PROBLEM SET 12 SOLUTIONS.

(1) Factor the following matrices into the form $A = L \cdot U$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 4 \\ 3 & 13 & 7 \end{bmatrix}$$

ANSWER:

(a)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ & 2 & 2 \\ & & 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 4 \\ 3 & 13 & 7 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ & 2 & 0 \\ & & 1 \end{bmatrix}$$

(2) Factor the following symmetric matrices into $A = L \cdot D \cdot L^T$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

ANSWER:

(a)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & & 1 & 3 \\ & & & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

(3) Given a permutation matrix

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

what is P^{-1}? Do you recognize this matrix?

ANSWER:

$$P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is P^T, the transpose of P

(4) There are only finitely many ($n!$) $n \times n$ permutation matrices. Use this fact to show that $P^r = I$ for some r.

ANSWER: We use the fact that any power of a permutation matrix is a permutation matrix. Therefore since there are only finitely many permutations, the sequence $P, P^2, P^3, P^4, P^5, \ldots$ must repeat at some point. But then $P^m = P^n$ for some choice of m, n with $m > n$ and so $P^{m-n} = I$.

(5) For what value(s) of c can you *not* factorize $A = L \cdot U$?

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & c & 7 \\ 0 & 1 & 3 \end{bmatrix}$$

ANSWER: Subtracting twice the first row from the second gives us

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & c-4 & -1 \\ 0 & 1 & 3 \end{bmatrix}.$$

If $c = 4$ then we cannot transform this matrix to an upper triangular one without first permuting the last two rows. So there is no LU decomposition only if $c = 4$.

(6) Compute the determinants of the following matrices.

$$\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & -2 \end{bmatrix}$$

ANSWER:

(a)

$$\begin{vmatrix} 1 & 2 \\ -4 & 3 \end{vmatrix} = 1 \cdot 3 - (-4) \cdot 2 = 11$$

(b)

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix} + 0 = -8$$

(c) The fastest thing to do here is to expand along the second row (or column) because it only has one nonzero entry. Then, lo and behold, we get the determinant from the previous problem

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & -2 \end{vmatrix} = 0 + 1 \cdot \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} + 0 + 0 = -8.$$

(7) Prove that $\det(A^{-1}) = \frac{1}{\det(A)}$.

ANSWER: This follows by taking the determinant of the equation

$$I = AA^{-1}$$

Then

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

So in fact

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Note that we are assuming that A is invertible so $\det(A)$ is nonzero and it makes sense to divide by it.

(8) Let

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 3 & \cdots & 3 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}.$$

Compute $\det(A)$.

ANSWER: The easiest way to compute the determinant is to expand along the first column. Then

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 3 & \cdots & 3 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & n \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 2 & \cdots & 2 \\ 0 & 3 & \cdots & 3 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & n \end{vmatrix} + 0 + 0 + \cdots + 0.$$

Iterating this computation we get that the determinant is

$$1 \cdot 2 \cdots (n-2) \cdot \begin{vmatrix} (n-1) & (n-1) \\ 0 & n \end{vmatrix} = n!$$

(9) If you have a block matrix

$$A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix},$$

what is $\det(A)$?

ANSWER: The determinant of A is equal to $\det(B)\det(C)$. the easiest way to prove this is to observe that

$$\begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix} = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}$$

The reason for doing this is that it is fairly straightforward using the technique of problem 8 (of repeatedly expanding along the a column) to show that

$$\det \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} = \det(B)$$

and

$$\det \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix} = \det(C)$$

Then it's immediate that

$$\det(A) = \det \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \det \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix} = \det(B)\det(C)$$