PROBLEM SET 15 SOLUTIONS

by Michael Allen

(1) Which of the following sets of vectors span $\mathbb{R}^3$?
   (a) $(1, 2, 0$ and $(0, -1, 1)$.
   No. Two vectors cannot span $\mathbb{R}^3$.
   (b) $(1, 1, 0)$, $(0, 1, -2)$, and $(1, 3, 1)$.
   Yes. The three vectors are linearly independent, so they span $\mathbb{R}^3$.
   (c) $(-1, 2, 3)$, $(2, 1, -1)$, and $(4, 7, 3)$.
   No. Only two of these vectors are linearly independent, and cannot span $\mathbb{R}^3$.
   (d) $(1, 0, 2)$, $(0, 1, 0)$, $(-1, 3, 0)$, and $(1, -4, 1)$.
   Yes. Three of these vectors are linearly independent, so they span $\mathbb{R}^3$.

(2) Which of the following sets of vectors span $P_3 = \{at^3 + bt^2 + ct + d\}$?
   (a) $t + 1$, $t^2 - t$, and t^3.
   No. The space is 4-D, but there are only three vectors.
   (b) $t^3 + t$ and $t^2 + 1$.
   No. Again, there are not enough vectors to span a 4-D space.
   (c) $t^2 + t + 1$, $t + 1$, 1, and t^3.
   Yes. These vectors are linearly independent and span P_3.
   (d) $t^3 + t^2$, $t^2 - t$, $2t + 4$, and $t^3 + 2t^2 + t + 4$.
   No. The fourth vector is the sum of the first three, so they cannot span P_3.

(3) Are the following sets of vectors linearly dependent or independent? If they are dependent, write one as a linear combination of the others.
   (a) $(1, 2, 0)$ and $(0, -1, 1)$ in $\mathbb{R}^3$.
   Independent.
   (b) $(-1, 2, 3)$, $(2, 1, -1)$, and $(4, 7, 3)$ in $\mathbb{R}^3$.
   Dependent. $2(-1, 2, 3) + 3(2, 1, -1) = (473)$
   (c) $(1, 2)$, $(2, 3)$, and $(8, -2)$ in $\mathbb{R}^2$.
   Dependent. $18(2, 3) - 28(1, 2) = (8, -2)$
   (d) $t^2 + 2t + 1$, $t^3 - t^2$, $t^3 + 1$, and $t^3 + t + 1$ in P_3.
   Dependent. $\frac{1}{2}(t^2 + 2t + 1) + \frac{1}{2}(t^3 - t^2) + \frac{1}{2}(t^3 + 1) = t^3 + t + 1$

(4) What is the dimension of the following spaces?
   (a) The set of 2×2 symmetric matrices, $A = A^T$.

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix},$$

   Three. Since the upper-left and lower-right corners must be equal, there are only three variables which can be changed.
   (b) The set of 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with $a + d = 0$.
Three. We can rewrite the constraint as

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix},$$

which makes it clear that the space is three dimensional.

(c) The set $\{(x, y, x - 3y, 2y - x) \mid x, y \in \mathbb{R}\}$ inside of $\mathbb{R}^4$.
Two. Since only x and y are free, this space is two dimensional.

(5) What is the column space and row space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 & -2 \\ 2 & -1 & 3 & -4 \\ -1 & 4 & 2 & 2 \end{bmatrix}?$$

If we use Gaussian Elimination on the matrix, we find that there are only two pivots, so we know both the column and row spaces are two dimensional. So, let's just take the first two columns and the first two rows (as long as they are linearly independent of each other).

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \quad x_1 \begin{bmatrix} 1 & 3 & 5 & -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 & -1 & 3 & -4 \end{bmatrix}$$

(6) Find an (infinite) basis for the space of all polynomials

$$\mathcal{P} = \{a_n x^n + z_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid \text{ for all } n\}.$$

$$\{a_n x^n | n = 0, 1, 2...\}$$

(7) Suppose $\{v_1, \ldots, v_n\}$ spans a vector space V, and suppose that v_n is a linear combination of v_1 through v_{n-1}. Then show that $\{v_1, \ldots, v_{n-1}\}$ spans V as well.
Since we can add to our span any vector which is a linear combination of the vectors already in the span, if follows easily that the two sets of vectors are equivalent, and $\{v_1, \ldots, v_{n-1}\}$ must span V.

(8) If A is a 4×6 matrix, show that the columns of A are linearly dependent.
Since there are only 4 rows, the matrix can at most be of rank 4. There are 6 columns in the matrix and only 4 of them can be linearly independent, at least two columns must be linear combinations of the others.

(9) Compute

$$\begin{bmatrix} .1 & .95 \\ .9 & .05 \end{bmatrix}^n,$$

for $n = 3$, 5 and 100 using methods from recitation.
We begin this problem by finding the eigenvalues and eigenvectors of the array from which will will construct our answer. These turn out to be:

$$\lambda_1 = 1 \qquad \lambda_2 = -0.85$$
$$v_1 = \begin{bmatrix} 19 \\ 18 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We decompose the columns of the matrix into eigenvectors, do a lot of algebra, and mathemagically, we arrive at the following answers:

$$A^3 = \begin{bmatrix} 0.215 & 0.829 \\ 0.785 & 0.171 \end{bmatrix} \quad A^5 = \begin{bmatrix} 0.298 & 0.741 \\ 0.702 & 0.259 \end{bmatrix} \quad A^{100} = \begin{bmatrix} 0.514 & 0.514 \\ 0.486 & 0.486 \end{bmatrix}$$