   (c) $\int_0^{\frac{3\pi}{2}} \sin(x)\ dx$
8. Compute the geometric area of the following functions on the corresponding intervals. These are the same functions and intervals as in the previous problem. Note the difference between geometric and algebraic area!
   (a) $f(x) = \cos(x)$ on $[0, 2\pi]$
   (b) $f(x) = x^4 + 2x^3 - 5x^2 - 6x = x(x-2)(x+1)(x+3)$ on $[-3, 2]$
   (c) $f(x) = \sin(x)$ on $[0, \frac{3\pi}{2}]$
9. Graph the following regions. Rotate them around the x-axis and compute their volumes.
   (a) The region below $f(x) = 5\sqrt{x}$, above $y = 0$ and to the left of $x = 4$.
   (b) The region below $f(x) = 9x - x^2$ above $y = 0$.
   (c) The region below $f(x) = 4x - x^2$ and above $g(x) = 3(x-2)^2$. (Hint: $f(x) = g(x)$ when $x = 1, 3$.)
10. Given vectors $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, draw these two vectors in $\mathbb{R}^2$.
    (a) Write the vector $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ as a linear combination of v and w.
    (b) Use Gram-Schmidt to make this into an orthonormal basis for $\mathbb{R}^n$.
11. Consider the following subsets of $\mathbb{R}^4$. Are they subspaces? For each subspace, write down a basis for it. What is it's dimension?
    (a) Vectors with all four entries equal to each other.
    (b) Vectors with the last entry equal to 2.
    (c) Vectors whose entries sum up to 0.
    (d) Vectors whose entries sum up to 1.
12. Solve the following system of equations.

$$\begin{cases} 2x + y + 2z = 3 \\ x - 2y - z = -2 \\ 4y - z = 1 \end{cases}$$

13. Given matrices, you should be able to multiply them and find their inverses (if they exist!).

14. Find the complete solution to

$$\begin{bmatrix} 1 & -1 & -2 & -3 & 0 & -2 & -2 \\ 2 & -2 & 3 & 8 & 0 & 3 & 1 \\ -1 & 1 & 0 & -1 & 1 & 2 & 3 \\ 0 & 0 & 2 & 4 & 0 & 2 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}.$$

15. Are the following sets of vectors bases of $\mathbb{R}^3$?
   (a) $(1, 2, 1)$ and $(-1, 0, 3)$.
   (b) $(2, 4, -1)$, $(-2, 0, 1)$ and $(0, 4, 0)$.
   (c) $(1, 2, 3)$, $(-1, 2, 0)$ and $(2, 0, 0)$.
   (d) $(2, -1, 0)$, $(2, 2, 4)$, $(1, -2, 4)$, and $(1, -1, 0)$.
16. For the sets of vectors in the previous problem that *are* bases, write down the change of basis matrices to change from the standard basis in $\mathbb{R}^3$ to that particular basis, and vice versa. Use your change of basis matrix to write $(2, -7, 9)$ as a linear combination of the basis vectors.